SPECTRAL THEORY OF THE ATIYAH-PATODI-SINGER OPERATOR ON COMPACT FLAT MANIFOLDS

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Abstract. We study the spectral theory of the Dirac-type boundary operator $\mathcal{D}$ defined by Atiyah, Patodi and Singer, acting on smooth even forms of a compact flat Riemannian manifold M. We give an explicit formula for the multiplicities of the eigenvalues of $\mathcal{D}$ in terms of values of characters of exterior representations of $\mathrm{SO}(n)$, where $n = \dim M$. As a consequence, we give large families of $\mathcal{D}$-isospectral flat manifolds that are non-homeomorphic to each other. Furthermore, we derive expressions for the eta series in terms of special values of Hurwitz zeta functions and, as a result, we obtain a simple explicit expression of the eta invariant.

1. Introduction

Let M be an oriented Riemannian manifold of dimension $n = 4h-1$ and denote by $\Omega(M)$ the space of differential forms on M. In this paper we will study the operator $\mathcal{D}$ defined on even forms by

$$\mathcal{D} : \Omega^{ev}(M) \to \Omega^{ev}(M) \qquad \mathcal{D}\phi = (-1)^{h+p+1}(*d - d*)\phi, \tag{1.1}$$

where $\Omega^{ev}(M) = \bigoplus_{p=0}^{2h-1} \Omega^{2p}(M)$ and $\phi \in \Omega^{2p}(M)$. We shall call this operator the *APS-boundary operator*, first considered in [?].

This operator is closely related to the signature operator. Indeed, there is a natural "Dirac-type" differential operator defined on forms by $d + d^*$ whose square is the Laplacian on forms $(d + d^*)^2 = dd^* + d^*d = \Delta_{\mathcal{F}}$. Let N be a compact Riemannian manifold. If N has dimension $n = 2\ell$, there is an involution τ on $\Omega(N)$ given by $\tau\phi = i^{p(p-1)+\ell} * \phi$ for $\phi \in \Omega^p(N)$ and $*$ is the Hodge-star operator. Denote by $\Omega^{\pm}(N) = \{\phi \in \Omega(N) \colon \tau\phi = \pm\phi\}$ the corresponding ±1-eigenspaces of τ. Since $(d + d^*)\tau = -\tau(d + d^*)$, we have that $(d + d^*)_{|\Omega^{\pm}(N)} \colon \Omega^{\pm}(N) \to \Omega^{\mp}(N)$. The operator $D_{\mathcal{S}} := (d + d^*)_{|\Omega^{+}(N)}$ is called the *signature operator* because, if N is closed ($\partial N = \varnothing$), by Hodge theory the index of $D_{\mathcal{S}}$ equals the signature of N. In fact, by the Hirzebruch index theorem, we further have $\mathrm{Ind}(D_{\mathcal{S}}) = \mathrm{Sign}(N) = \int_N L(p)$, where L is the Hirzebruch L-polynomial in the Pontrjagin classes.

However, if N is not closed, i.e. if $\partial N = M \neq \varnothing$, Atiyah, Patodi and Singer showed that under certain conditions on the boundary, the difference $\mathrm{Sign}(N) - \int_N L(p)$, called the defect of signature, is not necessarily zero. More precisely, one has that

$$\mathrm{Sign}(N) = \int_N L(p) - \eta_{\mathcal{D}}(0)$$

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where $\eta_{\mathcal{D}}(0)$ is the η-invariant (see (**??**)) associated to the operator $\mathcal{D}$ defined in (**??**). Here, $\mathcal{D} = (D_{\mathcal{S}})_{|M}$ is the restriction of the signature operator on N to the boundary M. It turns out that identifying $\Omega^{ev}(M)$ with $\Omega^+(N)_{|M}$, then $\mathcal{D}$ takes the form given in (**??**).

Let D be a self-adjoint elliptic differential operator of order d acting on smooth sections of a compact manifold M of dimension n. Then D has a discrete spectrum, denoted by $Spec_D(M)$, consisting of real eigenvalues λ with finite multiplicity d_λ, which accumulate only at infinity. The spectrum is said to be *asymmetric* if for some $\lambda \in Spec_D(M)$ one has that $d_\lambda \neq d_{-\lambda}$. Atiyah, Patodi and Singer introduced the so called *eta series*

$$\eta_D(s) = \sum_{0\neq\lambda\in Spec_D} sign(\lambda)\,|\lambda|^{-s} = \sum_{\lambda\in\mathcal{A}^+} \tfrac{d_\lambda - d_{-\lambda}}{\lambda^s}, \tag{1.2}$$

where $\mathcal{A} = \{\lambda \in Spec_D : d_\lambda \neq d_{-\lambda}\}$ is the *asymmetric spectrum* and $\mathcal{A}^+ = \mathcal{A} \cap \mathbb{R}^+$.

This series converges for $Re(s) > \frac{n}{d}$, and defines a holomorphic function $\eta_D(s)$ which has a meromorphic continuation to $\mathbb{C}$ having (possibly) simple poles at $s = n - k$, with $k \in \mathbb{N}_0$. Remarkably the residue of $\eta_D(s)$ at the origin vanishes (see [**?**], work of P. Gilkey, M. Wodzicki and also recently by R. Ponge). The number

$$\eta := \eta_D(0) \tag{1.3}$$

is a spectral invariant, globally defined, which is called *η-invariant* and gives a measure of the spectral asymmetry of A. It is also of interest the study of the *reduced eta invariant* defined by

$$\bar{\eta} := \frac{\eta + \dim\ker D}{2} \mod \mathbb{Z}\,.$$

In recent years the eta invariant has been studied by many authors in different contexts (for instance see work of P. Gilkey, W. Müller, S. Goette, U. Semmelmann, R. Meyerhoff, M. Ouyang and H. Moscovici – R. Stanton). For the spectral theory of Dirac type operators we mention the articles of C. Bär and the book of N. Ginoux ([**?**]). Also, the eta invariant has been computed in several particular cases (see for instance [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]).

The goal of this paper is to study the spectral theory of the Dirac-type APS-boundary operator $\mathcal{D}$ in (**??**) acting on smooth even forms of a general compact flat Riemannian manifold M of dimension $n = 2m + 1 = 4h - 1$ and to compute the corresponding eta invariant. Since any compact flat manifold M bounds ([**?**]), its eta invariant will give the defect of signature of any Riemannian manifold having M as its boundary. Furthermore, the cusp cross-sections of hyperbolic manifolds with cusps are compact flat manifolds, so their eta invariant is relevant in connection with the signature of such manifolds. There has been quite some research on the question of realizing any flat manifold as the boundary of a hyperbolic manifold (see [**?**], [**?**], [**?**], [**?**]). For instance, it is known that if a flat manifold is the boundary of a hyperbolic manifold with only one cusp, then its eta invariant is an integer ([**?**]).

In the present paper, as a first step, we develop the theory for a general flat torus (see Section **??**) giving explicitly a basis of eigenfunctions of mixed degree $(2p, n - 2p - 1)$, $(2p, n-2p+1)$, together with an expression for the multiplicities of the eigenvalues (Theorem **??**). We illustrate by showing the eigenfunctions in low dimensions $n = 7, 11$ for small eigenvalues (see Example **??**).

In Section **??**, we consider the case of a general flat Riemannian manifold M and derive a formula for the multiplicities of the eigenvalues of $\mathcal{D}$ (Theorem **??**). We then refine the formula expressing the multiplicities in terms of characters of exterior representations evaluated at elements of the maximal torus of $\mathrm{SO}(2m)$ and giving also an expression for the characters of such elements (see Theorem **??** and Proposition **??** respectively) that is very useful in explicit computations.

As a byproduct of the multiplicity formulas, we exhibit large families of $\mathcal{D}$-isospectral flat manifolds that are non-homeomorphic to each other (Proposition **??**). Namely, *any two* manifolds having holonomy groups $\mathbb{Z}_2^k$, for a fixed k, turn out to be $\mathcal{D}$-isospectral. This is connected to the isospectrality result in [**?**] in the case of the full Hodge-Laplace operator $\Delta = \Delta_{\mathcal{F}}$ acting on the space of forms of all degrees $\Omega(M)$.

Section **??** is devoted to the computation of the eta series and of the corresponding eta invariant. As a main result in the section we obtain an explicit expression of $\eta(s)$ in terms of Hurwitz zeta functions (Theorem **??**) and in particular it follows that $\eta(s)$ is an entire function, as expected, by flatness of M.

Formula (**??**) for $\eta(s)$ in Theorem **??** can be used effectively to compute the η-invariant in many cases, in particular for all flat manifolds with holonomy group $\mathbb{Z}_p$, p an odd prime. These examples, together with a comparison with Donnelly's formula in [**?**] and other applications of (**??**) will be developed in a future publication.

2. The operator $\mathcal{D}$ on a flat torus

In this section we will determine a complete set of eigenfunctions of $\mathcal{D}$ on a flat torus $T_\Lambda = \Lambda\backslash\mathbb{R}^n$, Λ a lattice in $\mathbb{R}^n$.

We assume throughout this paper that $n = 2m+1 = 4h-1$. We set $I_n = \{1,\dots,n\}$. If $J = \{j_1,\dots,j_p\} \subset I_n$ such that $j_1 < j_2 < \cdots < j_p$, we will use the standard notation

$$dx_J := dx_{j_1} \wedge \ldots \wedge dx_{j_p} \qquad dx_\varnothing = 1.$$

Similarly, we will use $e_J := e_{j_1} \wedge \ldots \wedge e_{j_p}$ for a basis $\{e_i\}$ of $\mathbb{R}^n$ and $e_\varnothing = 1$.

Let M be an oriented Riemannian manifold of odd dimension n. The Hodge star operator $* = *_p : \Omega^p(M) \to \Omega^{n-p}(M)$ is defined by $*dx_J = \sigma_J dx_{J^c}$ where σ_J is a sign determined by

$$dx_J \wedge dx_{J^c} = \sigma_J\, dx_{I_n} = \sigma_J\, d\,\mathrm{vol}. \tag{2.1}$$

For each p we have $*_{n-p} *_p = (-1)^{p(n-p)}\, Id$, hence $*^2 = Id$ since n is odd. Furthermore, the operator $*$ is an isometry with respect to the Hermitian inner product on $\Omega(M)\otimes\mathbb{C}$, and $(*_p)^* = (-1)^{p(n-p)} *_{n-p}$. If $d : \Omega^{p-1}(M) \to \Omega^p(M)$ denotes exterior differentiation, its adjoint d^* is given by $d^* = (-1)^{n(p+1)+1} * d\, *$ on $\Omega^p(M)$.

Since $\mathcal{D} = (-1)^{p+h+1}(*d - d*)$ on $2p$-forms, and

$$*d : \Omega^{2p}(M) \to \Omega^{n-2p-1}(M), \qquad d* : \Omega^{2p}(M) \to \Omega^{n-2p+1}(M),$$

using that $n = 4h-1$, it follows that $\mathcal{D}$ sends a $2p$-form to an even form of mixed type

$$\mathcal{D} : \Omega^{2p}(M) \to \Omega^{2(2h-p-1)}(M) \oplus \Omega^{2(2h-p)}(M).$$

It is convenient to introduce the linear endomorphism

$$S\phi = (-1)^{p+h+1}\phi \qquad \text{for } \phi \in \Omega^{2p}(M). \tag{2.2}$$

Then $\mathcal{D}$ can be written as $\mathcal{D} = TS$ with $T = *d - d*$. Note that S satisfies

$$S(*d) = -(*d)S, \qquad S(d*) = (d*)S. \tag{2.3}$$

We denote by Δ the Hodge-Laplacian $dd^* + d^*d$ acting on $\Omega(M)$ and by Δ_e the restriction of Δ to even forms. By using (**??**) one can check that

$$\mathcal{D}^2 = \Delta_e \tag{2.4}$$

and that $\mathcal{D}$ is self-adjoint since n is odd.

It is a standard fact that the eigenfunctions of Δ_e have the form

$$f_u(x)\,dx_J, \quad \text{where } f_u(x) := \mathrm{e}^{2\pi i u\cdot x},\ u \in \Lambda^* \text{ and } J \subset I_n,\ |J| \text{ even}, \tag{2.5}$$

with corresponding eigenvalue $4\pi^2\mu^2$ of multiplicity $d_{\mu,\Delta_e}(\Lambda) = 2^{n-1}|\Lambda^*_\mu|$ where

$$\Lambda^*_\mu = \{u \in \Lambda^* : \|u\| = \mu\}.$$

Hence, by (**??**), the eigenvalues of $\mathcal{D}$ must be of the form $\pm 2\pi\|u\|$ with $u \in \Lambda^*$ and furthermore, if $d^{\pm}_{\mu,\mathcal{D}}(\Lambda)$ denotes the multiplicity of the eigenvalue $\pm 2\pi\mu$, with $\mu > 0$, then

$$d^{+}_{\mu,\mathcal{D}}(\Lambda) + d^{-}_{\mu,\mathcal{D}}(\Lambda) = d_{\mu,\Delta_e}(\Lambda) = 2^{n-1}|\Lambda^*_\mu|. \tag{2.6}$$

Since $\mathcal{D}^2 = \Delta_e$, it is clear that the eigenfunctions of $\mathcal{D}$ with eigenvalue $\pm\lambda \neq 0$ can be written $\omega \pm \frac{1}{\lambda}\mathcal{D}\omega$, where $\Delta_e\omega = \lambda^2\omega$. If we take $\omega = f_u dx_J$ with $f_u(x)$ and J as in (**??**) then

$$d(f_u dx_J) = df_u \wedge dx_J = 2\pi i f_u\ du \wedge dx_J, \qquad du := \sum_{j\in J} u_j dx_j\,. \tag{2.7}$$

Thus, by computing $(Id \pm \frac{1}{\lambda}\mathcal{D})f_u dx_J$ with $\lambda = \pm 2\pi\|u\|$, using (**??**) and putting $\hat{u} = \frac{u}{\|u\|}$ for $u \in \Lambda^* \smallsetminus \{0\}$, since $d\hat{u} = \widehat{du}$, we obtain that a general eigenfunction of $\mathcal{D}$ has the form

$$f_u(x)\big\{dx_J \pm i(-1)^{h+\frac{|J|}{2}+1}\big(*(d\hat{u}\wedge dx_J) - d\hat{u}\wedge *dx_J\big)\big\}\,. \tag{2.8}$$

Let $J \subset I_n$ with $|J| = 2p$ and let $\mathcal{B} = \{e_1, \dots, e_n\}$ be the canonical basis in $\mathbb{R}^n$. We assume first that $u = \|u\|e_n$, that is, $\hat{u} = e_n$. Then we see that the expression in (**??**) simplifies considerably. Indeed, depending on J, one of the terms $*(du\wedge dx_J)$ or $du \wedge *(dx_J)$ vanishes and we obtain

$$f_{e_n}(x)\big(dx_J \mp iS*(dx_n \wedge dx_J)\big) = f_{e_n}(x)\,dx^{\pm}_J \qquad n \notin J,$$

$$f_{e_n}(x)\big(dx_J \mp iS(dx_n \wedge *dx_J)\big) = f_{e_n}(x)\,\widetilde{dx}^{\pm}_J \qquad n \in J,$$

where

$$\begin{aligned} dx^{\pm}_J &:= dx_J \mp i\,(-1)^{p+h}\,\sigma_J\ dx_{J^c\smallsetminus\{n\}}, && n \notin J,\\ \widetilde{dx}^{\pm}_J &:= dx_J \mp i\,(-1)^{p+h}\,\sigma_J\ dx_{J^c\cup\{n\}}, && n \in J, \end{aligned} \tag{2.9}$$

and σ_J is as defined in (**??**).

Now, for a general $u \in \Lambda^* \smallsetminus \{0\}$, we proceed as follows. We fix an ordered orthonormal basis $\mathcal{B}_u$ of $\mathbb{R}^n$ containing $\hat{u} = \frac{u}{\|u\|}$ in the last position

$$\mathcal{B}_u = \big\{e_{u,1}, \dots, e_{u,n-1}, e_{u,n} = \hat{u}\big\}, \tag{2.10}$$

and $C = C_u \in \mathrm{SO}(n)$ sending $\mathcal{B}_u$ to $\mathcal{B}$ such that $C_u e_{u,j} = e_j$ for $1 \le j \le n$. In particular, $C_u\hat{u} = e_n$. Thus, we have $C_u e_{u,J} = e_J$. On the other hand, since $(x_j \circ C_u)e_{u,i} = \delta_{i,j}$, then we have $C^*_u dx_j = d(x_j \circ C_u) = dx_{u,j}$ and therefore

$$C^*_u dx_J = dx_{u,J}.$$

Thus, for a general u we have that (**??**) takes the form

$$f_u(x)\,C^*_u\,dx^{\pm}_J, \quad n \notin J, \qquad \text{and} \qquad f_u(x)\,C^*_u\,\widetilde{dx}^{\pm}_J, \quad n \in J.$$

where $dx^{\pm}_J(u)$ and $\widetilde{dx}^{\pm}_J(u)$ are as in (**??**). We now normalize and name these eigenforms.

Definition 2.1. In the previous notations, for $u \in \Lambda^*$ and $J \subset I_n$, $|J|$ even, we define

$$\begin{aligned} \phi^{\pm}_{u,J}(x) &:= \tfrac{\mathrm{e}^{2\pi i u\cdot x}}{\sqrt{2v_\Lambda}} C_u^*\big(dx_J \mp i(-1)^{p+h}\sigma_J\, dx_{J^c\smallsetminus\{n\}}\big) \qquad n \notin J, \\ \psi^{\pm}_{u,J}(x) &:= \tfrac{\mathrm{e}^{2\pi i u\cdot x}}{\sqrt{2v_\Lambda}} C_u^*\big(dx_J \mp i\,(-1)^{p+h}\sigma_J\, dx_{J^c\cup\{n\}}\big) \qquad n \in J. \end{aligned} \tag{2.11}$$

where $v_\Lambda = \mathrm{vol}(\Lambda)$.

The forms $\phi^{\pm}_{u,J}(x)$ and $\psi^{\pm}_{u,J}(x)$ in (**??**) are eigenfunctions of $\mathcal{D}$ with eigenvalue $2\pi\|u\|$, and degrees of mixed types $(2p, n-2p-1)$ and $(2p, n-2p+1)$ respectively.

From now on, we will identify $T^*(\mathbb{R}^n)$ with $\mathbb{R}^n$ via the correspondence $dx_i \leftrightarrow e_i$. With this identification, dx_J corresponds to e_J and the expression in (**??**) becomes $d(f_u e_J) = 2\pi i f_u\, \mathcal{L}_u(e_J)$, where $\mathcal{L}_u$ denotes wedge multiplication by $u = \sum u_j e_j$. Also, $C_u^* dx_J$ turns into $C_u^{-1} e_J$.

Taking into account these identifications, we may alternatively write

$$\phi^{\pm}_{u,J}(x) = \tfrac{\mathrm{e}^{2\pi i u\cdot x}}{\sqrt{2v_\Lambda}}\, e^{\pm}_J(u) \qquad \text{and} \qquad \psi^{\pm}_{u,J}(x) = \tfrac{\mathrm{e}^{2\pi i u\cdot x}}{\sqrt{2v_\Lambda}}\, \tilde{e}^{\pm}_J(u) \tag{2.12}$$

where

$$\begin{aligned} e^{\pm}_J(u) &:= C_u^{-1}\, e^{\pm}_J := C_u^{-1}\,\big(e_J \mp i(-1)^{p+h}\sigma_J\, e_{J^c\smallsetminus\{n\}}\big), \qquad n \notin J, \\ \tilde{e}^{\pm}_J(u) &:= C_u^{-1}\, \tilde{e}^{\pm}_J := C_u^{-1}\,\big(e_J \mp i\,(-1)^{p+h}\sigma_J\, e_{J^c\cup\{n\}}\big), \qquad n \in J\,. \end{aligned} \tag{2.13}$$

We now show that there is some linear dependence among the $\phi^{\pm}_{u,J}(x)$'s and similarly among the $\psi^{\pm}_{u,J}(x)$'s.

Lemma 2.2. *Let $\{e_i\}_{i=1}^n$ be any orthonormal basis of $\mathbb{R}^n$. Let $J \subset I_n$ with $|J|$ even. Then $e^{\pm}_{J^c\smallsetminus\{j\}}(e_j) = c\, e^{\pm\epsilon}_J(e_j)$ for $j \notin J$ and $\tilde{e}^{\pm}_{J^c\cup\{j\}}(e_j) = \tilde{c}\, \tilde{e}^{\pm\tilde{\epsilon}}_J(e_j)$ for $j \in J$, where $c, \tilde{c} \in \{\pm i\}$, $\epsilon, \tilde{\epsilon} \in \{\pm 1\}$, with $\pm$ depending on n, j and J. In particular, using the basis $\mathcal{B}_u$ for each $u \in \Lambda^*_\mu$ we have $\phi^{\pm}_{u,J^c\smallsetminus\{n\}}(x) = c\,\phi^{\pm\epsilon}_{u,J}(x)$ for $n \notin J$ and $\psi^{\pm}_{u,J^c\cup\{n\}}(x) = \tilde{c}\,\psi^{\pm\tilde{\epsilon}}_{u,J}(x)$ for $n \in J$.*

Proof. It suffices to check the relations for the forms $e^{\pm}_J(e_j)$ and $\tilde{e}^{\pm}_J(e_j)$, respectively. If $n \notin J$, put $J' = J^c \smallsetminus \{n\}$. Then $n \notin J'$ and $J'' = J$. Similarly, if $n \in J$, put $J^* = J^c \cup \{n\}$. Then $n \notin J^*$ and $J^{**} = J$. This proves the first assertion. By (**??**) the lemma follows. □

For each $\mu > 0$, let

$$H^{\pm}_\mu(\Lambda) := \{\omega^{\pm} \in \Omega^{ev}(T_\Lambda) \colon \mathcal{D}\omega^{\pm} = \pm 2\pi\mu\,\omega^{\pm}\}, \tag{2.14}$$

the eigenspace of $\mathcal{D}$ on T_Λ, with corresponding eigenvalue $\pm 2\pi\mu$, and, in a similar way, let $H_0(\Lambda) = \{\omega \in \Omega^{ev}(T_\Lambda) : \mathcal{D}\omega = 0\}$ be the space of *even harmonics*. We now give a complete orthonormal system in $H^{\pm}_\mu(\Lambda)$ and $H_0(\Lambda)$ for a flat torus T_Λ.

Theorem 2.3. *Let $\Lambda \subset \mathbb{R}^n$ be a lattice with $n = 2m+1 = 4h-1$. If $\mu > 0$, for each $u \in \Lambda^*_\mu = \{u \in \Lambda^* : \|u\| = \mu\}$, let $\phi^{\pm}_{u,J}(x)$ and $\psi^{\pm}_{u,J}(x)$ be the forms defined in (**??**). Let*

$$\begin{aligned} \Phi^{\pm}_\mu &= \{\phi^{\pm}_{u,J}(x) \,:\, u \in \Lambda^*_\mu,\ n \notin J,\ |J| = 2p \text{ with } 0 \le p \le h-1\} \\ \Psi^{\pm}_\mu &= \{\psi^{\pm}_{u,J}(x) \,:\, u \in \Lambda^*_\mu,\ n \in J,\ |J| = 2p \text{ with } 1 \le p \le h-1 \text{ or } p = h,\ 1 \in J\}. \end{aligned} \tag{2.15}$$

Then $\Phi^{\pm}_\mu \cup \Psi^{\pm}_\mu$, is an orthonormal basis of $H^{\pm}_\mu(\Lambda)$. The multiplicities of the eigenvalues $\pm 2\pi\mu$ are given by

$$d^{+}_{\mu,\mathcal{D}}(\Lambda) = d^{-}_{\mu,\mathcal{D}}(\Lambda) = 2^{n-2}\,|\Lambda^*_\mu|. \tag{2.16}$$

Furthermore, if $\mu = 0$ then $\{e_J \,:\, |J| = 2p,\ 0 \le p \le \frac{n-1}{2}\}$ is an orthonormal basis of $H_0(\Lambda)$ and $d_{0,\mathcal{D}}(\Lambda) = 2^{n-1}$.

Proof. Assume first that $\mu = 0$. Since $\ker \mathcal{D} = \ker \mathcal{D}^2 = \ker \Delta_e$, the 0-eigenforms of $\mathcal{D}$ are the constant coefficient even forms e_J with $J \subset I_n$ and $|J| = 2p$, $0 \le p \le \frac{n-1}{2}$. Hence,

$$d_{0,\mathcal{D}}(\Lambda) = \sum_{q \text{ even}} \tbinom{n-1}{q} = 2^{n-1}.$$

Now, let $\mu > 0$. It is clear from their construction that $\phi^{\pm}_{u,J}(x)$ and $\psi^{\pm}_{u,J}(x)$ are eigenforms of $\mathcal{D}$ with eigenvalues $\pm 2\pi\mu$, where $\mu = \|u\|$ and $u \in \Lambda^*$. We now check that they have norm 1. Using that $*$ is an isometry with respect to the Hermitian inner product $\langle \cdot,\cdot \rangle$ on $\Lambda^*(\mathbb{R}^n) \otimes \mathbb{C}$, for $n \notin J$, we have

$$\langle \phi^{\pm}_{u,J}(x), \phi^{\pm}_{u,J}(x) \rangle = \tfrac{1}{2\, v_\Lambda} \int_{T_\Lambda} \mathrm{e}^{2\pi i (u-u)\cdot x} dx \, \big(\langle e_J, e_J \rangle + \langle e_{J\cup\{n\}}, e_{J\cup\{n\}} \rangle \big) = 1,$$

since $|J^c \smallsetminus \{n\}| \neq |J|$. Thus, $\|\phi^{\pm}_{u,J}(x)\| = 1$. The computations in the case of the $\psi^{\pm}_{u,J}(x)$'s are entirely similar.

We now show the orthogonality of the eigenforms in (**??**). Since

$$\langle f_u(x) e_J, f_{u'}(x) e_{J'} \rangle = \int_{T_\Lambda} \mathrm{e}^{2\pi i (u-u')\cdot x} dx \, \langle e_J, e_{J'} \rangle = \delta_{u,u'} \, v_\Lambda \, \langle e_J, e_{J'} \rangle,$$

$f_u(x)e_J$ and $f_{u'}(x)e_{J'}$ are orthogonal if $u \neq u'$ for any $J, J' \subset I$, thus we are left to consider eigenforms for a fixed $u \in \Lambda^*_\mu$. Also, since H^+_μ is orthogonal to H^-_μ, we have $\langle e^{\pm}_J(u), e^{\mp}_{J'}(u) \rangle = 0$, $\langle e^{\pm}_J(u), \tilde{e}^{\mp}_{J'}(u) \rangle = 0$ and $\langle \tilde{e}^{\pm}_J(u), \tilde{e}^{\mp}_J(u) \rangle = 0$. Furthermore, since $C_u \in \mathrm{SO}(n)$, we need only check the conditions

$$\langle e^{\pm}_J, e^{\pm}_{J'} \rangle = 0, \qquad \langle \tilde{e}^{\pm}_J, \tilde{e}^{\pm}_J \rangle = 0, \qquad \langle e^{\pm}_J, \tilde{e}^{\pm}_J \rangle = 0,$$

for every $J, J' \subset I$ with $J' \neq J$.

We shall work out the calculation for $\langle e^{\pm}_J, e^{\pm}_{J'} \rangle$; the remaining cases are proved similarly. Take $J, J' \subset I_{n-1}$, with $|J| = 2p$, $|J'| = 2q$. Then, by (**??**), we have

$$\begin{aligned} \langle e^{\pm}_J, e^{\pm}_{J'} \rangle &= \big\langle e_J \mp i(-1)^{p+h} \sigma_J \, e_{J^c \smallsetminus \{n\}}, e_{J'} \mp i(-1)^{p+h} \sigma_{J'} \, e_{J'^c \smallsetminus \{n\}} \big\rangle \\ &= \langle e_J, e_{J'} \rangle + \langle e_{J^c \smallsetminus \{n\}}, e_{J'^c \smallsetminus \{n\}} \rangle + i\epsilon \big(\langle e_J, e_{J'^c \smallsetminus \{n\}} \rangle + \langle e_{J^c \smallsetminus \{n\}}, e_{J'} \rangle \big), \end{aligned}$$

where $\epsilon = (-1)^{p+h}(\sigma_{J'} - \sigma_J)$.

Note that $|J^c \smallsetminus \{n\}| = n - 2p - 1 = 2(2h - p - 1)$ for $n \notin J$ and hence $|J^c \smallsetminus \{n\}| \neq |J|$. Thus, there are three cases to consider. First, if $2p \neq 2q$, then $2p \neq n - 2q - 1$ and we immediately obtain $\langle e^{\pm}_J, e^{\pm}_{J'} \rangle = 0$. Secondly, if $2p = 2q$, then $n - 2p - 1 = n - 2q - 1$ (and $2p \neq n - 2q - 1$). Since $n \notin J$ we get

$$\langle e^{\pm}_J, e^{\pm}_{J'} \rangle = \langle e_J, e_{J'} \rangle + \langle e_{J^c \smallsetminus \{n\}}, e_{J'^c \smallsetminus \{n\}} \rangle = 2 \langle e_J, e_{J'} \rangle = 2\delta_{J,J'}.$$

Finally, if $2p = n - 2q - 1$, then $2q = n - 2p - 1 = 2(2h - p - 1)$, and we have that

$$\langle e^{\pm}_J, e^{\pm}_{J'} \rangle = i\epsilon \big(\langle e_J, e_{J'^c \smallsetminus \{n\}} \rangle + \langle e_{J^c \smallsetminus \{n\}}, e_{J'} \rangle \big) = 0\,,$$

since we have $J' \neq J^c \smallsetminus \{n\}$, and this implies that both inner products equal 0.

Relative to the linear independence of the eigenfunctions, Lemma **??** implies that one should take half of the subsets $J \subset I$ with $|J| = 2p$. More precisely, for each u, in the case of the $\phi_{u,J}$'s, we take all J's such that $n \notin J$ and $|J| < |J^c \smallsetminus \{n\}|$, while in the case of the $\psi_{u,J}$'s we choose those J's such that $n \in J$ and $|J| < |J^c \cup \{n\}|$. Furthermore, in the special case when $n \in J$ and $p = h$, since $|J^c \cup \{n\}| = |J|$, we also require that $1 \in J$.

Now, we compute the multiplicities $d^{\pm}_{\mu,\mathcal{D}}(\Lambda) = \dim H^{\pm}_\mu(\Lambda)$. Note that $|\Phi^{\pm}_\mu| + |\Psi^{\pm}_\mu|$ equals

$$\Big(\# \Big\{ {\scriptstyle J \subset I_n \atop n \notin J} : {\scriptstyle |J| = 2p \atop 0 \le p \le h-1} \Big\} + \# \Big\{ {\scriptstyle J \subset I_n \atop n \in J} : {\scriptstyle |J| = 2p \atop 1 \le p \le h-1} \Big\} + \# \Big\{ {\scriptstyle J \subset I_n \atop 1, n \in J} : |J| = 2h \Big\} \Big) \, |\Lambda^*_\mu|.$$

Thus, using that $n-1=2(2h-1)$ and $\binom{n-2}{2h-2}=\binom{n-2}{\frac{n-1}{2}}=\frac{1}{2}\binom{n-1}{\frac{n-1}{2}}$ we obtain

$$\begin{aligned}|\Phi_\mu^\pm|+|\Psi_\mu^\pm| &= \Big(\sum_{p=0}^{h-1}\tbinom{n-1}{2p}+\sum_{p=1}^{h-1}\tbinom{n-1}{2p-1}+\tfrac{1}{2}\tbinom{n-2}{2h-2}\Big)|\Lambda_\mu^*| \\ &= \tfrac{1}{2}\Big(\sum_{p=0}^{n-1}\tbinom{n-1}{p}\Big)|\Lambda_\mu^*| = 2^{n-2}|\Lambda_\mu^*|\,.\end{aligned}$$

Finally, by the above equality, we see that $\Phi_\mu^+\cup\Psi_\mu^+\cup\Phi_\mu^-\cup\Psi_\mu^-$ has cardinality $2^{n-1}|\Lambda_\mu^*|$, hence by (**??**) it is an orthonormal basis of $H_\mu^\pm(\Lambda)$, and thus the theorem follows. □

Next, we will illustrate Theorem **??** by giving some examples in low dimensions.

Example 2.4. For $n=2m+1=4h-1$, by Theorem **??**, there is a basis of eigenforms of $\mathcal{D}$ on T_Λ having degrees of the following mixed types

$\phi_{u,J}^\pm(x)$, $n\notin J$	$(0,2m)$	$(2,2m-2)$	$\ldots$	$(m-1,m+1)$
$\psi_{u,J}^\pm(x)$, $n\in J$	$(2,2m)$	$(4,2m-2)$	$\ldots$	$(m+1,m+1)$

with $u\in\Lambda_\mu^*$ and $J\subset I_n$ (where in the case of the $\psi_{u,J}^\pm(x)$ of type $(m+1,m+1)$ we furthermore take $1\in J$).

We will next exhibit some explicit eigenforms on the canonical torus $\mathbb{T}_n=\mathbb{Z}^n\backslash\mathbb{R}^n$ in dimensions $n=7,11$ for $\mu=1$ corresponding to the non-zero eigenvalues $\pm 2\pi$.

• Dimension 7. Let $n=7$, hence $m=3$. The $\phi_{u,J}^\pm(x)$'s are of type $(0,6)$ and $(2,4)$ while the $\psi_{u,J}^\pm(x)$'s are of type $(2,6)$ and $(4,4)$. By (**??**), for each $u\in\Lambda^*$, the following choices of $J\subset I_7$ for $\phi_{u,J}^\pm(x)$ and $\psi_{u,J}^\pm(x)$ give independent eigenfunctions.

$\phi_{u,J}^\pm(x)$	$\lvert J\rvert=0$	$\lvert J\rvert=2$
$J\subset I_6$	$\varnothing$	{1,2}, {1,3}, {1,4}, {1,5}, {1,6},
		{2,3}, {2,4}, {2,5}, {2,6}, {3,4},
		{3,5}, {3,6}, {4,5}, {4,6}, {5,6}

and

$\psi_{u,J}^\pm(x)$	$\lvert J\rvert=2$	$\lvert J\rvert=4 \quad (1\in J)$
$J\subset I_7$	{1,7}, {2,7}, {3,7}	{1,2,3,7}, {1,2,4,7}, {1,2,5,7}, {1,2,6,7},
$7\in J$	{4,7}, {5,7}, {6,7}	{1,3,4,7}, {1,3,5,7}, {1,3,6,7},
		{1,4,5,7}, {1,4,6,7}, {1,5,6,7}

In the case of $\lambda=\pm 2\pi$, we have $\Lambda_1=\{\pm e_1,\ldots,\pm e_7\}$. Let $u=e_1$ and take the ordered basis

$$\mathcal{B}_u=\mathcal{B}_{e_1}=\{e_{u,1},\ldots,e_{u,6},e_{u,7}=e_1\}=\{e_7,e_2,\ldots,e_6,e_1\}\,.$$

Then, we choose C_u sending the basis $\mathcal{B}_u$ to the canonical basis $\mathcal{B}=\{e_1,\ldots,e_7\}$ as $C_u=\left(\begin{smallmatrix} & & 1\\ & Id_5 & \\ 1 & & \end{smallmatrix}\right)$, where Id_5 is the 5×5 identity matrix. One has that $C_u^{-1}=C_u$.

According to Theorem **??** and (**??**), the eigenfunctions associated to $u = e_1$ of type $\phi^{\pm}_{u,J}(x)$ are given by

$$\phi^{\pm}_{e_1,\varnothing}(x) = \tfrac{e^{2\pi i x_1}}{\sqrt{2}}\, C_u(1 \pm i\,\epsilon\, e_1 \wedge \ldots \wedge e_6) = \tfrac{e^{2\pi i x_1}}{\sqrt{2}}\,(1 \mp i\,\epsilon\, e_2 \wedge \ldots \wedge e_7)$$
$$\phi^{\pm}_{e_1,\{j_1,j_2\}}(x) = \tfrac{1}{\sqrt{2}}\, e^{2\pi i x_1}\, C_u\,(e_{j_1} \wedge e_{j_2} \pm i\,\epsilon'\, e_{j_3} \wedge e_{j_4} \wedge e_{j_5} \wedge e_{j_6})$$

where $J \subset I_6 = \{j_1, \ldots, j_6\}$. For example, if $J = \{1,2\}$ then

$$\phi^{\pm}_{e_1,\{1,2\}}(x) \quad = \quad \tfrac{e^{2\pi i x_1}}{\sqrt{2}}\,(-e_2 \wedge e_7 \pm i\,\epsilon'\, e_3 \wedge e_4 \wedge e_5 \wedge e_6).$$

Similarly, the $\psi^{\pm}_{u,J}(x)$'s are given by

$$\psi^{\pm}_{e_1,\{j_1,7\}}(x) = \tfrac{e^{2\pi i x_1}}{\sqrt{2}}\, C_u\,(e_{j_1} \wedge e_7 \pm i\,\tilde{\epsilon}\, e_{j_2} \wedge e_{j_3} \wedge e_{j_4} \wedge e_{j_5} \wedge e_{j_6} \wedge e_7)$$
$$\psi^{\pm}_{e_1,\{j_1,j_2,j_3,7\}}(x) = \tfrac{e^{2\pi i x_1}}{\sqrt{2}}\, C_u\,(e_{j_1} \wedge e_{j_2} \wedge e_{j_3} \wedge e_7 \pm i\,\tilde{\epsilon}'\, e_{j_4} \wedge e_{j_5} \wedge e_{j_6} \wedge e_7)$$

where $7 \in J \subset I_7 = \{j_1, \ldots, j_6, 7\}$. For instance, if $J = \{1,7\}$ and $J' = \{1,2,3,7\}$ we have

$$\psi^{\pm}_{e_1,J}(x) = -\tfrac{e^{2\pi i x_1}}{\sqrt{2}}\,(e_1 \wedge e_7 \pm i\,\tilde{\epsilon}\, e_2 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6 \wedge e_7)$$
$$\psi^{\pm}_{e_1,J'}(x) = -\tfrac{e^{2\pi i x_1}}{\sqrt{2}}\,(e_1 \wedge e_2 \wedge e_3 \wedge e_7 \pm i\,\tilde{\epsilon}'\, e_1 \wedge e_4 \wedge e_5 \wedge e_6).$$

In the expressions above, $\epsilon, \epsilon', \tilde{\epsilon}, \tilde{\epsilon}' \in \{\pm 1\}$ are signs depending on u, J that can be explicitly determined.

Relative to Lemma **??**, note that if we take $\psi^{\pm}_{e_1,J}(x)$ with $|J| = 4$, say $J = \{1,2,3,4\}$, we obtain $\phi^{\pm}_{e_1,J'}(x)$ or $\phi^{\mp}_{e_1,J'}(x)$ with $J' = J^c \smallsetminus \{7\}$, up to a scalar multiple. In fact,

$$\psi^{\pm}_{e_1,J}(x) = \tfrac{e^{2\pi i x_1}}{\sqrt{2}}\,(-e_2 \wedge e_3 \wedge e_4 \wedge e_7 \pm i\,\epsilon\, e_5 \wedge e_6)$$
$$\psi^{\pm}_{e_1,J'}(x) = \tfrac{e^{2\pi i x_1}}{\sqrt{2}}\,(e_5 \wedge e_6 \mp i\,\epsilon'\, e_2 \wedge e_3 \wedge e_4 \wedge e_7).$$

Thus we have that

$$\mp i\epsilon' \psi^{\pm}_{e_1,J'}(x) = \psi^{\pm(-\epsilon\epsilon')}_{e_1,J}(x).$$

Proceeding similarly for $\phi^{\pm}_{e_i,J}(x)$, $\phi^{\pm}_{-e_i,J}(x)$, $\psi^{\pm}_{e_i,J}(x)$ and $\psi^{\pm}_{-e_i,J}(x)$ for $1 \le i \le 7$, we get a basis for $H^{\pm}_1(\mathbb{T}_7)$. In this way, we get that

$$d^{\pm}_{1,\mathcal{D}}(\mathbb{Z}^7) = \Big(\tbinom{6}{0} + \tbinom{6}{2} + \tbinom{6}{1} + \tbinom{5}{2}\Big)|\Lambda_1| = (1 + 15 + 6 + 10) \cdot 14 = 2^5 \cdot 14\,,$$

which is the value obtained by using Theorem **??**.

• Dimension 11. Let $n = 11$, hence $m = 5$. The $\phi^{\pm}_{u,J}(x)$'s are of type $(0,10), (2,8), (4,6)$ while the $\psi^{\pm}_{u,J}(x)$'s are of type $(2,10), (4,8), (6,6)$. In this case, the $\phi^{\pm}_{u,J}(x)$'s are obtained using J with $|J| = 0, 2, 4$ and $11 \notin J$, while the $\psi^{\pm}_{u,J}(x)$'s by using J with $11 \in J$ and $|J| = 2, 4, 6$, and also $1 \in J$ if $|J| = 6$. In this case, we get

$$d^{\pm}_{1,\mathcal{D}}(\mathbb{Z}^{11}) = \Big(\tbinom{10}{0} + \tbinom{10}{2} + \tbinom{10}{4} + \tbinom{10}{1} + \tbinom{10}{3} + \tbinom{9}{4}\Big)\,|\Lambda_1| = 512\,|\Lambda_1| = 2^9|\Lambda_1|$$

as given in Theorem **??**. ◊

Example 2.5. Consider $\mathcal{D}$ acting on the 3-dimensional canonical torus $\mathbb{T}_3 = \mathbb{Z}^3\backslash\mathbb{R}^3$, i.e. $\Lambda = \mathbb{Z}^3 = \Lambda^*$ and $v_\Lambda = 1$. We will construct a basis of eigenforms of $\mathcal{D}$ associated to the three lowest eigenvalues $1, \sqrt{2}, \sqrt{3}$, and we will compute their corresponding multiplicities. More precisely, for each $u \in \Lambda^*_\mu$ we have

$$\phi^{\pm}_{u,\varnothing}(x) = \tfrac{e^{2\pi i u\cdot x}}{\sqrt{2}}\, C_u^{-1}\,(1 \pm i\,\epsilon\, e_1 \wedge e_2), \qquad \psi^{\pm}_{u,\{1,3\}}(x) = \tfrac{e^{2\pi i u\cdot x}}{\sqrt{2}}\, C_u^{-1}\,(e_1 \wedge e_3 \pm i\,\epsilon'\, e_2 \wedge e_3),$$

where ϵ, ϵ' are computable signs.

• Eigenvalue $\mu = 1$. We have that $\Lambda_\mu = \{\pm e_1, \pm e_2, \pm e_3\}$ and we choose the bases $\mathcal{B}_{e_1} = \{e_3, e_2, e_1\}$, $\mathcal{B}_{e_2} = \{e_1, e_3, e_2\}$ and $\mathcal{B}_{e_3} = \{e_1, e_2, e_3\}$ and corresponding matrices $C_{e_1} = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$, $C_{e_2} = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$, $C_{e_3} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$. Note that $C_{e_i}^{-1} = C_{e_i}$, $1 \le i \le 3$. We thus get

$$\begin{aligned}
\phi^{\pm}_{\epsilon e_1,\varnothing}(x) &= \tfrac{e^{2\pi i\epsilon x_1}}{\sqrt{2}}\,(1 \mp i\,\epsilon\, e_2\wedge e_3), &\qquad \psi^{\pm}_{\epsilon e_1,\{1,3\}}(x) &= \tfrac{e^{2\pi i\epsilon x_1}}{\sqrt{2}}(e_1\wedge e_2 \pm \epsilon i\, e_1\wedge e_3),\\
\phi^{\pm}_{\epsilon e_2,\varnothing}(x) &= \tfrac{e^{2\pi i\epsilon x_2}}{\sqrt{2}}\,(1 \pm i\,\epsilon\, e_1\wedge e_3), &\qquad \psi^{\pm}_{\epsilon e_2,\{1,3\}}(x) &= \tfrac{e^{2\pi i\epsilon x_2}}{\sqrt{2}}(e_1\wedge e_2 \pm \epsilon i\, e_2\wedge e_3),\\
\phi^{\pm}_{\epsilon e_3,\varnothing}(x) &= \tfrac{e^{2\pi i\epsilon x_3}}{\sqrt{2}}\,(1 \pm i\,\epsilon\, e_1\wedge e_2), &\qquad \psi^{\pm}_{\epsilon e_3,\{1,3\}}(x) &= \tfrac{e^{2\pi i\epsilon x_3}}{\sqrt{2}}(e_1\wedge e_3 \pm \epsilon i\, e_2\wedge e_3).
\end{aligned}$$

Then a basis of $H_1^{\pm}$ is given by

$$\big\{\phi^{\pm}_{\epsilon e_1,\varnothing}(x), \phi^{\pm}_{\epsilon e_2,\varnothing}(x), \phi^{\pm}_{\epsilon e_3,\varnothing}(x), \psi^{\pm}_{\epsilon e_1,\{1,3\}}(x), \psi^{\pm}_{\epsilon e_2,\{1,3\}}(x), \psi^{\pm}_{\epsilon e_3,\{1,3\}}(x)\big\}$$

and hence $d^{\pm}_{1,\mathcal{D}}(\Lambda) = 2|\Lambda_1| = 12$.

• Eigenvalue $\mu = \sqrt{2}$. We have $\Lambda_\mu = \{\epsilon(e_1 \pm e_2), \epsilon(e_1 \pm e_3), \epsilon(e_2 \pm e_3)\}$, with $\epsilon \in \{\pm 1\}$. Take the bases $\mathcal{B}_{e_1+e_2}$, $\mathcal{B}_{e_1+e_3}$ and $\mathcal{B}_{e_2+e_3}$ as in (**??**). For example, we take

$$\mathcal{B}_{\pm(e_i\pm e_j)} = \{e_k, \tfrac{\pm(e_i\mp e_j)}{\sqrt{2}}, \tfrac{\pm(e_i\pm e_j)}{\sqrt{2}}\},$$

for $i<j$ and $\{i,j,k\} = \{1,2,3\}$. Thus, for instance, we have

$$C_{e_1+e_2} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix}, \qquad C_{e_1+e_3} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad C_{e_2+e_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

and hence we obtain

$$\begin{aligned}
\phi^{\pm}_{\epsilon(e_1+\sigma e_2),\varnothing}(x) &= \tfrac{1}{\sqrt{2}}\,e^{2\pi i\epsilon(x_1+\sigma x_2)}\,\big(1 \pm \epsilon\tfrac{i}{\sqrt{2}}(e_2\wedge e_3 - \sigma\, e_1\wedge e_3)\big),\\
\phi^{\pm}_{\epsilon(e_1+\sigma e_3),\varnothing}(x) &= \tfrac{1}{\sqrt{2}}\,e^{2\pi i\epsilon(x_1+\sigma x_3)}\,\big(1 \pm \epsilon\tfrac{i}{\sqrt{2}}(e_2\wedge e_3 + \sigma\, e_1\wedge e_2)\big),\\
\phi^{\pm}_{\epsilon(e_2+\sigma e_3),\varnothing}(x) &= \tfrac{1}{\sqrt{2}}\,e^{2\pi i\epsilon(x_2+\sigma x_3)}\,\big(1 \mp \epsilon\tfrac{i}{\sqrt{2}}(e_1\wedge e_3 - \sigma\, e_1\wedge e_2)\big),
\end{aligned}$$

and

$$\begin{aligned}
\psi^{\pm}_{\epsilon(e_1+\sigma e_2),\{1,3\}}(x) &= \tfrac{1}{\sqrt{2}}\,e^{2\pi i\epsilon(x_1+\sigma x_2)}\,\big(e_1\wedge e_2 \pm \epsilon\tfrac{i}{\sqrt{2}}(e_1\wedge e_3 + \sigma\, e_2\wedge e_3)\big),\\
\psi^{\pm}_{\epsilon(e_1+\sigma e_3),\{1,3\}}(x) &= \tfrac{1}{\sqrt{2}}\,e^{2\pi i\epsilon(x_1+\sigma x_3)}\,\big(e_1\wedge e_3 \mp \epsilon\tfrac{i}{\sqrt{2}}(e_1\wedge e_2 + \sigma\, e_2\wedge e_3)\big),\\
\psi^{\pm}_{\epsilon(e_2+\sigma e_3),\{1,3\}}(x) &= \tfrac{1}{\sqrt{2}}\,e^{2\pi \epsilon i(x_2+\sigma x_3)}\,\big(e_2\wedge e_3 \mp \epsilon\tfrac{i}{\sqrt{2}}(e_1\wedge e_2 + \sigma\, e_1\wedge e_3)\big).
\end{aligned}$$

A basis of $H^{\pm}_{\sqrt{2}}$ is given by

$$\big\{\phi^{\pm}_{\epsilon(e_i+\sigma e_j),\varnothing}(x), \psi^{\pm}_{\epsilon(e_i+\sigma e_j),\{1,3\}}(x) : \epsilon,\sigma \in \{\pm 1\}, 1 \le i < j \le 3\big\}$$

and hence $d^{\pm}_{\sqrt{2},\mathcal{D}}(\Lambda) = 2|\Lambda_{\sqrt{2}}| = 24$.

• Eigenvalue $\mu = \sqrt{3}$. We have, $\Lambda_\mu = \big\{\epsilon_1 e_1 + \epsilon_2 e_2 + \epsilon_3 e_3 : \epsilon_1, \epsilon_2, \epsilon_3 \in \{\pm 1\}\big\}$. For $u = e_1 + e_2 + e_3$, we can take the orthonormal basis of $\mathbb{R}^3$ and the matrix

$$\mathcal{B}_{e_1+e_2+e_3} = \{\tfrac{e_1+e_2-2e_3}{\sqrt{6}}, \tfrac{e_1-e_2}{\sqrt{2}}, \tfrac{e_1+e_2+e_3}{\sqrt{3}}\}, \qquad C_{e_1+e_2+e_3} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Then, we compute

$$\begin{aligned}\phi^{\pm}_{e_1+e_2+e_3,\varnothing}(x) &= \tfrac{1}{\sqrt{2}}\, e^{2\pi i(x_1+x_2+e_3)}\big(1 \mp \tfrac{i}{\sqrt{3}}(e_1\wedge e_2 - e_1\wedge e_3 + e_2\wedge e_3)\big)\\ \psi^{\pm}_{e_1+e_2+e_3,\{1,3\}}(x) &= \tfrac{1}{\sqrt{2}}\, e^{2\pi i(x_1+x_2+x_3)}\Big(\big(\tfrac{e_1\wedge e_3}{\sqrt{2}} + \tfrac{e_2\wedge e_3}{\sqrt{2}}\big) \mp \tfrac{i}{\sqrt{3}}\big(\tfrac{2e_1\wedge e_2}{\sqrt{2}} + \tfrac{e_1\wedge e_3}{\sqrt{2}} - \tfrac{e_2\wedge e_3}{\sqrt{2}}\big)\Big)\end{aligned}$$

In general, we have the bases $\mathcal{B}_{\epsilon_1 e_1+\epsilon_2 e_2+\epsilon_3 e_3} = \{\frac{\epsilon_1 e_1+\epsilon_2 e_2-\epsilon_3 2e_3}{\sqrt{6}}, \frac{\epsilon_1 e_1-\epsilon_2 e_2}{\sqrt{2}}, \frac{\epsilon_1 e_1+\epsilon_2 e_2+\epsilon_3 e_3}{\sqrt{3}}\}$ and, by (**??**),

$$\big\{\phi^{\pm}_{\epsilon_1 e_1+\epsilon_2 e_2+\epsilon_3 e_3,\varnothing}(x), \psi^{\pm}_{\epsilon_1 e_1+\epsilon_2 e_2+\epsilon_3 e_3,\{1,3\}}(x) : \epsilon_1,\epsilon_2,\epsilon_3 \in \{\pm 1\}\big\}$$

is a basis of $H^{\pm}_{\sqrt{2}}$ and hence $d^{\pm}_{\sqrt{3},\mathcal{D}}(\Lambda) = 2|\Lambda_{\sqrt{2}}| = 16.$ $\Diamond$

3. Spectral theory of $\mathcal{D}$ for compact flat manifolds

We now shift our attention to the spectrum of $\mathcal{D}$ acting on an arbitrary compact flat manifold M of dimension $n = 2m+1 = 4h-1$ (i.e. $m = 2h-1 = \frac{n-1}{2}$) covered by a flat torus T_Λ (see Theorems **??** and **??**).

3.1. **Multiplicity formulas.** We will use similar methods as in [**?**]. We begin by proving two useful lemmas. For $\mu \ge 0$, recall that $\Lambda^*_\mu = \{u \in \Lambda^* : \|u\| = \mu\}$ and set $(\Lambda^*_\mu)^B = \{u \in \Lambda^*_\mu : Bu = u\}$.

Lemma 3.1. *Let $u \in (\Lambda^*_\mu)^B$, $J \subset \{1,\dots,n\}$ with $|J| = 2p$ and $e_J = e_{u,J}$ as defined in* (**??**). *Then we have*

$$\begin{aligned}\langle B^* e^{\pm}_J(u), e^{\pm}_J(u)\rangle &= \tfrac12\,\langle Be_J, e_J\rangle, && 0 \le p < h,\\ \langle B^* \tilde e^{\pm}_J(u), \tilde e^{\pm}_J(u)\rangle &= \tfrac12\,\langle Be_J, e_J\rangle \pm \tfrac{\delta_{p,h}}{4}\, i\,\langle (B-B^{-1})e_J, \hat u \wedge * e_J\rangle, && 0 < p \le h,\end{aligned}$$

where $e^{\pm}_J(u)$ and $\tilde e^{\pm}_J(u)$ are defined in (**??**).

Proof. Using that $n-2p-1 \ne 2p$ and that $B* = *B$ for $B \in \mathrm{SO}(n)$, we have

$$\begin{aligned}\langle B^* e^{\pm}_J(u), e^{\pm}_J(u)\rangle &= \tfrac14\langle B^{-1}\big(e_J \mp (-1)^{h+p} i * (\hat u\wedge e_J)\big), e_J \mp (-1)^{h+p} i * (\hat u \wedge e_J)\rangle\\ &= \tfrac14\big(\langle B^{-1}e_J, e_J\rangle + \langle B^{-1}(\hat u\wedge e_J), (\hat u \wedge e_J)\rangle\big)\\ &= \tfrac14\big(\langle Be_J, e_J\rangle + \langle \hat u \wedge B^{-1}e_J, \hat u\wedge e_J\rangle\big) = \tfrac12\langle Be_J, e_J\rangle.\end{aligned}$$

In the case of $\tilde e^{\pm}_J(u)$, for $1 \le p \le h-1$, we proceed in the same way. If $p = h$, we have that $n-2p+1 = 2p$, and hence

$$\begin{aligned}\langle B^* \tilde e^{\pm}_J(u), \tilde e^{\pm}_J(u)\rangle &= \tfrac14\,\{(\langle B^{-1}e_J, e_J\rangle + \langle \hat u \wedge B^{-1} * e_J, \hat u \wedge * e_J\rangle)\\ &\qquad \pm i\,(\langle B^{-1}(\hat u \wedge * e_J), e_J\rangle - \langle B^{-1}e_J, \hat u\wedge * e_J\rangle)\}\\ &= \tfrac14\,\big(2\langle Be_J, e_J\rangle \pm i\,\langle (B-B^{-1})e_J, \hat u \wedge * e_J\rangle\big),\end{aligned}$$

and the second identity follows. $\square$

Let $\mathbb{V}$ be an oriented real vector space of dimension $n-1 = 2m$. Since $n = 4h-1$, $m = 2h-1$. Consider the complex irreducible exterior representations $(\tau_p, \Lambda^p(\mathbb{V})_{\mathbb{C}})$, $0 \le p \le m-1$, and $(\tau^{\pm}_m, \Lambda^m_{\pm}(\mathbb{V}))_{\mathbb{C}}$ of $\mathrm{SO}(2m)$, where $\Lambda^m(\mathbb{V})_{\mathbb{C}} = \Lambda^m_+(\mathbb{V})_{\mathbb{C}} \oplus \Lambda^m_-(\mathbb{V})_{\mathbb{C}}$ and $\Lambda^m_{\pm}(\mathbb{V})_{\mathbb{C}}$ are the $\pm i$-eigenspaces of $*$ on $\Lambda^m(\mathbb{V})_{\mathbb{C}}$. We denote by τ_m the sum $\tau^+_m \oplus \tau^-_m$. Alternatively, $\Lambda^m_{\pm}(\mathbb{V})_{\mathbb{C}}$ can be seen as the ± 1-eigenspaces of the involution

$$\star = (-i)^m * = (-1)^h\, i\, *$$

on $\Lambda^m(\mathbb{V})_{\mathbb{C}}$. In fact, $\star^2 = -*^2 = (-1)^{m+1} Id = Id$. We denote by χ_p and $\chi^{\pm}_m$ respectively, the characters of the representations τ_p and $\tau^{\pm}_m$.

Now, put $\mathbb{V}_u := (\mathbb{R}u)^\perp$ for $u \in \mathbb{R}^n$. Let $\tau_{p,u}$ and $\tau^{\pm}_{m,u}$ be the exterior representations of $\mathrm{SO}(2m)$ on $\Lambda^p(\mathbb{V}_u)_{\mathbb{C}}$, $0 \le p \le m-1$, and $\Lambda^m_{\pm}(\mathbb{V}_u)_{\mathbb{C}}$, respectively, denoting their characters by $\chi_{p,u}$ and $\chi^{\pm}_{m,u}$. Also, we put

$$\mathrm{SO}(n-1,u) := \{B \in \mathrm{SO}(n) : Bu = u\} \simeq \mathrm{SO}(n-1).$$

If $B \in \mathrm{SO}(n)$ with $Bu = u$, the matrix of B in the basis $\mathcal{B}_u$ given in (**??**) is $B_u = \left(\begin{smallmatrix} B'_u & \\ & 1\end{smallmatrix}\right)$ with $B'_u \in \mathrm{SO}(n-1)$. From now on we will identify B_u with B'_u, for simplicity.

Lemma 3.2. *Let $u \in \mathbb{R}^n$, let $\mathcal{B}_u = \{e_{u,1}, \dots, e_{u,n-1}, e_{u,n} = \hat{u} = \frac{u}{\|u\|}\}$ be an ordered orthonormal basis of $\mathbb{R}^n$ and suppose that $B \in \mathrm{SO}(n-1,u)$. Then, we have*

$$\sum_{\substack{n \notin J \subset I_n \\ |J|=2p}} \langle Be_J, e_J\rangle = \chi_{2p,u}(B), \qquad \sum_{\substack{n \in J \subset I_n \\ |J|=2p}} \langle Be_J, e_J\rangle = \chi_{2p-1,u}(B), \tag{3.1}$$

and

$$\sum_{\substack{1,n \in J \\ |J|=2h}} \langle B(e_J \pm i\, e_n \wedge *e_J), e_J \pm i\, e_n \wedge *e_J\rangle = 2\,\chi^{\pm(-1)^h}_{m,u}(B), \tag{3.2}$$

where $n = 4h-1$.

Proof. For simplicity, we will set $e_J = e_{u,J}$. We first check the identities in (**??**). The set $\{e_J : |J| = q\}$ is an orthonormal basis of $\Lambda^q(\mathbb{R}^n)_{\mathbb{C}}$ for each q, $0 \le q \le m-1$. Hence $\{e_{J'}\}$, with $J' = J \smallsetminus \{n\}$, is an orthonormal basis of $\Lambda^q(\mathbb{V}_u)_{\mathbb{C}}$. Since $\langle Be_J, e_J\rangle = \langle Be_{J'}, e_{J'}\rangle\langle Be_n, e_n\rangle = \langle Be_{J'}, e_{J'}\rangle$, we have that

$$\sum_{\substack{n \notin J \subset I_n \\ |J|=2p}} \langle Be_J, e_J\rangle = \sum_{\substack{J \subset I_{n-1} \\ |J|=2p}} \langle Be_J, e_J\rangle = \chi_{2p,u}(B).$$

One can proceed similarly to show that the second sum in (**??**) equals $\chi_{2p-1,u}(B)$.

We now verify (**??**). If $J \subset I_n$ with $n \in J$, $|J|$ even, then

$$e_n \wedge *_n e_J = (*_{n-1} e_{J'}) \wedge e_n, \tag{3.3}$$

where, for each d, $*_d$ now denotes the star operator on $\Lambda^*(\mathbb{R}^d)$.

Indeed, if $J = \{j_1, \dots, j_{\ell-1}, n\}$ and $J^c = \{i_1, \dots, i_{n-\ell}\} \subset I_{n-1}$, then $*_n e_J = \mathrm{sgn}(\sigma)\, e_{J^c}$ and $*_{n-1} e_J = \mathrm{sgn}(\sigma')\, e_{J^c}$ where

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & \ell-1 & \ell & \ell+1 & \cdots & n \\ j_1 & j_2 & \cdots & j_{\ell-1} & n & i_1 & \cdots & i_{n-\ell}\end{pmatrix}, \qquad \sigma' = \begin{pmatrix} 1 & 2 & \cdots & \ell-1 & \ell & \ell+1 & \cdots & n-1 \\ j_1 & j_2 & \cdots & j_{\ell-1} & i_1 & i_2 & \cdots & i_{n-\ell}\end{pmatrix}.$$

Now, $e_{j_1} \wedge \cdots \wedge e_{j_{\ell-1}} \wedge e_n \wedge e_{i_1} \wedge \cdots \wedge e_{i_{n-\ell}} = \mathrm{sgn}(\sigma)\, e_1 \wedge \cdots \wedge e_n$, and since $n-\ell$ is odd, $e_{j_1} \wedge \cdots \wedge e_{j_{\ell-1}} \wedge e_{i_1} \wedge \cdots \wedge e_{i_{n-l}} = -\mathrm{sgn}(\sigma)\, e_1 \wedge \cdots \wedge e_{n-1}$, hence $-\mathrm{sgn}(\sigma) = \mathrm{sgn}(\sigma')$. In this way, $e_n \wedge *_n e_J = \mathrm{sgn}(\sigma)\, e_n \wedge e_{J^c} = -\mathrm{sgn}(\sigma)\, e_{J^c} \wedge e_n = \mathrm{sgn}(\sigma')\, e_{J^c} \wedge e_n = (*_{n-1} e_{J'}) \wedge e_n$, hence (**??**) follows.

Thus, if $|J| = m+1 = 2h$, by (**??**) and using that $*_{n-1} = (-1)^{h+1}\, i\, \star$, we have

$$e_J \pm i\, e_n \wedge *_n e_J = e_{J'} \wedge e_n \pm (-1)^h \star e_{J'} \wedge e_n = \big(e_{J'} \pm (-1)^h \star e_{J'}\big) \wedge e_n.$$

Thus, since $Bu = u$, the left hand side of (**??**) equals

$$\sum_{\substack{1 \in J \subset I_{n-1} \\ |J|=2h-1}} \big\langle B\big(e_J \,\pm\, (-1)^h \star e_J\big), e_J \,\pm\, (-1)^h \star e_J\big\rangle.$$

It is easy to check that the $e_J \pm (-1)^h \star e_J$ are eigenforms of $\star$ with eigenvalues $\pm(-1)^h$ and that $\|e_J \,\pm\, (-1)^h \star e_J\|^2 = 2$. One can also check that $e_J \,\pm\, (-1)^h \star e_J$ is orthogonal to

$e_K \pm (-1)^h \star e_K$ for $K \neq J$ and hence, $\{\frac{1}{\sqrt{2}}(e_J \pm (-1)^h \star e_J) : 1 \in J \subset I_{n-1}, |J| = m\}$ is an orthonormal basis of the $\pm(-1)^h$-eigenspaces $\Lambda^m_\pm(\mathbb{R}^{2m})$ of $\star$. Thus, the previous expression equals $2\,\mathrm{Tr}\,\tau^{\pm(-1)^h}_{m,u}(B)$, and hence (**??**) follows. □

Compact flat manifolds. We now recall some well-known facts on compact flat manifolds. A *flat* manifold is a Riemannian manifold with zero constant curvature. It is known that any such manifold is isometric to a quotient $M_\Gamma := \Gamma\backslash\mathbb{R}^n$, where Γ is a Bieberbach group, that is, a discrete cocompact torsion-free subgroup of $\mathrm{I}(\mathbb{R}^n)$, the isometry group of $\mathbb{R}^n$. Any element $\gamma \in \mathrm{I}(\mathbb{R}^n) = \mathrm{O}(n) \rtimes \mathbb{R}^n$ decomposes uniquely as $\gamma = BL_b$, where $B \in \mathrm{O}(n)$ and L_b denotes translation by $b \in \mathbb{R}^n$. The translations in Γ form a normal, maximal abelian subgroup of finite index, L_Λ, Λ a lattice in $\mathbb{R}^n$ that is B-stable, for each $BL_b \in \Gamma$. The restriction to Γ of the canonical projection $r : \mathrm{I}(\mathbb{R}^n) \to \mathrm{O}(n)$ given by $BL_b \mapsto B$ is a homomorphism with kernel L_Λ and $F := r(\Gamma)$ is a finite subgroup of $\mathrm{O}(n)$ called the point group. The group $\Lambda\backslash\Gamma \simeq F$ is called the *holonomy group* of Γ and gives the linear holonomy group of the Riemannian manifold M_Γ. We shall assume throughout this paper that M_Γ is orientable, i.e. $F \subset \mathrm{SO}(n)$. The action by conjugation of $\Lambda\backslash\Gamma$ on Λ defines an integral representation of F, called the *integral holonomy representation.* A flat manifold with holonomy group F will be called an F-manifold.

We are now in a position to state one of the main results in this section that gives the spectrum of $\mathcal{D}$ for an arbitrary compact flat manifold.

Theorem 3.3. *Let $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ be an orientable compact flat manifold of odd dimension $n = 2m+1 = 4h-1$ with holonomy group $F = \Lambda\backslash\Gamma$. Then, the non-zero eigenvalues of $\mathcal{D}$ are of the form $\pm 2\pi\mu$, $\mu = \|u\|$ where $u \in (\Lambda^*)^B$, $BL_b \in F$, with multiplicities given by*

$$d^{\pm}_{\mu,\mathcal{D}}(\Gamma) = \tfrac{1}{|F|} \sum_{BL_b \in F} \sum_{u \in (\Lambda^*_\mu)^B} e^{2\pi i u\cdot b} \Big(\sum_{p=0}^{m-1} \chi_{p,u}(B) \;+\; \chi^{\mp(-1)^h}_{m,u}(B) \Big). \tag{3.4}$$

Furthermore, we have

$$d_{0,\mathcal{D}}(\Gamma) = \tfrac{1}{|F|} \sum_{BL_b \in F} \sum_{p=0}^{2m} \chi_{p,u}(B)\,. \tag{3.5}$$

Proof. Suppose first that $\mu > 0$. Let $p^\pm_\mu : H^\pm_\mu(\Lambda) \to H^\pm_\mu(\Gamma) = (H^\pm_\mu(\Lambda))^\Gamma$ be the orthogonal projection given by $p^\pm_\mu = \frac{1}{|F|}\sum_{\gamma\in F} {\gamma^*}_{|H^\pm_\mu}$, with $H^\pm_\mu(\Gamma)$ as in (**??**). As a result,

$$d^{\pm}_{\mu,\mathcal{D}}(\Gamma) = \dim\,(H^\pm_\mu(\Lambda))^\Gamma = \mathrm{Tr}\,p^\pm_\mu = \tfrac{1}{|F|}\sum_{\gamma\in F} \mathrm{Tr}\,{\gamma^*}_{|H^\pm_\mu}. \tag{3.6}$$

If for each $u \in (\Lambda^*_\mu)^B$ we fix a basis $\mathcal{B}_u$ as in (**??**), by Theorem **??** we get the following expression for the traces of $\gamma^*_{|H^\pm_\mu}$

$$\mathrm{Tr}\,{\gamma^*}_{|H^\pm_\mu} = \sum_{u\in\Lambda^*_\mu} \Big(\sum_{J\in\mathcal{S}} \big\langle \gamma^*\phi^\pm_{u,J}(x), \phi^\pm_{u,J}(x)\big\rangle + \sum_{J\in\tilde{\mathcal{S}}} \big\langle \gamma^*\psi^\pm_{u,J}(x), \psi^\pm_{u,J}(x)\big\rangle \Big)$$

where

$$\mathcal{S} = \{J \subset I, n \notin J \,:\, |J| = 2p,\; 0 \le p \le h-1\},$$
$$\tilde{\mathcal{S}} = \{J \subset I, n \in J \,:\, |J| = 2p,\; 1 \le p \le h-1 \;\text{ or }\; 1 \in J,\; p = h\}.$$

Now, the action of $\gamma = BL_b \in \Gamma$ on $f_u(x)\, e_J \in \Omega(\mathbb{R}^n)$ –see (**??**), (**??**)– is given by

$$\gamma^*(f_u(x)\, e_J) = f_u(\gamma x)\, B^* e_J = \mathrm{e}^{2\pi i u\cdot(Bx+Bb)}\, B^* e_J = \mathrm{e}^{2\pi i\, B^{-1}u\cdot b} f_{B^{-1}u}(x)\, B^* e_J.$$

Therefore,

$$\begin{aligned}\langle \gamma^* \phi^{\pm}_{u,J}(x), \phi^{\pm}_{u,J}(x)\rangle &= \tfrac{2}{v_\Lambda}\big\langle \mathrm{e}^{2\pi i\, B^{-1}u\cdot b} f_{B^{-1}u}(x)\, B^* e^{\pm}_J(u), f_u(x)\, e^{\pm}_J(u)\big\rangle \\ &= 2\, \delta_{Bu,u}\, \mathrm{e}^{2\pi i\, B^{-1}u\cdot b}\, \langle B^* e^{\pm}_J(u), e^{\pm}_J(u)\rangle\end{aligned}$$

and similarly for $\psi^{\pm}_{u,J}(x)$. Thus, if $Bu \neq u$ we get 0 in the expression above. On the other hand, for $u \in \Lambda^*_\mu$ with $Bu = u$, we obtain

$$\langle \gamma^* \phi^{\pm}_{u,J}(x), \phi^{\pm}_{u,J}(x)\rangle = 2\, \mathrm{e}^{2\pi i u\cdot b}\, \langle B^* e^{\pm}_J(u), e^{\pm}_J(u)\rangle,$$

and similarly $\langle \gamma^* \psi^{\pm}_{u,J}(x), \psi^{\pm}_{u,J}(x)\rangle = 2\, \mathrm{e}^{2\pi i u\cdot b}\, \langle B^* \tilde{e}^{\pm}_J(u), \tilde{e}^{\pm}_J(u)\rangle$. In this way, we arrive at the following expression

$$\operatorname{Tr} \gamma^*_{|H^{\pm}_\mu} = 2 \sum_{u\in(\Lambda^*_\mu)^B} \mathrm{e}^{2\pi i u\cdot b} \Big(\sum_{J\in\mathcal{S}} \langle B^* e^{\pm}_J(u), e^{\pm}_J(u)\rangle + \sum_{J\in\tilde{\mathcal{S}}} \langle B^* \tilde{e}^{\pm}_J(u), \tilde{e}^{\pm}_J(u)\rangle\Big) \tag{3.7}$$

where $(\Lambda^*_\mu)^B = \{u \in \Lambda^*_\mu : Bu = u\}$.

By splitting the contributions of the terms involving $\tilde{e}^{\pm}_J(u)$ with $|J| = 2h$ in (**??**), and applying Lemma **??** conveniently, we obtain

$$\begin{aligned}\operatorname{Tr} \gamma^*_{|H^{\pm}_\mu} = \sum_{u\in(\Lambda^*_\mu)^B} e^{2\pi i u\cdot b} \Big\{ &\sum_{\substack{n\notin J\\ |J|=2p\\ 0\le p\le h-1}} \langle Be_{u,J}, e_{u,J}\rangle + \sum_{\substack{n\in J\\ |J|=2p\\ 1\le p\le h-1}} \langle Be_{u,J}, e_{u,J}\rangle \\ &+ \tfrac12 \sum_{\substack{1,n\in J\\ |J|=2h}} \big\langle B^{-1}(e_{u,J} \pm i\, e_{u,n}\wedge * e_{u,J}), e_{u,J} \pm i\, e_{u,n}\wedge * e_{u,J}\big\rangle \Big\}.\end{aligned}$$

Now, applying Lemma **??** we get

$$\operatorname{Tr} \gamma^*_{|H^{\pm}_\mu} = \sum_{u\in(\Lambda^*_\mu)^B} e^{2\pi i u\cdot b} \Big(\sum_{p=0}^{h-1} \chi_{2p,u}(B) + \sum_{p=1}^{h-1} \chi_{2p-1,u}(B) + \chi^{\pm(-1)^h}_{m,u}(B^{-1}) \Big).$$

Since $\chi^{\pm(-1)^h}_{m,u}(B^{-1}) = \chi^{\mp(-1)^h}_{m,u}(B)$, the result clearly follows from (**??**).

Now consider $\mu = 0$. By repeating the previous argument, using the orthonormal basis $\{e_J : |J| = 2p, 0 \le p \le m\}$ of $H_0(\Lambda)$, we have

$$d_{0,\mathcal{D}}(\Gamma) = \tfrac{1}{|F|} \sum_{\gamma\in F} \operatorname{Tr} \gamma^*_{|H_0} = \tfrac{1}{|F|} \sum_{BL_b\in F} \sum_{\substack{J\subset I_n, |J|=2p\\ 0\le p\le m}} \langle B^* e_J, e_J\rangle.$$

Thus, by way of Lemma **??**, we get that the inner sum equals

$$\sum_{p=0}^{m} \Big(\sum_{\substack{n\in J\\ |J|=2p}} \langle B^* e_J, e_J\rangle + \sum_{\substack{n\notin J\\ |J|=2p}} \langle B^* e_J, e_J\rangle \Big) = \sum_{p=1}^{m} \chi_{2p,u}(B) + \chi_{2p-1,u}(B) = \sum_{p=0}^{n-1} \chi_{p,u}(B),$$

and hence the first identity in (**??**) follows. □

3.2. **Character expressions.** Our next goal will be to give an alternative expression for the multiplicities $d^{\pm}_{\mu,\mathcal{D}}(\Gamma)$ in Theorem **??**, so that the traces occurring can be more explicitly computed. To this end, we will need some facts on the conjugacy classes of $\mathrm{SO}(n-1)$ in $\mathrm{SO}(n)$.

Conjugacy classes. Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ with $e_n = (0, \ldots, 0, 1)$. For each $u \in \mathbb{R}^n$, there exists $C = C_u \in \mathrm{SO}(n)$ such that $C\hat{u} = e_n$. Now, if $B \in \mathrm{SO}(n)$ and $Bu = u$ then CBC^{-1} fixes e_n, i.e. $CBC^{-1} \in \mathrm{SO}(n-1)$. Furthermore, if $D\hat{u} = e_n$, then DBD^{-1} and CBC^{-1} are conjugate in $\mathrm{SO}(n-1)$.

If $u \in \Lambda^*_\mu$, let $\mathcal{B}_u$ be as in (**??**). For each $0 \le p \le m$, if $\{e_{u,J} : J \subset I_{n-1},\ |J| = p\}$ is a basis of $\Lambda^p(\mathbb{V}_u)$, then $\{Ce_{u,J} : J \subset I_{n-1},\ |J| = p\}$ is a basis of $\Lambda^p(\mathbb{R}^{n-1})$. Furthermore

$$\chi_{p,u}(B) = \chi_p(CBC^{-1}). \tag{3.8}$$

where χ_p denotes the character of the p-exterior representation of $\mathrm{SO}(n-1)$.

To get explicit expressions for the traces, we will work in the maximal torus T_{n-1} of $\mathrm{SO}(n-1)$. There exists $g \in \mathrm{SO}(n-1)$ such that

$$g\, B_u\, g^{-1} = x(t_1, \ldots, t_m) := \mathrm{diag}\Big(\left[\begin{smallmatrix} \cos t_1 & -\sin t_1 \\ \sin t_1 & \cos t_1 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} \cos t_m & -\sin t_m \\ \sin t_m & \cos t_m \end{smallmatrix}\right] \Big) \in T_{n-1},$$

with $t_1, \ldots, t_m \in \mathbb{R}$.

Consider the matrix $\tilde{U} = \left(\begin{smallmatrix} U & \\ & -1 \end{smallmatrix}\right) \in \mathrm{SO}(n)$, where

$$U = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in \mathrm{O}(n-1). \tag{3.9}$$

Note that, if $x(t_1, \ldots, t_m) \in \mathrm{T}_{n-1}$, then $U\, x(t_1, \ldots, t_m)\, U^{-1} = x(-t_1, t_2, \ldots, t_m)$. Thus, $x(t_1, \ldots, t_m)$ and $x(-t_1, \ldots, t_m)$ are conjugate in $\mathrm{SO}(n)$, but not in $\mathrm{SO}(n-1)$, generically. In this way, $gB_ug^{-1} = x_{u,B} := x(\pm t_1(B), t_2(B), \ldots, t_m(B))$, and we have the identities

$$\chi_p(B_u) = \chi_p(gB_ug^{-1}) = \chi_p(x_{u,B}). \tag{3.10}$$

Given $B \in \mathrm{SO}(n)$ with $Bu = u$, choose $x_B \in \mathrm{T}_{n-1}$ conjugate to B in $\mathrm{SO}(n)$. By the previous comments, B is conjugate either to x_B or to Ux_BU^{-1} in $\mathrm{SO}(n-1)$. In this manner, for each $u \in (\mathbb{R}^n)^B$ we define the sign

$$\sigma_{u,B} = \begin{cases} 1 & \text{if } CBC^{-1} \sim x_B, \\ -1 & \text{if } CBC^{-1} \sim Ux_BU^{-1}, \end{cases} \tag{3.11}$$

where $\sim$ denotes conjugation in $\mathrm{SO}(n-1)$ and $C\hat{u} = e_n$. Note that,

$$\sigma_{-u,B} = -\sigma_{u,B} \qquad \text{and} \qquad \sigma_{tu,B} = \sigma_{u,B}, \quad t > 0. \tag{3.12}$$

We now see how the change of conjugacy class affects the traces in $\mathrm{SO}(n-1)$.

Lemma 3.4. *If $B \in \mathrm{SO}(n-1)$ with $n = 2m+1$ we have $\chi_p(Ux_BU^{-1}) = \chi_p(x_B)$ for $0 \le p \le m-1$ and $\chi^\pm_m(Ux_BU^{-1}) = \chi^\mp_m(x_B)$, where U is defined in* (**??**).

Proof. We use that $Ue_J = -e_J$, if $1 \in J$, and $Ue_J = e_J$, if $1 \notin J$. Since $\{e_J : J \subset I_{2m}, |J| = p\}$ is a basis of $\Lambda^p(\mathbb{R}^{2m})$ and $\langle Ux_BU^{-1}e_J, e_J\rangle = \langle x_Be_J, e_J\rangle$, the first identity in the lemma follows.

For the second one, we know that $\{\frac{1}{\sqrt{2}}(e_J \pm (-1)^h \star e_J) : 1 \in J \subset I_{2m}, |J| = m\}$ is an orthonormal basis of $\Lambda^m_\pm(\mathbb{R}^{2m})$. Since $V\star = -\star V$ for any $V \in \mathrm{O}(n-1) \smallsetminus \mathrm{SO}(n-1)$ then $U(e_J \pm (-1)^h \star e_J) = -(e_J \mp (-1)^h \star e_J)$. Then, we have

$$\begin{aligned} \chi^\pm_m(Ux_BU^{-1}) &= \sum_{1 \in J, |J| = m} \big\langle x_B\, U(e_J \pm (-1)^h \star e_J), U(e_J \pm (-1)^h \star e_J)\big\rangle \\ &= \sum_{1 \in J, |J| = m} \big\langle x_B(e_J \pm (-1)^h \star e_J), e_J \pm (-1)^h \star e_J\big\rangle = \chi^\mp_m(x_B), \end{aligned}$$

and the lemma follows. $\square$

We can now give an alternative expression for the multiplicities $d^{\pm}_{\mu,\mathcal{D}}(\Gamma)$ in (**??**) that is better suited for explicit calculations.

Theorem 3.5. *Let $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ be an orientable compact flat manifold of odd dimension $n = 2m+1 = 4h-1$ with holonomy group $F = \Lambda\backslash\Gamma$. Then, we have*

$$d^{\pm}_{\mu,\mathcal{D}}(\Gamma) = \tfrac{1}{|F|} \sum_{\substack{BL_b\in\Lambda\backslash\Gamma \\ u\in(\Lambda^*_\mu)^B}} e^{2\pi i u\cdot b}\Big(\sum_{p=0}^{m-1}\chi_p(x_B) + \chi_m^{\pm\sigma'_{u,B}}(x_B)\Big) \tag{3.13}$$

where $\sigma'_{u,B} = (-1)^{h+1}\sigma_{u,B}$ and x_B is a fixed element in T_{n-1} conjugate in $\mathrm{SO}(n)$ to B. In particular, if $F \subset T_{n-1}$ then we may take $x_B = B$ for every $BL_b \in \Gamma$.

Furthermore,

$$d_{0,\mathcal{D}}(\Gamma) = \tfrac{1}{|F|} \sum_{BL_b\in F}\sum_{p=0}^{2m}\chi_p(x_B). \tag{3.14}$$

Both $d^{\pm}_{\mu,\mathcal{D}}(\Gamma)$ and $d_{0,\mathcal{D}}(\Gamma)$ can be computed explicitly using (**??**) *and* (**??**) *below.*

Proof. We have that $\sigma_B(u) = 1$ or -1 depending on whether $x_{u,B} \sim x_B$, or $x_{u,B} \sim Ux_BU^{-1}$. Then, by (**??**), (**??**) and Lemma **??**, we have that $\chi_{p,u}(B) = \chi_p(x_{u,B}) = \chi_p(x_B)$ for $0 \le p \le m-1$, and $\chi^{\pm}_{m,u}(B) = \chi^{\pm}_m(x_{u,B}) = \chi_m^{\pm\sigma_{u,B}}(x_B)$. By substituting these identities in (**??**) and (**??**) we get the desired expression. The remaining assertion is clear. $\square$

Remark 3.6. The multiplicities given in Theorem **??** are expressed in terms of the original data $BL_b \in F$. However, the traces $\chi_{p,u}(B)$ and $\chi^{\pm}_{m,u}(B)$ show a dependence on $u \in \Lambda^*_\mu$. As we shall see, the characters $\chi_p(x_B)$, $\chi^{\pm}_m(x_B)$ in expression (**??**) can be explicitly computed in terms of the rotation angles $t_j(x_B)$, using (**??**) and (**??**) below.

Character formulas on the maximal torus. Here we give explicit formulas for the characters χ_p, $0 \le p \le n-1$, and $\chi^{\pm}_m$ on elements of the maximal torus T_{2m} of $\mathrm{SO}(2m)$.

Proposition 3.7. *Let $n = 2m+1$. The characters of the irreducible representations $\Lambda^p(\mathbb{R}^{2m})_{\mathbb{C}}$, $0 \le p \le 2m$, and $\Lambda^m_{\pm}(\mathbb{R}^{2m})_{\mathbb{C}}$ of $\mathrm{SO}(2m)$, on $x = x(t_1,\dots,t_m) \in T_{2m}$, are respectively given by*

$$\chi_p(x) = \sum_{\substack{\ell=0 \\ (-1)^{\ell+p}=1}}^{p} 2^\ell \tbinom{m-\ell}{\frac{p-\ell}{2}} \sum_{\{j_1,\dots,j_\ell\}\subset I_m}\Big(\prod_{h=1}^{\ell}\cos t_{j_h}\Big) \qquad (0 \le p \le m), \tag{3.15}$$

and by duality $\chi_{n-p}(x) = \chi_p(x)$ for $m+1 \le p \le 2m$. Also

$$\chi^{\pm}_m(x) = \Bigg(\sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{m} 2^{\ell-1}\tbinom{m-\ell}{\frac{m-\ell}{2}} \sum_{\{j_1,\dots,j_\ell\}\subset I_m}\Big(\prod_{h=1}^{\ell}\cos t_{j_h}\Big)\Bigg) \pm 2^{m-1} i^m \Big(\prod_{j=1}^{m}\sin t_j\Big), \tag{3.16}$$

where $I_p = \{1,\dots,p\}$.

Proof. The weight vectors of $\mathrm{SO}(2m)$ on $\Lambda^1(\mathbb{R}^{2m})_{\mathbb{C}} \cong \Lambda^1(\mathbb{C}^{2m})$ have the form $e_{2j-1} \pm ie_{2j}$ with corresponding weights $\pm\varepsilon_j(x(t_1,\dots,t_m)) = e^{\pm it_j}$, for $1 \le j \le m$. Thus, the character of this representation is given by:

$$\chi_1(x(t_1,\dots,t_m)) = \sum_{j=1}^{m} e^{it_j} + e^{-it_j} = 2\sum_{j=1}^{m} \cos t_j\,.$$

More generally, if $1 \le p \le m$, the weight vectors for the exterior representation of $\mathrm{SO}(2m)$ on $\Lambda^p(\mathbb{C}^{2m})$, have the form $(e_{2j_1-1} \pm ie_{2j_1}) \wedge (e_{2j_2-1} \pm ie_{2j_2}) \wedge \ldots \wedge (e_{2j_p-1} \pm ie_{2j_p})$ with corresponding weights $\sum_{i=1}^{p} \pm\varepsilon_{j_i}$, where $\{j_1,\dots,j_p\} \subset I_m = \{1,2,\dots,m\}$. We note that when, among the p chosen weight vectors, both of the vectors $e_{2j_i-1} \pm ie_{2j_i}$ occur for some i, then their two weights $\pm\varepsilon_i$ add up to 0. Thus, if we order the weight vectors by putting at the end all those coming in pairs, say $e_{2h_1-1} \pm ie_{2h_1},\dots,e_{2h_r-1} \pm ie_{2h_r}$, then the corresponding weights are of the form $\sum\limits_{1\le i\le \ell} \pm\varepsilon_{j_i}$, with $\ell = p-2r$, added over all possible choices of signs, for each subset $\{j_1,\dots,j_\ell\} \subset I_p = \{1,2,\dots,p\}$.

We now compute the multiplicity of each weight. We note that a weight of the form $\sum_{i=1}^{p} \pm\varepsilon_{j_i}$ can be obtained in a unique way, corresponding to the wedge product of p weight vectors (with a choice of a sign for each) associated to a p-set of angles t_j (no two of which come in pairs). Therefore their multiplicity is equal to 1. For $\ell = p-2r < p$, the multiplicity is higher than 1 and equal to the number of choices of r pairs of weight vectors with $1 \le r \le \frac{m-p}{2}$. That is, for a weight of the form $\sum_{i=1}^{\ell} \pm\varepsilon_{j_i}$, with $\ell = p-2r$ and $0 \le r \le \frac{m-p}{2}$, the multiplicity is $\binom{m-\ell}{\frac{p-\ell}{2}}$.

We shall need the identities

$$2^{\ell-1}\prod_{h=1}^{\ell}(\cos t_{j_h} \pm i^\ell \sin t_{j_h}) = \begin{cases} \sum\limits_{\mathrm{ev}} e^{i\sum\limits_{h=1}^{\ell} \pm t_{j_h}} & (+\text{ sign}) \\[2ex] \sum\limits_{\mathrm{odd}} e^{i\sum\limits_{h=1}^{\ell} \pm t_{j_h}} & (-\text{ sign})\,, \end{cases} \tag{3.17}$$

where $\sum_{\mathrm{ev}}$ (resp. $\sum_{\mathrm{odd}}$) stands for the sum over all possible choices of an even (resp. odd) number of $-$ signs. To check (**??**), we note that

$$2^\ell\prod_{h=1}^{\ell}\big(\cos t_{j_h} \pm i^\ell \sin t_{j_h}\big) = \prod_{h=1}^{\ell}(e^{it_{j_h}} + e^{-it_{j_h}}) \;\pm\; i^\ell i^{-\ell}\prod_{h=1}^{\ell}(e^{it_{j_h}} - e^{it_{j_h}})\,.$$

Now, in the $+$ case, if we compute the products in both summands in the r.h.s. in all possible ways, the products of ℓ factors $e^{\pm it_{j_i}}$ having an odd number of $-$ signs in the exponents cancel, hence we get twice the sum of products having an even number of $-$ signs and we thus get the identity in (**??**) in the $+$ case. The identity in the $-$ case is obtained in the same way.

If $p < m$, the representation $\Lambda^p(\mathbb{C}^{2m})$ is irreducible and the character $\chi_p(x(t_1,\dots,t_m))$ is the sum of all of its weights, i.e. of all exponentials of type $e^{\sum_{i=1}^{\ell} \pm it_{j_i}}$, with $\ell = p-2r$ for some $r \ge 0$, each one counted with its multiplicity. As explained before, for each fixed choice of $\{j_1,\dots,j_\ell\} \subset \{1,\dots,m\}$, the contribution is

$$\tbinom{m-\ell}{\frac{p-l}{2}}\sum e^{\sum_{h=1}^{\ell} \pm it_{j_h}} = 2^\ell\tbinom{m-\ell}{\frac{p-\ell}{2}}\prod_{h=1}^{\ell}\cos t_{j_h}\,,$$

obtained by adding both products in (**??**), since we have to consider all possible choices of signs. Thus we get the expression in (**??**).

In the case $p = m$, the representation splits as a sum of the two irreducible subrepresentations $\Lambda^m_\pm(\mathbb{C}^{2m})$ with highest weights $\sum_{i=1}^{m-1} \varepsilon_i \pm \varepsilon_m$. Thus, $\Lambda^m_+(\mathbb{C}^{2m})$ (resp. $\Lambda^m_-(\mathbb{C}^{2m})$) has weights with multiplicity one, of the form $\sum_{i=1}^m \pm\varepsilon_i$ over all sums having an even (resp. odd) number of minus signs. The remaining weights of $\Lambda^m_\pm(\mathbb{C}^{2m})$ have the form $\sum_{i=1}^{\ell} \pm \varepsilon_{j_i}$, for each $\{i_1, \ldots, i_\ell\} \subset I_m$ and *all* possible choices of signs, with multiplicities $\frac{1}{2}\binom{m-\ell}{\frac{m-\ell}{2}}$ for both $\Lambda^m_\pm(\mathbb{C}^{2m})$. In this way we get the expression in (**??**). □

We now list some useful facts on the values of the characters $\chi^\pm_m$.

Corollary 3.8. *For $x = x(t_1, \ldots, t_m) \in T_{2m}$ we have:*

(i) $\chi_0(x), \ldots, \chi_{m-1}(x) \in \mathbb{R}$ *and* $\overline{\chi^\pm_m(x)} = \chi^\mp_m(x)$.

(ii) $\big(\chi^+_m - \chi^-_m\big)(x) = (2i)^m \big(\prod_{j=1}^{m} \sin t_j \big)$.

(iii) $\chi^\pm_m(x) \in \mathbb{R}$ *if and only if m is even or $t_i \in \pi\mathbb{Z}$ for some $1 \le i \le m$.*

(iv) $\chi^+_m(x) = \chi^-_m(x)$ *if and only if $t_i \in \pi\mathbb{Z}$ for some $1 \le i \le m$. In particular, this holds for any x of order 2.*

Proof. Items (i), (ii) and (iii) are clear from (**??**) and (**??**). Now (iv) follows from (ii) since $\chi^+_m(x) = \chi^-_m(x)$ if the r.h.s. in (ii) equals 0, that is, if and only if $t_i \in \pi\mathbb{Z}$ for some $i = 1, \ldots, m$. If x is of order 2 then $x = x(\pm t_1, t_2, \ldots, t_m)$ with $t_i \in \pi\mathbb{Z}$ and $t_i = \pi$ for at least one i. □

3.3. **Symmetry and $\mathcal{D}$-isospectrality.** Suppose M_Γ is a n-dimensional orientable compact flat manifold with translation lattice Λ and holonomy group $F \simeq \Lambda\backslash\Gamma$. If $\gamma = BL_b \in \Gamma$ then $B \in \mathrm{SO}(n)$ and $B\Lambda = \Lambda$. Denote by $o(B)$ the order of B. Set

$$e_{\mu,\gamma}(\Gamma) = \sum_{u \in (\Lambda^*_\mu)^B} e^{2\pi i u\cdot b}. \tag{3.18}$$

We can now state a simple criterion for spectral symmetry.

Corollary 3.9. *Let M_Γ be an orientable compact flat manifold of odd dimension n with holonomy group $F \simeq \Lambda\backslash\Gamma$. If $\chi^+_m(B) = \chi^-_m(B)$ for every B with $n_B = 1$, then M_Γ has symmetric spectrum for $\mathcal{D}$ and, in this case,*

$$d^+_{\mu,\mathcal{D}}(\Gamma) = d^-_{\mu,\mathcal{D}}(\Gamma) = \tfrac{1}{2|F|} \sum_{\gamma = BL_b \in \Lambda\backslash\Gamma} e_{\mu,\gamma}(\Gamma) \cdot \sum_{p=0}^{2m} \chi_p(B) \tag{3.19}$$

Proof. We use (**??**). It is clear that $\chi^+_m(B) = \chi^-_m(B)$ for every $B \in F$ implies that $d^+_{\mu,\mathcal{D}}(\Gamma) = d^-_{\mu,\mathcal{D}}(\Gamma)$. Actually, we need the condition $\chi^+_m(B) = \chi^-_m(B)$ only for $B \in F_1$ with $o(B) \ge 3$. Indeed, $n_B \ge 1$ for every $BL_b \in \Gamma$. If $n_B \ge 2$ then $x_B \sim x(0, t_2, \ldots, t_m)$, and hence by (iv) of Corollary **??** we have $\chi^+_m(x_B) = \chi^-_m(x_B)$.

Now, $\chi^+_m(x_B) + \chi^-_m(x_B) = \chi_m(x_B)$ and by duality $\chi_p(x_B) = \chi_{n-p}(x_B)$ (since M_Γ is orientable). Thus, we have that $2\sum_{p=0}^{m-1} \chi_p(x_B) = \sum_{p=0}^{m-1} \chi_p(x_B) + \sum_{p=m+1}^{2m} \chi_p(x_B)$. Hence (**??**) follows from (**??**), with $e_{\mu,\gamma}(\Gamma)$ as given in (**??**). □

Remark 3.10. Note that the corollary is consistent with the fact that $d^+_{\mu,\mathcal{D}}(\Gamma) + d^-_{\mu,\mathcal{D}}(\Gamma) = d_{\mu^2,\Delta_e}(\Gamma)$, as it should be.

$\mathcal{D}$-isospectrality of $\mathbb{Z}_2^k$-manifolds. Here we will show that all $\mathbb{Z}_2^k$-manifolds are $\mathcal{D}$-isospectral to each other. This is a large class. Indeed, it has been shown in [?] that, if $k = n-1$, there are at least $2^{\frac{(n-1)(n-2)}{2}}$ manifolds of dimension n and holonomy group $\mathbb{Z}_2^{n-1}$ that are pairwise non-homeomorphic to each other.

We will next show that $\mathbb{Z}_2^k$-manifolds have symmetric $\mathcal{D}$-spectrum and that expression (**??**) for the multiplicities dramatically simplifies for the whole class of $\mathbb{Z}_2^k$-manifolds.

Proposition 3.11. *Let M_Γ be an n-dimensional $\mathbb{Z}_2^k$-manifold, $1 \le k \le n-1$, with lattice of translations Λ. Then M_Γ has symmetric $\mathcal{D}$-spectrum and*

$$d^{\pm}_{\mu,\mathcal{D}}(\Gamma) = 2^{n-k-2}|\Lambda^*_\mu| \quad (\mu > 0), \qquad d_{0,\mathcal{D}}(\Gamma) = 2^{n-k-1}. \tag{3.20}$$

Furthermore, for any fixed k, all $\mathbb{Z}_2^k$-manifolds with isospectral covering tori are mutually $\mathcal{D}$-isospectral. In particular, all $\mathbb{Z}_2^k$-manifolds covered by the same torus are $\mathcal{D}$-isospectral to each other.

Proof. Since M_Γ is a $\mathbb{Z}_2^k$-manifold, $B = B^{-1}$ for any $BL_b \in \Gamma$. Thus, by Lemma **??**, we have that $\langle B^* \tilde{e}^{\pm}_J(u), \tilde{e}^{\pm}_J(u)\rangle = \frac{1}{2}\langle Be_{u,J}, e_{u,J}\rangle$ with $|J| = 2p$ for every $1 \le p \le h$, and thus the difference between $d^+_{\mu,\mathcal{D}}(\Gamma)$ and $d^-_{\mu,\mathcal{D}}(\Gamma)$ disappear. Indeed, substituting this in (**??**), and following the computations, one gets (**??**) with $\chi^{\pm}_{m,u}(B)$ replaced by $\frac{1}{2}\chi_{m,u}(B)$ in this case. The symmetry of the spectrum of $\mathcal{D}$ can also be easily deduced from Corollary **??** or from the expressions of characters by using Theorem **??** and Corollary **??** (iv).

We now prove (**??**). Since $\mathcal{D}^2 = \Delta_e$, $\mathcal{D}$-isospectral implies Δ_e-isospectral. Now $\mathbb{Z}_2^k$-manifolds have symmetric $\mathcal{D}$-spectrum, hence $2d^+_{\mu,\mathcal{D}}(M) = 2d^-_{\mu,\mathcal{D}}(M) = d_{\mu^2,\Delta_e}(M)$. By Theorem 2.1 in [?], all $\mathbb{Z}_2^k$-manifolds having isospectral covering tori are mutually Δ_e-isospectral, that is $d_{\mu,\Delta_e}(M) = d_{\mu,\Delta_e}(M')$ for any pair M, M' of $\mathbb{Z}_2^k$-manifolds. From these two facts one obtains that $d^{\pm}_{\mu,\mathcal{D}}(M) = d^{\pm}_{\mu,\mathcal{D}}(M')$ thus proving the assertion. $\square$

Open question 3.12. Since $\mathcal{D}^2 = \Delta_e$, $\mathcal{D}$-isospectral implies Δ_e-isospectral. Proposition **??** shows the converse is trivially true for all $\mathbb{Z}_2^k$-manifolds, but we do not know if it holds in general for flat manifolds. We do not know a pair of Δ_e-isospectral compact flat manifolds (or even just compact Riemannian manifolds) that are not $\mathcal{D}$-isospectral. We note that for the spin Dirac operator D, by using different spin structures, it is not too difficult to give examples of Δ_{spin}-isospectral $\mathbb{Z}_2^k$-manifolds that are not D-isospectral (see [?], Example 4.4). Here, $D^2 = \Delta_{spin}$.

4. Eta series and eta invariants

For a compact flat manifold M_Γ, the eta series in (**??**) can be written in the form

$$\eta_{\mathcal{D}}(s,\Gamma) = \sum_{\mu \in \frac{1}{2\pi}\mathcal{A}^+} \frac{d^+_{\mu,\mathcal{D}}(\Gamma) - d^-_{\mu,\mathcal{D}}(\Gamma)}{(2\pi\mu)^s} \tag{4.1}$$

for $\mathrm{Re}(s) > 1$, where $\mathcal{A}^+ = \mathcal{A} \cap \mathbb{R}^+$ and $\mathcal{A} = \{\lambda \in Spec_{\mathcal{D}} : d_\lambda \ne d_{-\lambda}\}$, the *asymmetric* part of the spectrum. The multiplicities $d^{\pm}_{\mu,\mathcal{D}}(\Gamma)$ were computed in (**??**).

We know that any flat torus $T_\Lambda = \Lambda\backslash\mathbb{R}^n$ and any $\mathbb{Z}_2^k$-manifold M_Γ have symmetric $\mathcal{D}$-spectrum (see Theorem **??** and Proposition **??**) hence they have eta series and η-invariant equal to 0.

Our goal is to give an expression for the eta series for a general compact flat manifold.

Suppose M_Γ is an n-dimensional orientable compact flat manifold with translation lattice Λ and holonomy group $F \simeq \Lambda\backslash\Gamma$. If $\gamma = BL_b \in \Gamma$ then $B \in \mathrm{SO}(n)$ and $B\Lambda = \Lambda$. We will need some notations. Put

$$F_1 = F_1(\Gamma) := \{B \in F : n_B = 1\} \quad \text{where} \quad n_B := \dim\, (\mathbb{R}^n)^B. \tag{4.2}$$

If $x_B = x(t_1(B), \ldots, t_m(B)) \in \mathrm{T}_{n-1}$ is conjugate to B, we denote the rotation angles by $t_i(x_B)$ and put

$$F_1' = F_1'(\Gamma) := \{B \in F_1 : t_i(x_B) \notin \pi\mathbb{Z},\ 1 \le i \le m\} \tag{4.3}$$

Note that F_1' excludes the identity element and all elements of order 2.

For any $B \in F_1$, we pick $v_B \in \Lambda^*$ one of the two generators of the 1-dimensional lattice fixed by B, that is

$$(\Lambda^*)^B := \mathbb{Z}v_B. \tag{4.4}$$

Note that v_B depends on Λ, although we do not reflect this in the notation.

Lemma 4.1. *Let Γ be a Bieberbach group of $I(\mathbb{R}^n)$. For any $\gamma = BL_b \in F_1(\Gamma)$ we have $v_B \cdot b \in \frac{1}{o(B)}\mathbb{Z}$, where $o(B)$ is the order of B. That is, $v_B \cdot b = \frac{\ell_\gamma}{o(B)}$ with $\ell_\gamma \in \mathbb{Z}$.*

Proof. Put $b = b_+ + b'$ where b_+ and b' are the orthogonal projections of b onto $(\mathbb{R}^n)^B$ and $((\mathbb{R}^n)^B)^\perp$, respectively. Now,

$$(BL_b)^{o(B)} = B^{o(B)} L_{\sum_{j=0}^{o(B)-1} B^j b} = L_{o(B)b_+} \in L_\Lambda\,,$$

since $\sum\limits_{j=0}^{o(B)-1} B^j b' \in \ker(B-I) \cap \ker(B-I)^\perp$. Thus $b_+ \in \frac{1}{o(B)}\Lambda$, hence $v_B \cdot b = v_B \cdot b_+ \in \frac{1}{o(B)}\mathbb{Z}$, since $v_B \in \Lambda^*$. So, we can write $v_B \cdot b = v_B \cdot b_+ = \frac{\ell_\gamma}{o(B)}$ for some $\ell_\gamma \in \mathbb{Z}$, as asserted. $\square$

In the previous notations, we can now state the following theorem.

Theorem 4.2. *Let M_Γ be an orientable compact flat manifold with translation lattice Λ, holonomy group $F \simeq \Lambda\backslash\Gamma$ and dimension $n = 2m+1 = 4h-1$. Then, the eta function on M_Γ associated to $\mathcal{D}$ is given by*

$$\eta_{\mathcal{D}}(s) = -\tfrac{2^{m+1}}{|F|} \sum_{\substack{BL_b \in \Lambda\backslash\Gamma \\ B \in F_1'}} \tfrac{\sigma_{v_B,B}}{(2\pi\, \|v_B\|\, o(B))^s} \Big(\prod_{j=1}^m \sin t_j(x_B)\Big) \cdot \sum_{j=1}^{[\frac{o(B)-1}{2}]} \sin(\tfrac{2\pi j \ell_\gamma}{o(B)}) \Big(\zeta(s, \tfrac{j}{o(B)}) - \zeta(s, 1 - \tfrac{j}{o(B)})\Big), \tag{4.5}$$

where $o(B)$ is the order of B, ℓ_γ is as defined in Lemma **??** *and $\zeta(s,\alpha) = \sum\limits_{t=0}^{\infty} \frac{1}{(t+\alpha)^s}$ is the Hurwitz zeta function for $\alpha \in (0,1]$.*

Furthermore, the eta invariant is given by

$$\eta_{\mathcal{D}}(0) = -\tfrac{2^m}{|F|} \sum_{\substack{\gamma = BL_b \in \Lambda\backslash\Gamma \\ B \in F_1',\, \ell_\gamma \notin o(B)\mathbb{Z}}} \sigma_{v_B,B} \Big(\prod_{j=1}^m \sin t_j(x_B)\Big)\ \cot(\tfrac{\pi \ell_\gamma}{o(B)}). \tag{4.6}$$

Proof. We begin by looking at the difference $d^+_{\mu,\mathcal{D}}(\Gamma) - d^-_{\mu,\mathcal{D}}(\Gamma)$ in (**??**). By using (**??**) and (ii) of Corollary **??**, we get

$$\begin{aligned} d^+_{\mu,\mathcal{D}}(\Gamma) - d^-_{\mu,\mathcal{D}}(\Gamma) &= \tfrac{(-1)^{h+1}}{|F|} \sum_{\substack{BL_b\in\Lambda\backslash\Gamma \\ u\in(\Lambda^*_\mu)^B}} \sigma_{u,B}\, e^{2\pi i u\cdot b}\, \big(\chi^+_m - \chi^-_m\big)(x_B) \\ &= \tfrac{2^m\, i}{|F|} \sum_{BL_b\in\Lambda\backslash\Gamma} \Big(\prod_{j=1}^m \sin t_j(x_B)\Big) \sum_{u\in(\Lambda^*_\mu)^B} \sigma_{u,B}\, e^{2\pi i u\cdot b}, \end{aligned} \tag{4.7}$$

where we have used that $m = 2h-1$.

Since $\prod_{j=1}^m \sin t_j(x_B) = 0$ if and only if $t_j(x_B)\in\pi\mathbb{Z}$ for some $1\le j\le m$, we see that only the elements $\gamma = BL_b$ with $B\in F_1'$ (see (**??**)) can have a non-trivial contribution to the sum in (**??**). In fact, if $n_B>1$ then $n_B\ge 3$ (by orientability) and hence x_B is conjugate to $x(0,t_2,\dots,t_m)$. In case $n_B = 1$ and -1 is an eigenvalue of B, then x_B is conjugate to $x(\pi,t_2,\dots,t_m)$.

For elements $B\in F_1'$, one has $n_B = 1$, thus $(\Lambda^*)^B = \mathbb{Z}v_B$ with $v_B\in\Lambda^*$ as in (**??**), hence any $u\in(\Lambda^*)^B$ is of the form $u = \ell v_B$, with $\ell\in\mathbb{Z}$. In this way, we have that

$$\mathcal{A}^+ = \{2\pi\ell\,\|v_B\| : \ell\in\mathbb{N}, B\in F_1(\Gamma)\}$$

and, by (**??**) and (**??**), we get

$$\eta(s) = \tfrac{2^m\, i}{|F|(2\pi\|v_B\|)^s} \sum_{\substack{BL_b\in\Lambda\backslash\Gamma \\ B\in F_1'}} \Big(\prod_{j=1}^m \sin t_j(x_B)\Big) \sum_{\ell=1}^\infty \sum_{u\in\{\pm\ell\|v_B\|\}} \frac{\sigma_{u,B}\, e^{2\pi i u\cdot b}}{\ell^s}.$$

Using (**??**) and putting together the contributions of u and $-u$ we have

$$\sigma_{\ell v_B,B}\, e^{2\pi i\ell v_B\cdot b} + \sigma_{-\ell v_B,B}\, e^{-2\pi i\ell v_B\cdot b} = 2i\,\sigma_{v_B,B}\sin(2\pi\ell\, v_B\cdot b).$$

Thus, we arrive at the expression

$$\eta(s) = -\tfrac{2^{m+1}}{|F|} \sum_{\substack{BL_b\in\Lambda\backslash\Gamma \\ B\in F_1'}} \tfrac{\sigma_{v_B,B}}{(2\pi\|v_B\|)^s} \Big(\prod_{j=1}^m \sin t_j(x_B)\Big) \sum_{\ell=1}^\infty \frac{\sin(2\pi\ell v_B\cdot b)}{\ell^s} \tag{4.8}$$

Now, by Lemma **??**, for each $\gamma = BL_b\in\Gamma$, we have $v_B\cdot b = \frac{\ell_\gamma}{o_B}$ with $\ell_\gamma\in\mathbb{Z}$. Thus, writing $\ell = t\,o(B)+j$ with $t\in\mathbb{Z}$ and $0\le j\le o(B)-1$ we have that

$$\sum_{\ell=1}^\infty \frac{\sin(2\pi\ell v_B\cdot b)}{\ell^s} \;=\; \tfrac{1}{o(B)^s} \sum_{j=1}^{o(B)-1} \sin\big(\tfrac{2\pi j\ell_\gamma}{o(B)}\big) \sum_{t=0}^\infty \frac{1}{(t+\frac{j}{o(B)})^s}. \tag{4.9}$$

By substituting (**??**) in (**??**) we get

$$\eta(s) = -\tfrac{2^{m+1}}{|F|} \sum_{\substack{\gamma=BL_b\in\Lambda\backslash\Gamma \\ B\in F_1'}} \tfrac{\sigma_{v_B,B}}{(2\pi\,\|v_B\|\,o(B))^s} \Big(\prod_{j=1}^m \sin t_j(x_B)\Big) \sum_{j=1}^{o(B)-1} \sin\big(\tfrac{2\pi j\ell_\gamma}{o(B)}\big)\,\zeta\big(s,\tfrac{j}{o(B)}\big), \tag{4.10}$$

where $\zeta(s,\alpha) = \sum\limits_{t=0}^\infty \frac{1}{(t+\alpha)^s}$ for $0<\alpha\le 1$ and $\operatorname{Re} s>1$.

Note that, for any $k\in\mathbb{Z}$, one has that

$$\sin\big(\tfrac{2\pi(o(B)-j)k}{o(B)}\big) = -\sin\big(\tfrac{2\pi jk}{o(B)}\big).$$

By using this fact in (**??**) we get (**??**). Note that if $o(B)$ is even, there is a term coming alone in the sum over j, corresponding to $j = \frac{o(B)}{2}$. For this term we get $\sin(\frac{2\pi j \ell_\gamma}{o(B)}) = \sin(\pi \ell_\gamma) = 0$, since $\ell_\gamma \in \mathbb{Z}$. From the final formula (**??**) we deduce that $\eta(s)$ is an entire function, since $\zeta(s,\alpha)$ has a simple pole at $s = 1$. Note that only those $\gamma \in \Gamma$ such that $\ell_\gamma \notin o(B)\mathbb{Z}$ do contribute to the sum.

We will now verify the expression for the eta invariant. We have

$$\eta(0) = -\tfrac{2^{m+1}}{|F|} \sum_{\substack{\gamma = BL_b \in \Lambda\backslash\Gamma \\ B \in F_1'}} \sigma_{v_B,B} \Big(\prod_{j=1}^{m} \sin t_j(x_B) \Big) \sum_{j=1}^{o(B)-1} \sin(\tfrac{2\pi j \ell_\gamma}{o(B)})\, \zeta\big(0, \tfrac{j}{o(B)}\big).$$

By using that $\zeta(0,\alpha) = \frac{1}{2} - \alpha$ ([**?**]) and since $\sum_{j=1}^{p} \sin\big(\frac{2\pi jk}{p}\big) = 0$ for any $k, p \in \mathbb{N}$, we obtain, for γ such that $\ell_\gamma \notin o(B)\mathbb{Z}$,

$$\sum_{j=1}^{o(B)-1} \sin\big(\tfrac{2\pi j \ell_\gamma}{o(B)}\big)\big(\tfrac{1}{2} - \tfrac{j}{o(B)}\big) \quad = \quad -\tfrac{1}{o(B)} \sum_{j=1}^{o(B)-1} j\, \sin\big(\tfrac{2\pi j \ell_\gamma}{o(B)}\big) = -\tfrac{o(B)}{2} \cot\big(\tfrac{\pi k}{o(B)}\big),$$

where we have used the identity $\sum_{j=1}^{d-1} j\, \sin\big(\frac{2\pi jk}{d}\big) = -\frac{d}{2}\cot\big(\frac{\pi k}{d}\big)$, valid for integers k, d with $d \nmid k$ (see [**?**, (5.6)] for a proof). The desired formula for $\eta(0)$ thus follows. □

Remark 4.3. Consider the operator $\mathcal{D}_1 = d* : d\Omega^{2k-1} \subset \Omega^{2k} \to \Omega^{2k}$. The eigenfunctions of $\mathcal{D}_1$ are given by the $\psi^{\pm}_{u,J}(x)$ as in (**??**) with the J's restricted to those with $|J| = 2h$. Thus, by proceeding similarly as in the proof of Theorem **??**, and using (**??**) we get

$$d^{\pm}_{\mu,\mathcal{D}_1}(\Gamma) = \tfrac{1}{|F|} \sum_{\substack{BL_b \in \Lambda\backslash\Gamma \\ u \in (\Lambda^*_\mu)^B}} e^{2\pi i u\cdot b}\, \chi_m^{\pm\epsilon_{u,B}}(x_B). \tag{4.11}$$

where $\epsilon_{u,B} = \mp(-1)^{h+1}\sigma_{u,B}$ and

$$d_{0,\mathcal{D}_1}(\Gamma) = \tfrac{1}{|F|} \sum_{BL_b \in \Lambda\backslash\Gamma} \chi_m(x_B). \tag{4.12}$$

Thus, although $d^{\pm}_{\mu,\mathcal{D}}(\Gamma) \neq d^{\pm}_{\mu,\mathcal{D}_1}(\Gamma)$, we have $d^{+}_{\mu,\mathcal{D}}(\Gamma) - d^{-}_{\mu,\mathcal{D}}(\Gamma) = d^{+}_{\mu,\mathcal{D}_1}(\Gamma) - d^{-}_{\mu,\mathcal{D}_1}(\Gamma)$. Therefore, the eta series, as well as the eta invariants, for $\mathcal{D}$ and $\mathcal{D}_1$ are the same, i.e. $\eta(s) = \eta_1(s)$, $\eta(0) = \eta_1(0)$. These equalities are valid for general Riemannian manifolds, as was observed in [**?**, Prop. 4.20]. However, it should be pointed out that the reduced eta invariants, defined by $\bar\eta = \frac{1}{2}(\eta + \dim\ker D) \bmod \mathbb{Z}$, might be different, since one might have $d^{\pm}_{0,\mathcal{D}}(\Gamma) \neq d^{\pm}_{0,\mathcal{D}_1}(\Gamma)$.

References

[1] T. Apostol, *Introduction to analytic number theory*, Springer Verlag UTM, New York, 1998.

[2] M. F. Atiyah, V. K. Patodi, I. M. Singer, *Spectral asymmetry and Riemannian geometry I, II, III*, Math. Proc. Cambridge Philos. Soc. **77**, (43–69) 1975; **78** (405–432) 1975; **79**, (71–99) 1976.

[3] S. Bechtluft-Sachs, *The computation of η-invariants on manifolds with free circle action*, J. Funct. Anal. **174**, 2, (251–263) 2000.

[4] J. L. Cisneros-Molina, *The η-invariant of twisted Dirac operators of S^3/Γ*, Geom. Dedicata **84**, 1-3, (207–228) 2001.

[5] H. Donnelly. *Eta invariants for G-spaces*, Indiana University Math. J., vol. **224**, (161–170) 1976.

[6] P. B. Gilkey, *The eta invariant and the K-theory of odd-dimensional spherical space forms*, Invent. Math. **76**, 3, (421–453) 1984.

[7] P. B. Gilkey, *The geometry of spherical space form groups*, with an appendix by A. Bahri and M. Bendersky. Series in Pure Mathematics, 7. World Scientific Publishing, Teaneck, NJ, 1989.

[8] N. Ginoux, *The Dirac Spectrum.* Lecture Notes in Mathematics V. 1976. Springer Verlag, Berlin, Heidelberg, 2009.

[9] P. B. Gilkey, R. J. Miatello, R. A. Podestá, *The eta invariant and equivariant bordism of flat manifolds with cyclic holonomy group of odd prime order*, Ann. Glob. Anal. Geom. **37**, (275–306) 2010.

[10] S. Goette, *Equivariant η-invariants and η-forms*, J. Reine Angew. Math. **526**, (181–236) 2000.

[11] C. Guillarmou, S. Moroianu, J. Park,*Eta invariant and Selberg zeta function of odd type over convex co-compact hyperbolic manifolds*, Adv. Math. **225**, (2464–2516) 2010.

[12] G. Hamrick, D. Royster, *Flat Riemannian manifolds are boundaries*, Invent. Math. **66**, (405–413) 1982.

[13] D. D. Long, A. W. Reid, *On the geometric boundaries of hyperbolic 4-manifolds*, Geometry and Topology, **4**, (171–178) 2000.

[14] D. D. Long, A. W. Reid, *Constructing hyperbolic manifolds which bound geometrically*, Math. Res. Lett. **8**, 4, (443–455) 2001.

[15] D. D. Long, A. W. Reid, *All flat manifolds are cusps of hyperbolic orbifolds*, Algebraic and Geometric Topology **2**, (285–296) 2002.

[16] R. Meyerhoff, W. D. Neumann, *An asymptotic formula for the eta invariants of hyperbolic* 3-*manifolds*, Comment. Math. Helv. **67**, 1, (28–46) 1992.

[17] R. Meyerhoff, M. Ouyang, *The η-invariants of cusped hyperbolic* 3-*manifolds*, Canad. Math. Bull. **40**, 2, (204–213) 1997.

[18] R. J. Miatello, R. A. Podestá, *Spin structures and spectra of $\mathbb{Z}_2^k$-manifolds*, Math. Zeitschrift **247**, (319–335) 2004.

[19] R. J. Miatello, R. A. Podestá, *The spectrum of twisted Dirac operators on compact flat manifolds*, Trans. Amer. Math. Society **358**, 10, (4569–4603) 2006.

[20] R. J. Miatello, R. A. Podestá, J. P. Rossetti, *$\mathbb{Z}_2^k$-manifolds are isospectral on forms*, Math. Zeitschrift **258**, (301–317) 2008.

[21] W. Müller, *The eta invariant (some recent developments)*, Séminaire Bourbaki, Vol. 1993/94. Astérisque No. **227**, Exp. No. 787, 5, (335–364) 1995.

[22] L. I. Nicolaescu, *Eta invariants of Dirac operators on circle bundles over Riemann surfaces and virtual dimensions of finite energy SeibergŨWitten moduli spaces*, Israel J. Math. **114**, (61–123) 1999.

[23] B. E. Nimershiem, *All flat three-manifolds appear as cusps of hyperbolic four manifolds*, Topology and its Appl. **90**, (109–133) 1998.

[24] M. Q. Ouyang, *Geometric invariants for Seifert fibred 3-manifolds*, Trans. Amer. Math. Society **346**, 2, (641–659) 1994.

[25] M. Q. Ouyang, *On the eta-invariant of some hyperbolic* 3-*manifolds*, Topology Appl. **64**, 2, (149–164) 1995.

[26] A. Szczepański, *Problems on Bieberbach groups and flat manifolds*, Geom. Dedicata **120**, (111–118) 2006.

[27] A. Szczepański, *Eta invariants for flat manifolds*, `arXiv`:1001.1270, 2011.

[28] T. Yoshida, *The η-invariant of hyperbolic* 3-*manifolds*, Invent. Math. **81**, 3, (473–514) 1985.

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