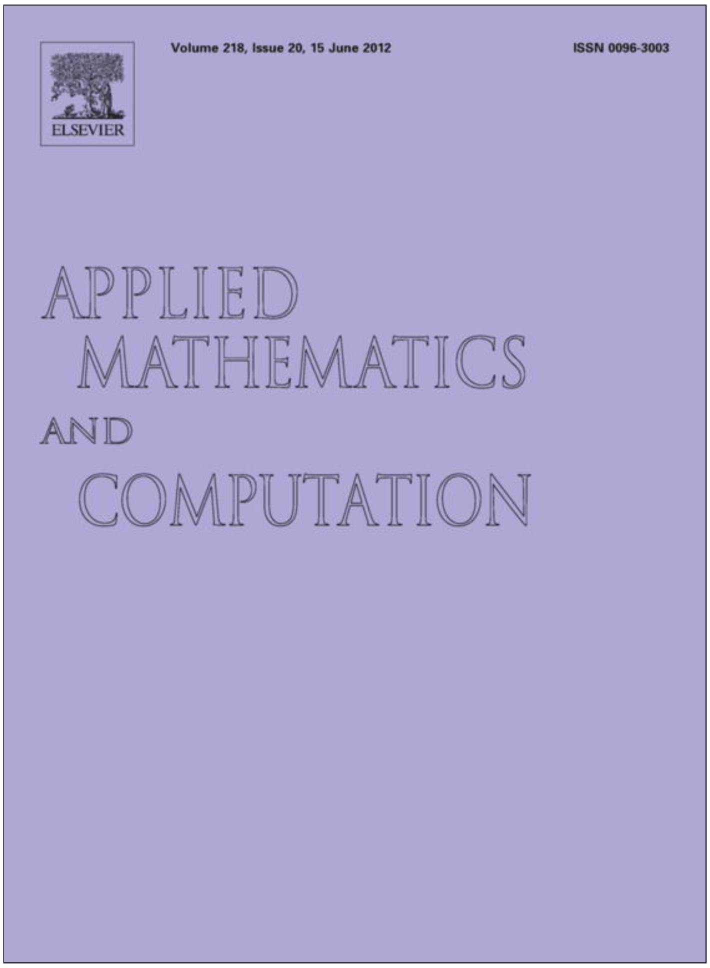
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The $K(m,n)$ equation with generalized evolution term studied by symmetry reductions and qualitative analysis

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ABSTRACT

In this paper we obtain symmetry reductions of the $K(m,n)$ equation with generalized evolution term. The reduction to ordinary differential equations comes from an optimal system of subalgebras. Some of these equations admit symmetries which lead to further reductions, and one of them comes out suitable for qualitative analysis. Its dynamical behavior is fully described and conservative quantities are stated.

1. Introduction

The $K(m,n)$ equation with generalized evolution term, was introduced by Biswas in [1] and it is given by

$$(u^l)_t + au^m u_x + b(u^n)_{xxx} = 0, \tag{1.1}$$

where $a, b \in \mathbb{R}^*$ and $l, m, n \in \mathbb{Z}^+$. The first term is the generalized evolution term, the second and the third terms represent the convection one and the dispersion one, respectively.

In [2], Bruzón and Gandarias presented a procedure to look for exact solutions of nonlinear ordinary differential equations (ODE's), which leads to solutions (not obtained in [1]) in terms of Jacobi elliptic functions for specific values of the parameters l, m, n, a and b of Eq. (1.1). This equation is a generalized form of the $K(m,n)$ equation, usually introduced as

$$u_t + a(u^m)_x + b(u^n)_{xxx} = 0 \tag{1.2}$$

and, in turn, of the Korteweg–de Vries (KdV) equation, where $l = m = n = 1$. On the other hand, Eq. (1.1) is equivalent to

$$v_t + \frac{a}{l} v^{\frac{m+1-l}{l}} v_x + b(v^{\frac{n}{l}})_{xxx} = 0,$$

after using the transformation $u = v^{\frac{1}{l}}$, so it is sufficient to consider the case $l = 1$ if just $\frac{m+1-l}{l};\ \frac{1}{l} \in \mathbb{Z}^+$.

Different variants or particular cases of the $K(m,n)$ equation are found in the literature [2–12]. Recently Chen and Li [3] have studied the simple peak solitary wave solutions of the osmosis $K(2,2)$ equation under the inhomogeneous boundary conditions and they have obtained all smooth, peaked and cusped solitary wave solutions of it. The modified KdV (mKdV) equations (Eq. (1.1) with $l = 1,\ m = 2$ and $n = 1$) and their solutions have also been studied intensively. Liu and Li ([4] and its references) considered an extended form of the mKdV equation of the form

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$$u_t + a_1 u_{xxx} + a_2 u_x + a_3 u u_x + a_4 u^2 u_x = 0,$$

the all exact solutions based on the Lie group method were given, and the bifurcations and traveling wave solutions were obtained. Rosenau and Hyman [5] studied the role of nonlinear dispersion in the formation of patterns in liquid drops of the nonlinear dispersive equations.

$$u_t + u^m u_x + (u^n)_{xxx} = 0$$

for $m > 0,\ 1 < n \leqslant 3$. They also introduced a class of solitary wave solutions with compact support, i.e. solutions with absence of infinite wings or absence of infinite tails, called *compactons*. In addition to compactons, Rosenau [6] proved that the nonlinear dispersive equations $K(m,n)$

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0,$$

which exhibit a number of remarkable dispersive effects, can support both: kinks and solitons with an infinite slope(s), periodic waves and dark solitons with cusp(s), all being manifestations of nonlinear dispersion in action. For $n < 0$ the enhanced dispersion at the tail may generate algebraically decaying patterns. Other solitary-wave solutions of $K(m,n)$ equations were also found by Rosenau in [7,8].

It is known that many integrable equations arise naturally from motions of plane or space curves. In [9,10] the authors investigated the possibility that the $K(m+1,m)$ and $K(m+2,m)$ models can be obtained from plane curves in certain geometries, which provides the geometric interpretations to $K(m,n)$ equations.

Existing techniques for solving nonlinear partial differential equations (PDE's) include: Inverse scattering transform, Wadati trace method, pseudo–spectral method, tanh–sech method, sine–cosine method, Riccati equation expansion method, exponential function method, etc. ([1] and references within it). In spite of the key role of these particular techniques used for solving the equations, one of their limitations is that they do not lay down the conserved quantities. This drawback is, for example, partially overcome in [1], where a 1-soliton solution of Eq. (1.1) is obtained by using the solitary wave ansatz, and a conserved quantity is calculated. Among the techniques, the methods of point transformations are a powerful tool. By means of the Theory of Symmetry Reductions [13,14] a single group reduction may transform a PDE with two independent variables into ODE's. Local symmetries admitted by a PDE are useful for finding invariant solutions. These solutions are obtained by using group invariants to reduce the number of independent variables. The basic idea of the technique is that, a reduction transformation exists when a differential equation is invariant under a Lie group of transformations. The machinery of the Lie group theory provides a systematic method to search for these special group invariant solutions. Although symmetry constraints are powerful in determining integrability of PDE's, not all of them yield exact solutions of the equations, as pointed out in [15].

It is an interesting and important problem how to generally explore integrability of nonlinear PDE's by integrable ODE's. There is a pretty general scheme to reduce PDE's into integrable ODE's. The separation of the time and space variables without using any structure associated with evolution equations is analyzed in [16], and an extension by means of the Frobenius integrable decompositions (FID) is introduced for partial differential equations in [17]. The resulting theory provides techniques which are applied in particular to the celebrated KdV and MKdV equations. The resulting integrable decompositions have exhibited many interesting solution relations with integrable ODE's, including those relations of traveling wave solutions with scalar differential equations and one-dimensional Hamiltonian systems. It also generalizes the Theory of Symmetry constraints in soliton theory, since it does not require any structure associated with the equations under investigation, such as Lax pairs for soliton equations and the symmetry property in symmetry constraints.

The dynamical systems theory [18–20] provides fundamental tools for dealing with ODE's, by qualitative analysis and conservative quantities. Previous works have used them to deal with ODE's coming from PDE's problems. In [21], solutions that present behaviors like sources, asymptotic plane waves, and blow up process at finite time have been characterized. In [22], singular perturbation theory has been applied for analyzing the solutions. In several works, Tang et al. have studied the traveling wave solutions of a given PDE according to different parametric conditions. In [11,12,23,24], bifurcations of phase portraits are discussed in detail, and although the conservative aspects of the system are not dealt with, a first integral (conserved quantity) is deduced. In particular, this study is applied to a generalized KdV equation in [12] and to $K(n,-n,2n)$ equations in [11].

In this work we consider the $K(m,n)$ equation with generalized evolution term (1.1). The paper is organized as follows: first, a complete calculus of the different reductions admitted by this equation is developed (Sections 2 and 3). Second, among the reduced equations, the most general case comes out suitable for qualitative analysis. Indeed, the reduced equation yields to a conservative system, and this allows us to make a complete characterization of its possible dynamical behaviors (Section 4).

2. Classical symmetries

To apply the classical method to Eq. (1.1) with $a, b \neq 0$ we consider the one-parameter Lie group of infinitesimal transformations in (x,t,u) given by

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2),\\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2),\\ u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \end{aligned}$$

where ϵ is the group parameter. We require that this transformation leaves invariant the set of solutions of Eq. (1.1). This yields to an overdetermined, linear system of equations for the infinitesimals $\xi(x,t,u)$, $\tau(x,t,u)$ and $\eta(x,t,u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{2.1}$$

Invariance of Eq. (1.1) under a Lie group of point transformations with infinitesimal generator (2.1), leads to a set of nineteen determining equations. Solving them, we obtain $\eta = \frac{\alpha}{u}^{n-1} + \frac{\beta}{u}$, for $\xi = \xi(x,t)$, $\tau = \tau(t)$, $\alpha = \alpha(x,t)$ and $\beta = \beta(x,t)$ related by the following conditions:

$$\begin{aligned}
&\frac{d\beta}{dx}n - \frac{d^2\xi}{dx^2} = 0,\\
&(n-1)\left(\frac{d\beta}{dx}n - \frac{d^2\xi}{dx^2}\right) = 0,\\
&\frac{d\beta}{dt}lu^{l+n} + \frac{d\alpha}{dt}lu^{l} + a\frac{d\beta}{dx}u^{n+m+1} + b\frac{d^3\beta}{dx^3}nu^{2n} + b\frac{d^3\alpha}{dx^3}nu^{n} + a\frac{d\alpha}{dx}u^{m+1} = 0,\\
&\frac{d\xi}{dt}lu^{l+n} + a\beta nu^{n+m+1} - a\beta mu^{n+m+1} - a\beta u^{n+m+1} - 2a\frac{d\xi}{dx}u^{n+m+1} - 3b\frac{d^2\beta}{dx^2}n^2u^{2n} + b\frac{d^3\xi}{dx^3}nu^{2n} + a\alpha nu^{m+1} - a\alpha mu^{m+1} - a\alpha u^{m+1} = 0,\\
&\beta lu^{n} - \beta nu^{n} - \frac{d\tau}{dt}u^{n} + 3\frac{d\xi}{dx}u^{n} + \alpha l - \alpha n = 0.
\end{aligned}$$

The solutions of this system depend on the parameters of Eq. (1.1). If a,b,n,l and m are arbitrary constants with $a,b \in \mathbb{R}^*$ and $n,l,m \in \mathbb{Z}^+$, the only symmetries admitted by Eq. (1.1) are defined by the infinitesimal generators

$$\mathbf{v}_1 = \partial_x, \quad \mathbf{v}_2 = \partial_t, \quad \mathbf{v}_3 = \left(\frac{n-m-1}{2}\right)x\partial_x + \left(\frac{2l+n-3(m+1)}{2}\right)t\partial_t + u\partial_u.$$

The cases for which Eq. (1.1) has extra symmetries are given in the Table 1.

For the sake of completeness, we next provide the generators of the nontrivial one-dimensional optimal system, similarity variables and similarity solutions (Table 2), the corresponding reduced equations (Table 3), where $\lambda, \mu, \in \mathbb{R}^*$ are arbitrary, and k_1 is an integration constant. In Table 3 equation ODE_0 has been derived by integrating once respect to z. The reduced ODE_5 and ODE_6 appear in Appendix A.

3. Symmetry reduction of the ODE's

In several cases, the reduced ODE's admit symmetries which lead to further order reductions by using again the techniques of Lie group theory. In Table 4 we list the symmetries corresponding to ODE_i, $i = 0,1,2$. The corresponding reductions

Table 1
Each row shows the infinitesimals where f satisfies $2bf_{xxx} + af_x + 2f_t = 0$.

k	Values	$\mathbf{v}_4^k$	$\mathbf{v}_\infty$
1	$l = n = 2,\ m = 1$	$\frac{1}{3}(x+at)\partial_x + t\partial_t$	$fu^{-1}\partial_u$
2	$l = n,\ m = 2n-1$	$t\partial_x + \frac{1}{a}u^{1-n}\partial_u$	

Table 2
Each row shows the infinitesimal generators U_i of the optimal systems, as well as their similarity variables z_i and similarity solutions u_i, where $\nu_1 = \frac{m-n+1}{3(m+1)-n-2l}$, $\nu_2 = \frac{2}{3(m+1)-n-2l}$.

i	Values	U_i	z_i	u_i
0	$l,n,m \in \mathbb{Z}^+$	$\lambda\mathbf{v}_1 + \mu\mathbf{v}_2$	$\mu x - \lambda t$	$h(z)$
1	$3(m+1) - n - 2l \neq 0$	$\mathbf{v}_3$	$xt^{-\nu_1}$	$t^{-\nu_2}h(z)$
2	$l = n = 2,\ m = 1$	$\mathbf{v}_3 + \mathbf{v}_4^1$	$xt^{-\frac{1}{3}} - \frac{a}{2}t^{\frac{2}{3}}$	$th(z)$
3	$l = n = 2,\ m = 2n-1$	$\lambda\mathbf{v}_2 + \mathbf{v}_4^2$	$\lambda x - \frac{t^2}{2}$	$\left(\frac{nt}{a\lambda} + h(z)\right)^{\frac{1}{n}}$
4	$l = n = 2,\ m = 2n-1$	$\lambda\mathbf{v}_4^2 + \mu\mathbf{v}_3$	$xt^{-\frac{1}{3}} - \frac{3\lambda}{2\mu}t^{\frac{2}{3}}$	$\frac{e^{-\frac{2h}{3n}}\left(3e^{\frac{2h}{3}}nt^{\frac{2}{3}}\lambda - 1\right)^{\frac{1}{n}}}{a^{\frac{1}{n}}\mu^{\frac{1}{n}}2^{\frac{1}{n}}t^{\frac{2}{3n}}}$
5	$3(m+1) - n - 2l = 0$	$\mathbf{v}_2 + \mathbf{v}_3$	$e^{(l-m-1)t}x$	he^{t}
6	$n - m - 1 = 0,\ l \neq n$	$\mathbf{v}_1 + \mathbf{v}_3$	$x - \frac{1}{l-m-1}\cos(t)$	he^{x}

Table 3
Each row shows the ODE's, $i = 1, 2, 3, 4$. ODE_5 and ODE_6 appear in Appendix A.

i	ODE_i
0	$h'' + \frac{n-1}{h}(h')^2 + \frac{a}{b\mu^2 n(m+1)}h^{m-n+2} - \frac{\lambda}{b\mu^3 n}h^{l-n+1} + \frac{k_1}{b\mu^3 n}h^{1-n} + k_1 = 0$
1	$bnh^{n+2}h''' - l\frac{m-n+1}{3(m+1)-n-2l}zh^{l+2}h' - \frac{2l}{3(m+1)-n-2l}h^{l+3} + b(n-2)(n-1)nh^n(h')^3 + 3(n-1)nh^{n+1}h'h'' - ah^{m+3}h' = 0$
2	$3bhh''' - zhh' + 9bh'h'' + 3h^2 = 0$
3	$abn\lambda^4 h''' + a^2\lambda^2 hh' + n^2 = 0$
4	$bh''' - \frac{1}{3}zh' - 2bh'h'' + \frac{4}{9}b(h')^3 - \frac{1}{2\mu n}e^{-\frac{2h}{3}}h' + 1 = 0$

Table 4
Each row shows the infinitesimals ODE_i, k_2 and k_3 are arbitrary constants.

ODE_i	j	Values	ξ	φ
$i = 0$	1	$n, m, l \in \mathbb{Z}^+$	k_2	0
$i = 0$	2	$n = 1,\ l = m+1,\ k_1 = 0,\ \alpha_3 = 0,$	$k_3 - k_2(l-1)x$	$2k_2u$
$i = 1$	3	$l = m+1,\ n \neq m+1,\ k_2 \neq -1$	$k_2\left(x + \frac{a}{m+1}\right)$	$\frac{3k_2}{n-m-1}u$
$i = 1$	4	$l = 2n,\ m = 2n-1$	$-\frac{7k_2}{a^8}(2nx + a)$	$\frac{42k_2}{a^8}u$
$i = 1$	5	$l = 2n,\ m = n-1$	k_2	0
$i = 1$	6	$n = m+1,\ l = 2(m+1)$	k_2	0
$i = 2$	7		0	$k_2u + \beta(x)$

(labeled with j) will be obtained from them; in some cases, we will also arrive to exact solutions. In relation to the reductions of ODE_0, it is useful to call $\alpha_1 = \frac{a}{b\mu^2 n(m+1)}$, $\alpha_2 = \frac{\lambda}{b\mu^3 n}$ and $\alpha_3 = \frac{k_1}{b\mu^3 n}$.

Reduction 1. From Table 4, for ODE_0 and $j = 1$, we can see that ODE_0 admits one group corresponding to the operator $W_1 = \partial_z$. We have that for the new variables $y = h$ and $\varphi = h'$ (invariants of the first prolongation of W_1) ODE_0 takes the form

$$\frac{\alpha_1}{y^{n-m-2}} - \frac{\alpha_2}{y^{n-l-1}} + \frac{\alpha_3}{y^{n-1}} + \frac{(n-1)\varphi^2}{y} + \varphi\frac{d\varphi}{dy} + k_1 = 0.$$

Reduction 2. For ODE_0 and $j = 2$, we can see that ODE_0 admits for $n = 1,\ l = m+1,\ l \neq 1$ two groups corresponding to the operators $W_1 = \partial_z,\ W_2 = z\partial_z + \frac{2h}{1-l}\partial_h$. It is easily checked that $[W_1, W_2] = W_1$, consequently, we can perform two order reductions. First, we use W_1, and we have that for the new variables $y = h$ and $\varphi = h'$ (invariants of the first prolongation of W_1) ODE_0 takes the form

$$\varphi\frac{d\varphi}{dy} + (\alpha_1 - \alpha_2)y^l = 0.$$

In terms of these variables the operator W_2 is given by $\widehat{W}_2$, where

$$\widehat{W}_2 = \frac{2}{1-l}y\partial_y + \left(\frac{2}{1-l} - 1\right)\varphi\partial_\varphi.$$

Second, using as new variables the invariants of the first prolongation of $\widehat{W}_2$, i.e. $w = \varphi y^{-\frac{1+l}{2}},\ \psi = \varphi' y^{\frac{1-l}{2}}$, we obtain

$$\psi = \frac{\alpha_2 - \alpha_1}{w}.$$

In terms of the previous variables

$$\frac{\varphi^2}{2} = (\alpha_2 - \alpha_1)\frac{y^{l+1}}{l+1} + c,$$
$$\frac{(h')^2}{2} = (\alpha_2 - \alpha_1)\frac{h^{l+1}}{l+1} + c.$$

For some values of l this equation can be solved in terms of the Jacobi elliptic functions.

In what follows, $\varphi' = \frac{d\varphi}{dy}$ and $\varphi'' = \frac{d^2\varphi}{dy^2}$.

Reduction 3. For ODE_1 and $j = 3$, we can see that ODE_1 admits one group corresponding to the operator $W_1 = \left(z + \frac{a}{m+1}\right)\partial_z + \frac{3}{n-m-1}h\partial_h$. We have that for the new variables (invariants of the first prolongation of W_1) $y = \left(z + \frac{a}{m+1}\right)^{-c}h$ and $\varphi = \left(z + \frac{a}{m+1}\right)^{1-c}h'$, ODE_1 takes the form

$$bn^2\varphi''y^2(3ny+p\varphi)^2+bn^2(p+3n)(2p+3n)\varphi y^2-3bn^2\varphi' y(3ny+p\varphi)(py+ny-np\varphi+p\varphi)$$
$$+bn^2p(\varphi')^2y^2(3ny+p\varphi)-p(p+n^2)y^{\frac{p}{n}+2}(2ny+p\varphi)-3b(n-1)n^2p(p+3n)\varphi^2y+b(n-2)(n-1)n^2p^2\varphi^3=0,$$

where

$$p=(k_1+1-n)n.$$

Reduction 4. For ODE_1 and $j=4$, we can see that ODE_1 admits one group corresponding to the operator $W_1=-\frac{7}{a^8}(2nz+a)\partial_z+\frac{42}{a^8}h\partial_h$. We have that for the new variables (invariants of the first prolongation of W_1) $y=(2nz+a)^{\frac{3}{n}}h$ and $\varphi=(2nz+a)^{\frac{3}{n}+1}h'$, ODE_1 takes the form

$$-bn\varphi''y^2(6y+\varphi)^2+3bn\varphi' y(6y+\varphi)(2ny+2y-n\varphi+\varphi)+\varphi y^2(y^n-8bn^3-36bn^2-36bn)$$
$$+6b(n-1)n(n+3)\varphi^2y-b(n-2)(n-1)n\varphi^3+4y^{n+3}-bn(\varphi')^2y^2(6y+\varphi)=0.$$

Reduction 5. For ODE_1 and $j=5$, we can see that ODE_1 admits for $l=2n,\ m=n-1,\ n\neq 1$, one group corresponding to the operator $W_1=\partial_z$. We have that for the new variables (invariants of the first prolongation of W_1) $y=h$ and $\varphi=h'$, ODE_1 takes the form

$$bn(\varphi^2\varphi''+\varphi^3(\varphi')^2)y^{n+2}-a\varphi y^{n+2}+3b(n-1)n\varphi^2\varphi' y^{n+1}+b(n-2)(n-1)n\varphi^3y^n+2y^{2n+3}=0.$$

Reduction 6. For ODE_1 and $j=6$, we can see that ODE_1 admits one group corresponding to the operator $W_1=\partial_z$. We have that for the new variables (invariants of the first prolongation of W_1) $y=h$ and $\varphi=h'$, ODE_1 takes the form

$$bk_1\varphi^2\varphi''y^{k_1+3}+b\varphi^2\varphi''y^{k_1+3}+bk_1\varphi^3(\varphi')^2y^{k_1+3}+b\varphi^3(\varphi')^2y^{k_1+3}+2y^{2k_1+5}-a\varphi y^{k_1+3}+3bk_1^2\varphi^2\varphi' y^{k_1+2}$$
$$+3bk_1\varphi^2\varphi' y^{k_1+2}+bk_1^3\varphi^3y^{k_1+1}-bk_1\varphi^3y^{k_1+1}=0.$$

Remark. For ODE_2 and $j=7,\ \xi=0$ and there do not exist possible reductions.

4. Qualitative study of ODE_0

Traveling wave solutions have been derived in [2] by means of ODE_0 (Section 2)

$$h''+\frac{n-1}{h}(h')^2+\alpha_1h^{m-n+2}-\alpha_2h^{l-n+1}+\alpha_3h^{1-n}+k_1=0. \tag{4.1}$$

This ODE also admits a qualitative study for the different values of $n,m,l\in\mathbb{Z}^+$. By means of suitable changes of variables, we state sufficient and/or necessary conditions on the parameters to identify different dynamical behaviors.

4.1. Case $n=1$

For $n=1$ Eq. (4.1) is

$$h''+\alpha_1h^{m+1}-\alpha_2h^l+\alpha_3+k_1=0. \tag{4.2}$$

Making the change of variables $x=h,\ \dot{x}=h'$, Eq. (4.2) becomes an analytical Hamiltonian system of the form

$$\begin{cases}\dot{x}=y,\\ \dot{y}=f(x),\end{cases} \tag{4.3}$$

with $f(x)=-\alpha_1x^{m+1}+\alpha_2x^l-(\alpha_3+k_1)$. The Hamiltonian of the system (4.3) is

$$H(x,y)=\frac{y^2}{2}+U(x),$$

where

$$U(x)=-\int_0^x f(s)ds=\frac{\alpha_1}{m+2}x^{m+2}-\frac{\alpha_2}{l+1}x^{l+1}+(\alpha_3+k_1)x$$

is the potential energy function. The trajectories of the system (4.3) lie on the c–level curves of $H(x,y)$

$$\frac{y^2}{2}+\frac{\alpha_1}{m+2}x^{m+2}-\frac{\alpha_2}{l+1}x^{l+1}+(\alpha_3+k_1)x=c$$

and they are symmetric with respect to the x–axis.

In order to describe qualitatively the phase portrait of the system (4.3) for the different possible situations, we first consider the case $f(x)\neq 0$ for all x. Fig. 1 shows the schematic phase portrait for the cases $f(x)>0$ and $f(x)<0$. In both, it is

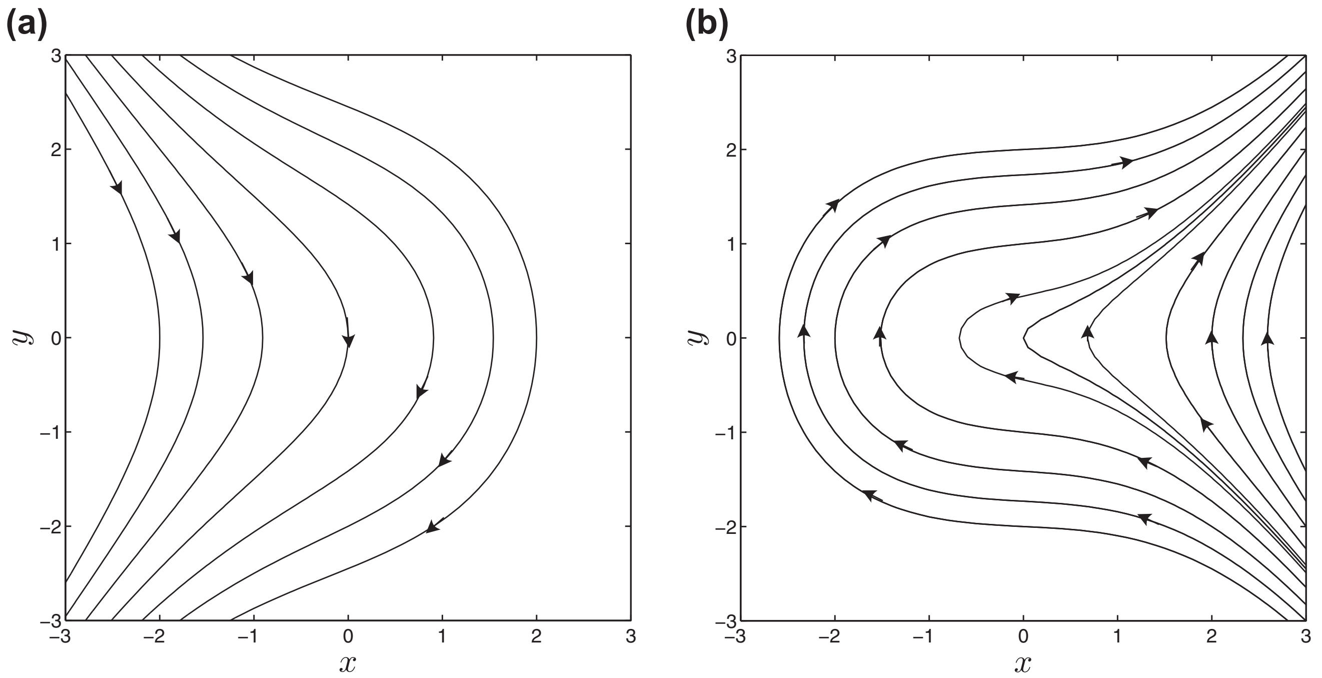

Fig. 1. (a) $f(x) = -\frac{3}{8}x^2 - 1$. (b) $f(x) = \frac{3}{10}x^2 + \frac{1}{10}$.

needed the degree r of polynomial function $f(x)$ to be even, $r = \max\{m+1, l\}$. We find that the parameter conditions are as follows:

(1) When $m+1 = l$, the necessary and sufficient conditions for $f(x) > 0$ are:
(i) if $\alpha_1 = \alpha_2 \Rightarrow \alpha_3 + k_1 < 0$,
(ii) if $\alpha_1 \neq \alpha_2 \Rightarrow \alpha_3 + k_1 < 0 \wedge \alpha_2 - \alpha_1 > 0$
and, for $f(x) < 0$ are:
(i) if $\alpha_1 = \alpha_2 \Rightarrow \alpha_3 + k_1 > 0$,
(ii) if $\alpha_1 \neq \alpha_2 \Rightarrow \alpha_3 + k_1 > 0 \wedge \alpha_2 - \alpha_1 < 0$.
(2) When $m+1 \neq l$, the necessary conditions for $f(x) > 0$ are:
(i) if $m+1 < l \Rightarrow \alpha_2 > 0 \wedge \alpha_3 + k_1 < 0$,
(ii) if $m+1 > l \Rightarrow \alpha_1 < 0 \wedge \alpha_3 + k_1 > 0$
and, for $f(x) < 0$ are:
(i) if $m+1 < l \Rightarrow \alpha_2 < 0 \wedge \alpha_3 + k_1 > 0$,
(ii) if $m+1 > l \Rightarrow \alpha_1 > 0 \wedge \alpha_3 + k_1 > 0$.

Second, we consider that f has real roots. Note that the fixed points of the system (4.3) all lie on the x-axis. $P = (x_*, 0)$ is a fixed point of (4.3) if and only if x_* is a critical point of $U(x)$, i.e. a zero of the polynomial function $f(x)$:

$$U'(x_*) = -f(x_*) = \alpha_1 x_*^{m+1} - \alpha_2 x_*^{l} + (\alpha_3 + k_1) = 0. \quad (4.4)$$

It can be pointed out the following [18]:

(i) If x_* is a strict local maximum of $U(x)$, P is a saddle for (4.3)

$$U''(x_*) = -f'(x_*) = \alpha_1(m+1)x_*^{m} - \alpha_2 l x_*^{l-1} < 0. \quad (4.5)$$

(ii) If x_* is a strict local minimum of $U(x)$, P is a center for (4.3)

$$U''(x_*) = -f'(x_*) = \alpha_1(m+1)x_*^{m} - \alpha_2 l x_*^{l-1} > 0. \quad (4.6)$$

(iii) If x_* is a horizontal inflection point of $U(x)$, P is a cusp for the system (4.3)

$$U''(x_*) = -f'(x_*) = \alpha_1(m+1)x_*^{m} - \alpha_2 l x_*^{l-1} = 0. \quad (4.7)$$

4.1.1. $m + 1 = l$

The system (4.3) becomes

$$\begin{cases} \dot{x} = y, \\ \dot{y} = (-\alpha_1 + \alpha_2)x^{l} - (\alpha_3 + k_1). \end{cases} \quad (4.8)$$

If $\alpha_1 = \alpha_2$, then (4.8) can be easily solved by $y(t) = -(\alpha_3 + k_1)t + c_1,\ x(t) = -(\alpha_3 + k_1)\frac{t^2}{2} + c_1 t + c_2,\ c_1, c_2 \in \mathbb{R}$.

If $\alpha_1 \neq \alpha_2$ and $\alpha_3 + k_1 \neq 0$, then Eq. (4.4) implies that x_* must verify $x_*^l = \frac{\alpha_3+k_1}{\alpha_2-\alpha_1}$, and so $U''(x_*) = -x_*^{-1} l(\alpha_3 + k_1)$. We see from Eq. (4.8) that for the equilibrium points of this system, the following conclusions hold.

(1) When l is odd,

(i) if $\alpha_3 + k_1 < 0 \wedge \alpha_2 - \alpha_1 > 0$ or $\alpha_3 + k_1 > 0 \wedge \alpha_2 - \alpha_1 > 0$, then

$$x_* = \sqrt[l]{\frac{\alpha_3 + k_1}{\alpha_2 - \alpha_1}} \quad \text{and} \quad U''(x_*) < 0, \quad \text{so } P \text{ is saddle.}$$

(ii) if $\alpha_3 + k_1 < 0 \wedge \alpha_2 - \alpha_1 < 0$ or $\alpha_3 + k_1 > 0 \wedge \alpha_2 - \alpha_1 < 0$, then

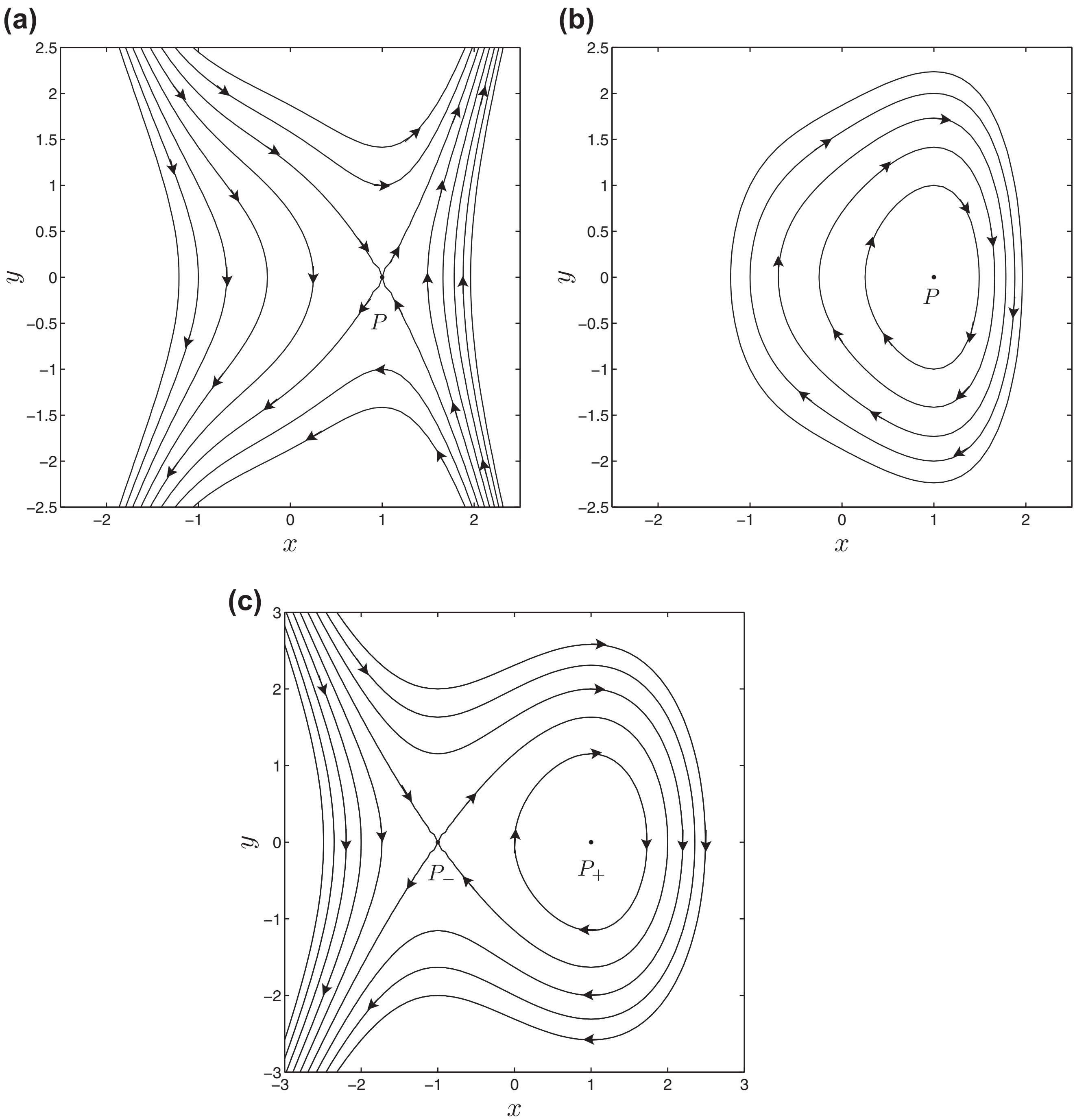

Fig. 2. (a) $f(x) = x^3 - 1,\ P = (1,0)$ is saddle. (b) $f(x) = -x^3 + 1,\ P = (1,0)$ is a center. (c) $f(x) = -x^2 + 1, P_- = (-1,0)$ is saddle and $P_+ = (1,0)$ is a center.

$$x_* = \sqrt[l]{\frac{\alpha_3 + k_1}{\alpha_2 - \alpha_1}} \quad \text{and} \quad U''(x_*) > 0, \quad \text{so } P \text{ is a center.}$$

(2) When l is even,
 (i) if $\alpha_3 + k_1 > 0 \wedge \alpha_2 - \alpha_1 > 0$, then

$$x_*^+ = +\sqrt[l]{\frac{\alpha_3 + k_1}{\alpha_2 - \alpha_1}} \quad \text{and} \quad U''(x_*^+) < 0, \quad \text{so } P_+ \text{ is saddle;}$$

$$x_*^- = -\sqrt[l]{\frac{\alpha_3 + k_1}{\alpha_2 - \alpha_1}} \quad \text{and} \quad U''(x_*^-) > 0, \quad \text{so } P_- \text{ is a center.}$$

 (ii) if $\alpha_3 + k_1 < 0 \wedge \alpha_2 - \alpha_1 < 0$, then

$$x_*^+ = +\sqrt[l]{\frac{\alpha_3 + k_1}{\alpha_2 - \alpha_1}} \quad \text{and} \quad U''(x_*^+) > 0, \quad \text{so } P_+ \text{ is a center;}$$

$$x_*^- = -\sqrt[l]{\frac{\alpha_3 + k_1}{\alpha_2 - \alpha_1}} \quad \text{and} \quad U''(x_*^-) < 0, \quad \text{so } P_- \text{ is saddle.}$$

 (iii) if $\alpha_3 + k_1 > 0 \wedge \alpha_2 - \alpha_1 < 0$ or $\alpha_3 + k_1 < 0 \wedge \alpha_2 - \alpha_1 > 0$, then there are no fixed points.

Some representative situations are sketched in Fig. 2, where the presence of periodic and homoclinic orbits is appreciated.

In this case, we can see also that $\widetilde{\alpha} = \alpha_3 + k_1$ is a parameter of the system (4.8), and when $\widetilde{\alpha} = 0$, it undergoes a local bifurcation of saddle-node type [19] at $(x_*, y_*) = (0, 0)$, which results a degenerate critical point of (4.8).

4.1.2. $m + 1 \neq l$

In this case f is a polynomial function of r degree, where $r = \max\{m + 1, l\}$, so f has at most r real roots.

From Eq. (4.4), if $\alpha_3 + k_1 \neq 0$ then $x_* \neq 0$, and so

$$\alpha_1 x_*^m = \alpha_2 x_*^{l-1} - \frac{\alpha_3 + k_1}{x_*}.$$

Replacing into the expression of $U''(x_*)$, we have

$$U''(x_*) = x_*^{-1}\left[\alpha_2(m + 1 - l)x_*^l - (m + 1)(\alpha_3 + k_1)\right],$$

so, if $x_*^l = \frac{(m+1)(\alpha_3+k_1)}{\alpha_2(m+1-l)}$, then $U''(x_*) = 0$ and $P = (x_*, 0)$ is a cusp for the system (4.3). The parameter conditions for the change of sign of $U''(x)$ are as follows:

(1) When $\alpha_2(m + 1 - l) > 0$,
 (i) if $x_* > 0 \wedge x_*^l > \frac{(m+1)(\alpha_3+k_1)}{\alpha_2(m+1-l)}$ or $x_* < 0 \wedge x_*^l < \frac{(m+1)(\alpha_3+k_1)}{\alpha_2(m+1-l)}$, then P is a center;
 (ii) if $x_* > 0 \wedge x_*^l < \frac{(m+1)(\alpha_3+k_1)}{\alpha_2(m+1-l)}$ or $x_* < 0 \wedge x_*^l > \frac{(m+1)(\alpha_3+k_1)}{\alpha_2(m+1-l)}$, then P is saddle.

(2) When $\alpha_2(m + 1 - l) < 0$,
 (i) if $x_* > 0 \wedge x_*^l < \frac{(m+1)(\alpha_3+k_1)}{\alpha_2(m+1-l)}$ or $x_* < 0 \wedge x_*^l > \frac{(m+1)(\alpha_3+k_1)}{\alpha_2(m+1-l)}$, then P is a center;
 (ii) if $x_* > 0 \wedge x_*^l > \frac{(m+1)(\alpha_3+k_1)}{\alpha_2(m+1-l)}$ or $x_* < 0 \wedge x_*^l > \frac{(m+1)(\alpha_3+k_1)}{\alpha_2(m+1-l)}$, then P is saddle.

In Fig. 3 we illustrate some of the possible phase portraits, where the presence of periodic and heteroclinic orbits is appreciated.

If $\alpha_3 + k_1 = 0$ then $x_* = 0$ is a critical point in any case (that coexists with one or two other equilibrium which are center or saddle according to the different system parameters values).

4.2. Case $n > 1$

Let us go back to Eq. (4.1), consider separately the solutions $h > 0$ and $h < 0$. In both cases, $x = h,\ y = h'h^{n-1}$ comes out a change of variables that convert Eq. (4.1) into the system

$$\begin{cases} \dot{x} = x^{1-n}y, \\ \dot{y} = g(x), \end{cases} \tag{4.9}$$

with $g(x) = -\alpha_1 x^{m+1} + \alpha_2 x^l - \alpha_3 - k_1 x^{n-1}$. As a consequence, the phase portrait is split into two invariant half planes $x > 0$ and $x < 0$. Note that for $n = 1$ the system (4.3) becomes a particular case of the system (4.9) -which is not analytical at $x = 0$. The system (4.9) is not of Hamiltonian type either, however it is conservative, since the quantity defined by the differentiable function E,

$$E(x,y)=\frac{y^2}{2}+\frac{\alpha_1}{m+n+1}x^{m+n+1}-\frac{\alpha_2}{l+n}x^{l+n}+\frac{\alpha_3}{n}x^n+\frac{k_1}{2n-1}x^{2n-1} \tag{4.10}$$

is constant on trajectories, i.e. $dE/dt=E_x\dot{x}+E_y\dot{y}=0$. Hence, the trajectories lie on the curves $E(x,y)=$ constant, and they are symmetric with respect to the x-axis.

Note that

$$E(x,y)=\frac{y^2}{2}+\widetilde{U}(x),$$

where

$$\widetilde{U}(x)=-\int_0^x s^n g(s)ds.$$

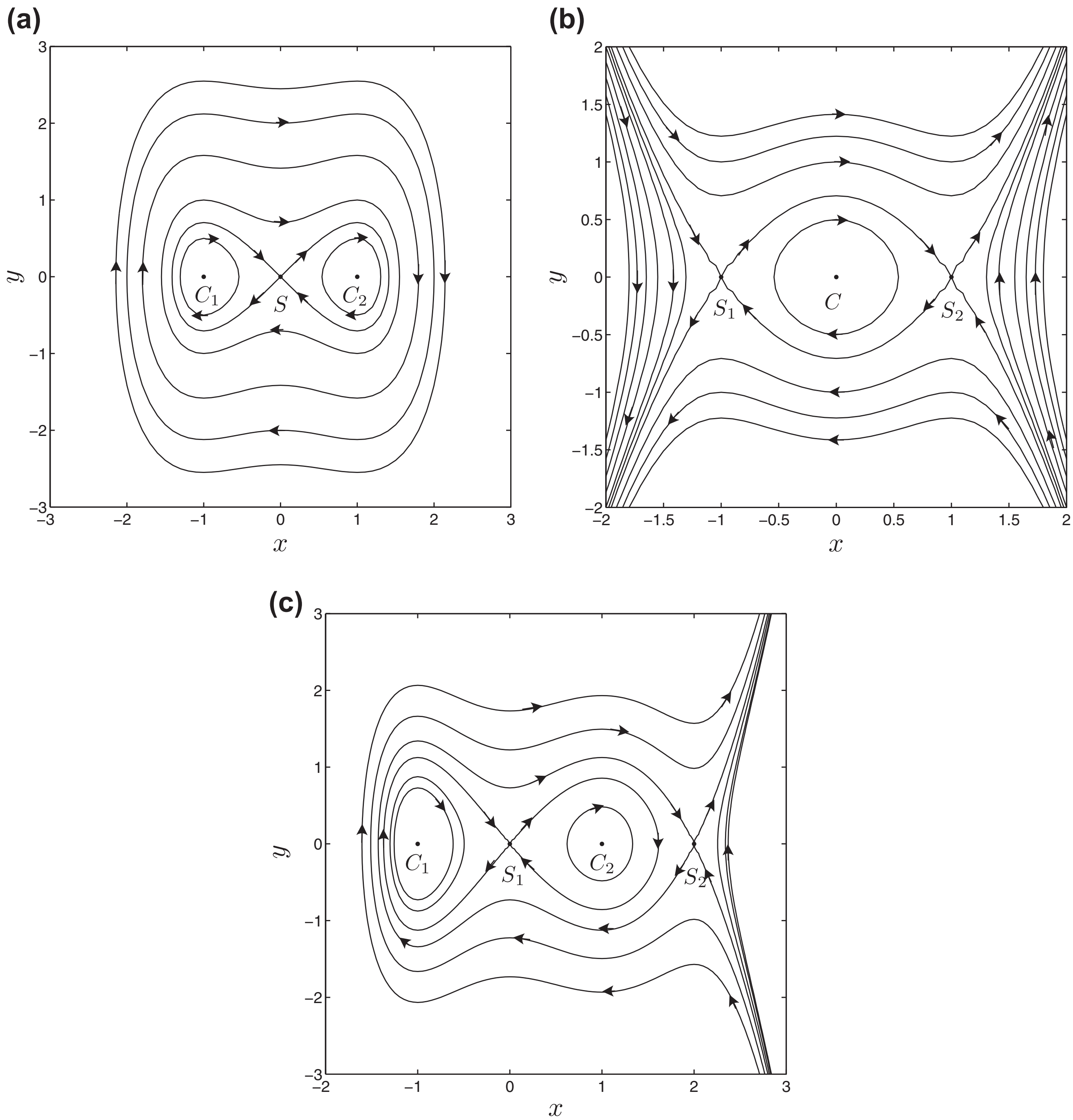

Fig. 3. (a) $f(x)=x^3-x$, $C_1=(-1,0)$ and $C_2=(1,0)$ are centers, and $S=(0,0)$ is saddle. (b) $f(x)=-x^3+x$, $S_1=(-1,0)$ and $S_2=(1,0)$ are saddle, and $C=(0,0)$ is a center. (c) $f(x)=x^4-2x^3-x^2+2x$, $C_1=(-1,0)$ and $C_2=(1,0)$ are centers, $S_1=(0,0)$ and $S_2=(2,0)$ are saddle.

The equilibrium points of (4.9) – if any – all lie on the x-axis and correspond to the critical point of $E(x,y)$, since

$$\frac{\partial E}{\partial x} = \widetilde{U}\prime(x) = -x^n g(x) = 0 \Longleftrightarrow \dot{y} = 0 \quad \text{and} \quad \frac{\partial E}{\partial y} = y = 0 \Longleftrightarrow \dot{x} = 0.$$

$P = (x_*, 0)$ is a fixed point of (4.9) if x_* is a critical point of $\widetilde{U}(x)$, i.e. a zero of the polynomial function $g(x)$. We can set down the analogous of properties (4.5), (4.6) and (4.7).

Theorem 4.1. *If x_* is a strict local maximum of the analytic function $\widetilde{U}(x)$, then P is a saddle for (4.9) if $x_* > 0$, and a center for (4.9) if $x_* < 0$. If x_* is a strict local minimum of the analytic function $\widetilde{U}(x)$, then P is a center for (4.9) if $x_* > 0$, and a saddle for (4.9) if $x_* < 0$. If x_* is a horizontal inflection point of $\widetilde{U}(x)$, then P is a cusp for the system (4.9).*

Proof. In fact, the Jacobian matrix of the linearized system (4.9) at $P = (x_*, 0)$ is

$$J(P) = \begin{bmatrix} 0 & x_*^{1-n} \\ g'(x_*) & 0 \end{bmatrix}.$$

Let $\Delta = \det(J(P)) = -x_*^{1-n} g\prime(x_*)$ be its determinant value. Since $\text{tr}(J) = 0$ the eigenvalues of the characteristic polynomial, $p(\delta)$, of J are of the form

$$\delta = \pm\sqrt{-\Delta}. \tag{4.11}$$

On the other hand, the second derivative of $\widetilde{U}$ is

$$\widetilde{U}''(x) = -nx^{n-1}g(x) - x^n g'(x)$$

so, $\widetilde{U}''(x_*) = -x_*^n g'(x_*)$. Note that, since the signs of the values $x_*^n g'(x_*)$ and $x_*^{-n} g'(x_*)$ are the same, then the difference between the sign of $\widetilde{U}''(x_*)$ and Δ depends on the sign of x_*. According to this, if x_* is a local maximum of $\widetilde{U}(x)$, i.e. $\widetilde{U}''(x_*) < 0$, then $\Delta < 0$ if $x_* > 0$, and by Eq. (4.11), P is a saddle for the system (4.9). If $x_* < 0$, then $\Delta > 0$, and by Eq. (4.11), P is a center for the linearized system of (4.9).

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} x \\ y \end{bmatrix}. \tag{4.12}$$

So P is either a center or a focus for (4.9). But since both, attractive fixed points and repellers, can not occur in a conservative systems [19], P results a nonlinear center for the system (4.9).

Analogously, if x_* is a local minimum of $\widetilde{U}(x)$, i.e. $\widetilde{U}''(x_*) > 0$, then $\Delta < 0$ if $x_* < 0$, and by Eq. (4.11), P is a saddle for the system (4.9). If $x_* > 0$, then $\Delta > 0$, and by Eq. (4.11), P is a center for the system (4.12), and so, a nonlinear center for the system (4.9). Finally, if x_* is a horizontal inflection point of $\widetilde{U}(x)$, i.e. $\widetilde{U}''(x_*) = 0$, then $\Delta = 0$, and by Eq. (4.11), P is a cusp for the system (4.9). □

With these tools at hand, the analysis of the case $n > 1$ keeps the essential features of the case $n = 1$. As $g(x)$ is a polynomial function, the condition for $g(x) \neq 0$ for all x may be stated as for f in the subsection 4.1. Four different types of phase portraits are possible according to the g-sign and the n-parity. If g has real roots, the phase portraits of the system (4.9) will look in each half plane, $x > 0, \; x < 0$, as the phase portraits of the system (4.3), according to: (i) how much positive real roots and negative ones does the function g have, (ii) which of them are centers or saddles, and (iii) the n-parity.

5. Conclusions

We have studied the one-dimensional $K(m,n)$ equation with generalized evolution term (1.1), by applying the Theory of Symmetry Reductions to differential equations. Using the characteristic equation, we have stated a complete classification (depending on the values of the parameters a, b, m, n and l) of the Lie symmetries admitted by (1.1) and we have found the similarity variables. Then, reduced forms of the original nonlinear ODE have been obtained as nonlinear differential equations. By applying the classical Lie method we have reduced the order of these ODE's.

Among the reduced equations obtained in Table 3, the ODE_0 has distinct features. The applied reduction is valid for arbitrary $n, m, l \in \mathbb{Z}^+$ and yields to a second order autonomous differential equation. Under a suitable change of variables (depending on if $n = 1$ or $n > 1$), this equation is transformed into an autonomous two-dimensional system which is able to be studied by means of Hamiltonian and conservative systems properties. Let us note that there is no restriction on the parameters values for computing the conserved quantity. Therefore, a rather complete scenario of the qualitative behaviors in terms of its parameter values has been obtained, with the typical presence of periodic, homoclinic and heteroclinic orbits which correspond [12] respectively to solitary wave solutions, kinks (or anti-kinks) and periodically traveling wave solutions of Eq. (1.1).

Interestingly, from the qualitative analysis a new reduced equation has arisen. In fact, writing the conservative quantities $H(x,y)$ or $E(x,y)$ in the original variables h and h', from Eq. (4.10), we have

$$\frac{1}{2}h'h^{2n-2}+\frac{\alpha_1}{m+n+1}h^{m+n+1}-\frac{\alpha_2}{l+1}h^{l+n}+\frac{\alpha_3}{n}h^n+\frac{k_1}{2n-1}h^{2n-1}=c,$$

which may be interpreted as a reduced first order equation of ODE_0. Note that, different from Reduction 1 and 2, this one is valid for arbitrary $n,m,l\in\mathbb{Z}^+$, and the reduced equation comes out autonomous. However, when $n=1,\ l=m+1$ and $k_1=0$, it coincides with the last reduction of Reduction 2 obtained in Section 3.

In ODE_0, the order is reduced by integrating once respect to z yielding to a two-dimensional dynamical system. This is not the case in any other ODE_i with $i>1$. These ODE's and even their worked reductions (Section 3), derive into higher dimensional dynamical systems; so, the qualitative study developed for ODE_0 may not be straightfully extended to those cases, and requires further research.

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Appendix A. EDO_5 and EDO_6

The reduced equations that fullfil Table 3 in Section 2 are:

for $i = 5$

$$\begin{aligned}&h^{3p}h'pz(p-m-1)-8bh^{3m+1}(h')^3p^3+36bh^{3m+1}(h')^3mp^2+12bh^{3m+2}h'h''p^2+24bh^{3m+1}(h')^3p^2\\&-54bh^{3m+1}(h')^3m^2p-36bh^{3m+2}h'h''mp-72bh^{3m+1}(h')^3mp+h^{3p+1}p-2bh^{3m+3}h'''p-30bh^{3m+2}h'h''p\\&-22bh^{3m+1}(h')^3p+27bh^{3m+1}(h')^3m^3+27bh^{3m+2}h'h''m^2+54bh^{3m+1}(h')^3m^2+3bh^{3m+3}h'''m+45bh^{3m+2}h'h''m\\&+33bh^{3m+1}(h')^3m+3bh^{3m+3}h'''+18bh^{3m+2}h'h''+6bh^{3m+1}(h')^3+ah^{2p+m+1}h'=0\end{aligned}$$

and, for $i=6$

$$\begin{aligned}&h^{p+1}h'pe^{\frac{(p^2+m^2+2m+1)z}{p-m-1}}+\Big\{\Big[h^m(-b(h')^3-3bh(h')^2-3bh^2h'-bh^3)m^3+h^m\Big((-3bhh'-3bh^2)h''-6bh(h')^2-9bh^2h'-3bh^3\Big)m^2+h^m\\&\times\Big(-bh^2h'''+(-3bhh'-6bh^2)h''+b(h')^3-3bh(h')^2-9bh^2h'-3bh^3\Big)m\\&+h^m\Big(-bh^2h'''-3bh^2h''+(-3b-a)h^2h'+(-b-a)h^3\Big)\Big]p\\&+h^m\Big(b(h')^3+3bh(h')^2+3bh^2h'+bh^3\Big)m^4+h^m\Big(\Big(3bhh'+3bh^2\Big)h''\\&+b(h')^3+9bh(h')^2+12bh^2h'+4bh^3\Big)m^3+h^m\Big(bh^2h'''+(6bhh'+9bh^2)h''\\&-b(h')^3+9bh(h')^2+18bh^2h'+6bh^3\Big)m^2+h^m\Big(2bh^2h'''+(3bhh'+9bh^2)h''-b(h')^3\\&+3bh(h')^2+(12b+a)h^2h'+(4b+a)h^3\Big)m+h^m\Big(bh^2h'''+3bh^2h''+(3b+a)h^2h'+(b+a)h^3\Big)\Big\}e^{\frac{(2m+2)pz}{p-m-1}}=0.\end{aligned}$$

References

[1] A. Biswas, 1-soliton solution of the $K(m,n)$ equation with generalized evolution, Phys. Lett. A 372 (2008) 4601–4602.
[2] M.S. Bruzón, M.L. Gandarias, Travelling wave solutions of the $K(m,n)$ equation with generalized evolution, Math. Methods Appl. Sci. (2010), http://dx.doi.org/10.1002/mma.1339. <http://www.interscience.wiley.com>.
[3] A. Chen, J. Li, Single peak solitary wave solutions for the osmosis $K(2,2)$ equation under inhomogeneous boundary condition, J. Math. Anal. Appl. 369 (2010) 758–766.
[4] H. Liu, J. Li, Lie symmetry analysis and exact solutions for the extended mKdV equation, Acta Appl. Math. 109 (2008) 1107–1119.
[5] P. Rosenau, J.M. Hyman, Compactons: solitons with finite wavelengths, Phys. Rev. Lett. 70 (5) (1993) 564–567.
[6] P. Rosenau, On nonanalytic solitary waves formed by a nonlinear dispersion, Phys. Lett. A 230 (1997) 305–318.
[7] P. Rosenau, Nonlinear dispersion and compact structures, Phys. Rev. Lett. 73 (13) (1994) 1737–1741.
[8] P. Rosenau, On a class of nonlinear dispersive-dissipative interactions, Physica D 230 (5–6) (1998) 535–546.
[9] C.Z. Qu, S. Zhang, Q. Zhang, Integrability of models arising from motions of plane curves, Z. Naturforsch. 58a (2003) 75–83.
[10] K.S. Chou, C.Z. Qu, Integrable equations arising from motions of plane curves. II, J. Nonlinear Sci. 13 (2003) 487–517.
[11] S. Tang, W. Huang, Bifurcations of travelling wave solutions for the $K(n,-n,2n)$ equations, Appl. Math. Comput. 203 (2008) 39–49.
[12] S. Tang, M. Li, Bifurcations of travelling wave solutions in a class of generalized KdV equation, Appl. Math. Comput. 177 (2006) 589–596.
[13] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover, New York, 1972.
[14] P. Olver, Applications of Lie Groups to Differential Equations, Springer-Verlag, New York, 1993.
[15] W.X. Ma, M. Chen, Do symmetry constraints yield exact solutions?, Chaos Solitons Fract 32 (2007) 1513–1517.

[16] W.X. Ma, H. Wu, Time–space integrable decompositions of nonlinear evolution equations, J. Math. Anal. Appl. 324 (2006) 134–149.
[17] W.X. Ma, H. Wu, J. He, Partial differential equations possesing Frobenius integrable decompositions, Phys. Lett. A 364 (2007) 29–32.
[18] L. Perko, Differential Equations and Dynamical Systems, second ed., Springer-Verlag, 1996.
[19] S.H. Strogatz, Nonlinear Dynamics and Chaos, Perseus Publishing, Cambridge Massachusetts, 1994.
[20] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, 2003.
[21] J.L. Romero, M.L. Gandarias, E. Medina, Symmetries, periodic plane waves and blow-up of λ–ω systems, Physica D 147 (2000) 259–272.
[22] V. Camacho, R.D. Guy, J. Jacobsen, Travelling waves and shocks in a viscoelastic generalization of Burgers' equation, SIAM J. Appl. Math. 68 (5) (2008).
[23] S. Tang, X. Huang, W. Huang, Bifurcations of travelling wave solutions for the generalized KP–BBM equation, Appl. Math. Comput. 216 (2010) 2881–2890.
[24] Y. Xie, B. Zhou, S. Tang, Bifurcations of travelling wave solutions for the generalized (2+1)-dimensional Boussinesq–Kadomtsev–Petviashvili equation, Appl. Math. Comput 217 (6) (2010) 2433–2447.