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Spin and center of mass in axially symmetric Einstein–Maxwell spacetimes

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Abstract

We give a definition and derive the equations of motion for the center of mass and angular momentum of an axially symmetric, isolated system that emits gravitational and electromagnetic radiation. A central feature of this formulation is the use of Newman–Unti cuts at null infinity that are generated by worldlines of the spacetime. We analyze some consequences of the results and comment on the generalization of this work to general asymptotically flat spacetimes.

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1. Introduction

The notion of center of mass for an isolated system is very important in Newtonian theory. It is used to define the linear and intrinsic angular momentum of the system, both conserved observables in the theory. However, its generalization to general relativity (GR) has proved to be a non-trivial task.

A major obstacle to a relativistic definition is the fact that energy or momentum cannot be local quantities. The energy–momentum tensor of a system does not take into account the fact that gravitational waves carry away energy and momentum and nevertheless are solutions of the Ricci flat equations. Thus, one must search for global definitions of these quantities. Also one must bear in mind that, unless the spacetime is stationary, energy or momentum of an isolated system are not conserved in GR due to the emission of gravitational radiation.

Fortunately there are ways to overcome, at least in principle, these difficulties. One has available in the literature the notion of an asymptotically flat spacetime, the appropriate framework to analyze isolated systems in GR. For those spacetimes, one can define the notion of Bondi mass and linear momentum and write down equations of motion linking their time evolution with the emitted gravitational radiation. All that is needed then is to relate these variables to a suitable definition of center of mass, characterized by a worldline R^a , such that $P^a = MR^a +$ radiative corrections. The problem is how to select this worldline.

There are two approaches in the literature that select the center of mass worldline in a fiducial space using the available structure at null infinity.

One approach that has been carried out by Newman and collaborators is based on the introduction of asymptotically shear free null congruences, i.e. congruences such that at \mathcal{I} , we have vanishing shear. At null infinity, this congruence appears to come from a point in the Minkowski space. This asymptotically shear free condition yields a family of complex two-surfaces, ‘good cuts’, that are constructed from special solutions of the Good Cut equation. This family is characterized by complex worldlines in a fiducial holographic space. Furthermore, the vanishing of the complex mass dipole term at \mathcal{I} singles out a particular worldline of this family. Its real part gives the center of mass, while the imaginary part is, by definition, the intrinsic angular momentum per unit mass. Assuming a quadrupole radiation and using any available definition of total angular momentum (all of them coincide for quadrupole radiation), one obtains equations of motion coupling center of mass, intrinsic angular momentum and radiation. (A complete description is available in *Living Reviews* [1]).

The other approach has been done by Moreschi [2]. In this case, one first defines the notion of ‘supermomentum’ at \mathcal{I} and then asks for a special family of cuts, called nice cuts, where the supermomentum only has $l = 0, 1$ spherical harmonics decomposition. Again one obtains another holographic solution space and one special worldline in this space is selected, via a similar condition as above, as the center of mass. Furthermore, Moreschi defines the notion of total angular momentum [3] and its restriction to the center of mass worldline yields the intrinsic angular momentum of the system.

Both formulations coincide at a linear level if the gravitational radiation is pure quadrupole. In spite of the clever ideas used in both approaches to define a global notion of center of mass, there are some drawbacks that one must mention.

It is not true that light coming from distant isolated systems is asymptotically shear free. In fact, the shear of light coming from these sources is used to define weak lensing effects in GR. Most important, neither the nice cuts nor the good cuts are future light cones from points inside the spacetime. One can write down the equation that must be satisfied by any light cone cut at \mathcal{I} coming from a worldline in the spacetime [4]. None of these approaches satisfy this equation up to second and higher orders.

The idea of this work is to define the center of mass as a special worldline on the spacetime. One knows that a light cone cut of null infinity, the intersection of the future light cone from a point with null infinity, can be used to describe any point or worldline of the spacetime [5]. Thus, worldlines of the spacetime yield light cone cuts that are locally Newman–Unti cuts [6]. The vanishing of the mass dipole moment on these cuts then selects the special center of mass worldline.

The intrinsic angular momentum or spin is then defined as the restriction of the angular momentum to the center of mass. The only caveat is that so far, a satisfactory definition of angular momentum for non-stationary spacetimes without symmetries is not available. To overcome this difficulty, we will consider here axially symmetric spacetimes since in this case there is a suitable definition of angular momentum using a Komar integral that is conserved. It is worth mentioning that neither the Adamo–Kozameh–Newman (AKN) nor the Moreschi formulation of angular momentum yield the Komar formula when restricted to axially symmetric spacetimes. Thus, our work is clearly different from previous approaches.

This work is divided in seven sections and two appendices. Section 2 is devoted to the mathematical tools and definitions needed for this work. Readers familiar with the Newman–Penrose formulation and asymptotic flatness may skip this section. The light cone cuts of null infinity are introduced in sections 3. In particular, we show how either the good cut sections or nice sections used by AKN or Moreschi do not come from points of the spacetime. In

section 4, we introduce the notion of angular momentum for axially symmetric Einstein–Maxwell spacetimes. Sections 5 and 6 constitute the core of the paper. We define the center of mass and intrinsic angular momentum and relate them with others defined at null infinity like Bondi mass or momentum. We derive their equations of motion, give some examples and compare our results with those obtained by AKN. Finally, in the conclusions we summarize our work and outline a generalization of this approach to arbitrary spacetimes.

2. Foundations

There are many results that are needed for this work. In this section, we introduce several of the key ideas and the basic tools that are indispensable for our later discussions. Derivations of these results are given in the references.

2.1. Asymptotically flat spacetime and \mathcal{S}^+

The notion of asymptotically flatness is the adequate tool to analyze the gravitational and electromagnetic radiation coming from an arbitrary compact source. A spacetime is called asymptotically flat if the curvature tensor vanishes as it approaches infinity along the future-directed null geodesics of the spacetime. All these geodesics end up at what is referred to as future null infinity, \mathcal{S}^+ , the future boundary of the spacetime [1, 7]. One can introduce a natural set of coordinates in the neighborhood of \mathcal{S}^+ called Bondi coordinates $(u_B, r, \zeta, \bar{\zeta})$. In this system, the Bondi time u_B labels a special family of null surfaces whose intersections with \mathcal{S} are two spheres, r is the affine parameter along each null geodesic of the constant u_B surface and $\zeta = e^{i\varphi} \cot \frac{\theta}{2}$ is the complex stereographic angle that labels the null geodesics of the null surface.

Associated with the Bondi coordinates, there is a null tetrad system denoted by $(l_a, n_a, m_a, \bar{m}_a)$. The first tetrad vector l_a is defined as [7]

$$l_a = \nabla_a u_B. \quad (2.1)$$

Thus, $l^a = g^{ab} \nabla_b u_B$ is a null vector tangent to the geodesics of the surface. For the second tetrad vector, we pick a null vector n^a normalized to

$$n_a l^a = 1. \quad (2.2)$$

The tetrad is finally completed with the choice of a complex null vector m^a orthogonal to l^a and n^a

$$m_a \bar{m}^a = -1. \quad (2.3)$$

The spacetime metric is then written as [7]

$$g_{ab} = l_a n_b + n_a l_b - m_a \bar{m}_b - \bar{m}_a m_b. \quad (2.4)$$

There is a great deal of tetrad freedom, but the most important to us is a different choice of the original $u_B = \text{const.}$ cuts of \mathcal{S}^+ so that

$$u_B = Z(u, \zeta, \bar{\zeta}), \quad (2.5)$$

where $Z(u, \zeta, \bar{\zeta})$ is a real function. Let us denote by T the inverse function Z , so

$$u = T(u_B, \zeta, \bar{\zeta}). \quad (2.6)$$

It is easy to show that $\dot{T} = \frac{1}{Z}$, then the rest of the coordinate system and the tetrad system are then constructed as before.

Another important notion is that of spin weight [7]. A quantity η that transforms as $\eta \rightarrow e^{is\lambda}\eta$ under a rotation $m^a \rightarrow e^{i\lambda}m^a$ is said to have a spin weight s . One can also define spin weighted differential operators $\bar{\partial}$ and $\bar{\bar{\partial}}$ by

$$\bar{\partial}f = P^{1-s} \frac{\partial(P^s f)}{\partial\zeta} \quad (2.7)$$

$$\bar{\bar{\partial}}f = P^{1+s} \frac{\partial(P^{-s} f)}{\partial\bar{\zeta}}, \quad (2.8)$$

where f has a spin weight s and P is the conformal factor defining a metric on the sphere

$$ds^2 = \frac{4d\zeta d\bar{\zeta}}{P^2}. \quad (2.9)$$

The operators $\bar{\partial}$ and $\bar{\bar{\partial}}$ raise and lower the spin weight by one respectively. Note that for axial symmetry, $\bar{\partial}$ and $\bar{\bar{\partial}}$ act as derivatives on θ since all functions will not depend on ϕ . In equations (2.7) and (2.8), the conformal factor P is arbitrary. However, in Bondi coordinates, the conformal factor is restricted to

$$P = P_0 = 1 + \zeta\bar{\zeta}. \quad (2.10)$$

2.2. The Newman–Penrose formalism

Although the Newman–Penrose (NP) formalism is the basic working tool for our analysis, we will simply give an outline of the formulation and leave the reference [7] for details. We focus in the general form of the asymptotically flat solutions of Einstein–Maxwell equations in Bondi coordinates.

Using the tetrad components [7, 8]

$$\lambda^a{}_c = (l^a, n^a, m^a, \bar{m}^a); \quad c = 1, 2, 3, 4 \quad (2.11)$$

as the basic variables, the metric, equation (2.4) can be written as

$$g^{ab} = \eta^{cd} \lambda^a{}_c \lambda^b{}_d \quad (2.12)$$

with

$$\eta^{cd} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \quad (2.13)$$

The Ricci rotation coefficients $\gamma^c{}_{df}$ are defined by [7, 8]

$$\gamma^c{}_{df} = \lambda^a{}_d \lambda^b{}_f \nabla_a \lambda^c{}_b. \quad (2.14)$$

The 12 spin coefficients are defined as combinations of the $\gamma^c{}_{df}$:

$$\begin{aligned} \alpha &= \frac{1}{2}(\gamma_{124} - \gamma_{344}); & \lambda &= -\gamma_{244}; & \kappa &= \gamma_{131} \\ \beta &= \frac{1}{2}(\gamma_{123} - \gamma_{343}); & \mu &= -\gamma_{243}; & \rho &= \gamma_{134} \\ \gamma &= \frac{1}{2}(\gamma_{122} - \gamma_{342}); & \nu &= -\gamma_{242}; & \sigma &= \gamma_{133} \\ \varepsilon &= \frac{1}{2}(\gamma_{121} - \gamma_{341}); & \pi &= -\gamma_{241}; & \tau &= \gamma_{132}. \end{aligned} \quad (2.15)$$

The third basic variable in the NP formalism is the Weyl tensor or, equivalently, the following five complex tetrad components of the Weyl tensor:

$$\begin{aligned} \psi_0 &= -C_{abcd} l^a m^b l^c m^d; & \psi_1 &= -C_{abcd} l^a n^b l^c m^d \\ \psi_2 &= -\frac{1}{2}(C_{abcd} l^a n^b l^c n^d - C_{abcd} l^a n^b m^c \bar{m}^d) \\ \psi_3 &= C_{abcd} l^a n^b n^c \bar{m}^d; & \psi_4 &= -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d. \end{aligned} \quad (2.16)$$

When an electromagnetic field is present, we include the complex tetrad components of the Maxwell field

$$\phi_0 = F_{ab}l^a m^b; \quad \phi_1 = \frac{1}{2}F_{ab}(l^a n^b + m^a \bar{m}^b); \quad \phi_2 = F_{ab}n^a \bar{m}^b, \quad (2.17)$$

into the equations [1, 7].

Using Sachs Peeling theorem [9], one can show that the Weyl and Maxwell scalars behave as

$$\begin{aligned} \psi_0 &= \psi_0^0 r^{-5} + O(r^{-6}), \\ \psi_1 &= \psi_1^0 r^{-4} + O(r^{-5}), \\ \psi_2 &= \psi_2^0 r^{-3} + O(r^{-4}), \\ \psi_3 &= \psi_3^0 r^{-2} + O(r^{-3}), \\ \psi_4 &= \psi_4^0 r^{-1} + O(r^{-2}), \\ \phi_0 &= \phi_0^0 r^{-3} + O(r^{-4}), \\ \phi_1 &= \phi_1^0 r^{-2} + O(r^{-3}), \\ \phi_2 &= \phi_2^0 r^{-1} + O(r^{-2}), \end{aligned} \quad (2.18)$$

where the quantities with a zero superscript are function only of $(u_B, \zeta, \bar{\zeta})$. The spin coefficients and metric variables are given as [1, 7].

$$\begin{aligned} \kappa &= \pi = \varepsilon = 0; & \rho &= \bar{\rho}; & \tau &= \bar{\alpha} + \beta \\ \rho &= -r^{-1} - \sigma^0 \bar{\sigma}^0 r^{-3} + O(r^{-5}) \\ \sigma &= \sigma^0 r^{-2} + [(\sigma^0)^2 \bar{\sigma}^0 - \psi_0^0/2]r^{-4} + O(r^{-5}) \\ \alpha &= \alpha^0 r^{-1} + O(r^{-2}) \\ \beta &= \beta^0 r^{-1} + O(r^{-2}) \\ \gamma &= \gamma^0 - \psi_2^0 (2r^2)^{-1} + O(r^{-3}) \\ \mu &= \mu^0 r^{-1} + O(r^{-2}) \\ \lambda &= \lambda^0 r^{-1} + O(r^{-2}) \\ \nu &= \nu^0 + O(r^{-1}) \end{aligned} \quad (2.19)$$

with the following relationships among the r-independent functions:

$$\begin{aligned} \xi^{0\zeta} &= -P_0; & \bar{\xi}^{0\zeta} &= 0; & \xi^{0\bar{\zeta}} &= 0; & \bar{\xi}^{0\bar{\zeta}} &= -P_0, \\ \alpha^0 &= -\bar{\beta}^0 = -\frac{\zeta}{2}; & \gamma^0 &= \nu^0 = 0; & \omega^0 &= -\bar{\delta}\sigma^0, \\ \lambda^0 &= \dot{\sigma}^0; & \mu^0 &= U^0 = -1; & \psi_4^0 &= -\dot{\sigma}^0; & \psi_3^0 &= \bar{\delta}\dot{\sigma}^0, \\ \psi_2^0 - \bar{\psi}_2^0 &= \bar{\delta}^2\sigma^0 - \bar{\delta}^2\bar{\sigma}^0 + \bar{\sigma}^0\lambda^0 - \sigma^0\bar{\lambda}^0. \end{aligned} \quad (2.20)$$

Finally, we return to the choice of the null tetrad. If we start from the rescaled metric $g^{ab*} = Z'^2 g^{ab}$, then the tangent vector to the generators of \mathcal{S} is rescaled as $n^{a*} = Z' n^a$. Using this null vector, we can define all the other vectors of the new tetrad as

$$l_a^* = \frac{1}{Z'} \left(l_a - \frac{L}{r} \bar{m}_a - \frac{\bar{L}}{r} m_a + \frac{L\bar{L}}{r^2} n_a \right) \quad (2.21)$$

$$n_a^* = \frac{1}{Z'} n_a \quad (2.22)$$

$$m_a^* = \frac{1}{Z'} \left(m_a - \frac{L}{r} n_a \right) \quad (2.23)$$

$$\bar{m}_a^* = \frac{1}{Z'} \left(\bar{m}_a - \frac{\bar{L}}{r} n_a \right), \quad (2.24)$$

where

$$L(u_B, \zeta, \bar{\zeta}) = -\frac{\bar{\partial}_{(u_B)} T}{\dot{T}} = \bar{\partial}_{(u)} Z(u, \zeta, \bar{\zeta})|_{u=T(u_B, \zeta, \bar{\zeta})}.$$

In the above, $\bar{\partial}_{(u_B)}$ or $\bar{\partial}_{(u)}$ means to apply the eth operator keeping u_B or u constant respectively. From this tetrad, we can define the new Weyl scalar [1, 10]

$$\psi_1^{0*}(u, \zeta, \bar{\zeta}) = Z'^3 [\psi_1^0 - 3L\psi_2^0 + 3L^2\psi_3^0 - L^3\psi_4^0](u_B, \zeta, \bar{\zeta}) \quad (2.25)$$

from which we can write the following approximation:

$$\psi_1^{0*} = \psi_1^0 - 3L\psi_2^0 \quad (2.26)$$

if we keep up to linear terms in Z' and/or L .

2.3. Evolution equations and physical definitions

Using the peeling theorem, the radial part of the Einstein equations can be integrated leaving only the Bianchi identities at \mathcal{S} as the unsolved equations. Some of those equations are used to relate Weyl scalars with the free Bondi data σ^0 , i.e. [7]

$$\psi_2^0 - \bar{\psi}_2^0 = \bar{\partial}^2 \sigma^0 - \bar{\partial}^2 \bar{\sigma}^0 + \bar{\sigma}^0 \dot{\sigma}^0 - \sigma^0 \dot{\bar{\sigma}}^0, \quad (2.27)$$

$$\psi_3^0 = \bar{\partial} \dot{\bar{\sigma}}^0, \quad (2.28)$$

$$\psi_4^0 = -\dot{\bar{\sigma}}^0. \quad (2.29)$$

From equation (2.27), we can define the mass aspect [1]

$$\Psi = \psi_2^0 + \bar{\partial}^2 \bar{\sigma}^0 + \sigma^0 \dot{\bar{\sigma}}^0 \quad (2.30)$$

which satisfies the reality condition

$$\Psi = \bar{\Psi}. \quad (2.31)$$

Finally, the evolution equations (Bianchi identities) are given by [7]

$$\dot{\psi}_1^0 = -\bar{\partial}\Psi + \bar{\partial}^3 \bar{\sigma}^0 + \bar{\partial}\sigma^0 \dot{\bar{\sigma}}^0 + 3\sigma^0 \bar{\partial} \dot{\bar{\sigma}}^0 + \frac{4G}{c^4} \phi_1^0 \bar{\phi}_2^0, \quad (2.32)$$

$$\dot{\psi}_2^0 = -\bar{\partial}^2 \dot{\bar{\sigma}}^0 - \sigma^0 \dot{\bar{\sigma}}^0 + \frac{2G}{c^4} \phi_2^0 \bar{\phi}_2^0, \quad (2.33)$$

$$\dot{\phi}_1^0 = -\bar{\partial}\phi_2^0, \quad (2.34)$$

$$\dot{\phi}_0^0 = -\bar{\partial}\phi_1^0 + \sigma^0 \phi_2^0. \quad (2.35)$$

Using the mass aspect Ψ instead of ψ_2^0 in the second of the asymptotic Bianchi identities, one obtains

$$\dot{\Psi} = \dot{\sigma}^0 \dot{\bar{\sigma}}^0 + \frac{2G}{c^4} \phi_2^0 \bar{\phi}_2^0. \quad (2.36)$$

Note that in the above equation the gravitational radiation σ^0 and the electromagnetic radiation ϕ_2^0 determine the mass aspect Ψ . In addition, we can define the Bondi mass and linear momentum as [7]

$$M = -\frac{c^2}{8\pi\sqrt{2}G} \int \Psi \, d\Omega \quad (2.37)$$

$$P^i = -\frac{c^3}{8\pi\sqrt{2}G} \int \Psi l^i \, d\Omega, \quad (2.38)$$

and one can easily see that the Bondi mass decreases as a result of the emitted radiation.

3. Light cone cuts of null infinity

A very important tool in our definition of center of mass is the notion of a light cone cut of null infinity [5], which is the basic variable in the null surface formulation (NSF) of GR [4, 11]. We first give a brief review of the local and global properties of light cone cuts for a generic spacetime and then we apply an approximation valid for vacuum spacetimes in the definition of center of mass.

A light cone cut of null infinity is defined as the intersection of the future light cone from a point x^a with \mathcal{I}^+ . Using standard Bondi coordinates $(u_B, \zeta, \bar{\zeta})$, one can locally describe such intersection as

$$u_B = Z(x^a, \zeta, \bar{\zeta}). \quad (3.1)$$

Using the $\bar{\partial}$ and ∂ operators on the sphere holding constant x^a , one can show that Z satisfies

$$\bar{\partial}^2 Z = \sigma_B - \sigma_x. \quad (3.2)$$

The rhs of the above equation is the difference between the Bondi shear and the shear from the point x^a at null infinity. While σ_B is the gravitational radiation reaching \mathcal{I}^+ , σ_x is determined by solving the geometrical optics equation and functionally depends on the Weyl and Ricci tensor. Except for a special class of spacetimes and for very special worldlines, σ_x will always be non-vanishing at null infinity.

For a flat spacetime, σ_x vanishes and one can always choose a Bondi system where $\sigma_B = 0$. One thus has

$$\bar{\partial}^2 Z = 0, \quad (3.3)$$

whose regular solution is given by

$$Z_0 = x^a \ell_a. \quad (3.4)$$

with

$$\ell_a = (Y_{0,0}, Y_{1,i}).$$

3.1. Global properties

In a flat space, each cut is a global section of \mathcal{I}^+ described by a function on the sphere which is a linear combination of the $l = 0$ and $l = 1$ spherical harmonics. However, for a generic space-time, the cuts will not define global sections of \mathcal{I}^+ . It is clear that, due to the presence of Weyl curvature or matter, null cones will generically develop caustics and self-intersection. Thus, the cuts associated with these cones will have self intersections, singularities, and a global behavior that might be difficult to describe.

It turns out that both the homotopic properties of these cuts, i.e. how many times they wrap around themselves and the generic singularities of light cone cuts, i.e. singularities that cannot be removed by small perturbations are remarkably simple [12, 13]. Below we summarize the main global properties of light cone cuts.

- The light cone cuts are projections from globally smooth 2-dim Legendre submanifolds of the projective cotangent bundle of \mathcal{I}^+ . Denoting by $(\eta, \bar{\eta})$ the coordinates on the sphere of null directions above each point of the spacetime, by (y^i, p_j) with $p_j n^j = 1$, the local coordinates of the projective cotangent bundle, the Legendre submanifold is given by $(y^i(x^a, \eta, \bar{\eta}), p_j(x^a, \eta, \bar{\eta}))$. In Bondi coordinates, the projection is given by

$$\begin{aligned} u_B &= Z(x^a, \eta, \bar{\eta}) \\ \zeta &= \mathcal{Z}(x^a, \eta, \bar{\eta}). \end{aligned}$$

- They have a finite number of singularities and those singularities can be classified as either cusps or swallowtails.

It follows from the above properties that the light cone cuts are smooth maps from the sphere of null directions to \mathcal{S}^+ and local sections at null infinity. In fact, a cut can define a global section on \mathcal{S}^+ by selecting the points of the cut that cannot be joined by a timelike curve with the apex x^a . This closed 2-surface is described by a finite number of local sections. Physically, this arises from the bending of rays that get close to the sources since they arrive at a later time than those who escape away from the sources. Thus, each cut intersects itself before reaching a singularity. After the self-intersection (this is not a singularity), the points of the cut that contain the singular points lie in the future of the 2-surface that is singularity free. (see [14] for an explicit 2+1 description of an axisymmetric spacetime).

Finally, if $x^a(u)$ describes parametrically a worldline the closed 2-surfaces $u = \text{const.}$ (after we remove the points that are timelike connected with x^a) are topological spheres and any subset of these and can be used to introduce local Newman–Unti coordinates.

3.2. Comparing light cone cuts with other sections of \mathcal{S}^+

Now we would like to compare the light cone cuts at null infinity with the sections that are used in either the AKN or the Moreschi definition of center of mass and angular momentum. To do so, one must assume that the spacetime is Ricci flat in the neighborhood of \mathcal{S}^+ and that in absence of gravitational radiation the cuts are given by equation (3.4).

It has recently been shown [4] that the light cone cuts Z satisfy

$$\bar{\partial}^2 \partial^2 Z = \bar{\partial}^2 \sigma_B(Z, \zeta, \bar{\zeta}) + \bar{\partial}^2 \bar{\sigma}_B(Z, \zeta, \bar{\zeta}) + \int_{\infty}^Z \dot{\sigma}_B \dot{\bar{\sigma}}_B du|_{l \geq 2} + F[Z^2], \quad (3.5)$$

where only spherical harmonics with $l \geq 2$ are taken into account in the integral term and where $F[Z^2]$ is quadratic in Z and vanishes as the apex of the cone approaches null infinity. The first three terms on the rhs depend on the free data σ_B and it shows that the light cone cuts are determined from its knowledge. The last term appears from the back scattering and acts as a timelike source. Note that the spacetimes points have disappeared from the equation. They are recovered as the solution space, i.e. the constants of integration of the NSF equation.

One can write down a perturbative series

$$Z = Z_0 + Z_1 + Z_2 + \dots, \quad (3.6)$$

where each term in the series is determined from the previous one and the free data $\sigma_B(u_b, \zeta, \bar{\zeta})$. The first two terms satisfy

$$\begin{aligned} \bar{\partial}^2 \partial^2 Z_0 &= 0, \\ \bar{\partial}^2 \partial^2 Z_1 &= \bar{\partial}^2 \sigma_B(Z_0, \zeta, \bar{\zeta}) + \bar{\partial}^2 \bar{\sigma}_B(Z_0, \zeta, \bar{\zeta}). \end{aligned}$$

The first term is simply the flat cut, the second term is Huygens, i.e. it is determined from the data given on the flat cut. Note that the second term on the rhs shows that $\sigma_x \neq 0$ even at a linearized level.

It is important to compare equation (3.5) with either the good cut equation or the nice section equation used in the AKN definition or the Moreschi definition of center of mass respectively.

The good cut equation reads

$$\bar{\partial}^2 Z_N = \sigma_B(Z_N, \zeta, \bar{\zeta}), \quad (3.7)$$

and consequently the good cuts are shear free (see equation (3.2)). Since any cut coming from a point on the spacetime will have shear, we conclude that the solution space of the good cut equation is given by points that are not in the spacetime but rather in a different holographic space.

A similar observation can be made for the nice section equation, namely

$$\bar{\partial}^2 \partial^2 Z_M = \int_{\infty}^{Z_M} \dot{\sigma}_B \dot{\bar{\sigma}}_B du|_{l \geq 2}, \quad (3.8)$$

Note that the sections of this equation are not cuts coming from points even at the linear level. The solution space of this equation does not lie in the spacetime. Only in absence of gravitational radiation the three solution spaces coincide and it is given by Minkowski space.

Moreover, the way in which the Bondi shear $\sigma_B(u_B, \zeta, \bar{\zeta})$ enters into the equation for the cuts is crucial in determining the equation of motion for the center of mass, giving in fact three different equations. We claim our formulation gives the correct answer.

Finally, it is also important to mention that once a solution to the good cut or the nice section equation is given it is not clear by how much this section differs from a cut coming from a point of the spacetime. An alternative way of phrasing the same question is, in what sense the holographic spaces of solutions to those equations render the real spacetime they are trying to describe? If they are not the same, in which sense are the holographic spaces different from the spacetime?

The light cone cuts, on the other hand, do come from real points of the spacetime. Even if we perform a perturbation calculation to obtain the light cone cuts, it is clear what is being approximated. The price one has to pay are the caustics and self-intersections, but even those features can be dealt with following the steps outlined above.

3.3. Approximations and assumptions

We now return to the assumptions and approximations that will be used in this work.

- (1) We will assume that the zeroth-order contribution to the center of mass cut is given by (3.4), with $R^a(u)$ the worldline of the center of mass.
- (2) Since the equation of motion of the center of mass will only keep second-order terms in either the gravitational or the electromagnetic radiation, we will only need the first-order perturbation of the flat cuts which are regular solutions to (3.7).

As we will see from the equations that determine the center of mass worldline, one only needs the zeroth-order cut and the linearized Weyl scalars $\text{Re}[\psi_1^0]$ and $\text{Re}[\psi_2^0]$ (which are essentially $\Sigma m_i \mathbf{r}_i$ and Σm_i respectively) to define the center of mass at the lowest level. This calculation gives

$$M_T \mathbf{R} = \sum m_i \mathbf{r}_i,$$

a desired result.

If more perturbative terms are needed, equation (3.5) gives us in principle a method to obtain those terms. However, a number of technical difficulties can make this perturbation calculation very cumbersome, one of those being how to deal with caustics and self-intersections. Equation (3.5) is given in a caustic free neighborhood and it cannot be solved globally on the sphere. There is a method to overcome this problem, namely to write an equation on the sphere of null directions, obtain regular solutions and then find the points where the function $\zeta = \mathcal{L}(x^a, \eta, \bar{\eta})$ fails to be injective. For practical purposes, we do not address those issues in this work.

On the other hand, the zeroth-order cut is still a valid approximation if one is interested in gravitational or electromagnetic tails off a Schwarzschild background [15]. In this case, the center of mass is given by the stationary worldline $(u, 0, 0, 0)$ and the emission of radiation produces an acceleration of the center of mass that will be related to the radiation via the equations of motion described below.

4. Angular momentum

The definition of angular momentum in GR has proven to be a major problem which so far does not have a satisfactory solution. Basically the problem lies at identifying a canonical origin at null infinity. However, for vacuum axially symmetric spacetimes, one can use the Komar integral associated with the rotation Killing field $\xi_{(\varphi)}^a$ and write a conserved quantity

$$J^z = \frac{1}{16\pi} \lim_{S_t \rightarrow \infty} \oint_{S_t} \nabla^a \xi_{(\varphi)}^b dS_{ab} = \text{const.} \quad (4.1)$$

We now want to extend this definition to include the contribution of the electromagnetic radiation.

Using Stokes theorem and the fact that $\xi_{(\varphi)}^a$ is a Killing field, we have

$$\oint_{\partial\Sigma} \nabla^a \xi_{(\varphi)}^b dS_{ab} = 2 \int_{\Sigma} R_{ab} \xi_{(\varphi)}^b d\Sigma^a, \quad (4.2)$$

where $\partial\Sigma$ is the boundary of the hypersurface Σ and consists of two 2-surfaces, S_+ and S_- . Since the Killing vector is tangent to \mathcal{I} , ie $\xi_{(\varphi)}^b l_b = 0$, we can replace the Ricci tensor by the stress energy tensor T_{ab} , in the above, i.e.

$$\oint_{\partial\Sigma} \nabla^a \xi_{(\varphi)}^b dS_{ab} = 16\pi \int_{\Sigma} T_{ab} \xi_{(\varphi)}^b d\Sigma^a. \quad (4.3)$$

Inserting the stress-energy tensor of electromagnetic field $T_{ab} = \frac{1}{4\pi} (F_{ac} F_b^c - \frac{1}{4} g_{ab} F^{cd} F_{cd})$ in the rhs. of the above equation yields

$$\oint_{\partial\Sigma} \nabla^a \xi_{(\varphi)}^b dS_{ab} = 4 \int_{\Sigma} F_a^c F_{bc} \xi_{(\varphi)}^b d\Sigma^a \quad (4.4)$$

$$= 4 \int_{\Sigma} F_a^c (\nabla_b A_c - \nabla_c A_b) \xi_{(\varphi)}^b d\Sigma^a. \quad (4.5)$$

Since we can choose the Maxwell potential to have axial symmetry and the Maxwell field is pure radiation, we have in addition

$$\xi_{(\varphi)}^b \nabla_b A_c + A_b \nabla_c \xi_{(\varphi)}^b = 0$$

and

$$\nabla^c F_{ac} = 0.$$

Thus,

$$\oint_{\partial\Sigma} \nabla^a \xi_{(\varphi)}^b dS_{ab} = -4 \int_{\Sigma} \nabla_c (A_b \xi_{(\varphi)}^b F_a^c) d\Sigma^a. \quad (4.6)$$

Using Stokes theorem once again, we finally obtain

$$-4 \int_{\Sigma} \nabla_c (A_b \xi_{(\varphi)}^b F_a^c) d\Sigma^a = -2 \oint_{\partial\Sigma} A_b \xi_{(\varphi)}^b F^{ac} dS_{ac}. \quad (4.7)$$

which shows that

$$\oint_{S_+} [\nabla^a \xi_{(\varphi)}^b + 2A_c \xi_{(\varphi)}^c F^{ab}] dS_{ab} = \oint_{S_-} [\nabla^a \xi_{(\varphi)}^b + 2A_c \xi_{(\varphi)}^c F^{ab}] dS_{ab}. \quad (4.8)$$

Since the boundaries are arbitrary, the integral is constant on null infinity. We thus redefine the angular momentum equation (4.1), to include electromagnetic field, as

$$J_T^z = \frac{1}{16\pi} \lim_{S_t \rightarrow \infty} \oint_{S_t} [\nabla^a \xi_{(\varphi)}^b + 2A_c \xi_{(\varphi)}^c F^{ab}] dS_{ab}, \quad (4.9)$$

where the integral is taken over any 2-surface. The first integrand is the original gravitational term, whereas the second one is the electromagnetic contribution. Using the N-P formalism [7], one can write equation (4.9) as (see appendix A),

$$J_T^z = \frac{\sqrt{2}c^3}{8G} [\text{Im}(\psi_1^0 - \sigma^0 \bar{\delta} \bar{\sigma}^0)|_{l=1} + 2 \text{Im}(A^0 \phi_1^0)|_{l=1}], \quad (4.10)$$

where $A^0(u_B, \zeta, \bar{\zeta})$ is the Maxwell potential free data related to the electromagnetic radiation via

$$\bar{\phi}_2^0 = A^0.$$

The new conserved quantity J_T^z will be called total angular momentum for any axially symmetric Einstein–Maxwell spacetime. It is worth mentioning that neither in the AKN nor in the Moreschi approach, the conserved quantity (4.9) is used to define angular momentum. Since their definitions of angular momentum do not give conserved quantities, it is difficult to see the physical meaning of those definitions.

5. Center of mass

By assumption the spacetime is axially symmetric, we will thus assume that the center of mass is given by a worldline $R^a(u)$ along the axis of symmetry, i.e. along the z -axis. We recall that in section 2.2, we introduced a null tetrad based on outgoing null hypersurfaces $u = \text{const}$. We will then assume that this family of hypersurfaces has been generated by the future light cones of $R^a(u)$. The intersection of these light cones with \mathcal{S}^+ yield Newman–Unti coordinates $(u, \zeta, \bar{\zeta})$. The basic idea is to start with the mass dipole term at \mathcal{S} in a Bondi frame, use equation (2.26) to write down the transformation equation to a Newman–Unti frame and demand that the mass dipole term vanishes on the $u = \text{const}$. slices. (A similar idea has been used before in another approach for axially symmetric spacetimes [16] but as we will see below it yields different results.)

5.1. Analysis and definition

In a Bondi frame, the mass dipole momentum for asymptotically flat spacetime is defined to be the real part of the $l = 1$ component of $\text{Re}[\psi_1^0]$. We extend this definition to a Newman–Unti frame and define the mass dipole momentum as the $l = 1$ component of $\text{Re}[\psi_1^{0*}]$.

The basic idea to obtain the center of mass is to start by imposing the condition that on the $u = \text{const}$ cuts generated by the worldline $R^a(u)$, the mass dipole momentum vanishes. Then, using the relation (2.26) and expanding ψ_1^0 in a tensorial spherical harmonic basis as

$$\psi_1^0 = \psi_1^{0i}(u_B) Y_{1i}^1 + \psi_1^{0ij}(u_B) Y_{2ij}^1 + \dots,$$

one obtains a relationship between $\text{Re}[\psi_1^{0i}(u_B)]$ and the center of mass worldline \dot{R}^i .

Following this prescription, and using equation (2.26), on a $u = \text{const}$. slice we impose

$$\text{Re}[\psi_1^0(Z, \zeta, \bar{\zeta}) - 3\bar{\delta}(Z)\psi_2^0(Z, \zeta, \bar{\zeta})]_{u=\text{const}}^i = 0, \quad (5.1)$$

where we have replaced u_B by the function $u_B = Z(R^a(u), \zeta, \bar{\zeta})$. Furthermore, using a slow motion approximation and keeping up to first-order terms in the velocity of the center of mass, we write

$$Z(R^a(u), \zeta, \bar{\zeta}) = u + \delta u,$$

where we assume δu is small. Thus, we make a Taylor expansion of

$$\text{Re}[\psi_1^{0*}(u, \zeta, \bar{\zeta})] = \text{Re}[\psi_1^0(u + \delta u, \zeta, \bar{\zeta}) - 3\bar{\partial}\delta u\psi_2^0(u + \delta u, \zeta, \bar{\zeta})],$$

decompose each term in spherical harmonics and demand that on the $u = \text{const.}$ cut, the $l = 1$ part of this series vanishes. The Taylor expansion yields

$$\text{Re}[\psi_1^{0*}(u, \zeta, \bar{\zeta})] = \text{Re}[\psi_1^0(u, \zeta, \bar{\zeta}) + \psi_1^{0'}(u, \zeta, \bar{\zeta})\delta u - 3\bar{\partial}\delta u\psi_2^0(u, \zeta, \bar{\zeta})],$$

where we have omitted second-order terms in δu . Taking the $l = 1$ part of the above expression and putting it equal to zero yields the following expression:

$$\text{Re}[\psi_1^0(u)]^i = \text{Re}[(\bar{\partial}\Psi - \bar{\partial}^3\bar{\sigma}^0)\delta u]^i + 3\text{Re}[\bar{\partial}\delta u(\Psi - \bar{\partial}^2\bar{\sigma}^0)]^i, \quad (5.2)$$

i.e. the real, $l = 1$ part of ψ_1^0 can be written in terms of the center of mass and other Weyl scalars at null infinity.

Inserting the following tensorial spin-s harmonics expansion [19]:

$$Z = u + \delta u$$

$$\delta u = -\frac{1}{2}R^i(u)Y_{1i}^0(\zeta) + x^{ij}(u)Y_{2ij}^0(\zeta) + x^{ijk}(u)Y_{3ijk}^0(\zeta)$$

$$\bar{\partial}\delta u = R^i(u)Y_{1i}^1(\zeta) - 6x^{ij}(u)Y_{2ij}^1(\zeta) - 12x^{ijk}(u)Y_{3ijk}^1(\zeta)$$

$$\sigma_B = \sigma^{ij}(u_B)Y_{2ij}^2(\zeta) + \sigma^{ijk}(u_B)Y_{3ijk}^2(\zeta)$$

$$\psi_1^0 = \psi_1^{0i}(u_B)Y_{1i}^1(\zeta) + \psi_1^{0ij}(u_B)Y_{2ij}^1(\zeta) + \psi_1^{0ijk}(u_B)Y_{3ijk}^0(\zeta)$$

$$\Psi = -\frac{2\sqrt{2}G}{c^2}M(u_B) - \frac{6G}{c^3}P^i(u_B)Y_{1i}^0(\zeta) + \Psi^{ij}(u_B)Y_{2ij}^0(\zeta) + \Psi^{ijk}(u_B)Y_{3ijk}^0(\zeta)$$

$$\phi_0^0 = \phi_0^{0i}(u_B)Y_{1i}^1(\zeta) + \phi_0^{0ij}(u_B)Y_{2ij}^1(\zeta)$$

$$\phi_1^0 = Q(u_B) + \phi_1^{0i}(u_B)Y_{1i}^0(\zeta) + \phi_1^{0ij}(u_B)Y_{2ij}^0(\zeta)$$

$$\phi_2^0 = \phi_2^{0i}(u_B)Y_{1i}^{-1}(\zeta) + \phi_2^{0ij}(u_B)Y_{2ij}^{-1}(\zeta)$$

in equation (5.2) yields

$$D^i = MR^i - \frac{96}{5\sqrt{2}c}x^{ij}P^j + \frac{c^2}{7\sqrt{2}G}[72x^{jk}(\sigma_R^{ijk} - \Psi^{ijk}) + 360x^{ijk}(\Psi^{jk} - \sigma_R^{jk})] \quad (5.3)$$

where we have defined

$$D^i(u) \equiv -\frac{c^2}{6\sqrt{2}G}\text{Re}[\psi_1^0(u)]^i.$$

(Since we assume axial symmetry, $R^i(u)$ only has a z component. Likewise, all the high order tensors are symmetric, diagonal and trace-free.)

We now take a small digression to concentrate on the light cone cut function $Z = u + \delta u$ defined as the intersection of the future lightcone from a worldline $x^a(u)$ with \mathcal{I}^+ . The function Z dynamically depends on the matter and radiation content of the spacetime via the solution of the Einstein equations, i.e. the light cone cut function is dynamical variable and we do not make any *a priori* assumption about its behavior. Z satisfies the equation

$$\bar{\partial}^2 Z = \Lambda(Z, \bar{\partial}Z, \bar{\partial}\bar{\partial}Z, \zeta, \bar{\zeta}),$$

and Λ satisfies the Einstein's equations (it vanishes for a flat spacetime). The freedom in the solution is given by a combination of $l = 0, 1$ spherical harmonics since they are annihilated by the $\bar{\partial}^2$ operator.

One can write this freedom as

$$Z_0 = t(u) + x^i(u)Y_{1i}^0(\zeta, \bar{\zeta}) = u + x^i(u)Y_{1i}^0,$$

where in the last equality we have thrown away quadratic and higher order terms in $v^i(u)$. The function Λ only contains $l = 2$ and higher spherical harmonics decomposition which are completely determined from the Einstein's equations. For example, if we assume a vacuum spacetime in the neighborhood of null infinity, the linearized equation for Λ is given by

$$\bar{\partial}^2 \Lambda = \bar{\partial}^2 \bar{\sigma}_B(Z_0, \zeta, \bar{\zeta}) + \bar{\partial}^2 \sigma_B(Z_0, \zeta, \bar{\zeta}). \tag{5.4}$$

It follows from the above equation that given any point x^a of the spacetime, the $l = 2$ and higher terms of Z are completely determined from $\sigma_B(u, \zeta, \bar{\zeta})$. For example, up to linear order terms in σ_B and/or x^i we have

$$x^{ij} = \frac{1}{12} \sigma_R^{ij} \tag{5.5}$$

$$x^{ijk} = \frac{1}{60} \sigma_R^{ijk}, \tag{5.6}$$

where the subscript R means the real part of the complex quantities. (If we keep bilinear terms of σ_B and x^i in 5.4, then the above terms also depend on $x^i(u)$.) Inserting x^{ij} and x^{ijk} in equation (5.2) gives an explicit relationship between $x^i(u)$ and $\text{Re}[\psi_1^0(u)]^i$. For the particular assumption given in (5.5) and (5.6), we obtain

$$D^i = MR^i - \frac{8}{5\sqrt{2}c} \sigma_R^{ij} P^j - \frac{6c^2}{7\sqrt{2}G} [\sigma_R^{ijk} \Psi^{jk} - \sigma_R^{jk} \Psi^{ijk}]. \tag{5.7}$$

It follows from the Bianchi identities that the time evolution of Ψ is quadratic in $\dot{\sigma}_B$. Assuming σ_B, Ψ^{ij} and Ψ^{ijk} vanish at $u_B = -\infty$ and keeping up to second-order terms in σ_B or R^i , we obtain the following expression:

$$D^i + \frac{8}{5\sqrt{2}c} \sigma_R^{ij} P^j = MR^i. \tag{5.8}$$

We have thus obtained an explicit relationship between the center of mass R^i and the Weyl scalars defined at null infinity. Even if we assume a more involved field equation for Z giving a functional dependence of x^{ij} and x^{ijk} on R^i as well as on σ_B , equation (5.2) will give an algebraic expression relating R^i with the Weyl scalars at null infinity from which one can solve for the center of mass worldline.

Note also that if the radiation is not present in (5.8) and we compute the linearized version of D^i arising from several sources with masses m_A and positions X_A^i , we obtain the expression

$$\sum_A m_A X_A^i = MR^i,$$

a familiar expression for the center of mass definition.

To obtain a relation between the velocity of the center of mass and the Bondi linear momentum, we take a time derivative of (5.8). Using again the Bianchi identities yields the following expression

$$P^i + \frac{4}{5c} \dot{\sigma}_R^{ij} P^j - \frac{3c^2}{14G} (\sigma^{ijk} \dot{\sigma}^{jk} - \sigma^{jk} \dot{\sigma}^{ijk})_R - \frac{1}{3c} (\phi_1^0 \bar{\phi}_2^0)_R^i = \frac{M}{\sqrt{2}} V^i. \tag{5.9}$$

(In the above, we have omitted the terms \dot{M} , \dot{P}^i , $\dot{\Psi}^{ij}$ and $\dot{\Psi}^{ijk}$ as they are quadratic in σ_B and ϕ_2^0 .) Since all the vector quantities are aligned with the symmetry axis and the tensor variables are symmetric and trace free, we can obtain a simplified form of the equations as

$$D^z + \frac{8}{5\sqrt{2}c}\sigma_R^{zz}P^z = MR^z \quad (5.10)$$

$$P^z + \frac{4}{5c}\dot{\sigma}_R^{zz}P^z - \frac{3c^2}{14G}(\sigma^{zjk}\dot{\sigma}^{jk} - \sigma^{jk}\dot{\sigma}^{zjk})_R - \frac{1}{3c}(\phi_1^0\bar{\phi}_2^0)_R^z = \frac{M}{\sqrt{2}}V^z. \quad (5.11)$$

These equations provide explicit relations between D^z , P^z and R^z , V^z . It is important to note that all derivatives are taken with respect to the Bondi time u_B . Defining the time coordinate as $t = \sqrt{2}u_B$, it is possible to rewrite all the equations in term of t using $\dot{W} = \sqrt{2}\partial_t W = \sqrt{2}\dot{W}$. In this way, the equation for the linear momentum take the well known form $P^z = M\dot{R}^z + \dots = Mv^z + \dots$

The equation for the linear momentum can also be written as a sum of different parts as

$$P^z = P_M^z + P_G^z + P_{EM}^z, \quad (5.12)$$

with

$$P_M^z = \left(1 - \frac{4}{5c}\dot{\sigma}_R^{zz}\right) \frac{M}{\sqrt{2}}V^z, \quad (5.13)$$

$$P_G^z = \frac{3c^2}{14G}(\sigma^{zjk}\dot{\sigma}^{jk} - \sigma^{jk}\dot{\sigma}^{zjk})_R, \quad (5.14)$$

$$P_{EM}^z = \frac{1}{3c}(\phi_1^0\bar{\phi}_2^0)_R^z, \quad (5.15)$$

emphasizing the role of each contribution to the total linear momentum. The leading term in (5.15) is proportional to the charge times the time derivative of the electric dipole contribution. If one further assumes that the dipole contribution is due to a charged particle with worldline R^i , one recovers a known result, the Abraham–Lorentz momentum [18]. However, we are here concerned with astrophysical compact objects and this term usually vanishes.

5.2. Equation of motion

Taking a time derivative of (5.11) and inserting the Bianchi identity

$$\dot{P}^z = -\frac{6c^2}{14G}(\dot{\sigma}^{zjk}\dot{\sigma}^{jk})_R + \frac{1}{3}(\dot{\phi}_2^0\bar{\phi}_2^0)_R^z \quad (5.16)$$

gives the equation of motion for the center of mass,

$$M\left(\dot{V}^z - \frac{4}{5c}\ddot{\sigma}_R^{zz}V^z\right) = -\frac{\sqrt{2}c^2}{14G}[3(\sigma^{zjk}\dot{\sigma}^{jk} - \sigma^{jk}\dot{\sigma}^{zjk})_R + 2(\dot{\sigma}^{zjk}\dot{\sigma}^{jk})_R] - \frac{\sqrt{2}}{3}[(\phi_2^0\bar{\phi}_2^0)_R^z + c^{-1}(\phi_1^0\bar{\phi}_2^0)_R^z]. \quad (5.17)$$

For completeness, we also give the mass loss equation

$$\dot{M} = -\frac{c}{2\sqrt{2}G}\left[\frac{1}{5}\dot{\sigma}^{ij}\dot{\sigma}^{ij} + \frac{6}{7}\dot{\sigma}^{ijk}\dot{\sigma}^{ijk}\right] - \frac{\sqrt{2}}{6c}\bar{\phi}_2^{0i}\phi_2^{0i}. \quad (5.18)$$

From the rhs of (5.17), we define the notion of gravitational and electromagnetic forces, i.e.

$$F_G \equiv -\frac{\sqrt{2}c^2}{14G}[3(\sigma^{zjk}\dot{\sigma}^{jk} - \sigma^{jk}\dot{\sigma}^{zjk})_R + 2(\dot{\sigma}^{zjk}\dot{\sigma}^{jk})_R] \quad (5.19)$$

$$F_{EM} \equiv -\frac{\sqrt{2}}{3} [(\phi_2^0 \bar{\phi}_2^0)^z + c^{-1} (\phi_1^0 \bar{\phi}_2^0)^z_R]. \quad (5.20)$$

Note that the gravitational force vanishes when either the quadrupole or octupole moment vanish. Since the electromagnetic force is negligible for most situations, the center of mass will have no acceleration when one of these two moments vanish. On the other hand, since the mass loss equation has a separate contribution from the quadrupole and octupole moments, the net effect in this situation will be a reduction of the gravitational mass of the system while the center of mass remains at rest in a suitable Bondi frame. However, most head-on collisions between compact objects will produce quadrupole and octupole radiation terms and there will be a net acceleration of the center of mass.

It also follows from equation (5.17) that there are no runaway solutions. The functions $\ddot{\sigma}_R^{zz}$, F_G and F_{EM} decrease to zero as $u \rightarrow \infty$. Asymptotically, this equation gives a constant velocity if the total radiation is finite. Thus, the motion of the center of mass does not have runaway behavior.

6. Applications

In this section, we will first check that the formalism developed gives the correct answer for the cases where definitions have already been given. We will also compare our work with a similar approach to analyze similarities and differences between them. We begin by applying this formalism to the case of a stationary and axially symmetric spacetime.

6.1. Stationary and axially symmetric spacetime

We first consider the Kerr metric for which we have $\psi_4^0 = \psi_3^0 = 0$. Moreover, as the spacetime is stationary, all asymptotic scalars do not depend on u_B . In particular, $\sigma = \sigma(\zeta, \bar{\zeta})$ and we can find a reference frame where $\sigma = 0$. From equation (2.36), we obtain $P^i = \Psi^{ij} = \Psi^{ijk} = 0$ for all i, j, k and from equation (2.30) we obtain $\psi_2^0 \propto M$. Furthermore, from equation (5.4) it is easy to show that $x^{ij} = x^{ijk} = 0$ and that,

$$\psi_1^{0z} = \frac{6G}{c^2} M \left(-\sqrt{2} R^z + i \frac{4}{3\sqrt{2}c} a \right), \quad (6.1)$$

where a is the angular parameter. The real part of ψ_1^{0z} yields equation (5.8), whereas from equation (4.10) we have

$$J_T^z = aM, \quad (6.2)$$

which corresponds to the angular momentum of the Kerr spacetime. The Kerr–Newman case is very similar to Kerr. Note that in equation (4.10) $A_c m^c = 0$, so we obtain the same equations

$$\psi_1^{0z} = \frac{6G}{c^2} M \left(-\sqrt{2} R^z + i \frac{4}{3\sqrt{2}c} a \right) \quad (6.3)$$

$$P^x = P^y = P^z = 0 \quad (6.4)$$

$$J_T^z = aM. \quad (6.5)$$

Although these equations are identical, the evolution of the mass center is different for Kerr–Newman spacetime due to the presence of electromagnetic fields.

6.2. Massive explosions or head-on collisions

We consider here either a massive explosion, like type I supernova, or a massive head-on collision. We also assume that initially the center of mass is at rest. Immediately after the explosion or collision, the acceleration of the center of mass will be given by

$$M\dot{V}^z = -\frac{\sqrt{2}c^2}{14G} [3(\sigma^{zjk}\dot{\sigma}^{jk} - \sigma^{jk}\dot{\sigma}^{zjk})_R + (\dot{\sigma}^{zjk}\dot{\sigma}^{jk} + \dot{\sigma}^{jk}\dot{\sigma}^{zjk})] \\ - \frac{\sqrt{2}}{3} [(\phi_2^0\bar{\phi}_2^0)^z + c^{-1}(\phi_1^0\bar{\phi}_2^0)^z].$$

Note that if either the quadrupole or octupole term vanish there is no gravitational contribution to the acceleration. Any collision will have a quadrupole term, but only collisions between uneven masses will also have an octupole contribution. Likewise, the electromagnetic force will be dominated by the radiation term since for most astrophysical objects $\phi_1^0 = 0$.

Although total angular momentum is conserved, the coupling between gravitational and electromagnetic angular momentum gives a transfer mechanism by which the system can gain or lose intrinsic angular momentum. Consider for simplicity that initially the system does not have angular momentum. After the explosion or collision, the system will acquire an intrinsic gravitational angular momentum if electromagnetic radiation is emitted, i.e. from

$$J_T^z = J_G^z + \frac{\sqrt{2}c^3}{8G} [2\text{Im}(A^0\phi_1^0)|_{l=1}] = 0,$$

the electromagnetic angular momentum creates an intrinsic gravitational angular momentum in the opposite direction of the electromagnetic one. This effect could be important in charged isolated systems like the positron cloud discovered by the COMPTON detector in GRO, but will be negligible for most cases.

6.3. Comparison with the AKN approach

In this subsection, we will compare our results with those obtained by AKN. It is important to do so since both approaches offer explicit relationships between asymptotically defined quantities like Bondi momentum and kinematical variables like the velocity of the center of mass. Also in both approaches there are explicit formulae relating the acceleration of the center of mass and the emitted gravitational radiation. The question is whether or not both approaches yield similar results and if not which one gives physically consistent answers. Both approaches yield the same results for stationary spacetimes. Thus, we will assume that the spacetime contains gravitational radiation.

Our first observation is that the AKN definition of angular momentum for vacuum spacetimes

$$J_{\text{AKN}}^z = \frac{\sqrt{2}c^3}{12G} [\text{Im}(\psi_1^0)]|_{l=1} \quad (6.6)$$

is not conserved whereas the Komar formula (4.10) is. Only when the gravitational radiation is pure quadrupole both formulae agree. Even for this particular case, the AKN formulation does not give a simple relationship between intrinsic and total angular momentum. Defining the intrinsic angular momentum via

$$S^z \equiv \frac{\sqrt{2}}{2} Mc\xi_I^z,$$

with ξ_I^z being the imaginary part of the complex center of mass worldline, one can show that the total angular momentum in the AKN formalism is given by

$$J_{\text{AKN}}^z = S^z - \frac{3\sqrt{2}}{10} \frac{c^2}{GM} S^z \sigma_R^{zz} + \frac{1}{20c} \dot{S}^z \sigma_R^{zz} + \frac{3}{10} \frac{c^3}{G} 2R^z \sigma_I^{zz} - \frac{9}{10} M V^z \sigma_I^{zz}. \quad (6.7)$$

This relationship is highly non-trivial and unexpected. Since the orbital part of the angular momentum vanishes when the center of mass vector R^i and the velocity V^i are aligned along the z -axis, one expects that the intrinsic and total angular momentum should be equal for an axially symmetric spacetime. In addition, since $J_{\text{AKN}}^z = \text{const.}$ (when only quadrupole radiation is considered), the above relation gives an intrinsic angular momentum that is not conserved. This is not what one should expect for an axially symmetric spacetime.

In our approach, we obtain

$$J_T^z = S^z = \text{const.}, \quad (6.8)$$

a result that follows from the Komar formula if one uses a Bondi and a Newman–Unti cut as the boundary $\partial\Sigma$. Generalizing (6.7) to include octupole terms does not help either since in this case neither the total nor the intrinsic angular momentum are conserved. On the other hand, the Komar formula (6.8) is true for a general form of gravitational radiation.

One can also compare the relation between the Bondi momentum P^z and the velocity V^z in both approaches. Again we consider a vacuum spacetime in the neighborhood of null infinity and assume that the gravitational radiation only has quadrupole terms. Directly from [1], we write

$$P^z = \frac{M}{\sqrt{2}} V^z - \frac{9}{20} \frac{M}{c} (\dot{V}^z \sigma_R^{zz} + V^z \dot{\sigma}_R^{zz}) - \frac{1}{20} \frac{M}{c} (\ddot{\xi}_I^z \sigma_I^{zz} + \dot{\xi}_I^z \dot{\sigma}_I^{zz}) - \frac{6}{10\sqrt{2}} \frac{c^2}{G} (2R^z \sigma_R^{zz} + \xi_I^z \sigma_I^{zz}). \quad (6.9)$$

In this work, the equivalent equation is given by

$$P^z = \frac{M}{\sqrt{2}} V^z \left(1 - \frac{4}{5c} \dot{\sigma}_R^{zz} \right). \quad (6.10)$$

Since in both cases (and only for quadrupole radiation) one has

$$P^z = \text{const.},$$

it is interesting to consider the situation of a head-on collision of equal masses (with or without spin). In this case, one expects that $V^z = 0$, i.e. one expects that the center of mass does not move before or after the quadrupole radiation is emitted. This is the result obtained with our definition of center of mass when we use (6.10) together with $P^z = 0$. The AKN approach, on the other hand gives a non vanishing acceleration even if R^z and V^z initially vanish. At this order of approximation, the AKN approach does not give a consistent result.

The above results clearly show the consistency of our approach in contrast to the AKN approach.

It is also worth mentioning a recent result obtained as an application of the AKN formulation to type II metrics [17]. Although it suffers from the same weaknesses as the general version, i.e. fiducial world lines defined on a different manifold and a definition of angular momentum that does not yield the Komar formula when axial symmetry is imposed, the work incorporated Maxwell fields and in principle could be applied to axially symmetric spacetimes. Thus, it is worth making some comments on this work.

- Regarding the issue of angular momentum, only a linear approximation for quadrupole radiation is considered. This is the reason why it appears that angular momentum is conserved. When higher order terms are included and also higher multipole terms are taken into account, the formula for angular momentum does not give a conserved quantity.

- Only gravitational deviations from Reissner–Nordstrom are discussed and the definition of angular momentum does not include a Maxwell part. In contrast, our derivation for total angular momentum is given for any axially symmetric Einstein–Maxwell spacetime.
- Since the work applies to any type II solutions, the formula for angular momentum should have contained the orbital part, i.e. RxP. This can be obtained very easily when considering up to second-order terms and it is the reason why the orbital angular momentum does not appear in the formula. However, if one adds this term then the angular momentum is not conserved.
- Our definition of linear momentum also differs from the one in [17] as it can be seen by comparing equation (5.11) with the analogous formula in this paper.

7. Conclusions

Using the available geometric structure of asymptotically flat spacetimes together with conservation laws that arise when those spacetimes are axially symmetric, we have introduced the notion of linear and angular momentum for Einstein–Maxwell spaces.

Using the light cone equation, we have been able to identify worldlines inside the spacetime with Newman–Unti cuts at null infinity. The center of mass worldline R^a is then selected by imposing the condition that the mass dipole moment at null infinity vanishes when restricted to the center of mass NU cut. Using the available Bianchi identities at \mathcal{I}^+ , one obtains a relationship between the center of mass velocity and the Bondi momentum as well as the equation of motion of R^a .

Several nice highlights of this approach are as follows.

- A definition of angular momentum when electromagnetic fields are present.
- A definition of center of mass worldline and velocity which are algebraically related to radiation fields at null infinity.
- Definitions of gravitational and electromagnetic forces in terms of radiation fields.
- Appropriate behavior of the equations of motion (no runaway solutions).
- A natural relationship between intrinsic and total angular momentum (they are the same in this case).
- Consistent behavior when applied to particular cases of collisions with emission of quadrupole radiation.

The equations of motion could be used in astrophysical situations when the system has axial symmetry to predict the motion of the center of mass from the emitted radiation or to predict the amount of radiation if the velocity and acceleration of the center of mass is given. This could be the case in head-on collisions or supernova explosions.

The main ideas upon we have based our approach are as follows.

- (1) Center of mass should be a worldline of the spacetime.
- (2) Angular momentum should give us the Komar formula when axial symmetry is imposed.

Since neither the Adamo–Kozameh–Newman (AKN) nor the Moreschi approach satisfy these hypothesis, our formulation is clearly different from them. Nevertheless, since the AKN formulation offers explicit formulae, we have compared our results with them and shown that our formulation yields physically consistent results while the AKN does not.

The formalism is ready to be generalized for spaces without symmetries. In the generalization, we expect some new features that are absent in axially symmetric spaces. Although at the moment there is no definition of angular momentum that has been universally accepted, it appears that the so-called linkages of Geroch and Winicour fit nicely with our generalization. This will be left for future work.

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Appendix A. Komar integral and angular momentum

In vacuum spaces, the Komar integral of the Killing field $\xi_{(\varphi)}^b$ yields a definition of the z-component of the angular momentum,

$$J^z = \frac{1}{16\pi} \lim_{S_t \rightarrow \infty} \oint_{S_t} \nabla^a \xi_{(\varphi)}^b dS_{ab}. \tag{A.1}$$

One can explicitly integrate this equation in the N-P formalism to obtain a formula at \mathcal{S} in terms of the spin coefficients. We first write the Killing vector field $\xi_{(\varphi)}^b$ as a combination of the null tetrad vectors as

$$\xi_{(\varphi)}^b = \xi_l^b + \bar{\xi}_m m^b + \xi_m \bar{m}^b + \xi_n n^b, \tag{A.2}$$

where

$$\xi_l = -\frac{i \sin \theta}{\sqrt{2}} r(\omega - \bar{\omega}) \tag{A.3}$$

$$\xi_m = -\frac{i \sin \theta}{\sqrt{2}} r \tag{A.4}$$

$$\xi_n = 0, \tag{A.5}$$

where the two-dimensional surface area can also be expressed as

$$dS_{ab} = -2n_{[a}l_{b]}r^2 \sin \theta d\theta d\varphi, \tag{A.6}$$

thus the Komar integral can be written as

$$J^z = -\frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_0^\pi \int_0^{2\pi} \nabla^a \xi_{(\varphi)}^b (n_a l_b - l_a n_b) r^2 \sin \theta d\varphi d\theta. \tag{A.7}$$

Using equations (2.15), (2.19) and (2.2), (2.3) and writing this equation up to order $O(r^{-2})$ we obtain

$$\nabla^a \xi_{(\varphi)}^b (n_a l_b - l_a n_b) = -\frac{\sqrt{2} \sin \theta}{r^2} \text{Im}[\psi_1^0 - \sigma^0 \bar{\delta} \bar{\sigma}^0]. \tag{A.8}$$

Thus, the Komar integral can be written as

$$J^z = \frac{\sqrt{2}}{8} \int_{-1}^1 \text{Im}[\psi_1^0 - \sigma^0 \bar{\delta} \bar{\sigma}^0] \sin \theta d(\cos \theta), \tag{A.9}$$

where we have used the axial symmetry to integrate in the azimuth direction. Finally, this integral gives the following definition of angular momentum [20]

$$J^z \propto \text{Im}[\psi_1^0 - \sigma^0 \bar{\delta} \bar{\sigma}^0]_{|l=1}. \tag{A.10}$$

We can follow a similar calculation with equation (4.9)

$$J_T^z = \frac{1}{16\pi} \lim_{S_t \rightarrow \infty} \oint_{S_t} [\nabla^a \xi_{(\varphi)}^b + 2A_c \xi_{(\varphi)}^c F^{ab}] dS_{ab}. \tag{A.11}$$

Using the fact that [7]

$$F^{ab} = 2\phi_0 \bar{m}^{[a} n^{b]} + \phi_1 (n^{[a} l^{b]} + m^{[a} \bar{m}^{b]}) + 2\phi_2 l^{[a} m^{b]}, \tag{A.12}$$

the second integral can be put in the form

$$\frac{1}{16\pi} \lim_{S_r \rightarrow \infty} \oint_{S_r} 2A_c \xi_{(\varphi)}^c F^{ab} dS_{ab} = \frac{1}{8} \lim_{r \rightarrow \infty} \int_0^\pi 2A_c \xi_{(\varphi)}^c \phi_1^0 \sin \theta d\theta. \quad (\text{A.13})$$

Using the tetrad decomposition of the vector killing field $\xi_{(\varphi)}^b$ at \mathcal{I}^+ , and the fact that A_c and ϕ_1^0 are real, one can define the electromagnetic angular momentum J_{EM} as

$$\begin{aligned} J_{\text{EM}} &= \frac{1}{8} \lim_{r \rightarrow \infty} \int_0^\pi 2A_c \xi_{(\varphi)}^c \phi_1^0 \sin \theta d\theta \\ &= -\frac{1}{8} \lim_{r \rightarrow \infty} \int_{-1}^1 i\sqrt{2} A_c r (m^c - \bar{m}^c) \phi_1^0 \sin \theta d(\cos \theta) \\ &= \frac{2\sqrt{2}}{8} \lim_{r \rightarrow \infty} r \text{Im}(A_c m^c \phi_1^0)|_{l=1}. \end{aligned}$$

Furthermore, it can be shown that,

$$\lim_{r \rightarrow \infty} r A_c m^c = A^0,$$

where $A^0(u_B, \zeta, \bar{\zeta})$ is the free Maxwell potential data related to the electromagnetic radiation via

$$\bar{\phi}_2^0 = \dot{A}^0.$$

Thus, the total angular momentum is finally expressed as

$$J_T^z = \frac{\sqrt{2}c^3}{8G} [\text{Im}(\psi_1^0 - \sigma^0 \bar{\delta} \bar{\sigma}^0)|_{l=1} + 2 \text{Im}(A^0 \phi_1^0)|_{l=1}].$$

Appendix B. Tensorial spin-s harmonics products

We present a table of tensorial harmonics products which complete the list of product [19].

Products of the form $Y_{li}^s Y_{3jkl}^s$

$$Y_{li}^{-1} Y_{3jkl}^2 = \frac{5}{21} F_{ijkl}^{2(1)} - \frac{2}{21} G_{ijkl}^{2(1)} - \frac{1}{24} i\sqrt{2} F_{ijkl}^{3(1)} - \frac{1}{56} F_{ijkl}^{4(1)} \quad (\text{B.1})$$

$$Y_{li}^0 Y_{3jkl}^1 = \frac{20}{21} F_{ijkl}^{2(1)} - \frac{8}{21} G_{ijkl}^{2(1)} - \frac{1}{12} i\sqrt{2} F_{ijkl}^{3(1)} + \frac{5}{28} F_{ijkl}^{4(1)} \quad (\text{B.2})$$

$$Y_{li}^1 Y_{3jkl}^0 = -\frac{10}{7} F_{ijkl}^{2(1)} + \frac{4}{7} G_{ijkl}^{2(1)} + \frac{1}{2} i\sqrt{2} F_{ijkl}^{3(1)} + \frac{5}{14} F_{ijkl}^{4(1)} \quad (\text{B.3})$$

$$Y_{li}^0 Y_{3jkl}^0 = \frac{10}{7} F_{ijkl}^{2(0)} - \frac{4}{7} G_{ijkl}^{2(0)} + \frac{1}{7} F_{ijkl}^{4(0)} \quad (\text{B.4})$$

$$Y_{li}^{-1} Y_{3jkl}^1 = \frac{5}{21} F_{ijkl}^{2(0)} - \frac{2}{21} G_{ijkl}^{2(0)} + \frac{1}{24} i\sqrt{2} F_{ijkl}^{3(0)} - \frac{1}{56} F_{ijkl}^{4(0)} \quad (\text{B.5})$$

$$Y_{li}^1 Y_{3jkl}^{-1} = \frac{5}{21} F_{ijkl}^{2(0)} - \frac{2}{21} G_{ijkl}^{2(0)} - \frac{1}{24} i\sqrt{2} F_{ijkl}^{3(0)} - \frac{1}{56} F_{ijkl}^{4(0)} \quad (\text{B.6})$$

where

$$F_{ijkl}^{2(s)} = \delta_{ij} Y_{2kl}^s + \delta_{ik} Y_{2jl}^s + \delta_{il} Y_{2jk}^s$$

$$G_{ijkl}^{2(s)} = \delta_{jk} Y_{2il}^s + \delta_{kl} Y_{2ij}^s + \delta_{jl} Y_{2ik}^s$$

$$F_{ijkl}^{3(s)} = \epsilon_{ijm} Y_{3klm}^s + \epsilon_{ikm} Y_{3jlm}^s + \epsilon_{ilm} Y_{3jkm}^s$$

$$F_{ijkl}^{4(s)} = Y_{4ijkl}^s$$

with the superscript $s = 0, 1$.

Products of the form $Y_{2ij}^s Y_{3klm}^s$

$$Y_{2ij}^1 Y_{3klm}^0 = -\frac{1}{12} F_{ijklm}^{1(1)} + \frac{24}{35} G_{ijklm}^{1(1)} + \frac{24}{35} H_{ijklm}^{1(1)} - \frac{2}{7} i \sqrt{2} F_{ijklm}^{2(1)} + \frac{5}{7} i \sqrt{2} G_{ijklm}^{2(1)} \\ + \frac{2}{15} F_{ijklm}^{3(1)} + \frac{1}{15} G_{ijklm}^{3(1)} + \frac{1}{15} H_{ijklm}^{3(1)} + \frac{1}{14} i \sqrt{2} F_{ijklm}^{4(1)} + \frac{2}{21} F_{ijklm}^{5(1)} \quad (\text{B.7})$$

$$Y_{2ij}^0 Y_{3klm}^1 = \frac{1}{12} F_{ijklm}^{1(1)} - \frac{24}{35} G_{ijklm}^{1(1)} - \frac{24}{35} H_{ijklm}^{1(1)} + \frac{1}{7} i \sqrt{2} F_{ijklm}^{2(1)} - \frac{5}{14} i \sqrt{2} G_{ijklm}^{2(1)} \\ + \frac{1}{5} F_{ijklm}^{3(1)} + \frac{1}{10} G_{ijklm}^{3(1)} + \frac{1}{10} H_{ijklm}^{3(1)} - \frac{1}{28} i \sqrt{2} F_{ijklm}^{4(1)} + \frac{1}{14} F_{ijklm}^{5(1)} \quad (\text{B.8})$$

$$Y_{2ij}^{-2} Y_{3klm}^3 = \frac{1}{14} F_{ijklm}^{1(1)} - \frac{1}{35} G_{ijklm}^{1(1)} - \frac{1}{35} H_{ijklm}^{1(1)} - \frac{1}{168} i \sqrt{2} F_{ijklm}^{2(1)} \\ + \frac{5}{336} i \sqrt{2} G_{ijklm}^{2(1)} - \frac{1}{180} F_{ijklm}^{3(1)} - \frac{1}{360} G_{ijklm}^{3(1)} - \frac{1}{360} H_{ijklm}^{3(1)} \\ - \frac{1}{1680} i \sqrt{2} F_{ijklm}^{4(1)} + \frac{1}{5040} F_{ijklm}^{5(1)} \quad (\text{B.9})$$

$$Y_{2ij}^2 Y_{3klm}^{-1} = -\frac{1}{7} F_{ijklm}^{1(1)} + \frac{2}{35} G_{ijklm}^{1(1)} + \frac{2}{35} H_{ijklm}^{1(1)} - \frac{1}{28} i \sqrt{2} F_{ijklm}^{2(1)} \\ + \frac{5}{56} i \sqrt{2} G_{ijklm}^{2(1)} + \frac{1}{15} F_{ijklm}^{3(1)} + \frac{1}{30} G_{ijklm}^{3(1)} + \frac{1}{30} H_{ijklm}^{3(1)} \\ - \frac{1}{84} i \sqrt{2} F_{ijklm}^{4(1)} - \frac{1}{168} F_{ijklm}^{5(1)} \quad (\text{B.10})$$

$$Y_{2ij}^{-1} Y_{3klm}^2 = \frac{2}{7} F_{ijklm}^{1(1)} - \frac{4}{35} G_{ijklm}^{1(1)} - \frac{4}{35} H_{ijklm}^{1(1)} + \frac{1}{30} F_{ijklm}^{3(1)} + \frac{1}{60} G_{ijklm}^{3(1)} \\ + \frac{1}{60} H_{ijklm}^{3(1)} + \frac{1}{120} i \sqrt{2} F_{ijklm}^{4(1)} - \frac{1}{210} F_{ijklm}^{5(1)} \quad (\text{B.11})$$

$$Y_{2ij}^0 Y_{3klm}^0 = \frac{36}{7} F_{ijklm}^{1(0)} - \frac{72}{35} G_{ijklm}^{1(0)} - \frac{72}{35} H_{ijklm}^{1(0)} + \frac{4}{15} F_{ijklm}^{3(0)} + \frac{2}{15} G_{ijklm}^{3(0)} \\ + \frac{2}{15} H_{ijklm}^{3(0)} + \frac{1}{21} F_{ijklm}^{5(0)} \quad (\text{B.12})$$

$$Y_{2ij}^1 Y_{3klm}^{-1} = \frac{4}{7} F_{ijklm}^{1(0)} - \frac{8}{35} G_{ijklm}^{1(0)} - \frac{8}{35} H_{ijklm}^{1(0)} + \frac{1}{42} i \sqrt{2} F_{ijklm}^{2(0)} \\ - \frac{5}{84} i \sqrt{2} G_{ijklm}^{2(0)} + \frac{1}{90} F_{ijklm}^{3(0)} + \frac{1}{180} G_{ijklm}^{3(0)} + \frac{1}{180} H_{ijklm}^{3(0)} \\ - \frac{1}{168} i \sqrt{2} F_{ijklm}^{4(0)} - \frac{1}{252} F_{ijklm}^{5(0)} \quad (\text{B.13})$$

$$Y_{2ij}^{-1} Y_{3klm}^1 = \frac{4}{7} F_{ijklm}^{1(0)} - \frac{8}{35} G_{ijklm}^{1(0)} - \frac{8}{35} H_{ijklm}^{1(0)} - \frac{1}{42} i \sqrt{2} F_{ijklm}^{2(0)} \\ + \frac{5}{84} i \sqrt{2} G_{ijklm}^{2(0)} + \frac{1}{90} F_{ijklm}^{3(0)} + \frac{1}{180} G_{ijklm}^{3(0)} + \frac{1}{180} H_{ijklm}^{3(0)} \\ + \frac{1}{168} i \sqrt{2} F_{ijklm}^{4(0)} - \frac{1}{252} F_{ijklm}^{5(0)} \quad (\text{B.14})$$

$$Y_{2ij}^{-2} Y_{3klm}^2 = \frac{1}{14} F_{ijklm}^{1(0)} - \frac{1}{35} G_{ijklm}^{1(0)} - \frac{1}{35} H_{ijklm}^{1(0)} - \frac{1}{168} i \sqrt{2} F_{ijklm}^{2(0)} \\ + \frac{5}{336} i \sqrt{2} G_{ijklm}^{2(0)} - \frac{1}{180} F_{ijklm}^{3(0)} - \frac{1}{360} G_{ijklm}^{3(0)} - \frac{1}{360} H_{ijklm}^{3(0)} \\ - \frac{1}{1680} i \sqrt{2} F_{ijklm}^{4(0)} + \frac{1}{5040} F_{ijklm}^{5(0)} \quad (\text{B.15})$$

$$Y_{2ij}^2 Y_{3klm}^{-2} = \frac{1}{14} F_{ijklm}^{1(0)} - \frac{1}{35} G_{ijklm}^{1(0)} - \frac{1}{35} H_{ijklm}^{1(0)} + \frac{1}{168} i \sqrt{2} F_{ijklm}^{2(0)} \\ - \frac{5}{336} i \sqrt{2} G_{ijklm}^{2(0)} - \frac{1}{180} F_{ijklm}^{3(0)} - \frac{1}{360} G_{ijklm}^{3(0)} - \frac{1}{360} H_{ijklm}^{3(0)} \\ + \frac{1}{1680} i \sqrt{2} F_{ijklm}^{4(0)} + \frac{1}{5040} F_{ijklm}^{5(0)} \quad (\text{B.16})$$

where (with $s = 0, 1$)

$$\begin{aligned}
F_{ijklm}^{1(s)} &= (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})Y_{1m}^s + (\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})Y_{1k}^s + (\delta_{im}\delta_{jk} + \delta_{ik}\delta_{jm})Y_{1l}^s \\
G_{ijklm}^{1(s)} &= (\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{mk} + \delta_{jm}\delta_{kl})Y_{1i}^s + (\delta_{ik}\delta_{lm} + \delta_{il}\delta_{mk} + \delta_{im}\delta_{kl})Y_{1j}^s \\
H_{ijklm}^{1(s)} &= \delta_{ij}(\delta_{kl}Y_{1m}^s + \delta_{lm}Y_{1k}^s + \delta_{mk}Y_{1l}^s) \\
F_{ijklm}^{2(s)} &= (\delta_{lm}\epsilon_{ikf} + \delta_{km}\epsilon_{ilf} + \delta_{lk}\epsilon_{imf})Y_{2jf}^s + (\delta_{lm}\epsilon_{jkf} + \delta_{km}\epsilon_{jlf} + \delta_{lk}\epsilon_{imf})Y_{2if}^s \\
G_{ijklm}^{2(s)} &= \delta_{il}(\epsilon_{jmf}Y_{2kf}^s + \epsilon_{jkf}Y_{2mf}^s) + \delta_{im}(\epsilon_{jkf}Y_{2lf}^s + \epsilon_{jlf}Y_{2kf}^s) \\
&\quad + \delta_{ik}(\epsilon_{jlf}Y_{2mf}^s + \epsilon_{jmf}Y_{2lf}^s) + \delta_{jl}(\epsilon_{imf}Y_{2kf}^s + \epsilon_{ikf}Y_{2mf}^s) \\
&\quad + \delta_{jm}(\epsilon_{ikf}Y_{2lf}^s + \epsilon_{ilf}Y_{2kf}^s) + \delta_{jk}(\epsilon_{ilf}Y_{2mf}^s + \epsilon_{imf}Y_{2lf}^s) \\
F_{ijklm}^{3(s)} &= \delta_{ij}Y_{3klm}^s \\
G_{ijklm}^{3(s)} &= \delta_{jm}Y_{3ikl}^s + \delta_{jk}Y_{3ilm}^s + \delta_{jl}Y_{3ikm}^s + \delta_{im}Y_{3jkl}^s + \delta_{ik}Y_{3jlm}^s + \delta_{il}Y_{3jkm}^s \\
H_{ijklm}^{3(s)} &= (\epsilon_{ikf}\epsilon_{jln} + \epsilon_{jkf}\epsilon_{iln} + \epsilon_{ilf}\epsilon_{jkn} + \epsilon_{jlf}\epsilon_{ikn})Y_{3fnm}^s \\
&\quad + (\epsilon_{ilf}\epsilon_{jmn} + \epsilon_{jlf}\epsilon_{imn} + \epsilon_{imf}\epsilon_{jln} + \epsilon_{jmf}\epsilon_{iln})Y_{3fnk}^s \\
&\quad + (\epsilon_{imf}\epsilon_{jkn} + \epsilon_{jmf}\epsilon_{ikn} + \epsilon_{ikf}\epsilon_{jmn} + \epsilon_{kmf}\epsilon_{imn})Y_{3fnl}^s \\
F_{ijklm}^{4(s)} &= \epsilon_{ikf}Y_{4fjlm}^s + \epsilon_{ilf}Y_{4fjkm}^s + \epsilon_{imf}Y_{4fjkl}^s + \epsilon_{jkf}Y_{4film}^s + \epsilon_{jlf}Y_{4fikm}^s + \epsilon_{jmf}Y_{4fikl}^s \\
F_{ijklm}^{5(s)} &= Y_{5ijklm}^s
\end{aligned}$$

Products of the form $Y_{3ijk}^s Y_{3lmn}^s$

$$\begin{aligned}
Y_{3ijk}^0 Y_{3lmn}^1 &= \frac{6}{7}i\sqrt{2}F_{ijklmn}^{1(1)} - \frac{30}{7}i\sqrt{2}G_{ijklmn}^{1(1)} + \frac{1}{3}F_{ijklmn}^{2(1)} + \frac{1}{3}G_{ijklmn}^{2(1)} \\
&\quad - \frac{2}{21}H_{ijklmn}^{2(1)} + \frac{9}{7}K_{ijklmn}^{2(1)} - \frac{5}{18}i\sqrt{2}F_{ijklmn}^{3(1)} + \frac{2}{9}i\sqrt{2}G_{ijklmn}^{3(1)} \\
&\quad + \frac{2}{9}i\sqrt{2}H_{ijklmn}^{3(1)} + \frac{15}{154}F_{ijklmn}^{4(1)} + \frac{15}{154}G_{ijklmn}^{4(1)} + \frac{25}{154}H_{ijklmn}^{4(1)} \\
&\quad - \frac{5}{252}i\sqrt{2}F_{ijklmn}^{5(1)} + \frac{5}{132}F_{ijklmn}^{6(1)}
\end{aligned} \tag{B.17}$$

$$\begin{aligned}
Y_{3ijk}^{-2} Y_{3lmn}^3 &= \frac{1}{140}i\sqrt{2}F_{ijklmn}^{1(1)} - \frac{1}{28}i\sqrt{2}G_{ijklmn}^{1(1)} + \frac{1}{72}F_{ijklmn}^{2(1)} + \frac{1}{72}G_{ijklmn}^{2(1)} \\
&\quad - \frac{1}{252}H_{ijklmn}^{2(1)} + \frac{3}{56}K_{ijklmn}^{2(1)} + \frac{1}{216}i\sqrt{2}F_{ijklmn}^{3(1)} - \frac{1}{270}i\sqrt{2}G_{ijklmn}^{3(1)} \\
&\quad - \frac{1}{270}i\sqrt{2}H_{ijklmn}^{3(1)} - \frac{1}{616}F_{ijklmn}^{4(1)} - \frac{1}{616}G_{ijklmn}^{4(1)} - \frac{5}{1848}H_{ijklmn}^{4(1)} \\
&\quad - \frac{1}{6048}i\sqrt{2}F_{ijklmn}^{5(1)} + \frac{1}{15840}F_{ijklmn}^{6(1)}
\end{aligned} \tag{B.18}$$

$$\begin{aligned}
Y_{3ijk}^2 Y_{3lmn}^{-1} &= -\frac{1}{14}i\sqrt{2}F_{ijklmn}^{1(1)} + \frac{5}{14}i\sqrt{2}G_{ijklmn}^{1(1)} + \frac{1}{12}F_{ijklmn}^{2(1)} + \frac{1}{12}G_{ijklmn}^{2(1)} \\
&\quad - \frac{1}{42}H_{ijklmn}^{2(1)} + \frac{9}{28}K_{ijklmn}^{2(1)} + \frac{1}{77}F_{ijklmn}^{4(1)} + \frac{1}{77}G_{ijklmn}^{4(1)} \\
&\quad + \frac{5}{231}H_{ijklmn}^{4(1)} - \frac{1}{336}i\sqrt{2}F_{ijklmn}^{5(1)} - \frac{1}{528}F_{ijklmn}^{6(1)}
\end{aligned} \tag{B.19}$$

$$\begin{aligned}
Y_{3ijk}^{-1} Y_{3lmn}^2 &= \frac{1}{14}i\sqrt{2}F_{ijklmn}^{1(1)} - \frac{5}{14}i\sqrt{2}G_{ijklmn}^{1(1)} + \frac{1}{12}F_{ijklmn}^{2(1)} + \frac{1}{12}G_{ijklmn}^{2(1)} \\
&\quad - \frac{1}{42}H_{ijklmn}^{2(1)} + \frac{9}{28}K_{ijklmn}^{2(1)} + \frac{1}{77}F_{ijklmn}^{4(1)} + \frac{1}{77}G_{ijklmn}^{4(1)} \\
&\quad + \frac{5}{231}H_{ijklmn}^{4(1)} + \frac{1}{336}i\sqrt{2}F_{ijklmn}^{5(1)} - \frac{1}{528}F_{ijklmn}^{6(1)}
\end{aligned} \tag{B.20}$$

$$\begin{aligned}
Y_{3ijk}^0 Y_{3lmn}^0 &= -\frac{48}{7}F_{ijklmn}^0 + \frac{120}{7}G_{ijklmn}^0 + \frac{4}{3}F_{ijklmn}^{2(0)} + \frac{4}{3}G_{ijklmn}^{2(0)} \\
&\quad - \frac{8}{21}H_{ijklmn}^{2(0)} + \frac{36}{7}K_{ijklmn}^{2(0)} + \frac{9}{77}F_{ijklmn}^{4(0)} + \frac{9}{77}G_{ijklmn}^{4(0)} + \frac{15}{77}H_{ijklmn}^{4(0)} + \frac{5}{231}F_{ijklmn}^{6(0)}
\end{aligned} \tag{B.21}$$

$$\begin{aligned}
Y_{3ijk}^{-1} Y_{3lmn}^1 &= -\frac{4}{7} F_{ijklmn}^0 + \frac{10}{7} G_{ijklmn}^0 - \frac{1}{14} i \sqrt{2} F_{ijklmn}^{1(0)} + \frac{5}{14} i \sqrt{2} G_{ijklmn}^{1(0)} \\
&+ \frac{1}{12} F_{ijklmn}^{2(0)} + \frac{1}{12} G_{ijklmn}^{2(0)} - \frac{1}{42} H_{ijklmn}^{2(0)} + \frac{9}{28} K_{ijklmn}^{2(0)} \\
&+ \frac{5}{216} i \sqrt{2} F_{ijklmn}^{3(0)} - \frac{1}{54} i \sqrt{2} G_{ijklmn}^{3(0)} - \frac{1}{54} i \sqrt{2} H_{ijklmn}^{3(0)} \\
&+ \frac{1}{616} F_{ijklmn}^{4(0)} + \frac{1}{616} G_{ijklmn}^{4(0)} + \frac{5}{1848} H_{ijklmn}^{4(0)} \\
&- \frac{5}{3024} i \sqrt{2} F_{ijklmn}^{5(0)} - \frac{5}{3696} F_{ijklmn}^{6(0)}
\end{aligned} \tag{B.22}$$

$$\begin{aligned}
Y_{3ijk}^{-2} Y_{3lmn}^2 &= -\frac{2}{35} F_{ijklmn}^0 + \frac{1}{7} G_{ijklmn}^0 - \frac{1}{70} i \sqrt{2} F_{ijklmn}^{1(0)} + \frac{1}{14} i \sqrt{2} G_{ijklmn}^{1(0)} \\
&+ \frac{1}{432} i \sqrt{2} F_{ijklmn}^{3(0)} - \frac{1}{540} i \sqrt{2} G_{ijklmn}^{3(0)} - \frac{1}{540} i \sqrt{2} H_{ijklmn}^{3(0)} \\
&- \frac{1}{880} F_{ijklmn}^{4(0)} - \frac{1}{880} G_{ijklmn}^{4(0)} - \frac{1}{528} H_{ijklmn}^{4(0)} \\
&- \frac{1}{7560} i \sqrt{2} F_{ijklmn}^{5(0)} + \frac{1}{18480} F_{ijklmn}^{6(0)}
\end{aligned} \tag{B.23}$$

where

$$\begin{aligned}
F_{ijklmn}^0 &= \delta_{ij}(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{nl} + \delta_{kn}\delta_{lm}) + \delta_{jk}(\delta_{il}\delta_{mn} + \delta_{im}\delta_{nl} + \delta_{in}\delta_{lm}) \\
&+ \delta_{ki}(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{nl} + \delta_{jn}\delta_{lm}) \\
G_{ijklmn}^0 &= \delta_{il}(\delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km}) + \delta_{jl}(\delta_{km}\delta_{in} + \delta_{kn}\delta_{im}) + \delta_{kl}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) \\
F_{ijklmn}^{1(s)} &= (\delta_{ij}\delta_{lm}\epsilon_{knf} + \delta_{ij}\delta_{ln}\epsilon_{kmf} + \delta_{ij}\delta_{nm}\epsilon_{klf} + \delta_{jk}\delta_{lm}\epsilon_{inf} + \delta_{jk}\delta_{ln}\epsilon_{imf} \\
&+ \delta_{jk}\delta_{nm}\epsilon_{ilf} + \delta_{ki}\delta_{lm}\epsilon_{jnf} + \delta_{ki}\delta_{ln}\epsilon_{jmf} + \delta_{ki}\delta_{nm}\epsilon_{jlf}) Y_{1f}^s \\
G_{ijklmn}^{1(s)} &= (\delta_{il}\delta_{jm}\epsilon_{knf} + \delta_{im}\delta_{jn}\epsilon_{klf} + \delta_{in}\delta_{jl}\epsilon_{kmf} + \delta_{jl}\delta_{km}\epsilon_{inf} + \delta_{jm}\delta_{kn}\epsilon_{ilf} \\
&+ \delta_{jm}\delta_{kl}\epsilon_{imf} + \delta_{kl}\delta_{im}\epsilon_{jnf} + \delta_{km}\delta_{in}\epsilon_{jlf} + \delta_{kn}\delta_{il}\epsilon_{jmf}) Y_{1f}^s \\
F_{ijklmn}^{2(s)} &= (\delta_{mn}\delta_{kl} + \delta_{ml}\delta_{kn} + \delta_{ln}\delta_{km}) Y_{2ij}^s + (\delta_{mn}\delta_{il} + \delta_{ml}\delta_{in} + \delta_{ln}\delta_{im}) Y_{2jk}^s \\
&+ (\delta_{mn}\delta_{jl} + \delta_{ml}\delta_{jn} + \delta_{ln}\delta_{jm}) Y_{2ik}^s \\
G_{ijklmn}^{2(s)} &= (\delta_{ij}\delta_{kl} + \delta_{jk}\delta_{il} + \delta_{ik}\delta_{jl}) Y_{2mn}^s + (\delta_{ij}\delta_{km} + \delta_{jk}\delta_{im} + \delta_{ik}\delta_{jm}) Y_{2ln}^s \\
&+ (\delta_{ij}\delta_{kn} + \delta_{jk}\delta_{in} + \delta_{ik}\delta_{jn}) Y_{2lm}^s \\
H_{ijklmn}^{2(s)} &= (\delta_{jk}\delta_{mn} + \delta_{jm}\delta_{kn} + \delta_{jn}\delta_{km}) Y_{2il}^s + (\delta_{jk}\delta_{ln} + \delta_{jn}\delta_{kl} + \delta_{jl}\delta_{kn}) Y_{2im}^s \\
&+ (\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}) Y_{2in}^s + (\delta_{ik}\delta_{mn} + \delta_{im}\delta_{kn} + \delta_{in}\delta_{km}) Y_{2jl}^s \\
&+ (\delta_{ik}\delta_{ln} + \delta_{im}\delta_{kl} + \delta_{il}\delta_{kn}) Y_{2jm}^s + (\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl}) Y_{2jn}^s \\
&+ (\delta_{ji}\delta_{mn} + \delta_{jm}\delta_{in} + \delta_{jn}\delta_{im}) Y_{2kl}^s + (\delta_{ji}\delta_{ln} + \delta_{jn}\delta_{il} + \delta_{jl}\delta_{in}) Y_{2km}^s \\
&+ (\delta_{ji}\delta_{lm} + \delta_{jl}\delta_{im} + \delta_{jm}\delta_{il}) Y_{2kn}^s \\
K_{ijklmn}^{2(s)} &= (\epsilon_{ilf}\epsilon_{jmg}\delta_{kn} + \epsilon_{imf}\epsilon_{jng}\delta_{kl} + \epsilon_{inf}\epsilon_{jlg}\delta_{km} + \epsilon_{jlf}\epsilon_{kmg}\delta_{in} + \epsilon_{jmf}\epsilon_{kng}\delta_{il} \\
&+ \epsilon_{jnf}\epsilon_{klg}\delta_{im} + \epsilon_{klf}\epsilon_{img}\delta_{jn} + \epsilon_{kmf}\epsilon_{ing}\delta_{jl} + \epsilon_{knf}\epsilon_{ilg}\delta_{jm}) Y_{2fg}^s \\
F_{ijklmn}^{3(s)} &= \delta_{il}(\epsilon_{jmf} Y_{3fkn}^s + \epsilon_{jnf} Y_{3fkm}^s + \epsilon_{kmf} Y_{3fjn}^s + \epsilon_{knf} Y_{3fjm}^s) + \delta_{im}(\epsilon_{jnf} Y_{3fkl}^s \\
&+ \epsilon_{jlf} Y_{3fkn}^s + \epsilon_{knf} Y_{3fjl}^s + \epsilon_{klf} Y_{3fjn}^s) + \delta_{in}(\epsilon_{jlf} Y_{3fkm}^s + \epsilon_{jmf} Y_{3fkl}^s + \epsilon_{klf} Y_{3fjm}^s \\
&+ \epsilon_{kmf} Y_{3fjl}^s) + \delta_{jl}(\epsilon_{kmf} Y_{3fin}^s + \epsilon_{knf} Y_{3fim}^s + \epsilon_{imf} Y_{3fkn}^s + \epsilon_{inf} Y_{3fkm}^s) \\
&+ \delta_{jm}(\epsilon_{knf} Y_{3fil}^s + \epsilon_{klf} Y_{3fin}^s + \epsilon_{inf} Y_{3fkl}^s + \epsilon_{ilf} Y_{3fkn}^s) + \delta_{jn}(\epsilon_{klf} Y_{3fim}^s \\
&+ \epsilon_{kmf} Y_{3fil}^s + \epsilon_{ilf} Y_{3fkm}^s + \epsilon_{imf} Y_{3fkl}^s) + \delta_{kl}(\epsilon_{imf} Y_{3fjn}^s + \epsilon_{inf} Y_{3fjm}^s \\
&+ \epsilon_{jmf} Y_{3fin}^s + \epsilon_{jnf} Y_{3fim}^s) + \delta_{km}(\epsilon_{inf} Y_{3fjl}^s + \epsilon_{ilf} Y_{3fjn}^s + \epsilon_{jnf} Y_{3fil}^s) \\
&+ \epsilon_{jlf} Y_{3fin}^s + \delta_{kn}(\epsilon_{ilf} Y_{3fjm}^s + \epsilon_{imf} Y_{3fjl}^s + \epsilon_{jlf} Y_{3fim}^s + \epsilon_{jmf} Y_{3fil}^s)
\end{aligned}$$

$$\begin{aligned}
G_{ijklmn}^{3(s)} &= \delta_{ij}(\epsilon_{klf}Y_{3fmn}^s + \epsilon_{kmf}Y_{3fln}^s + \epsilon_{knf}Y_{3flm}^s) + \delta_{ik}(\epsilon_{jlf}Y_{3fmm}^s + \epsilon_{jmf}Y_{3fln}^s \\
&\quad + \epsilon_{jnf}Y_{3flm}^s) + \delta_{jk}(\epsilon_{ilf}Y_{3fmm}^s + \epsilon_{imf}Y_{3fln}^s + \epsilon_{inf}Y_{3flm}^s) \\
H_{ijklmn}^{3(s)} &= \delta_{lm}(\epsilon_{inf}Y_{3fjk}^s + \epsilon_{jn f}Y_{3fik}^s + \epsilon_{knf}Y_{3fij}^s) + \delta_{ln}(\epsilon_{imf}Y_{3fjk}^s + \epsilon_{jmf}Y_{3fik}^s \\
&\quad + \epsilon_{knf}Y_{3fij}^s) + \delta_{nm}(\epsilon_{ilf}Y_{3fjk}^s + \epsilon_{jlf}Y_{3fik}^s + \epsilon_{klf}Y_{3fij}^s) \\
F_{ijklmn}^{4(s)} &= \delta_{ij}Y_{4klmn}^s + \delta_{kj}Y_{4ilmn}^s + \delta_{ki}Y_{4jlmn}^s \\
G_{ijklmn}^{4(s)} &= \delta_{lm}Y_{4nij}^s + \delta_{mn}Y_{4lik}^s + \delta_{nl}Y_{4mij}^s \\
H_{ijklmn}^{4(s)} &= \epsilon_{ilf}\epsilon_{jmg}Y_{4fgkn}^s + \epsilon_{jlf}\epsilon_{kmg}Y_{4fgin}^s + \epsilon_{klf}\epsilon_{img}Y_{4fgjn}^s \\
&\quad + \epsilon_{imf}\epsilon_{jng}Y_{4fgkl}^s + \epsilon_{jmf}\epsilon_{kng}Y_{4fgil}^s + \epsilon_{kmf}\epsilon_{ing}Y_{4fgjl}^s \\
&\quad + \epsilon_{inf}\epsilon_{jlg}Y_{4fgkm}^s + \epsilon_{jn f}\epsilon_{klg}Y_{4fgim}^s + \epsilon_{knf}\epsilon_{ilg}Y_{4fgjm}^s \\
F_{ijklmn}^{5(s)} &= \epsilon_{ilf}Y_{5fjkmn}^s + \epsilon_{jlf}Y_{5fikmn}^s + \epsilon_{klf}Y_{5fijmn}^s + \epsilon_{imf}Y_{5fjkl n}^s + \epsilon_{inf}Y_{5fjkl m}^s \\
&\quad + \epsilon_{jmf}Y_{5fikln}^s + \epsilon_{jn f}Y_{5fiklm}^s + \epsilon_{kmf}Y_{5fijln}^s + \epsilon_{knf}Y_{5fijlm}^s \\
F_{ijklmn}^{6(s)} &= Y_{6ijklmn}^s
\end{aligned}$$

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