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# Integrable systems and Poisson–Lie $T$-duality: A finite dimensional example

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ABSTRACT

We study the deep connection between integrable models and Poisson–Lie $T$-duality working on a finite dimensional example constructed on $SL(2,\mathbb{C})$ and its Iwasawa factors $SU(2)$ and $B$. We shown the way in which the Adler–Kostant–Symes theory and collective dynamics combine to solve the equivalent systems by solving the factorization problem of an exponential curve in $SL(2,\mathbb{C})$. It is shown that the Toda system embraces the dynamics of the systems on $SU(2)$ and $B$.



## 1. Introduction

The search for dualities in theoretical physics is motivated by the hope of finding a couple of related theories in which one of them is, in some sense, easily solved and the solutions to the second one is attained from the solution of the former system. *Poisson–Lie $T$-duality* is a nice example in this direction: it is built on phase spaces having a rich structure entailing a close connection with integrable model, exploiting the inherent self-dual character of Poisson–Lie groups in order to relate a couple of sigma models having targets on the factors of a Drinfeld double Lie group [1]. In Refs. [2,3], PL $T$-duality was accurately encoded in a Hamiltonian scheme ruled by some Hamiltonian actions of the double Lie group $G$ on the cotangent bundle of its factors, where $T$-duality transformations are provided by the associated momentum maps targeting on the same coadjoint orbit. Moreover, it was realized that collective dynamics on these Hamiltonian $G$-spaces underpins the dynamic correspondence between these models. In those references, $G$ was taken as the centrally extended Drinfeld double of a loop group and $T$-duality comes to relate sigma models built on each factor of it. This scheme also reveals the role played by a WZNW model whose reduced phase space, the shared coadjoint orbit, embraces the dynamics of both sigma models.

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In all these systems, compatible dynamics are ruled by *collective Hamiltonians*. Thus, the natural setting is infinite dimensional: it is provided by phase spaces modelled on cotangent bundles of loop groups, and the momentum maps are associated with the centrally extended action of the double group. In spite of this, the essential issues of $T$-duality can be clearly sketched in a finite dimensional context, avoiding the specific difficulties of the infinite dimensional case.

The current work is aimed to stress the intrinsic connection of the Poisson–Lie $T$-duality with integrable systems, working in a finite dimensional framework, allowing us to concentrate on the structural facts behind this connection. We describe the geometric structure underlying the Hamiltonian version of this duality, following Refs. [2,3], by considering a complex Lie group $G$ and its Iwasawa decomposition in the compact factor $K$ and the soluble one, $B$. As an alternative to the standard scheme built on Hamiltonian $G$-spaces, we introduce a wider version of $T$-duality in order to include schemes based on the Hamiltonian action of the Iwasawa factors, giving rise to duality classes of Hamiltonian $K$- or $B$-spaces. This leads straightforwardly to the *Adler–Kostant–Symes* (AKS) theory for integrable systems [4], through the introduction of collective dynamics. An explicit example is constructed in full detail working on $\mathrm{SL}(2,\mathbb{C})$ and its factors in the Iwasawa decomposition, namely $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{SU}(2)\times B$, involving three Hamiltonian $B$-spaces: $T^*\mathrm{SU}(2)$, $T^*B$ and $\mathbb{R}^2$. The respective dually related dynamical systems are a dressing invariant system in $\mathrm{SU}(2)$, a kind of generalized top on $B$, and a Toda model on $\mathbb{R}^2$. This last system plays an analogous role to that played by the WZNW in loop group case, embracing the dynamics of the other systems. Then, we use the *AKS* theory to show explicitly the integrability of these systems constructing the solution in each case, and providing a precise meaning for the Poisson–Lie $T$-duality transformations. By passing to the Lagrangian framework, we show the equivalence between systems described by bilinear forms on the corresponding tangent bundles, so that the constructed duality relates different *targets geometries*.

It is important to point out that most of the results can be translated, with some cares, to the infinite dimensional case and the underlying structure works in any case. Whatever the case, we can consider the finite dimensional case as a restriction of the loop group one to the constant map from $S^1$ to a Lie group.

This work is organized as follows: in Section 2 we give a description of the geometric setting for the Hamiltonian approach to PL $T$-duality and its relation with the theory of integrable models, in particular with the AKS theory. In Section 3, we describe the main features related to Iwasawa decomposition and coadjoint orbits; in Section 4 the involved phase spaces are presented, describing its symmetry properties; the $T$-duality scheme is described in Section 5; in Section 6 we apply explicitly the AKS Theory to solve the systems, and in Section 7 the compatible dynamics is analyzed from Hamiltonian and Lagrangian point of view. Finally, the conclusions are included in Section 8.

## 2. Geometric setting for Poisson–Lie *T*-duality

The standard Hamiltonian approach to PL $T$-duality, as introduced in [2,3], considers a Lie group $G$ which can be written as a product of two subgroups $K$ and $B$, so that all of them are endowed with a Poisson–Lie structure and their Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{b}$, such that $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{b}$, turn in Lie bialgebras. Hence, the PL $T$-duality is built up on Hamiltonian $G$-spaces: the group $G$ acts on the cotangent bundle of its factors, giving rise to momentum maps with nontrivial intersections in $\mathfrak{g}^*$. In the loop group case, this is warranted by taking the central extension of $G$ or of its Lie algebra $\mathfrak{g}$, providing intersections with a rich class of coadjoint orbits inside. However, this seems to be a very specific situation, in general it happens that the momentum maps have no nontrivial intersection, as it is the case in finite dimension.

Handling this problem in a general fashion lead us to propose a wider scheme for PL $T$-duality by considering $T$-dual equivalence classes constructed alternatively on Hamiltonian $G, K$ or $B$-spaces. As we shall show below, the main facts underlying the standard PL $T$-duality remain the same: the canonical transformation between systems on the factors $K$ and $B$ arises from the symmetries involving their *Poisson–Lie structure*. In this way, one is able to built up PL $T$-dual equivalence classes attached to coadjoint orbits in $\mathfrak{g}^*$, $\mathfrak{k}^*$ or $\mathfrak{b}^*$. In addition, this wider framework allows to make contact with the *AKS* theory for integrable systems.

So, let us consider the Lie group $G$ and its Iwasawa decomposition $G = KB$, where $K$ is the compact factor and $B$ is the solvable one. The abstract framework we use here also includes the Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{b}$, which correspond to the Lie groups $G, K, B$ so that $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{b}$, and $\mathfrak{g}$ is equipped with a nondegenerate symmetric bilinear form $(\,,\,)_{\mathfrak{g}}$ turning $\mathfrak{k}$ and $\mathfrak{b}$ into isotropic subspaces. This allows the identification $\mathfrak{b}^* \simeq \mathfrak{k}$ and $\mathfrak{k}^* \simeq \mathfrak{b}$. The projectors are denoted by $\Pi_K : G \to K$, $\Pi_B : G \to B$; moreover, the symbols $\Pi_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$, $\Pi_{\mathfrak{b}} : \mathfrak{g} \to \mathfrak{b}$ are meant to indicate the projections into the summands of the Lie algebra decomposition induced by the factorization.

Let us describe a PL $T$-duality scheme based on the action of one of the factors, $B$ in this case, instead of the action of $G$. It will involve the Poisson manifold $(\mathfrak{b}^*, \{,\}_{\mathfrak{b}^*})$, where $\{,\}_{\mathfrak{b}^*}$ is the Kirillov–Kostant bracket, and the symplectic manifolds $(T^*K, \omega_K)$ and $(T^*B, \omega_B)$, with $\omega_K, \omega_B$ standing for the canonical symplectic forms on each phase spaces, respectively. All the cotangent bundles are regarded in body coordinates, so they are trivialized by left translations.

The phase space $T^*B \cong B\times\mathfrak{b}^*$ turns in a Hamiltonian $B$-space by the action $\tau : B\times(B\times\mathfrak{b}^*) \longrightarrow B\times\mathfrak{b}^*$ obtained as the lift of the action of $B$ on itself by left translations

$$\tau\left(\tilde{g}, (\tilde{h}, \tilde{\eta})\right) = (\tilde{g}\tilde{h}, \tilde{\eta})$$

for $\tilde{g}, \tilde{h} \in B$, $\tilde{\eta} \in \mathfrak{b}^*$, with Ad-equivariant momentum map $\lambda : B\times\mathfrak{b}^* \longrightarrow \mathfrak{b}^*$

$$\lambda(\tilde{h}, \tilde{\eta}) = \mathrm{Ad}^*_{\tilde{h}^{-1}}\tilde{\eta}.$$

On the other side, $T^*K \cong K \times \mathfrak{k}^*$, becomes in a Hamiltonian $B$-space by virtue of the *Poisson–Lie structure* of $K$ inherited from the Iwasawa decomposition $G = KB$. In fact, we introduce the action $\mathsf{dr} : B \times (K \times \mathfrak{k}^*) \longrightarrow K \times \mathfrak{k}^*$

$$\mathsf{dr}\left(\tilde{b}, (g, \eta)\right) = \left(g^{\tilde{b}}, \mathrm{Ad}_{\tilde{b}^g}\eta\right) \tag{1}$$

for $\tilde{b} \in B, g \in K, \eta \in \mathfrak{k}^*$ which is obtained by lifting the *dressing action* of $B$ on $K$ to the cotangent bundle. This action, introduced in [5,6], works as follows: by writing each element $l \in G$ as $l = g\tilde{h}$, with $g \in K$ and $\tilde{h} \in B^*$, the product $\tilde{h}g$ in $G$ can be expressed as $\tilde{h}g = g^{\tilde{h}}\tilde{h}^g$, with $g^{\tilde{h}} \in K$ and $\tilde{h}^g \in B$. The *dressing action* of $B$ on $K$ is defined as $\mathsf{Dr} : B \times K \longrightarrow K$, such that $\mathsf{Dr}\left(\tilde{h}, g\right) := \Pi_K \tilde{h}g = g^{\tilde{h}}$. Infinitesimally, $\xi \in \mathfrak{b}$ is mapped onto the tangent vector

$$(\xi_{K\times\mathfrak{k}^*})_{(g,\eta)} = \left(g^{\xi}, \left[\mathrm{Ad}^*_g\xi, \eta\right]\right)$$

at $(g, \eta) \in K \times \mathfrak{k}^*$; here are assumed the identifications explained above, so that $\mathrm{Ad}^*_g : \mathfrak{b} \to \mathfrak{b}$ and the bracket is the Lie bracket in $\mathfrak{b}$. The momentum map $\phi : K \times \mathfrak{k}^* \longrightarrow \mathfrak{b}^*$ is

$$\phi(g, \eta) = g\left(g^{-1}\right)^{\eta}$$

having in mind that the right hand side belongs to $\mathfrak{k} \simeq \mathfrak{b}^*$. In order to avoid confusion, these identifications will be explicitly shown in the specific case addressed in the following sections.

The momentum maps $\phi$ and $\lambda$ turn $(K \times \mathfrak{k}^*, \omega_K)$ and $(B \times \mathfrak{b}^*, \omega_B)$ in symplectic realizations of the Poisson manifold $(\mathfrak{b}^*, \{,\}_{\mathfrak{b}^*})$, as depicted in the diagram

$$\begin{array}{ccc} K \times \mathfrak{k}^* & & B \times \mathfrak{b}^* \\ & \searrow^{\phi} \quad {}^{\lambda}\swarrow & \\ & \mathfrak{b}^* & \end{array} \tag{2}$$

which is the basic geometric scheme underlying PL $T$-duality. Seeking for compatible dynamics drives to the realm of *collective Hamiltonian systems* [7], meaning that a Hamiltonian function $\mathsf{h} \in C^\infty(\mathfrak{b}^*)$ is the masterpiece governing both the PL $T$-dual systems on $K \times \mathfrak{k}^*$ and $B \times \mathfrak{b}^*$. In fact, the corresponding pull backs by the momentum maps $\phi$ and $\lambda$, namely $\mathsf{h} \circ \phi \in C^\infty(K \times \mathfrak{k}^*)$ and $\mathsf{h} \circ \lambda \in C^\infty(B \times \mathfrak{b}^*)$, produce the desired compatible dynamics.

These systems are said to be in *collective Hamiltonian form* and to understand its geometric meaning we work on a generic Hamiltonian $B$-space $(M, \omega)$, with an Ad-equivariant momentum map $J : M \to \mathfrak{b}^*$ associated with the symplectic action $\varphi : B \times M \longrightarrow M$ of the Lie group $B$, and taking the *collective Hamiltonian* $H = \mathsf{h} \circ J$. In terms of the *orbit map* through $m \in M, \varphi_m : B \longrightarrow M/\varphi_m(b) := \varphi(b, m)$, the infinitesimal generators can be written as $X_M(m) = (\varphi_m)_* X$, for $X \in \mathfrak{b}$ and $X_M \in \mathfrak{X}(M)$. Hence, introducing the *Legendre transformation* of $\mathsf{h}$, namely the linear map $\mathcal{L}_{\mathsf{h}} : \mathfrak{b}^* \to \mathfrak{b}$ defined as $\langle \xi, \mathcal{L}_{\mathsf{h}}(\eta)\rangle_{\mathfrak{g}} = \langle \mathrm{d}\mathsf{h}|_\eta, \xi\rangle$, for any $\xi \in \mathfrak{b}^*$, we may write the Hamiltonian vector field of $H$ as

$$V_H|_m = (\varphi_m)_*\left[\mathcal{L}_{\mathsf{h}} \circ J\right](m)$$

and its image by $J$ is tangent to the coadjoint orbit through $J(m)$

$$J_*|_m V_H = -\left(\mathrm{ad}^B_{\mathcal{L}_{\mathsf{h}}(J(m))}\right)^* J(m).$$

In other words, the Hamiltonian vector field $V_H$ is mapped on the tangent space of a coadjoint orbits in $\mathfrak{b}^*$. If $m(t)$ denotes the trajectory of the Hamiltonian system through $m(0) = m, \dot{m}(t) = V_H|_{m(t)}$, the images $\gamma(t) = J(m(t))$ lies completely on the coadjoint orbit through $J(m)$, where the equation of motion is

$$\dot{\gamma}(t) = -\left(\mathrm{ad}^B_{\mathcal{L}_{\mathsf{h}}(\gamma(t))}\right)^* \gamma(t) \tag{3}$$

that corresponds to a Hamiltonian system on the coadjoint orbits on $\mathfrak{b}^*$, with Hamiltonian function $\mathsf{h}$.

**Proposition.** *Let $\gamma : \mathbb{R} \to \mathfrak{b}^*$ be the solution curve of Eq. (3) with initial condition $\gamma(0) = J_M(m)$, and select a curve $b(t)$ in $B$ such that*

$$\gamma(t) = \left(\mathrm{Ad}^B_{b^{-1}(t)}\right)^* J_M(m). \tag{4}$$

*Then, among these curves there exists a solution of the differential equation on $B$*

$$\dot{b}(t)b^{-1}(t) = \mathcal{L}_{\mathsf{h}}(\gamma(t)), \qquad b(0) = n_0 \in B_{J_M(m)} \tag{5}$$

*where $B_{J_M(m)}$ is the stabilizer group of the point $J_M(m)$ under the coadjoint action of $B$ on $\mathfrak{b}^*$.*

**Proof.** Let us suppose that $b : \mathbb{R} \to B$ satisfies Eq. (3) through Eq. (4), and take $n : \mathbb{R} \to B_{J_M(m)}$ such that $b(t)n(t)$ solves the differential equation (5). Then

$$\frac{\mathrm{d}(b(t)n(t))}{\mathrm{d}t}(b(t)n(t))^{-1} = \mathcal{L}_{\mathsf{h}}(\gamma(t))$$

or equivalently

$$\dot{n}(t)n^{-1}(t) = \mathrm{Ad}^B_{b^{-1}(t)}\mathcal{L}_{\mathsf{h}}(\gamma(t)) - b^{-1}(t)\dot{b}(t).$$

We have to verify that the right hand side of this expression belongs to $\mathfrak{b}_{J_M(m)}$, the Lie algebra of the stabilizer subgroup $B_{J_M(m)}$. Taking into account that $b$ satisfies Eq. (4), we have

$$\dot{b}(t)b^{-1}(t) = \mathcal{L}_{\mathsf{h}}(\gamma(t)) + M(t)$$

for some curve $M : \mathbb{R} \to \mathfrak{b}_{\gamma(t)}$. Furthermore, we have that $X \in \mathfrak{b}_{\gamma(t)}$ iff $\left(\mathrm{ad}^B_X\right)^* \gamma(t) = 0$, and this means

$$0 = \left(\mathrm{ad}^B_X\right)^* \left(\mathrm{Ad}^B_{b^{-1}(t)}\right)^* J_M(m) = \left(\mathrm{Ad}^B_{b^{-1}(t)}\right)^* \left(\mathrm{ad}^B_{\mathrm{Ad}_{b^{-1}(t)}X}\right)^* J_M(m).$$

Then $X \in \mathfrak{b}_{\gamma(t)}$ iff $\mathrm{Ad}^B_{b^{-1}(t)}X \in \mathfrak{b}_{J_M(m)}$. Therefore, $M(t) = \mathrm{Ad}^B_{b(t)}N(t)$ for some curve $N : \mathbb{R} \to \mathfrak{b}_{J_M(m)}$, and finally

$$\dot{n}(t)n^{-1}(t) = N(t) \in \mathfrak{b}_{J_M(m)}$$

as we want to show. □

Hence, $m(t) = \varphi(b(t), m)$ is the solution to the original Hamiltonian system. Moreover, if $\mathfrak{b}$ is supplied with an invariant nondegenerate bilinear form $( , ) : \mathfrak{b} \times \mathfrak{b} \longrightarrow \mathbb{K}$, and denoting $\tilde{\gamma} : \mathbb{R} \to \mathfrak{b}$ the image of $\gamma : \mathbb{R} \to \mathfrak{b}^*$ through the induced isomorphism $\mathfrak{b}^* \to \mathfrak{b}$, the equation of motion turns into the Lax form

$$\frac{\mathrm{d}\tilde{\gamma}(t)}{\mathrm{d}t} = [\tilde{\gamma}(t), \mathcal{L}_{\mathsf{h}}(\gamma(t))]. \tag{6}$$

### 2.1. Relation with AKS method

The success of the method described above relies on the integrability of the Eq. (3). The AKS theory [4] gives a family of integrable Hamiltonians associated to $\mathrm{Ad}^{G*}$-invariant functions on $\mathfrak{g}^*$. First, we have the identification $\mathfrak{k}^\circ \simeq \mathfrak{b}^*$ by the map $\eta \in \mathfrak{b}^* \mapsto \eta \circ \Pi_{\mathfrak{b}}$. It allows us to define a $B$-action via

$$\tau^B_b(\mu) := \left(\left(\mathrm{Ad}^G_b\right)^* \mu\right) \circ \Pi_{\mathfrak{b}} \quad \forall b \in B,\ \mu \in \mathfrak{k}^\circ.$$

The orbit $\mathcal{O}^B_\mu \subset \mathfrak{k}^\circ$ for this action through $\mu$ is a symplectic manifold; in fact, for $\eta \in \mathcal{O}^B_\mu$ we have that

$$T_\eta\left(\mathcal{O}^B_\mu\right) = \left\{\left(\mathrm{ad}^G_X\right)^* \eta \circ \Pi_{\mathfrak{b}} : X \in \mathfrak{b}\right\}$$

and the symplectic structure is given by

$$\left\langle \omega, \left(\mathrm{ad}^G_{X_1}\right)^* \eta \circ \Pi_{\mathfrak{b}} \otimes \left(\mathrm{ad}^G_{X_2}\right)^* \eta \circ \Pi_{\mathfrak{b}} \right\rangle_\eta = \langle \eta, [X_1, X_2] \rangle .$$

This structure will be used in proving the following result.

**Theorem.** *Let $f \in C^\infty(\mathfrak{g}^*)$ be an $\mathrm{Ad}^*$-invariant function, and let the restriction $\mathsf{h} := f|\mathcal{O}^B_{J_M(m)}$ be the Hamiltonian function for the system defined on $\mathcal{O}^B_{J_M(m)} \subset \mathfrak{k}^\circ \simeq \mathfrak{b}^*$. Hence, the solution of this system with initial condition $\eta(0) = J_M(m)$ is*

$$\eta(t) = \left(\mathrm{Ad}^G_{k(t)}\right)^* J_M(m)$$

*where $k : \mathbb{R} \to K$ is the $K$-factor in the decomposition of the element $g(t) = \exp\left(t\mathcal{L}_f(J_M(m))\right)$.*

**Proof.** Let $\eta$ be an arbitrary element in the orbit $\mathcal{O}^B_{J_M(m)}$ defined above. In this case, using the associated Legendre transformation $\mathcal{L}_f : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ that allows to identify $T^*_\eta(\mathfrak{g}^*) \simeq \mathfrak{g}$, we have

$$\left\langle \mathrm{d}f, \left(\mathrm{ad}^G_X\eta\right)^* \circ \Pi_{\mathfrak{b}} \right\rangle_\eta = \left\langle \left(\mathrm{ad}^G_X\eta\right)^* \circ \Pi_{\mathfrak{b}}, \mathcal{L}_f(\eta) \right\rangle = \left\langle \eta, \left[X, \Pi_{\mathfrak{b}}\mathcal{L}_f(\eta)\right] \right\rangle$$

so that the Hamiltonian vector field associated to $\mathsf{h}$ is given by

$$V_{\mathsf{h}}|_\eta = -\left(\mathrm{ad}^G_{\Pi_{\mathfrak{b}}\mathcal{L}_f(\eta)}\eta\right)^* \circ \Pi_{\mathfrak{b}}.$$

Because of the $\mathrm{Ad}^*$-invariance of $f$, we have that $\left(\mathrm{ad}^G_{\mathcal{L}_f(\eta)}\right)^* \eta = 0$, and we can rewrite it as

$$V_{\mathsf{h}}|_\eta = \left(\mathrm{ad}^G_{\Pi_{\mathfrak{k}}\mathcal{L}_f(\eta)}\right)^* \eta \tag{7}$$

by taking into account that $\left(\mathrm{Ad}^G_k\right)^* \mathfrak{k}^\circ \subset \mathfrak{k}^\circ$ for all $k \in K$.

On the other side, the curve in $\mathfrak{k}^\circ$ defined through

$$\eta(t) = \left(\mathrm{Ad}^G_{k(t)}\right)^* J_M(m)$$

has tangent vector field given by

$$\dot{\eta}(t) = \left.\frac{\vec{\mathrm{d}}}{\mathrm{d}t}\right|_t \left[\left(\mathrm{Ad}^G_{k(t)}\right)^* J_M\,(m)\right] = \left(\mathrm{ad}^G_{k^{-1}(t)\dot{k}(t)}\right)^* \eta(t).$$

Now, considering the integral curve $g(t)$ of the right invariant vector field $\mathcal{L}_f\,(J_M(m))\,g(t)$, written in terms of the decomposition curves $g\,(t) = k(t)b(t)$ we obtain

$$\dot{g}(t)g^{-1}(t) = \left.\frac{\mathrm{d}\,(kb)}{\mathrm{d}t}(kb)^{-1}\right|_t = \mathrm{Ad}^G_{k(t)}\left(k^{-1}(t)\dot{k}(t) + \dot{b}(t)b^{-1}(t)\right)$$

meaning that $\mathrm{Ad}^G_{k^{-1}(t)}\mathcal{L}_f\,(J_M(m)) = k^{-1}(t)\dot{k}\,(t) + \dot{b}(t)b^{-1}(t)$, and therefore

$$\Pi_{\mathfrak{k}}\left(\mathrm{Ad}^G_{k^{-1}(t)}\mathcal{L}_f\,(J_M(m))\right) = k^{-1}(t)\dot{k}(t).$$

By using $\mathrm{Ad}^*$-invariance for $f$ again, we have that

$$\mathrm{Ad}^G_{k^{-1}(t)}\mathcal{L}_f\,(J_M\,(m)) = \mathcal{L}_f\left(\left[\mathrm{Ad}^G_{k(t)}\right]^* J_M(m)\right) = \mathcal{L}_f\,(\eta(t))$$

implying that $k^{-1}(t)\dot{k}(t) = \Pi_{\mathfrak{k}}\left(\mathcal{L}_f\,(\eta(t))\right)$. Comparing with Eq. (7) we can conclude that $\eta(t)$ has $V_{\mathsf{h}}$ as tangent vector field. □

The $\mathrm{Ad}^*$-invariance implies the identity $\left(\mathrm{ad}^G_{\mathcal{L}_f(\eta)}\right)^*\eta = 0$, for all $\eta \in \mathfrak{g}^*$, and so

$$\left[\mathrm{Ad}^G_{\exp t\mathcal{L}_f(J_M(m))}\right]^* J_M(m) = J_M(m)$$

meaning that

$$\left[\mathrm{Ad}^G_{k(t)}\right]^* J_M\,(m) = \left[\mathrm{Ad}^G_{b^{-1}(t)}\right]^* J_M(m)$$

and assuming $J_M(m) \in \mathfrak{k}^\circ$ it is clear that we can take $b : \mathbb{R} \to B$ (the $B$-factor of $\exp t\mathcal{L}_f\,(J_M(m))$) as the solution curve in Eq. (3). In such case it is necessary to find the differential equation for the $B_{J_M(m)}$-factor $n$ (Cf. proof of the proposition below Eq. (3)). But as was previously shown

$$\mathcal{L}_f\,(\eta(t)) = k^{-1}(t)\dot{k}(t) + \dot{b}(t)b^{-1}(t), \quad \eta(t) = \mathrm{Ad}^G_{k^{-1}(t)}J_M(m),$$

so $\dot{b}(t)b^{-1}(t) = \Pi_{\mathfrak{b}}\mathcal{L}_f\,(\eta(t))$.

On the other side, for all $\xi, \eta \in \mathfrak{k}^\circ$ we have that

$$\left\langle \mathcal{L}_f\,(\eta)\,,\xi\right\rangle = \left\langle \Pi_{\mathfrak{b}}\mathcal{L}_f\,(\eta)\,,\xi\right\rangle$$

$$\left\langle \xi, \mathcal{L}_f\,(\eta)\right\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}f\,(\eta + t\xi)\right|_{t=0} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{h}\,(\eta + t\xi)\right|_{t=0} = \langle \xi, \mathcal{L}_{\mathsf{h}}\,(\eta)\rangle$$

meaning that $\mathcal{L}_{\mathsf{h}}\,(\eta) = \Pi_{\mathfrak{b}}\mathcal{L}_f\,(\eta)$ and then $\dot{b}(t)b^{-1}(t) = \mathcal{L}_{\mathsf{h}}\,(\eta)$; therefore

$$\dot{n}(t)n^{-1}(t) = 0$$

and the $B_{J_M(m)}$-factor is constant.

### 2.2. Summary

The setting consist of a factorizable Lie group $G = KB$ and a Hamiltonian $B$-space $M$. The collective motion associated to the restriction to $\mathfrak{k}^\circ \simeq \mathfrak{b}^*$ of an $\left(\mathrm{Ad}^G\right)^*$-invariant function $f$ gives rise to a collective Hamiltonian system on $M$, which can be thus solved algebraically as follows:

1. Factorize the straight curve $t \mapsto \exp t\mathcal{L}_f\,(J_M(m)) = k(t)b(t)$.
2. The solution curve on $M$ for the Hamiltonian system defined by $H := (f\,|\mathfrak{k}^\circ) \circ J_M$ is given by

$$t \mapsto \varphi\,(b(t)n_0, m)$$

for some element $n_0 \in B_{J_M(m)}$.

## 3. Iwasawa decomposition of $\mathrm{SL}(2, \mathbb{C})$ and coadjoint orbits

We now specialize the above abstract structure to $G = \mathrm{SL}(2, \mathbb{C})$ and its Iwasawa decomposition $\mathrm{SL}(2, \mathbb{C}) \cong \mathrm{SU}(2) \times B$, where $B$ is the solvable group of $2 \times 2$ complex upper triangular matrices, with real positive diagonals and determinant equal to 1. Let us address the construction of an explicit example of $T$-dual systems in this framework.

In order to start with, we consider the maximal Abelian subalgebra $\mathfrak{h} = \mathbb{C}\langle\sigma_3\rangle$ of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, with the root system $\Delta := \{-\alpha, \alpha\}$, where $\alpha \in \mathfrak{h}^*$ is given by $\alpha(\sigma_3) = 2$. The associated decomposition is $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{h} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, with

$$\mathfrak{g}_{\pm\alpha} := \mathbb{C}\left\langle \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)\right\rangle.$$

For the positive root set $\Delta_+ = \{\alpha\}$ we define $\mathfrak{n} := \bigoplus_{\beta\in\Delta_+} \mathfrak{g}_\beta = g_\alpha$. Then we may find a decomposition as expected for $\mathfrak{sl}_2(\mathbb{C})^{\mathbb{R}}$ by taking $\mathfrak{k} = \mathfrak{su}_2$ and $\mathfrak{b} := \mathfrak{a} \oplus \mathfrak{n}^{\mathbb{R}}$, where $\mathfrak{a} := \mathbb{R}\langle\sigma_3\rangle = \mathrm{i}\mathfrak{t}$, being $\mathfrak{t} := \mathfrak{h} \cap \mathfrak{su}_2$ a real form for $\mathfrak{h}$,

$$\mathfrak{sl}_2(\mathbb{C})^{\mathbb{R}} = \mathfrak{su}_2 \oplus \mathfrak{b}.$$

With this election for $\mathfrak{h}$, $\mathfrak{b}$ is the subalgebra of upper triangular matrices with real diagonal and null trace, and $\mathfrak{k}$ is the real subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ of antihermitian matrices.

Alternatively, one would may choose for instance $\mathfrak{h}' := \mathbb{C}\langle\sigma_1\rangle$, changing the roots $\alpha'$ and the spaces $\mathfrak{g}_{\alpha'}$, so that $\mathfrak{b}'$ is no longer composed of upper triangular matrices. However, by change of basis (the one which diagonalize $\sigma_1$) will turn $\mathfrak{b}'$ into triangular matrices again. The compact real form is obtained as usual, defining

$$\mathfrak{u}_n := \sum_{\alpha\in\Delta} \mathbb{R}(\mathrm{i}H_\alpha) + \sum_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}\mathrm{i}(X_\alpha + X_{-\alpha})$$

once $\mathfrak{h}$ is fixed.

The Killing form for $\mathfrak{sl}_2(\mathbb{C})$ is $\kappa(X, Y) := \mathrm{tr}\,(\mathrm{ad}\,(X)\,\mathrm{ad}\,(Y)) = 4\mathrm{tr}\,(XY)$, the restrictions to $\mathfrak{su}_2$, $\mathfrak{a}$, and $\mathfrak{n}$ are negative defined, positive defined, and 0, respectively. We consider the nondegenerate symmetric bilinear form on $\mathfrak{sl}_2(\mathbb{C})$

$$(X, Y)_{\mathfrak{sl}_2} = -\frac{1}{4}\,\mathrm{Im}\;\kappa(X, Y) \tag{8}$$

which turns $\mathfrak{b}$ and $\mathfrak{k}$ into isotropic subspaces. Also, we take the basis

$$X_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad X_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

for $\mathfrak{su}_2$, and

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathrm{i}E = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}, \qquad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in $\mathfrak{b}$. Then, the crossed product are

$$\begin{array}{lll} (X_1, E)_{\mathfrak{sl}_2} = -1 & (X_2, E)_{\mathfrak{sl}_2} = 0 & (X_3, E)_{\mathfrak{sl}_2} = 0 \\ (X_1, \mathrm{i}E)_{\mathfrak{sl}_2} = 0 & (X_2, \mathrm{i}E)_{\mathfrak{sl}_2} = 1 & (X_3, \mathrm{i}E)_{\mathfrak{sl}_2} = 0 \\ (X_1, H)_{\mathfrak{sl}_2} = 0 & (X_1, H)_{\mathfrak{sl}_2} = 0 & (X_3, H)_{\mathfrak{sl}_2} = -2 \end{array} \tag{9}$$

allowing for the identification $\psi : \mathfrak{su}_2 \to \mathfrak{b}^*$ given by

$$\psi(X_1) = -\mathbf{e}, \qquad \psi(X_2) = \tilde{\mathbf{e}}, \qquad \psi(X_3) = -2\mathbf{h} \tag{10}$$

where $\left\{\mathbf{e}, \tilde{\mathbf{e}}, \mathbf{h}\right\} \subset \mathfrak{b}^*$ is the dual basis to $\{E, \mathrm{i}E, H\} \subset \mathfrak{b}$.

This map allows to carry the Poisson structure of $\mathfrak{b}^*$ to $\mathfrak{su}_2$. In terms of the dual basis $\{\mathbf{x}_k\} \subset \mathfrak{su}_2^*$, $\langle \mathbf{x}_k, X_j\rangle = \delta_{kj}$, so for $f \in C^\infty(\mathfrak{su}_2)$ we have that $\mathrm{d}f(X_k) = \frac{\partial f}{\partial \mathbf{x}_k}$ and the Poisson bracket reads as

$$\{f, g\} = \left(\frac{\partial f}{\partial \mathbf{x}_1}\frac{\partial g}{\partial \mathbf{x}_3} - \frac{\partial f}{\partial \mathbf{x}_3}\frac{\partial g}{\partial \mathbf{x}_1}\right)\mathbf{x}_1 + \left(\frac{\partial f}{\partial \mathbf{x}_2}\frac{\partial g}{\partial \mathbf{x}_3} - \frac{\partial f}{\partial \mathbf{x}_3}\frac{\partial g}{\partial \mathbf{x}_2}\right)\mathbf{x}_2.$$

The Hamiltonian vector fields are then

$$V_g = \frac{\partial g}{\partial \mathbf{x}_3}\left(\mathbf{x}_1\frac{\partial}{\partial \mathbf{x}_1} + \mathbf{x}_2\frac{\partial}{\partial \mathbf{x}_2}\right) - \left(\mathbf{x}_1\frac{\partial g}{\partial \mathbf{x}_1} + \mathbf{x}_2\frac{\partial g}{\partial \mathbf{x}_2}\right)\frac{\partial}{\partial \mathbf{x}_3}.$$

With the identification $X_k = \frac{\partial}{\partial \mathbf{x}_k}$, we get

$$X_{\mathbf{x}_1} = \mathbf{x}_1 X_3, \qquad X_{\mathbf{x}_2} = \mathbf{x}_2 X_3, \qquad X_{\mathbf{x}_3} = -\mathbf{x}_1 X_1 - \mathbf{x}_2 X_2,$$

from where it can be determined the *symplectic leaves*, which are divided in two uniparametric families, namely,

- *Symplectic leaves of dimension* 0: *each leaf is a point* $\alpha X_3$, $\alpha \in \mathbb{R}$, *on the vertical axis of* $\mathfrak{su}_2 \simeq \mathbb{R}^3$,

– *Symplectic leaves of dimension* 2: *each leaf is a vertical semiplane*

$$\mathcal{O}_\theta = \left\{ (xX_\theta + zX_3) \in \mathfrak{su}_2 / x \in \mathbb{R}_{>0}, z \in \mathbb{R}, \theta \in S^1 \right\} \tag{11}$$

*where* $X_\theta = \cos\theta X_1 + \sin\theta X_2$.

The zero dimensional orbits lack of interest for our purpose, so let us focus our attention on the two dimensional ones. They are semiplanes orthogonal to the plane $X_1, X_2$, spanned radially from the $X_3$ axis like the pages of a book, without touching it, and characterized by the angle $\theta$ between the $X_1$ axis and the intersection of the leaf with the $X_1, X_2$ plane.

To write out the explicit form of the $B$ action on $\mathfrak{su}_2$, we parametrize an arbitrary element $\tilde{b} \in B$ as

$$\tilde{b} = \begin{bmatrix} a & b + \mathrm{i}c \\ 0 & a^{-1} \end{bmatrix} \tag{12}$$

with $a \in \mathbb{R}_{>0}$ and $b, c \in \mathbb{R}$. Then, $\psi \circ \mathrm{Ad}^*_{\tilde{b}^{-1}} \circ \psi^{-1} : \mathfrak{su}_2 \longrightarrow \mathfrak{su}_2$, in the basis $\{X_1, X_2, X_3\} \subset \mathfrak{su}_2$ gives

$$\begin{aligned} (\psi \circ (\mathrm{Ad}_{\tilde{b}^{-1}})^* \circ \psi^{-1}) X_1 &= \frac{b}{a} X_3 + a^{-2} X_1 \\ (\psi \circ (\mathrm{Ad}_{\tilde{b}^{-1}})^* \circ \psi^{-1}) X_2 &= -\frac{c}{a} X_3 + a^{-2} X_2 \\ (\psi \circ (\mathrm{Ad}_{\tilde{b}^{-1}})^* \circ \psi^{-1}) X_3 &= X_3 \end{aligned} \tag{13}$$

so that on the orbit $\mathcal{O}_\theta$ it acts as

$$(\psi \circ (\mathrm{Ad}_{\tilde{b}^{-1}})^* \circ \psi^{-1}) (xX_\theta + zX_3) = xa^{-2} X_\theta + \left( z + \frac{x}{a} (b\cos\theta - c\sin\theta) \right) X_3.$$

Hence, the *stabilizer* of $X \in \mathcal{O}_\theta$ is the *normal subgroup* $B_\theta \subset B$ composed by the matrices

$$\tilde{b}_\theta := \begin{pmatrix} 1 & d(\sin\theta + \mathrm{i}\cos\theta) \\ 0 & 1 \end{pmatrix} \tag{14}$$

with $d \in \mathbb{R}$. The Lie algebra $\mathfrak{b}_\theta$ is generated by the element

$$E_\theta = \sin\theta E + \cos\theta (\mathrm{i}E)$$

and, consequently, $\mathfrak{b}/\mathfrak{b}_\theta$ is spanned by the images in the quotient of the elements

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{E}_\theta = \begin{pmatrix} 0 & (\cos\theta - \mathrm{i}\sin\theta) \\ 0 & 0 \end{pmatrix} = \cos\theta E - \sin\theta (\mathrm{i}E).$$

### *3.1. Orbits and Bruhat decomposition*

Let us now describe an issue which will be of central importance in defining the dualizable subspaces in cotangent bundle of the compact factor $K$. As it was mentioned in (17), the action of the solvable factor $B$ on this phase space arises from the lift of the dressing action and its orbits on $K$ are the dressing orbits.

The dressing orbits of the Poisson–Lie structure associated to the Iwasawa decomposition [6] in a semisimple group can be described by using the *Bruhat decomposition* [8,9]. Let us begin with a compact Lie group $K$; let $G$ be its complexification. For $G = KB$, the Iwasawa decomposition associated to $K$, let us choose in the Lie algebra $\mathfrak{k}$ a maximal abelian subalgebra $\mathfrak{t}$; then $\mathfrak{h} := \mathfrak{t} + \mathrm{i}\mathfrak{t}$ is a Cartan subalgebra for $\mathfrak{g}$. Let us fix some ordering of the roots associated to $\mathfrak{h}$. For example, if $K = \mathrm{SU}(n)$ then $G = \mathrm{SL}(n, \mathbb{C})$ and we can choose the order in the roots such that $B$ is the set of upper triangular matrices with real diagonal entries. Let $T \subset K$ be the connected subgroup associated to $\mathfrak{t}$.

**Lemma.** *The set*

$$T \cdot B := \{tb : t \in T, b \in B\}$$

*is a Lie subgroup of* $G$*; moreover, we have that* $T \cdot B = B \cdot T$.

**Proof.** Because $tBt^{-1} \subset B$ for all $t \in T$, we have that $T \cdot B$ is a subgroup of $G$ and $T \cdot B = B \cdot T$; if $(c_n) \subset T \cdot B$ is a sequence convergent in $G$, we have sequences $(a_n) \subset T$, $(b_n) \subset B$ such that $c_n = a_n b_n$ for all $n \in \mathbb{N}$. Now, because $T$ is compact, there exists a convergent subsequence $(a_{n_k})$, and $a_{n_k} \to a \in T$. Thus the sequence $b_{n_k} = (a_{n_k})^{-1} c_{n_k}$ has all its terms in $B$, and it is convergent in $G$, due to the continuity of the group operations. But $B$ is closed in $G$, thus $b_{n_k} \to b \in B$. Therefore $c_n \to ab \in T \cdot B$, and $T \cdot B$ is a closed subgroup in $G$. □

In the example considered above, $T$ is the set composed of diagonal matrices whose nonvanishing entries are elements of $S^1$; then $B_+ := B \cdot T$ is the group of upper triangular matrices. Let $N(T)$ the *normalizer* of $T$: It consists of the elements $k \in K$ such that $kTk^{-1} \subset T$; then the group $W := N(T)/T$ is the *Weyl group* of $K$.

**Theorem** (*Bruhat Decomposition*)**.** *The group G can be decomposed as*

$$G = \coprod_{w\in W} B_+ w B_+.$$

In order to use this decomposition, a set of representatives must be chosen for the elements of $W$. For example, $W$ for $\mathrm{SU}(n)$ is the set of permutations of $n$ elements, and representatives for two-cycles are

$$s_i := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

with the permutation matrix in the $i, i+1$-entries. The disjoint sets in the Bruhat decomposition gives a kind of cellular decomposition with a unique open cell plus lower dimensional submanifolds. In the SU(2) case, representatives for the Weyl groups members are the identity matrix and the element

$$\sigma := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

so the open submanifold is the set

$$B_+\sigma B_+ = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{C}) : c \neq 0 \right\}.$$

Note that the Bruhat decomposition yields to the decomposition

$$G = \coprod_{\substack{w\in W\\ t\in T}} tBwB$$

by using the fact that $wTw^{-1} \subset T$ for every $w \in N(T)$. On $\mathrm{SL}(2,\mathbb{C})$ this decomposition can be written as

$$\mathrm{SL}(2,\mathbb{C}) = \left( \coprod_{t\in T} t\cdot B \right) \amalg \left( \coprod_{t\in T} t\cdot B^w \right)$$

where $B^w$ is the subset of $\mathrm{SL}(2,\mathbb{C})$ composed of those matrices with its lower-left element strictly negative.

By definition, the dressing orbits in $K$ are the sets $\pi_K(Bk)$ for $k \in K$. With the previous decomposition at hands, it is possible to characterize the orbits of the $B$-action on $K$: In fact, if $w \in W$, let us denote by $\Sigma_w$ the $B$-orbit through $w$: $\Sigma_w = \pi_K(Bw)$ (by fixing a set of representatives in $K$ for the element $w \in W$). Then we have the following result.

**Proposition.** *The orbits of the B-action on K can be parametrized by $T \times W$: That is, every orbit can be written as $t\cdot\Sigma_w$ for some $(w,t) \in W \times T$.*

**Proof.** Let us denote by $\mathcal{O}_k$ the $B$-orbit through $k \in K$; then by using the previous decomposition we can write $k = tb_1wb_2$ for some $b_1, b_2 \in B$, $t \in T$ and $w \in W$. Therefore

$$\mathcal{O}_k = \pi_K(Bk) = \pi_K(Btb_1w) = \pi_K(tBw) = t\pi_K(Bw) = t\cdot\Sigma_w$$

where it was used that $tB = Bt$. $\square$

In the case $G = \mathrm{SL}(2,\mathbb{R})$, the orbits are

$$\Sigma_1 := \pi_{\mathrm{SU}(2)}(B\cdot \mathrm{id}) = \{\mathrm{id}\}$$

and

$$\Sigma_{-1} := \pi_{\mathrm{SU}(2)}\left( B \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} \alpha & b \\ -b & \bar{\alpha} \end{bmatrix} : \alpha \in \mathbb{C}, b \in \mathbb{R}^+, |\alpha|^2 + b^2 = 1 \right\}.$$

So the orbits of the $B$-action on SU(2) are the zero dimensional ones

$$t\cdot\Sigma_1 = \left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \right\}$$

and the two dimensional, given by

$$t \cdot \Sigma_{-1} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} : \arg\beta = \arg t + \pi \right\}.$$

A comment on the choice of representatives for the elements of $W$ is in order: Any other choice just gives another parametrization $W \times K \to \{\mathcal{O}_k : k \in K\}$.

## 4. Hamiltonian *B*-spaces

In this section we describe some Hamiltonian $B$-spaces related to the two dimensional symplectic leaves $\mathcal{O}_\theta$ (11) which in turn will assemble the $T$-duality scheme.

### 4.1. Two dimensional symplectic leaves $\mathcal{O}_\theta \subset \mathfrak{su}_2$

The semiplanes $\mathcal{O}_\theta \subset \mathfrak{su}_2$ turn in symplectic manifolds when endowed with the pullback by $\psi : \mathfrak{su}_2 \to \mathfrak{b}^*$ of the Kirillov–Kostant structure on the coadjoint orbits in $\mathfrak{b}^*$

$$\left\langle \omega_{\mathcal{O}_\theta}, \bar{\psi}_* \mathrm{ad}^*_X \psi(Z) \otimes \bar{\psi}_* \mathrm{ad}^*_Y \psi(Z) \right\rangle = (Z, [X, Y])_{\mathfrak{su}_2}$$

where $\bar{\psi} : \mathfrak{b}^* \to \mathfrak{su}_2$ is the inverse mapping of $\psi$. They also are Hamiltonian $B$-spaces under the action (13).

### 4.2. $\mathbb{R}^2$ as a phase space

Given the phase space $\mathbb{R}^2$ with coordinates $(q, p)$, there exist a family of embeddings which can be interpreted as the momentum map associated with some action of $B$ on $\mathbb{R}^2$, as explained in the following proposition.

**Proposition.** *The maps* $\rho : B \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined as*

$$\rho\left(\tilde{b}, (q, p)\right) = \left( q - \frac{1}{\mu} \ln a, p - 2\mu \frac{\varepsilon}{a} \exp(2\mu q) \left(b \cos\theta - c \sin\theta\right) \right) \tag{15}$$

*are a family of transitive actions of* $B$ *on* $\mathbb{R}^2$, *for* $(q, p) \in \mathbb{R}^2$, $\tilde{b} \in B$ *as given in* (12), $\theta \in [0, 2\pi]$, $\mu \in \mathbb{R}_{>0}$ *and* $\varepsilon \in \mathbb{R}$, *arbitrary parameters. Moreover, regarding* $\mathbb{R}^2$ *as a symplectic space endowed with the canonical symplectic form* $\omega_\circ = \mathrm{d}q \wedge \mathrm{d}p$, $\left(\mathbb{R}^2, \omega_\circ\right)$, *it becomes in an homogeneous Hamiltonian B-space with associated equivariant momentum map* $\sigma_\theta : \mathbb{R}^2 \hookrightarrow \mathfrak{b}^*$

$$\sigma_\theta(q, p) = -\frac{1}{\mu} p\mathbf{h} + \varepsilon \exp(2\mu q)(\cos\theta \mathbf{e} - \sin\theta \tilde{\mathbf{e}}).$$

*For each fixed value of* $\theta$, *the induced map* $\tilde{\sigma}_\theta : \mathbb{R}^2 \hookrightarrow \mathfrak{su}_2$

$$\tilde{\sigma}_\theta(q, p) := \psi^{-1} \circ \sigma_\theta(q, p) = \frac{1}{2\mu} pX_3 - \varepsilon \exp(2\mu q)(\cos\theta X_1 + \sin\theta X_2)$$

*is a symplectic isomorphism between* $\left(\mathbb{R}^2, \omega_\circ\right)$ *and* $(\mathcal{O}_\theta, \psi^* \omega_{KK})$, *where* $\omega_{KK}$ *is the Kirillov–Kostant symplectic form.*

**Proof.** It is straightforward to check that $\rho$ is a transitive action and that it is Hamiltonian. The infinitesimal generator associated to $\tilde{X} \in \mathfrak{b}$,

$$\tilde{X} = uE + v(\mathrm{i}E) + wH = \begin{bmatrix} w & u + \mathrm{i}v \\ 0 & -w \end{bmatrix}$$

can be calculated from the expression

$$\mathrm{e}^{t\tilde{X}} = \begin{bmatrix} \mathrm{e}^{tw} & \frac{1}{w} \sinh(tw)\,(u + \mathrm{i}v) \\ 0 & \mathrm{e}^{-tw} \end{bmatrix}$$

giving

$$\tilde{X}_{\mathbb{R}^2}|_{(q,p)} = \left( -\frac{1}{\mu} w, -2\varepsilon\mu \exp(2\mu q)\,(u\cos\theta - v\sin\theta) \right).$$

The contraction of this vector with the symplectic form is

$$\imath_{\tilde{X}_{\mathbb{R}^2}} (\mathrm{d}q \wedge \mathrm{d}p) = d\left\langle -\frac{1}{\mu} p\mathbf{h} + \varepsilon \exp(2\mu q)(\cos\theta \mathbf{e} - \sin\theta \tilde{\mathbf{e}}), \tilde{X} \right\rangle$$

from where we get the momentum map $\sigma_\theta(q, p)$.

The last statement is a direct consequence of the equivariance property. Under the action $\psi \circ \mathrm{Ad}^*_{\tilde{b}^{-1}} \circ \psi^{-1} : \mathfrak{su}_2 \longrightarrow \mathfrak{su}_2$ it behaves as

$$\left(\psi \circ \mathrm{Ad}^*_{\tilde{b}^{-1}} \circ \psi^{-1}\right) \tilde{\sigma}(q,p) = -\frac{1}{2}\left(p + \frac{2\varepsilon \exp(2\mu q)}{a}\left(b\cos\theta - c\sin\theta\right)\right) X_3 - \frac{\varepsilon \exp(2\mu q)}{a^2}\left(\cos\theta X_1 + \sin\theta X_2\right)$$

satisfying the equivariant property $\tilde{\sigma}_\theta(\rho_{\tilde{b}}(q,p)) = \left(\psi \circ \mathrm{Ad}^*_{\tilde{b}^{-1}} \circ \psi^{-1}\right) \tilde{\sigma}(q,p)$. □

### 4.3. The cotangent bundle of B

Let us consider the phase space $T^*B = B \times \mathfrak{b}^*$, trivialized by left translation, endowed with the canonical symplectic form $\tilde{\omega}_\circ$. It is a Hamiltonian $B$-space by the Hamiltonian action of $B$ on $B \times \mathfrak{b}^*$

$$\lambda : B \times \left(B \times \mathfrak{b}^*\right) \longrightarrow \left(B \times \mathfrak{b}^*\right) / \lambda\left(\tilde{h}, \left(\tilde{b}, \tilde{\eta}\right)\right) = \left(\tilde{h}\tilde{b}, \tilde{\eta}\right)$$

for $\tilde{h}, \tilde{b} \in B, X \in \mathfrak{b}^*$, with associated momentum map $\mu : B \times \mathfrak{b}^* \longrightarrow \mathfrak{b}^*/\mu\left(\tilde{b}, \tilde{\eta}\right) = \mathrm{Ad}^*_{\tilde{b}^{-1}}\tilde{\eta}$. The corresponding map $\tilde{\mu} : B \times \mathfrak{b}^* \longrightarrow \mathfrak{su}_2$ with $\tilde{\mu} = \psi^{-1} \circ \mu$ is

$$\tilde{\mu}\left(\tilde{b}, \tilde{\eta}\right) = \left(\bar{\psi} \circ \mathrm{Ad}^*_{\tilde{b}^{-1}} \circ \psi\right) \psi\left(\tilde{\eta}\right)$$

that has the explicit form

$$\tilde{\mu}\left(\tilde{b}, \tilde{\eta}_{\mathbf{e}}\mathbf{e} + \tilde{\eta}_{\tilde{\mathbf{e}}}\tilde{\mathbf{e}} + \tilde{\eta}_{\mathbf{h}}\mathbf{h}\right) = -a^{-2}\tilde{\eta}_{\mathbf{e}}X_1 + a^{-2}\tilde{\eta}_{\tilde{\mathbf{e}}}X_2 - \left(\frac{1}{2}\tilde{\eta}_{\mathbf{h}} + \frac{b\tilde{\eta}_{\mathbf{e}} + c\tilde{\eta}_{\tilde{\mathbf{e}}}}{a}\right) X_3 \tag{16}$$

where we parametrized an element $\tilde{b} \in B$ as

$$\tilde{b} = \begin{pmatrix} a & b + \mathrm{i}c \\ 0 & -a \end{pmatrix}$$

with $a \in \mathbb{R}^+$, $b, c \in \mathbb{R}$.

### 4.4. The cotangent bundle of SU(2)

The third phase space we consider here is the cotangent bundle of the remaining factor of the factorization of $\mathrm{SL}(2,\mathbb{C})$, namely $T^*\mathrm{SU}(2)$. To stand the notation to be used in rest of this work, we parametrize an element $g \in \mathrm{SU}(2)$ as

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

where the bar over the complex entries $\alpha, \beta$ is meant to indicate the complex conjugate. We regard $T^*\mathrm{SU}(2)$ trivialized by left translation, $T^*\mathrm{SU}(2) = \mathrm{SU}(2) \times \mathfrak{su}_2^*$, endowed with the canonical symplectic form $\omega_\circ$. In order to turn it into a Hamiltonian $B$-space, we are given the dressing action $\mathsf{d} : B \times \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2)$ which arises from the factorization $\mathrm{SL}(2,\mathbb{C}) = \mathrm{SU}(2) \times B$ such that, for $\tilde{b} \in B$ and $g \in \mathrm{SU}(2)$, $\mathsf{Dr}\left(\tilde{b}, g\right) = \Pi_{\mathrm{SU}(2)}\left(\tilde{b}g\right) = g^{\tilde{b}}$. It is lifted to $\mathrm{SU}(2) \times \mathfrak{su}_2^*$ as explained in the following proposition.

**Theorem.** *The action* $\mathsf{d} : B \times \mathrm{SU}(2) \longrightarrow \mathrm{SU}(2)$, *defined above as* $\mathsf{d}_{\tilde{b}}(g) = g^{\tilde{b}}$, *lift to the cotangent bundle in body coordinates,* $\mathrm{SU}(2) \times \mathfrak{su}_2^*$, *as*

$$\hat{\mathsf{d}}_{\tilde{b}}(g, \eta) = \left(g^{\tilde{b}}, \left(\psi^* \circ \mathrm{Ad}_{\tilde{b}g} \circ \bar{\psi}^*\right)\eta\right). \tag{17}$$

*It is a symplectic action with* Ad-*equivariant momentum map* $\varphi : \mathrm{SU}(2) \times \mathfrak{su}_2^* \longrightarrow \mathfrak{b}^*$

$$\varphi(g, \eta) = \psi\left(\Pi_{\mathfrak{su}_2}\left(\mathrm{Ad}^G_g \bar{\psi}^*(\eta)\right)\right) \tag{18}$$

*where* $\bar{\psi}^* : \mathfrak{su}_2^* \longrightarrow \mathfrak{b}$ *is the pullback of the bijection* $\mathfrak{b}^* \xrightarrow{\bar{\psi}} \mathfrak{su}_2$.

**Proof.** We get the action on the left trivialized cotangent bundle from the relation

$$\left\langle \left(g^{\tilde{b}}, \eta\right), \left(\mathsf{d}_{\tilde{b}}\right)_{*B}(g, X) \right\rangle = \left\langle \eta, X^{\tilde{b}g} \right\rangle$$

where $X = g^{-1}\dot{g}$ and

$$X^{\tilde{b}} = \left.\frac{\mathrm{d}\left(g^{-1}g(t)\right)^{\tilde{b}}}{\mathrm{d}t}\right|_{t=0} = \mathrm{Ad}^*_{\tilde{b}^g} X$$

by using the relations $(gh)^{\tilde{b}} = g^{\tilde{b}} h^{\tilde{b}^g}$.

Then, the infinitesimal generator $\tilde{Z}_{\mathrm{SU}(2)\times\mathfrak{su}_2^*}|_{(g,\eta)} = \frac{\mathrm{d}}{\mathrm{d}t}\left.\mathrm{d}^{\mathrm{SU}(2)\times\mathfrak{su}_2^*}_{\mathrm{e}^{t\tilde{Z}}}(g,\eta)\right|_{t=0}$ is, for $\tilde{Z} \in \mathfrak{b}$,

$$\tilde{Z}_{\mathrm{SU}(2)\times\mathfrak{su}_2^*}|_{(g,\eta)} = \left(g^{\tilde{Z}}, \psi^*\left(\left[\mathrm{Ad}^*_g \tilde{Z}, \bar{\psi}^*(\eta)\right]\right)\right)$$

where $g^{\tilde{Z}} = d(g^{\mathrm{e}^{t\tilde{Z}}})/\mathrm{d}t|_{t=0}$. From this vector field on $\mathrm{SU}(2) \times \mathfrak{su}_2^*$ we compute the momentum map since $\left\langle \eta, g^{-1}g^{\tilde{Z}}\right\rangle = \left\langle \varphi(g,\eta), \tilde{Z}\right\rangle$, using that $\Pi_{\mathfrak{su}_2}\mathrm{Ad}^G_{g^{-1}}\tilde{Z} = g^{-1}g^{\tilde{Z}}$ and the bijection $\psi : \mathfrak{su}_2 \longrightarrow \mathfrak{b}^*$, its the adjoint $\psi^* : \mathfrak{b} \longrightarrow \mathfrak{su}_2^*$ and its inverse $\bar{\psi}^* : \mathfrak{su}_2^* \longrightarrow \mathfrak{b}$, obtaining

$$\left\langle \varphi(g,\eta), \tilde{Z}\right\rangle_{\mathfrak{g}} = \left\langle \psi\left(\Pi_{\mathfrak{su}_2}\mathrm{Ad}^G_g \bar{\psi}^*(\eta)\right), \tilde{Z}\right\rangle.$$

Since it arises as the lifting of a symmetry on the base space SU (2), it is naturally *equivariant*. □

As in the previous sections, we shall consider momentum maps valued on $\mathfrak{su}_2$, so we define

$$\begin{aligned} &\tilde{\varphi} \cong \bar{\psi} \circ \varphi : \mathrm{SU}(2) \times \mathfrak{su}_2^* \longrightarrow \mathfrak{su}_2 \\ &\tilde{\varphi}(g,\eta) = \Pi_{\mathfrak{su}_2}\left(\mathrm{Ad}^G_g \bar{\psi}^*(\eta)\right) = g\left(g^{-1}\right)^{\bar{\psi}^*(\eta)} \end{aligned} \tag{19}$$

where we used that $\mathrm{Ad}^G_g \bar{\psi}^*(\eta) = g\left(g^{-1}\right)^{\bar{\psi}^*(\eta)} + \bar{\psi}^*\left(\mathrm{Ad}^*_{g^{-1}}\eta\right)$. Observe that the momentum map associated with the dressing action is the Maurer–Cartan form applied to the infinitesimal generator at each point.

## 5. *T*-duality

The *T*-duality scheme involves the three Hamiltonian *B*-spaces described above, linking them with $\mathfrak{su}_2 \overset{\psi}{\cong} \mathfrak{b}^*$ by equivariant arrows

$$\begin{array}{ccccc} T^*\mathrm{SU}(2) & & \mathbb{R}^2 & & T^*B \\ & \searrow^{\tilde{\varphi}} & \downarrow^{\tilde{\sigma}} & \swarrow^{\tilde{\mu}} & \\ & & \mathfrak{su}_2 & & \end{array}$$

It is worth to remark that *T*-duality is not symplectic equivalence on the full phase space. Indeed, each symplectic equivalence class is defined by a coadjoint orbit $\mathcal{O}_\theta$ (11) and its elements are some symplectic submanifolds contained in SU (2) $\times\ \mathfrak{su}_2^*$ and $B \times \mathfrak{b}^*$, which are called *dualizable subspaces*. They can be defined as the leaves of some foliation in the pre-images of $\mathcal{O}_\theta$ through the maps $\tilde{\mu}, \tilde{\varphi}, \tilde{\sigma}_\theta$, as it will be explained below.

Let us consider the three fibrations on $\mathcal{O}_\theta$

$$\begin{array}{ccccc} \tilde{\varphi}^{-1}(\mathcal{O}_\theta) & & \mathbb{R}^2 & & \tilde{\mu}^{-1}(\mathcal{O}_\theta) \\ & \searrow^{\tilde{\varphi}} & \downarrow^{\tilde{\sigma}} & \swarrow^{\tilde{\mu}} & \\ & & \mathcal{O}_\theta & \hookrightarrow & \mathfrak{su}_2 \end{array}$$

where $\tilde{\mu}^{-1}(\mathcal{O}_\theta) \subset B \times \mathfrak{b}^*$ and $\tilde{\varphi}^{-1}(\mathcal{O}_\theta) \subset \mathrm{SU}(2) \times \mathfrak{su}_2^*$ are *coisotropic submanifolds*, and $\tilde{\sigma}_\theta^{-1}(\mathcal{O}_\theta) \cong \mathbb{R}^2$ is a symplectic space. Let us take a closer look of them. The tangent spaces of the fibers are the kernels of the corresponding differential momentum map, and their symplectic orthogonal are the tangent spaces to the orbits of *B* through each point. Then, collective Hamiltonians on $B \times \mathfrak{b}^*$, SU (2) $\times\ \mathfrak{su}_2^*$ and $\mathbb{R}^2$ furnish the compatible dynamics having Hamiltonian vector fields tangent to the *B*-orbits. The equivariant momentum maps carry them over a Hamiltonian vector field tangent to the coadjoint orbit $\mathcal{O}_\theta$. This is the main idea underlying *T*-duality, establishing a correspondence between Hamiltonian vector fields, so the correspondence between integral curves is defined up to a shifting of the initial condition.

Let us work out the *dualizable space* in each case.

### *5.1. Dualizable subspaces in $B \times \mathfrak{b}^*$*

Let us denote by $\bar{\tau}_B : B \times \mathfrak{b}^* \to B$ to the canonical projection. Sizing up the set obtained by the intersection between $\tilde{\mu}^{-1}(\mathcal{O}_\theta)$ and the fiber $\bar{\tau}_B^{-1}\left(\tilde{b}\right)$

$$\mu_1^{-1}(\mathcal{O}_\theta) \cap \bar{\tau}_B^{-1}\left(\tilde{b}\right) = \left\{\left(\tilde{b}, \eta\right) \in \left\{\tilde{b}\right\} \times \mathfrak{b}^* : \bar{\psi}\left(\mathrm{Ad}^*_{\tilde{b}^{-1}}\tilde{\eta}\right) \in \mathcal{O}_\theta\right\}$$

one realizes that $\tilde{\mu}^{-1}(\mathcal{O}_\theta) = B \times \psi(\mathcal{O}_\theta)$. It is a coisotropic submanifold and the null distribution of the presymplectic form $\tilde{\omega}_\circ|_{\tilde{\mu}^{-1}\mathcal{O}_\theta}$ is spanned by the infinitesimal generators associated to the Lie algebra of the *stabilizer* subgroup $B_\theta$. A more precise description of this set is

$$\tilde{\mu}^{-1}\mathcal{O}_\theta = \left\{\left(\tilde{b}, \tilde{\eta}_{\mathbf{h}}\mathbf{h} + \tilde{\eta}_{\tilde{\mathbf{e}}_\theta}\tilde{\mathbf{e}}_\theta\right) / \tilde{b} \in B, \tilde{\eta}_{\tilde{\mathbf{e}}_\theta} \in \mathbb{R}^+, \tilde{\eta}_{\mathbf{h}} \in \mathbb{R}\right\} = B \times \psi(\mathcal{O}_\theta) \tag{20}$$

where we introduced the dual basis $\left\{\mathbf{e}_\theta, \tilde{\mathbf{e}}_\theta, \mathbf{h}\right\} \subset \mathfrak{b}^*$ with

$$\tilde{\mathbf{e}}_\theta = \cos\theta\mathbf{e} - \sin\theta\tilde{\mathbf{e}}, \qquad \mathbf{e}_\theta = \sin\theta\mathbf{e} + \cos\theta\tilde{\mathbf{e}}.$$

Observe that $\mathbf{e}_\theta = -\psi(X_\theta)$.

In order to determine the presymplectic form on $\tilde{\mu}^{-1}\mathcal{O}_\theta$, we left trivialize the canonical vector bundles on $B$; thus we have the identifications

$$T^*T^*B \simeq \underbrace{B \times \mathfrak{b}^*}_{\text{base}} \times \mathfrak{b}^* \times \mathfrak{b} \qquad TT^*B \simeq \overbrace{B \times \mathfrak{b}^*}^{\text{base}} \times \mathfrak{b} \times \mathfrak{b}^*$$

and it yields to the following expression for the canonical 2-form on $T^*T^*B$

$$\omega_\circ|_{(\tilde{b},\tilde{\eta})}\left((\xi_1, \rho_1), (\xi_2, \rho_2)\right) = -\rho_1(\xi_2) + \rho_2(\xi_1) + \tilde{\eta}([\xi_1, \xi_2]).$$

Having in mind that $\tilde{\mu}^{-1}\mathcal{O}_\theta = E_\theta^{-1}(0)$ and expanding it in the given basis, we can express the canonical form restricted to this submanifold as

$$\tilde{\omega}_\circ|_{\tilde{\mu}^{-1}\mathcal{O}_\theta}\left(\tilde{b}, \tilde{\eta}\right) = 2\tilde{\eta}_{\tilde{\mathbf{e}}_\theta}\left(\tilde{b}^{-1}\mathbf{h} \wedge \tilde{b}^{-1}\tilde{\mathbf{e}}_\theta\right) - \tilde{E}_\theta \wedge \tilde{b}^{-1}\mathbf{e}_\theta - H \wedge \tilde{b}^{-1}\mathbf{h}.$$

As a map from $T_{(\tilde{b},\tilde{\eta})}\tilde{\mu}^{-1}\mathcal{O}_\theta \longrightarrow T^*_{(\tilde{b},\tilde{\eta})}\tilde{\mu}^{-1}\mathcal{O}_\theta$, it assigns to a vector $V = \left(\tilde{b}\left(v_H H + v_{E_\theta}E_\theta + v_{\tilde{E}_\theta}\tilde{E}_\theta\right), \xi_{\mathbf{e}_\theta}\mathbf{e}_\theta + \tilde{\xi}_{\tilde{\mathbf{e}}_\theta}\tilde{\mathbf{e}}_\theta + \tilde{\xi}_{\mathbf{h}}\mathbf{h}\right)$ the Hamiltonian 1-form

$$\imath_V\,\tilde{\omega}_\circ|_{\tilde{\mu}^{-1}\mathcal{O}_\theta} = \left(2\tilde{\eta}_{\tilde{\mathbf{e}}_\theta}v_H - \tilde{\xi}_{\tilde{\mathbf{e}}_\theta}\right)\tilde{b}^{-1}\tilde{\mathbf{e}}_\theta - \left(2\tilde{\eta}_{\tilde{\mathbf{e}}_\theta}v_{\tilde{E}_\theta} + \tilde{\xi}_{\mathbf{h}}\right)\tilde{b}^{-1}\mathbf{h} + v_{\tilde{E}_\theta}\tilde{E}_\theta + v_H H.$$

#### 5.1.1. Gauge fixing and canonical coordinates

The evolution of the system is contained in the coisotropic submanifold $\tilde{\mu}^{-1}(\mathcal{O}_\theta) \subset B \times \mathfrak{b}^*$. Without doing explicit mention of this fact from now on, we will use in the current section the identification

$$B \times \mathfrak{b}^* \overset{\mathrm{id}\times\bar{\psi}}{\simeq} B \times \mathfrak{su}_2.$$

As it is known [7], the leaf of the null foliation through a point $\left(\tilde{b}, X\right)$ in $B \times \mathfrak{su}_2$ coincides with the orbit $B_{\tilde{\zeta}} \cdot \left(\tilde{b}, X\right)$ of the isotropy group $B_{\tilde{\zeta}}$ of the element $\tilde{\zeta} := \tilde{\mu}\left(\tilde{b}, \tilde{\eta}\right)$. We also know that $B_{\tilde{\zeta}} = B_{\tilde{\zeta}'}$ for every pair of elements $\tilde{\zeta}, \tilde{\zeta}' \in \mathcal{O}_\theta$ in the same orbit, so we denote it as $B_\theta$ and its elements where described in Eq. (14). Therefore the leaves of the null foliation for the presymplectic structure on $B \times \mathcal{O}_\theta$ are the subsets

$$B_\theta \cdot \left(\begin{bmatrix} a & b + \mathrm{i}c \\ 0 & a^{-1} \end{bmatrix}, X\right) = \left\{\left(\begin{bmatrix} a & \mathrm{id}\,(b + \mathrm{i}c)\,\mathrm{e}^{-\mathrm{i}\theta} \\ 0 & a^{-1} \end{bmatrix}, X\right) : d \in \mathbb{R}\right\}.$$

Then any slice for the action of $B_\theta$ on $B \times \mathcal{O}_\theta$ gives a symplectic submanifold of $B \times \mathfrak{su}_2$. As such slice we can consider the submanifold $\mathcal{S}$ parametrized by $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^2$ through the map

$$\Psi : (t, a; v_3, v_\theta) \mapsto \left(\begin{bmatrix} a & -t\mathrm{e}^{-\mathrm{i}\theta} \\ 0 & a^{-1} \end{bmatrix}, -\frac{1}{2}v_3X_3 - v_\theta X_\theta\right)$$

where $X_\theta := \cos\theta X_1 + \sin\theta X_2$. By taking into account the dual basis $\{\mathbf{x}_i\} \subset \mathfrak{su}_2^*$ and the form $\mathbf{x}_\theta := -\sin\theta\mathbf{x}_1 + \cos\theta\mathbf{x}_2$, the constraints for $\mathcal{S}$ are

$$\begin{cases} F_1(\tilde{b}, X) := \mathbf{x}_\theta(X) = 0 \\ F_2(\tilde{b}, X) := b\sin\theta + c\cos\theta = 0. \end{cases}$$

**Proposition.** *The pullback of the canonical* 1*-form along* $\Psi$ *is given by*

$$(\theta_0|\mathcal{S})\,|_{(t,a;v_3,v_\theta)} = -a^{-2}(tv_\theta - v_3)\,\mathrm{d}a - a^{-1}v_\theta\mathrm{d}t.$$

*Then the canonical* 2*-form on* $\mathcal{S}$ *is*

$$\omega_{\mathcal{S}} = -a^{-2}\left(t\mathrm{d}v_\theta - \mathrm{d}v_3\right)\wedge\mathrm{d}a + 2a^{-2}v_\theta\mathrm{d}a\wedge\mathrm{d}t - a^{-1}\mathrm{d}v_\theta\wedge\mathrm{d}t.$$

*Furthermore it can be written as*

$$\omega_{\mathcal{S}} = -\mathrm{d}p_a\wedge\mathrm{d}a - \mathrm{d}p_t\wedge\mathrm{d}t$$

*where* $p_a\left(a, t; v_3, v_\theta\right) := a^{-2}\left(tv_\theta - v_3\right), p_t\left(a, t; v_3, v_\theta\right) := a^{-1}v_\theta$.

**Proof.** It is immediate to show that

$$\left[\Psi\left(t, a; v_3, v_\theta\right)\right]^{-1}\Psi_*\left(\frac{\partial}{\partial a}\right) = \left(\begin{bmatrix} a^{-1} & -a^{-2}t\mathrm{e}^{-\mathrm{i}\theta} \\ 0 & -a^{-1}\end{bmatrix}; 0\right)$$
$$\left[\Psi\left(t, a; v_3, v_\theta\right)\right]^{-1}\Psi_*\left(\frac{\partial}{\partial t}\right) = \left(\begin{bmatrix} 0 & -a^{-1}\mathrm{e}^{-\mathrm{i}\theta} \\ 0 & 0\end{bmatrix}; 0\right).$$

It implies

$$\begin{aligned}\left(\theta_0|\mathcal{S}\right)|_{(t,a;v_3,v_\theta)} = &\left(\left[\Psi\left(t, a; v_3, v_\theta\right)\right]^{-1}\Psi_*\left(\frac{\partial}{\partial a}\right), -\frac{1}{2}v_3X_3 - v_\theta X_\theta\right)_{\mathfrak{sl}(2,\mathbb{C})}\mathrm{d}a \\ &+\left(\left[\Psi\left(t, a; v_3, v_\theta\right)\right]^{-1}\Psi_*\left(\frac{\partial}{\partial t}\right), -\frac{1}{2}v_3X_3 - v_\theta X_\theta\right)_{\mathfrak{sl}(2,\mathbb{C})}\mathrm{d}t\end{aligned}$$

so, we can write

$$\left(\theta_0|\mathcal{S}\right) = -a^{-2}\left(t\tilde{E}_\theta - aH\right)\mathrm{d}a - a^{-1}\tilde{E}_\theta\mathrm{d}t.$$

The rest follows by exterior differentiation. □

**Corollary.** *The constraints describing the submanifold* $\mathcal{S}$ *are of second order.*

**Proof.** For each pair $(a, t)\in\mathbb{R}^+\times\mathbb{R}$, the map

$$\left(v_3, v_\theta\right)\mapsto\left(a^{-2}\left(tv_\theta - v_3\right), a^{-1}v_\theta\right)$$

is nonsingular, and maps $\omega_{\mathcal{S}}$ on a nondegenerate 2-form, as the previous proposition shows. Then $\omega_{\mathcal{S}}$ is nondegenerate, and $\mathcal{S}$ is a symplectic submanifold. □

### *5.2. Dualizable subspaces in* $\mathrm{SU}(2)\times\mathfrak{su}_2^*$

In this case, the *dualizable spaces* are the symplectic leaves of the coisotropic submanifold $\tilde{\varphi}^{-1}\left(\mathcal{O}_\theta\right)$. To get some insight about this set, we consider the infinitesimal generators of the dressing action of $B$ on SU(2) associated with the basis $\{E, (\mathrm{i}E), H\}\subset\mathfrak{b}$

$$\begin{aligned} g^{-1}g^H &= -\mathrm{i}\left(\alpha\bar{\beta} - \bar{\alpha}\beta\right)X_1 - \left(\alpha\bar{\beta} + \bar{\alpha}\beta\right)X_2 \\ g^{-1}g^{(\mathrm{i}E)} &= -\frac{1}{2}\left(\beta^2 + \bar{\beta}^2\right)X_1 - \frac{1}{2}\mathrm{i}\left(\beta^2 - \bar{\beta}^2\right)X_2 - \frac{1}{2}\left(\bar{\alpha}\bar{\beta} + \alpha\beta\right)X_3 \\ g^{-1}g^E &= -\frac{1}{2}\mathrm{i}\left(\beta^2 - \bar{\beta}^2\right)X_1 + \frac{1}{2}\left(\beta^2 + \bar{\beta}^2\right)X_2 - \frac{1}{2}\mathrm{i}\left(\alpha\beta - \bar{\alpha}\bar{\beta}\right)X_3\end{aligned} \tag{21}$$

where $\psi:\mathfrak{su}_2\to\mathfrak{b}^*$ was given in Eq. (10). It is worth remarking that they can be also obtained from the coboundary Poisson bivector on SU(2)

$$\pi_{\mathrm{SU}(2)}(g) = \frac{1}{4}\left(gX_2\otimes gX_1 - gX_1\otimes gX_2 - X_2g\otimes X_1g + X_1g\otimes X_2g\right) \tag{22}$$

as

$$g^{-1}g^{\bar{\psi}^*(\eta)} = L_{g^{-1}*}\left[\left(\mathrm{id}\otimes\eta\circ R_{g^{-1}*}\right)\pi_{\mathrm{SU}(2)}\left(g\right)\right]$$

so

$$\tilde{\varphi}\left(g, \eta\right)\equiv g\left(g^{-1}\right)^{\bar{\psi}^*(\eta)} = L_{g*}\left[\left(\mathrm{id}\otimes\eta\circ R_{g*}\right)\pi_{\mathrm{SU}(2)}\left(g^{-1}\right)\right].$$

The dual map $\bar{\psi}^*:\mathfrak{su}_2^*\to\mathfrak{b}$ for the dual base $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}\subset\mathfrak{su}_2^*$ gives

$$\bar{\psi}^*(\mathbf{x}_1) = -E,\qquad\bar{\psi}^*(\mathbf{x}_2) = \mathrm{i}E,\qquad\bar{\psi}^*(\mathbf{x}_3) = -\frac{1}{2}H$$

which allows us to write

$$\bar{\psi}^*(\eta) = -\eta_1 E + \eta_2(\mathrm{i}E) - \frac{1}{2}\eta_3 H = \begin{bmatrix} -\frac{1}{2}\eta_3 & -\eta_1 + i\eta_2 \\ 0 & \frac{1}{2}\eta_3 \end{bmatrix}$$

so we get the explicit form of the momentum map $\tilde{\varphi} : \mathrm{SU}(2) \times \mathfrak{su}_2^* \longrightarrow \mathfrak{su}_2$

$$\begin{aligned} g\left(g^{-1}\right)^{\bar{\psi}^*(\eta)} = {} & \left(-\operatorname{Im}\left(\beta^2\right)(\eta_1\cos\theta - \eta_2\sin\theta) - \operatorname{Re}\left(\beta^2\right)(\eta_1\sin\theta + \eta_2\cos\theta) - \eta_3\operatorname{Im}\left(\alpha\beta\mathrm{e}^{\mathrm{i}\theta}\right)\right)X_\theta \\ & + \left(\operatorname{Im}\left(\beta^2\right)(\eta_1\sin\theta + \eta_2\cos\theta) - \operatorname{Re}\left(\beta^2\right)(\eta_1\cos\theta - \eta_2\sin\theta) - \eta_3\operatorname{Re}\left(\alpha\beta\mathrm{e}^{\mathrm{i}\theta}\right)\right)X_\theta^* \\ & - \frac{\mathrm{i}}{2}\left(\bar{\alpha}\beta(\eta_1 + \mathrm{i}\eta_2) + \alpha\bar{\beta}(\eta_1 - \mathrm{i}\eta_2)\right)X_3 \end{aligned} \tag{23}$$

where $X_\theta = X_1\cos\theta + X_2\sin\theta$ and $X_\theta^* = -X_1\sin\theta + X_2\cos\theta$. The orbit $\mathcal{O}_\theta = \{x(\cos\theta X_1 + \sin\theta X_2) + zX_3 / x \in \mathbb{R}_{>0}, z \in \mathbb{R}\}$ can be characterized by means the dual basis $\left\{\mathbf{x}_\theta, \mathbf{x}_\theta^*, \mathbf{x}_3\right\} \subset \mathfrak{su}_2^*$, defining $\mathbf{x}_\theta = (\cos\theta\mathbf{x}_1 + \sin\theta\mathbf{x}_2)$ and $\mathbf{x}_\theta^* = (-\sin\theta\mathbf{x}_1 + \cos\theta\mathbf{x}_2)$, so that

$$\mathcal{O}_\theta = \left(\mathbf{x}_\theta^*\right)^\circ \cap \mathbf{x}_\theta^{-1}(\mathbb{R}_{>0}).$$

In this way, we get for (23),

$$\begin{aligned} \left\langle \mathbf{x}_\theta^*, g\left(g^{-1}\right)^{\bar{\psi}^*(\eta)}\right\rangle = {} & -\frac{\mathrm{i}}{2}\left(\beta^2 - \bar{\beta}^2\right)(\eta_1\sin\theta + \eta_2\cos\theta) - \frac{1}{2}\left(\beta^2 + \bar{\beta}^2\right)(\eta_1\cos\theta - \eta_2\sin\theta) \\ & - \frac{1}{2}\eta_3\left(\bar{\alpha}\bar{\beta}\mathrm{e}^{-\mathrm{i}\theta} + \alpha\beta\mathrm{e}^{\mathrm{i}\theta}\right). \end{aligned}$$

The annihilator is obtained from the condition $\left\langle \mathbf{x}_\theta^*, g\left(g^{-1}\right)^{\bar{\psi}^*(\eta)}\right\rangle = 0$, implying that $\operatorname{Re}\left(\left(\beta^2\eta_+ + \eta_3\alpha\beta\right)\mathrm{e}^{\mathrm{i}\theta}\right) = 0$, where $\eta_+ := \eta_1 + i\eta_2$. The remaining restriction is $\left\langle \mathbf{x}_\theta, g\left(g^{-1}\right)^{\bar{\psi}^*(\eta)}\right\rangle > 0$ that is equivalent to $\operatorname{Im}\left(\left(\beta^2\eta_+ + \eta_3\alpha\beta\right)\mathrm{e}^{\mathrm{i}\theta}\right) < 0$. Thus have shown the following statement.

**Proposition.** *$\tilde{\varphi}^{-1}(\mathcal{O}_\theta)$ is determined by the constraints*

$$\operatorname{Re}\left(\left(\beta^2\eta_+ + \eta_3\alpha\beta\right)\mathrm{e}^{\mathrm{i}\theta}\right) = 0 \tag{24}$$

$$\operatorname{Im}\left(\left(\beta^2\eta_+ + \eta_3\alpha\beta\right)\mathrm{e}^{\mathrm{i}\theta}\right) < 0 \tag{25}$$

*on the components of* $(g, \eta) \in \mathrm{SU}(2) \times \mathfrak{su}_2^*$.

As explained above, the dualizable subspaces are the symplectic leaves in $\tilde{\varphi}^{-1}(\mathcal{O}_\theta)$, which coincide with the orbits of the action $\hat{\mathrm{d}}$, see (17). Despite the rather obscure description of $\tilde{\varphi}^{-1}(\mathcal{O}_\theta)$, determining these orbit looks simpler working out separately the factors in $T\mathrm{SU}(2)$ and $\mathfrak{su}_2^*$.

To start with, we work out the projection of $T\tilde{\varphi}^{-1}(\mathcal{O}_\theta)$ on the factor $T\mathrm{SU}(2)$. Here, we make use of the digression 3.1 to conclude that there are two kind of dressing orbits in SU(2): the zero dimensional ones determined by the points with $\beta = 0$, and the two dimensional ones determined by $\vartheta = \arg\beta$, for $\beta \neq 0$, so they are two dimensional spheres $S^2$. The zero dimensional case are trivial dualizable subspaces, so we focus our attention on the last ones.

The infinitesimal generators (21) involve for each $g \in \mathrm{SU}(2)$ the linear transformation associated to $\mathrm{d} : \mathfrak{b} \longrightarrow T\mathrm{SU}(2) \cong \mathrm{SU}(2) \times \mathfrak{su}_2$ relating the basis $\{H, E, \mathrm{i}E\}$ and $\{X_1, X_2, X_3\}$, which has a nontrivial *kernel* spanned by the vector

$$\tilde{X}_\circ(g) = H - \mathrm{i}\frac{1}{|\beta|^2}\left(\alpha\beta - \bar{\alpha}\bar{\beta}\right)(\mathrm{i}E) + \frac{1}{|\beta|^2}\left(\alpha\beta + \bar{\alpha}\bar{\beta}\right)E.$$

On the other side, since $(\operatorname{Im}\mathrm{d})^\circ = \ker\mathrm{d}^\top$, where $\mathrm{d}^\top : \mathrm{SU}(2) \times \mathfrak{su}_2^* \longrightarrow \mathfrak{b}^*$ is the *transpose* of d, we make the identification $\operatorname{Im}\mathrm{d} = \left(g, \hat{\pi}\right)^\circ$, with $\left(g, \hat{\pi}\right) \in \ker\mathrm{d}^\top$ being the generator of $\ker\mathrm{d}^\top$ with

$$\hat{\pi} = -\frac{1}{|\beta|^2}\operatorname{Re}\left(\alpha\bar{\beta}\right)\mathbf{x}_1 - \frac{1}{|\beta|^2}\operatorname{Im}\left(\alpha\bar{\beta}\right)\mathbf{x}_2 + \mathbf{x}_3 \tag{26}$$

written it in terms of the dual basis $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} \subset \mathfrak{su}_2^*$. So, $\hat{\pi}$ annihilates the vectors (21) which spans the tangent space of each dressing orbit.

The dressing action of $B$ on SU(2) left invariant $\vartheta = \arg\beta$, so it is a suitable parameter to characterize the dressing orbits in SU(2), and the Lie derivative along $\vartheta$ gives a normal vector to them. Thus, let us consider this tangent vector

$$V_0|_g = \left(g, -g^{-1}\frac{\partial g}{\partial\vartheta}\right) = \left(g, -\frac{1}{2}\left(\alpha\bar{\beta}+\bar{\alpha}\beta\right)X_1 - \frac{1}{2}\mathrm{i}\left(\bar{\alpha}\beta-\alpha\bar{\beta}\right)X_2 + |\beta|^2 X_3\right). \tag{27}$$

One may easily verify that

$$\left\langle \hat{\pi}, -g^{-1}\frac{\partial g}{\partial\vartheta}\right\rangle = 1.$$

On the other side, the projection of $T\tilde{\varphi}^{-1}(\mathcal{O}_\theta)$ on the factor $\mathfrak{su}_2^*$ is foliated by the orbits of the action

$$\Pi_{\mathfrak{su}_2^*}\hat{\mathrm{d}}_{\tilde{b}}(g,\eta) = \left(\psi^*\circ \mathrm{Ad}_{\tilde{b}g}\circ\bar{\psi}^*\right)\eta$$

which left invariant the $\mathbf{x}_3$ component of $\eta$, turning $\eta_3 = \langle\eta, X_3\rangle$ in a good parameter for the corresponding orbits.

Therefore, we introduce the projectors

$$\mathrm{P}_0 : T\mathrm{SU}(2) \longrightarrow \Pi_{T\mathrm{SU}(2)}T\tilde{\varphi}^{-1}(\mathcal{O}_\theta)\,/\mathrm{P}_0 = \mathrm{Id} - V_0\left(\hat{\pi}\circ\left(L_{g^{-1}}\right)_*\right)$$
$$\mathrm{P}_3 : T\mathfrak{su}_2^* \longrightarrow \Pi_{\mathfrak{su}_2^*}T\tilde{\varphi}^{-1}(\mathcal{O}_\theta)\,/\mathrm{P}_3 = \mathrm{Id} - \mathbf{x}_3X_3$$

so that $\mathrm{P} = P_0\times P_3$ is the projector onto the $\hat{\mathrm{d}}$ orbits.

As mentioned before, the submanifold $\tilde{\varphi}^{-1}(\mathcal{O}_\theta)$ is a presymplectic one when endowed with the restriction of the canonical form $\omega_\circ$ of $T^*\mathrm{SU}(2) \cong \mathrm{SU}(2)\times\mathfrak{su}_2^*$, and its symplectic leaves are the orbits of the action $\hat{\mathrm{d}}$ (17). We use the above projectors to write down this restricted symplectic form on each orbit starting from the relation

$$\left\langle\omega_\circ^R, (v,\rho)\otimes(w,\lambda)\right\rangle_{(g,\eta)\in\tilde{\varphi}^{-1}(\mathcal{O}_\theta)} = \left\langle\omega_\circ, \mathrm{P}(v,\rho)\otimes\mathrm{P}(w,\lambda)\right\rangle_{(g,\eta)\in\tilde{\varphi}^{-1}(\mathcal{O}_\theta)}$$

to get the expression

$$\omega_\circ^R = \omega_\circ - L_{g^{-1}*}\hat{\pi}\wedge\left(g^{-1}V_0 - |\beta|^2X_3 + g^{-1}\mathrm{ad}^*_{g^{-1}V_0}\eta\right) + X_3\wedge g^{-1}\mathbf{x}_3 \tag{28}$$

from where we obtain the Dirac bracket

$$\{\mathcal{F},\mathcal{G}\}^D_{\tilde{\varphi}^{-1}(\mathcal{O}_\theta)}(g,\eta) = \{\mathcal{F},\mathcal{G}\}(g,\eta) - \left\langle g\mathbf{d}\mathcal{F}, \left\langle\hat{\pi},\delta\mathcal{G}\right\rangle X_3\right\rangle + \left\langle g\mathbf{d}\mathcal{G}, \left\langle\hat{\pi},\delta\mathcal{F}\right\rangle X_3\right\rangle + \left\langle\eta - \frac{\left\langle\eta, g^{-1}V_0\right\rangle}{|\beta|^2}\mathbf{x}_3, [\delta\mathcal{F},\delta\mathcal{G}]\right\rangle$$

and the Hamiltonian vector field

$$V_{\mathcal{G}} = V^\circ_{\mathcal{G}} - \left(\left\langle\hat{\pi},\delta\mathcal{G}\right\rangle X_3, \mathrm{ad}^*_{\delta\mathcal{G}}\left(\eta - \frac{\left\langle\eta, g^{-1}V_0\right\rangle}{|\beta|^2}\mathbf{x}_3\right) - \langle g\mathbf{d}\mathcal{G}, X_3\rangle\,\hat{\pi}\right) \tag{29}$$

which is tangent to the $\hat{\mathrm{d}}$-orbits in $\tilde{\varphi}^{-1}(\mathcal{O}_\theta)$, as expected.

## 6. AKS integrability scheme and dynamics on factors of SL(2, ℂ)

Let us work out the dynamical setting on the coadjoint orbit in $\mathfrak{sl}_2^*\mathbb{C}$, which we shall identify $\mathfrak{sl}_2\mathbb{C}$ through the invariant nondegenerate bilinear form $(X,Y)_{\mathfrak{sl}_2}$, (8). Following the AKS scheme, we choose an Ad-invariant function on a coadjoint orbit of SL (2, ℂ), so that its restriction to the coadjoint orbit of $B$ in $\mathfrak{b}^* \hookrightarrow \mathfrak{sl}_2^*\mathbb{C}$ gives a nontrivial dynamics embracing also the dynamics on the cotangent bundles $T^*\mathrm{SU}(2)$ and $T^*B$ [10,11]. In particular, the Ad-invariant function $f : \mathfrak{sl}_2\mathbb{C} \longrightarrow \mathbb{R}$

$$f(\mathbf{X}) := -\frac{1}{16}\,\mathrm{Re}\;\kappa(\mathbf{X},\mathbf{X})$$

is related to the Hamiltonian function of the Toda model, as we shall see below. Its Legendre transform, $\mathcal{L}_f : \mathfrak{sl}_2\mathbb{C} \longrightarrow \mathfrak{sl}_2^*\mathbb{C}$, is

$$\left\langle\mathcal{L}_f(\mathbf{X}), \mathbf{Y}\right\rangle \equiv \frac{\mathrm{d}}{\mathrm{d}t}f(\mathbf{X}+t\mathbf{Y})\bigg|_{t=0} = -\frac{1}{8}\,\mathrm{Re}\,\kappa(\mathbf{X},\mathbf{Y})$$

valid for all $\mathbf{Y}\in\mathfrak{sl}(2,\mathbb{C})$. As an element of $\mathfrak{sl}(2,\mathbb{C})$ through the invariant product $(\cdot,\cdot)_{\mathfrak{sl}_2}$, it is given by $\left(\tilde{\mathcal{L}}_f(\mathbf{X}), \mathbf{Y}\right)_{\mathfrak{sl}_2} = -\frac{1}{4}\,\mathrm{Im}\;\kappa(\tilde{\mathcal{L}}_f(\mathbf{X}), \mathbf{Y})$ and the ℂ-linearity of $\kappa$ enables to make the identification $\mathrm{Re}(\kappa(\mathbf{X},\mathbf{Y})) = \mathrm{Im}(\kappa(\mathrm{i}\mathbf{X},\mathbf{Y}))$. Hence, from the definition of the bilinear product $(\cdot,\cdot)_{\mathfrak{sl}_2}$, we get

$$\mathrm{Im}\,\kappa\left(\tilde{\mathcal{L}}_f(\mathbf{X}) - \frac{\mathrm{i}}{2}\mathbf{X}, \mathbf{Y}\right) = 0, \qquad \forall\mathbf{Y}\in\mathfrak{sl}(2,\mathbb{C})$$

and, because of the nondegeneracy of $(\cdot,\cdot)_{\mathfrak{sl}_2}$, we conclude that

$$\tilde{\mathcal{L}}_f(\mathbf{X}) = \frac{\mathrm{i}}{2}\mathbf{X}.$$

Applying the AKS scheme to $T^*\mathrm{SU}(2)$ and $T^*B$, regarded as $B$-Hamiltonian spaces, requires that $\psi^*(\Pi_{\mathfrak{b}}\mathbf{X})$ be a character of $\mathfrak{su}_2$. The only chance is $\Pi_{\mathfrak{b}}\mathbf{X} = 0$, so $\mathbf{X} \equiv X \in \mathfrak{su}_2$ and the Hamiltonian vector field associated with $f$ has integral curves given by the *fundamental flux* $t \mapsto \exp t\tilde{\mathcal{L}}_f(X)$: if $X = a_1X_1 + a_2X_2 + a_3X_3$,

$$\exp t\tilde{\mathcal{L}}_f(X) = \cosh\left(t\frac{\|X\|}{2}\right) + \mathrm{i}X\sinh\left(t\frac{\|X\|}{2}\right).$$

Here $\|X\| = \sqrt{\det X}$, and from now on we consider $\det X = 1$, equivalent to $a_1^2+a_2^2+a_3^2 = 1$. Thus, the solutions to collective $B$-Hamiltonian systems are orbits of a curve in $B$ obtained from the factorization of the curve $\exp t\tilde{\mathcal{L}}_f(X)$ on $\mathrm{SU}(2)\times B$ [10,11]. In fact, the curve in $\mathrm{SL}(2,\mathbb{C})$

$$\exp t\tilde{\mathcal{L}}_f(X) = \begin{bmatrix} \cosh(t/2) - a_3\sinh(t/2) & -(a_1 - \mathrm{i}a_2)\sinh(t/2) \\ -(a_1 + \mathrm{i}a_2)\sinh(t/2) & \cosh(t/2) + a_3\sinh(t/2) \end{bmatrix} \tag{30}$$

can be factorized out as

$$\exp t\tilde{\mathcal{L}}_f(X) = g(t)\tilde{b}(t)$$

with $g(t) \subset \mathrm{SU}(2)$ and $\tilde{b}(t) \subset B$ given by

$$g(t) = \begin{bmatrix} \dfrac{\cosh(t/2) - a_3\sinh(t/2)}{\sqrt{\cosh t - a_3\sinh t}} & \dfrac{(a_1 - \mathrm{i}a_2)\sinh(t/2)}{\sqrt{\cosh t - a_3\sinh t}} \\ -\dfrac{(a_1 + \mathrm{i}a_2)\sinh(t/2)}{\sqrt{\cosh t - a_3\sinh t}} & \dfrac{\cosh(t/2) - a_3\sinh(t/2)}{\sqrt{\cosh t - a_3\sinh t}} \end{bmatrix} \tag{31}$$

$$\tilde{b}(t) = \begin{bmatrix} \sqrt{\cosh t - a_3\sinh t} & \dfrac{-(a_1 - \mathrm{i}a_2)\sinh t}{\sqrt{\cosh t - a_3\sinh t}} \\ 0 & \left(\sqrt{\cosh t - a_3\sinh t}\right)^{-1} \end{bmatrix}. \tag{32}$$

Hence, the solution curves of suitable $B$-Hamiltonian systems are given by the orbits of $\tilde{b}(t)$ in each space, as we shall describe below.

## 7. Dynamics on *B*-spaces

$T$-duality relates dynamical systems on the three Hamiltonian $B$-spaces, namely $T^*B$, $T^*\mathrm{SU}(2)$ and $\mathbb{R}^2$. The restriction of the $\mathrm{Ad}^*$-invariant function $f$ to $(\mathfrak{su}_2)^\circ \cong \mathfrak{su}_2$ induces, on each of these spaces, collective systems whose solution can be found through the AKS method. Accordingly with it, we just need to know the form of the action of the Lie group $B$ on the spaces under consideration to find out the solution to the equation of motion.

Hamiltonian systems modelled on the cotangent bundles of a Lie group $G$ are characterized, in body coordinates, by the canonical symplectic structure which, besides the Hamilton function, defines equations through the Hamiltonian vector field $V_{\mathcal{H}}|_{(g,\eta)} = \left(g\delta\mathcal{H}, \mathrm{ad}^*_{\delta\mathcal{H}}\eta - g\mathbf{d}\mathcal{H}\right)$, for the function $\mathcal{H} \in C^\infty(G\times\mathfrak{g}^*)$, where $\mathrm{d}\mathcal{H} = (\mathbf{d}\mathcal{H}, \delta\mathcal{H})$, with $\mathbf{d}\mathcal{H} \in T^*_gG$ and $\delta\mathcal{H} \in T^*_\eta\mathfrak{g}^*$, so

$$g^{-1}\dot{g} = \delta\mathcal{H}$$
$$\dot{\eta} = \mathrm{ad}^*_{\delta\mathcal{H}}\eta - g\mathbf{d}\mathcal{H}.$$

As explained, we consider a function $\mathsf{h} : \mathfrak{su}_2 \longrightarrow \mathbb{R}$ and, in each case, the Hamiltonian functions of the respective dynamical systems are obtained by composing them with the corresponding momentum maps.

### 7.1. Dynamics on $\mathbb{R}^2$

The action of $B$ on $\mathbb{R}^2$, Eq. (15), is

$$\rho\left(\tilde{b},(q,p)\right) = \left(q - \frac{1}{\mu}\ln a, p - 2\mu\frac{\varepsilon}{a}\exp(2\mu q)\,(b\cos\theta - c\sin\theta)\right)$$

where

$$\tilde{b} = \begin{bmatrix} a & b+\mathrm{i}c \\ 0 & a^{-1} \end{bmatrix}$$

with associated momentum map $\tilde{\sigma}_\theta$

$$\tilde{\sigma}_\theta(q,p) = \frac{1}{2\mu}pX_3 - \varepsilon\exp(2\mu q)(\cos\theta X_1 + \sin\theta X_2).$$

The collective Hamiltonian here is

$$\mathcal{H}_{\mathbb{R}^2}(q,p) = -\frac{1}{16}\operatorname{Re}\kappa\left(\tilde{\sigma}_\theta(q,p),\tilde{\sigma}_\theta(q,p)\right) = \frac{1}{2}\left(\frac{1}{4\mu^2}p^2 + 2\varepsilon^2\exp(4\mu q)\right).$$

By choosing the point $X_\circ := n_\circ(\cos\theta X_1 + \sin\theta X_2) + m_\circ X_3$, $n_\circ^2 + m_\circ^2 = 1$, in $\mathcal{O}_\theta \subset \mathfrak{su}_2$, the solution curve through the initial point $(q_\circ, p_\circ)$ with $\tilde{\sigma}_\theta(q_\circ,p_\circ) = X_\circ$ is given by the action of the curve $\tilde{b}(t)$, obtained in Eq. (32),

$$t \mapsto \rho\left(\tilde{b}(t),(q_\circ,p_\circ)\right) = \left(q_\circ - \frac{1}{\mu}\ln a(t), p_\circ - \frac{2\mu\varepsilon}{a(t)}e^{2\mu q_\circ}\left(b(t)\cos\theta - c(t)\sin\theta\right)\right)$$

where

$$\begin{aligned} a(t) &= \sqrt{\cosh t - a_3\sinh t} \\ b(t) &= \frac{-a_1\sinh t}{\sqrt{\cosh t - a_3\sinh t}} \\ c(t) &= \frac{a_2\sinh t}{\sqrt{\cosh t - a_3\sinh t}} \end{aligned}$$

and

$$\begin{aligned} a_1 &= -\varepsilon\exp(2\mu q)\cos\theta \\ a_2 &= -\varepsilon\exp(2\mu q)\sin\theta \\ a_3 &= \frac{1}{2\mu}p \end{aligned}$$

so

$$\begin{aligned}\rho\left(\tilde{b}(t),(q_\circ,p_\circ)\right) = \Bigg(&q_\circ - \frac{1}{2\mu}\ln\left(\cosh t - \frac{1}{2\mu}p\sinh t\right), \\ &p_\circ - 2\mu\frac{\varepsilon^2}{\left(\cosh t - \frac{1}{2\mu}p\sinh t\right)}\exp(2\mu(q_\circ + q))\sinh t\Bigg).\end{aligned}$$

In order to have $\tilde{\sigma}_\theta(q_\circ,p_\circ) = X_\circ$, the following relations must hold

$$\left.\begin{aligned} m_\circ &= \frac{1}{2\mu}p_\circ \\ n_\circ &= -\varepsilon\exp(2\mu q_\circ) \end{aligned}\right\} \Longrightarrow \left(\frac{1}{2\mu}p_\circ\right)^2 + \varepsilon^2\exp(4\mu q_\circ) = 1.$$

Thus, the curve $\rho\left(\tilde{b}(t),(q_\circ,p_\circ)\right)$ becomes in

$$\rho\left(\tilde{b}(t),(q_\circ,p_\circ)\right) = \left(q_\circ - \frac{1}{2\mu}\ln\left(\cosh t - \frac{1}{2\mu}p_\circ\sinh t\right), -2\mu\frac{\sinh t - \frac{1}{2\mu}p_\circ\cosh t}{\cosh t - \frac{1}{2\mu}p_\circ\sinh t}\right).$$

For the particular values $\mu = \frac{1}{2}$ and $\varepsilon = \pm\sqrt{2}$ in $\mathcal{H}_{\mathbb{R}^2}(q,p)$, we obtain the *Toda Hamiltonian*

$$\mathcal{H}_{\text{Toda}}(q,p) = \frac{1}{2}p^2 + \exp(2q) \tag{33}$$

whose Hamilton equation

$$\begin{cases}\dot{q} = p \\ \dot{p} = -2\exp(2q)\end{cases}$$

are solved by the curves

$$q(t) = -\ln(\cosh t - p_\circ\sinh t), \qquad p(t) = -\frac{\sinh t - p_\circ\cosh t}{\cosh t - p_\circ\sinh t}.$$

*A little note about parameters. In the previous setting it was possible to solve the Toda equations of motion in case in which the energy of the system is equal to* 1*, however it is possible to choose the parameters in order to solve the system at any other (positive of course) energy.*

The Lagrangian corresponding to the Toda Hamiltonian (33) is

$$\mathcal{L}_{\text{Toda}}(q,\dot{q}) = \frac{1}{2}\dot{q}^2 - \exp(2q). \tag{34}$$

*7.2. Dynamics on $T^*B$*

In terms of the momentum map $\tilde{\mu}\left(\tilde{b},\tilde{\eta}\right)=\left(\bar{\psi}\circ\mathrm{Ad}^*_{\tilde{b}^{-1}}\circ\psi\right)\psi\left(\tilde{\eta}\right)$, given in Eq. (16), the collective Hamiltonian on $T^*B$ is then

$$\begin{aligned}\mathcal{H}_B\left(\tilde{b},\tilde{\eta}\right)&=-\frac{1}{16}\kappa\left(\tilde{\mu}\left(\tilde{b},X\right),\tilde{\mu}\left(\tilde{b},X\right)\right)\\&=\frac{1}{2a^4}\left(\tilde{\eta}_{\mathbf{e}}^2+\tilde{\eta}_{\tilde{\mathbf{e}}}^2\right)+\frac{1}{2}\left(\frac{1}{2}\tilde{\eta}_{\mathbf{h}}+\frac{b\tilde{\eta}_{\mathbf{e}}+c\tilde{\eta}_{\tilde{\mathbf{e}}}}{a}\right)^2.\end{aligned}$$

The evolution curve passing through the initial point $\left(\tilde{b}_\circ,X_\circ\right)$ with $X_\circ=(n_\circ\cos\theta X_1+n_\circ\sin\theta X_2)+m_\circ X_3$ is

$$t\mapsto\tilde{\lambda}\left(\tilde{b}(t),\left(\tilde{b}_\circ,\tilde{\eta}_\circ\right)\right)=\left(\begin{bmatrix}a(t)&z(t)\\0&(a(t))^{-1}\end{bmatrix}\cdot\begin{bmatrix}a_\circ&z_\circ\\0&a_\circ\end{bmatrix},X_\circ\right)$$

where the curve $\tilde{b}(t)$ is that of Eq. (32). Explicitly, it means the curve

$$\begin{aligned}&\tilde{\lambda}\left(\tilde{b}(t),\left(\tilde{b}_\circ,\tilde{\eta}_\circ\right)\right)\\&=\left(\begin{bmatrix}a_\circ\sqrt{\cosh t-m_\circ\sinh t}&\dfrac{a_\circ z_\circ\cosh t-\left(m_\circ a_\circ z_\circ+n_\circ\mathrm{e}^{-\mathrm{i}\theta}\right)\sinh t}{a_\circ\sqrt{\cosh t-m_\circ\sinh t}}\\0&\dfrac{1}{a_\circ\sqrt{\cosh t-m_\circ\sinh t}}\end{bmatrix},X_\circ\right).\end{aligned}$$

The Lagrangian version of this model can be retrieved from the first Hamilton equation

$$\tilde{b}^{-1}\dot{\tilde{b}}=\delta\mathcal{H}_B$$

that in explicit form is

$$\begin{aligned}&a\dot{b}-\dot{a}b=\tilde{\eta}_{\mathbf{e}}a^{-2}\\&a\dot{c}-\dot{a}c=\tilde{\eta}_{\tilde{\mathbf{e}}}a^{-2}\\&a^{-1}\dot{a}=\frac{1}{2}\left(\tilde{\eta}_{\mathbf{e}}ba^{-1}+\tilde{\eta}_{\tilde{\mathbf{e}}}ca^{-1}+\frac{1}{2}\tilde{\eta}_{\mathbf{h}}\right).\end{aligned}$$

Hence, the Lagrangian is obtained as

$$\tilde{L}_B\left(\tilde{b},\dot{\tilde{b}}\right)=\left\langle\tilde{\eta},\tilde{b}^{-1}\dot{\tilde{b}}\right\rangle-\mathcal{H}_B\left(\tilde{b},\tilde{\eta}\right)$$

that after some calculation gives

$$\tilde{L}_B\left(\tilde{b},\dot{\tilde{b}}\right)=\frac{1}{2}\left(b\dot{a}-a\dot{b}\right)^2+\frac{1}{2}\left(c\dot{a}-a\dot{c}\right)^2+2\left(a^{-1}\dot{a}\right)^2. \tag{35}$$

This Lagrangian reduce on $\tilde{\mu}^{-1}\mathcal{O}_\theta$ to

$$\tilde{L}_B^{\mathrm{red}}\left(\tilde{b},\dot{\tilde{b}}\right)=\frac{1}{2}\left(t\dot{a}-a\dot{t}\right)^2+2\left(\frac{\dot{a}}{a}\right)^2.$$

Since

$$\dot{\tilde{b}}\tilde{b}^{-1}=\left(a\dot{b}-\dot{a}b\right)E+\left(a\dot{c}-\dot{a}c\right)(\mathrm{i}E)+a^{-1}\dot{a}H$$

and introducing the linear map $\mathbb{K}:=\hat{\kappa}_{\mathfrak{su}_2}^{-1}\circ\psi^*:\mathfrak{b}\longrightarrow\mathfrak{su}_2$, given by

$$\mathbb{K}E=\frac{1}{8}X_1,\qquad\mathbb{K}(\mathrm{i}E)=-\frac{1}{8}X_2,\qquad\mathbb{K}H=\frac{1}{4}X_3$$

we may write the Lagrangian function (35) using the symmetric bilinear form (8) on $\mathfrak{sl}_2$ as

$$\tilde{L}_B\left(\tilde{b},\dot{\tilde{b}}\right)=-4\left(\mathbb{K}\dot{\tilde{b}}\tilde{b}^{-1},\dot{\tilde{b}}\tilde{b}^{-1}\right)_{\mathfrak{sl}_2} \tag{36}$$

that resembles a generalized top Lagrangian on $B$.

*7.3. Dynamics on $T^*\mathrm{SU}(2)$*

We consider the Hamiltonian function

$$\mathcal{H}_{\mathrm{SU}(2)}(g,\eta) = -\frac{1}{16}\kappa(\tilde{\varphi}(g,\eta),\tilde{\varphi}(g,\eta)). \quad (37)$$

In order to get the Hamilton equation of motion, we need to calculate the differential $\left(\mathbf{d}\mathcal{H}_{\mathrm{SU}(2)},\delta\mathcal{H}_{\mathrm{SU}(2)}\right)$. In doing so, we use the expression for the differential of the momentum map $\tilde{\varphi}$

$$\tilde{\varphi}_*(gX,\xi) = -\left(\mathrm{id}\otimes\mathrm{Ad}^*_{g^{-1}}\xi\right)\pi^R(g) + \left(\mathrm{id}\otimes\mathrm{Ad}^*_{g^{-1}}\eta\right)(\mathrm{id}\otimes\mathrm{ad}_X)\,\pi^R(g) - \left(\mathrm{Ad}_g\otimes\eta\right)\delta_{\mathfrak{su}_2}(X)$$

where $\delta_{\mathfrak{su}_2}:\mathfrak{su}_2\longrightarrow\mathfrak{su}_2\otimes\mathfrak{su}_2$ is the coboundary coalgebra structure of $\mathfrak{su}_2$ and $\pi^R(g) = \left(R_{g^{-1}*}\otimes R_{g^{-1}*}\right)\pi_{\mathrm{SU}(2)}(g)$. Thus, the differential of the Hamilton function are

$$g\mathbf{d}\mathcal{H}_{\mathrm{SU}(2)} = \frac{1}{8}\left[\hat{\kappa}_{\mathfrak{su}_2}\tilde{\varphi}(g,\eta),\mathrm{Ad}^*_{g^{-1}}\left(\bar{\psi}^*(\eta)\right)\right]_{\mathfrak{su}_2^*}$$

$$\delta\mathcal{H}_{\mathrm{SU}(2)} = \frac{1}{8}\mathrm{Ad}_{g^{-1}}\left(\hat{\kappa}_{\mathfrak{su}_2}(\tilde{\varphi}(g,\eta))\otimes\mathrm{id}\right)\pi^R(g).$$

Observe that

$$\delta\mathcal{H}_{\mathrm{SU}(2)} = \frac{1}{8}\mathrm{Ad}_{g^{-1}}\left(\hat{\kappa}_{\mathfrak{su}_2}(\tilde{\varphi}(g,\eta))\otimes\mathrm{id}\right)\pi^R(g) = g^{-1}g^{\bar{\psi}^*\left(\hat{\kappa}_{\mathfrak{su}_2}(\tilde{\varphi}(g,\eta))\right)}$$

so $\left\langle\hat{\pi},\delta\mathcal{H}_{\mathrm{SU}(2)}\right\rangle = 0$.

Now, using the expression for the Hamiltonian vector field on the $\hat{\mathrm{d}}$-orbits (29)

$$V_{\mathcal{G}} = V^{\circ}_{\mathcal{G}} - \left(\left\langle\hat{\pi},\delta\mathcal{G}\right\rangle X_3,\mathrm{ad}^*_{\delta\mathcal{G}}\left(\eta - \frac{\left\langle\eta,g^{-1}V_0\right\rangle}{|\beta|^2}\mathbf{x}_3\right) - \langle g\mathbf{d}\mathcal{G},X_3\rangle\,\hat{\pi}\right)$$

where $V^{\circ}_{\mathcal{G}} = \left(g\delta\mathcal{G},\mathrm{ad}^*_{\delta\mathcal{G}}\lambda - g\mathbf{d}\mathcal{G}\right)$ is the Hamiltonian vector field associated with the canonical Poisson bracket in $T\mathrm{SU}(2)$. We get

$$V_{\mathcal{H}_{\mathrm{SU}(2)}} = \left(g\delta\mathcal{H}_{\mathrm{SU}(2)},\mathrm{ad}^*_{\delta\mathcal{H}_{\mathrm{SU}(2)}}\left(\frac{\left\langle\eta,g^{-1}V_0\right\rangle}{|\beta|^2}\mathbf{x}_3\right) - g\mathbf{d}\mathcal{H}_{\mathrm{SU}(2)}\right)$$

which, obviously, satisfy $\mathrm{P}V_{\mathcal{H}_{\mathrm{SU}(2)}} = V_{\mathcal{H}_{\mathrm{SU}(2)}}$, for the projector $\mathrm{P} = \left(\mathrm{Id} - V_0\hat{\pi}\right)\oplus(\mathrm{Id} - \mathbf{x}_3X_3)$ on the $\hat{\mathrm{d}}$-orbits. So the reduced equation of motion are

$$\begin{cases} g^{-1}\dot{g} = g\delta\mathcal{H}_{\mathrm{SU}(2)} \\ \dot{\eta} = \dfrac{1}{|\beta|^2}\left\langle\eta,g^{-1}V_0\right\rangle\mathrm{ad}^*_{\delta\mathcal{H}_{\mathrm{SU}(2)}}\mathbf{x}_3 - g\mathbf{d}\mathcal{H}_{\mathrm{SU}(2)}. \end{cases}$$

Observe that the equation for $g$, including the explicit form of $\delta\mathcal{H}_{\mathrm{SU}(2)}$ given above in terms of $\pi^R(g)$, resembles the equation of motion of a finite dimensional Poisson sigma model.

The solution curve is generated by the action of the curve $\tilde{b}(t)$, Eq. (32), through the dressing action $\hat{\mathrm{d}}:B\times\mathrm{SU}(2)\times\mathfrak{su}_2^*\longrightarrow\mathrm{SU}(2)\times\mathfrak{su}_2^*$. Having in mind the explicit form of $\tilde{\varphi}(g,\eta)$, we consider the initial point $(g_\circ,\eta_\circ)$ such that $\tilde{\varphi}(g_\circ,\eta_\circ) = X_\circ := n_\circ(\cos\theta X_1 + \sin\theta X_2) + m_\circ X_3$, $n_\circ^2 + m_\circ^2 = 1$, in $\mathcal{O}_\theta\subset\mathfrak{su}_2$. A suitable election for the initial condition is

$$g_\circ(\psi_\circ,\phi_\circ) = \begin{pmatrix} 0 & -\mathrm{e}^{\mathrm{i}(\phi_\circ-\psi_\circ)} \\ \mathrm{e}^{-\mathrm{i}(\phi_\circ-\psi_\circ)} & 0 \end{pmatrix}, \qquad \eta_\circ = \eta_{3\circ}\begin{pmatrix} -\sin 2\psi_\circ \\ \cos 2\psi_\circ \\ 1 \end{pmatrix}$$

with

$$\eta_{3\circ}^2\cos^2(\theta + 2\phi_\circ) = 1$$

that gives a solution curves in each $\hat{\mathrm{d}}$-orbit

$$(g(t),\eta(t)) = \left(g_\circ^{\tilde{b}(t)},\left(\psi^*\circ\mathrm{Ad}_{(\tilde{b}(t))^{g_\circ}}\circ\bar{\psi}^*\right)\eta_\circ\right)$$

with

$$g_\circ^{\tilde{b}(t)} = \frac{1}{\sqrt{\lambda^2\sinh^2 t + 1}}\begin{pmatrix} -\lambda\mathrm{e}^{-\mathrm{i}(\phi_\circ-\psi_\circ+\sigma)}\sinh t & -\mathrm{e}^{\mathrm{i}(\phi_\circ-\psi_\circ)} \\ \mathrm{e}^{-\mathrm{i}(\phi_\circ-\psi_\circ)} & -\lambda\mathrm{e}^{\mathrm{i}(\phi_\circ-\psi_\circ+\sigma)}\sinh t \end{pmatrix}$$

$$\begin{aligned} \eta(t) \;=\; & -\eta_3\left(\sin(2\psi)(\cosh t + a_3\sinh t) + \lambda\cos(\sigma - 2(\phi_\circ - \psi_\circ))\sinh t\right)\mathbf{x}_1 \\ & + \eta_3\left(\cos(2\psi)(\cosh t + a_3\sinh t) + \lambda\sin(\sigma - 2(\phi_\circ - \psi_\circ))\sinh t\right)\mathbf{x}_2 + \eta_3\mathbf{x}_3. \end{aligned}$$

Here, we wrote the parameters $a_1, a_2$ in the curve $\tilde{b}(t)$, Eq. (32), as: $a_1 = \lambda\cos\sigma$ and $a_2 = \lambda\sin\sigma$.

Let us to obtain the Lagrangian version of this system. The Legendre transformation in this case is singular, it can be partially retrieved from the first Hamilton equation written as

$$g^{-1}\dot{g} = -g^{-1}g^{\bar{\psi}^*\left(\hat{\kappa}_{\mathfrak{su}_2}(\tilde{\varphi}(g,\eta))\right)}.$$

Explicitly, this equation are

$$\bar{\alpha}\dot{\beta} - \beta\dot{\bar{\alpha}} = -\beta^2\left(\tilde{\varphi}_1(g,\eta) + i\tilde{\varphi}_2(g,\eta)\right) + \bar{\alpha}\beta\tilde{\varphi}_3(g,\eta)$$
$$\bar{\alpha}\dot{\alpha} + \beta\dot{\bar{\beta}} = -\frac{1}{2}\left(\alpha\beta - \bar{\alpha}\bar{\beta}\right)\tilde{\varphi}_1(g,\eta) - \frac{1}{2}\mathrm{i}\left(\bar{\alpha}\bar{\beta} + \alpha\beta\right)\tilde{\varphi}_2(g,\eta)$$

where we denoted $\tilde{\varphi}(g,\eta) = \sum_{i=1}^{3}\tilde{\varphi}_i(g,\eta)X_i$. This system of equation can be solved for two components of $\eta$, for instance $\eta_1$ and $\eta_2$, as a function of the velocities and $\eta_3$,

$$\eta_1 = \mathrm{i}\frac{\left(\alpha\bar{\beta} - \beta\bar{\alpha}\right)}{2\beta\bar{\alpha}\,|\beta|^2}\left(\bar{\alpha}\dot{\beta} - \beta\dot{\bar{\alpha}}\right) - \mathrm{i}\frac{\left(|\beta|^4 + |\beta|^2 + \beta^2\bar{\alpha}^2\right)}{2\beta\bar{\alpha}\,|\beta|^4}\left(\bar{\alpha}\dot{\alpha} + \beta\dot{\bar{\beta}}\right) - \frac{\left(|\alpha|^2|\beta|^2 + \bar{\alpha}^2\beta^2\right)}{2\beta\bar{\alpha}\,|\beta|^2}\eta_3$$
$$\eta_2 = \frac{\left(\alpha\bar{\beta} + \beta\bar{\alpha}\right)}{2\beta\bar{\alpha}\,|\beta|^2}\left(\bar{\alpha}\dot{\beta} - \beta\dot{\bar{\alpha}}\right) - \frac{\left(|\beta|^4 + |\beta|^2 - \beta^2\bar{\alpha}^2\right)}{2\beta\bar{\alpha}\,|\beta|^4}\left(\bar{\alpha}\dot{\alpha} + \beta\dot{\bar{\beta}}\right) + \mathrm{i}\frac{\left(|\alpha|^2|\beta|^2 - \beta^2\bar{\alpha}^2\right)}{2\beta\bar{\alpha}\,|\beta|^2}\eta_3.$$

Then, the Lagrangian function is defined as

$$L_{\mathrm{SU}(2)}(g,\dot{g}) = \left\langle \eta, g^{-1}\dot{g}\right\rangle - \mathscr{H}(g,\eta)$$

where $\hat{\pi}$ was given in (26). Replacing $\eta_1, \eta_2$ by the above relations, we obtain the Lagrangian

$$L_{\mathrm{SU}(2)}(g,\dot{g}) = \frac{1}{2}\left(g^{-1}\dot{g}, g^{-1}\dot{g}\right)_{\mathfrak{su}_2} + \eta_3\left\langle\hat{\pi}, g^{-1}\dot{g}\right\rangle \tag{38}$$

where we have introduced the metric in trivialized tangent bundle $\mathrm{SU}(2)\times\mathfrak{su}_2$ given by

$$\left(g^{-1}\dot{g}, g^{-1}\dot{g}\right)_{\mathfrak{su}_2} = -\frac{1}{8\,|\beta|^2}\kappa\left(g^{-1}\dot{g}, g^{-1}\dot{g}\right).$$

Observe that $\eta_3$ appears as a Lagrange multiplier realizing the constraint

$$\left\langle\hat{\pi}, g^{-1}\dot{g}\right\rangle \equiv \frac{\mathrm{i}}{2\,|\beta|^2}\left(\bar{\beta}\dot{\beta} - \beta\dot{\bar{\beta}}\right) = 0$$

which in terms of the Euler angles reduces to

$$\dot{\psi} - \dot{\phi} = 0$$

showing that the dynamics is naturally restricted on dressing orbit, as expected.

By introducing $\mathrm{A}(g) := 8\,|\beta|^2\,\eta_3\hat{\kappa}_{\mathfrak{su}_2}^{-1}\left(\hat{\pi}\right) \in \mathfrak{su}_2$, and after handling the quadratic form, we may write the Lagrangian as

$$L_{\mathrm{SU}(2)}(g,\dot{g}) = \frac{1}{2}\left(g^{-1}\dot{g} - \mathrm{A}(g), g^{-1}\dot{g} - \mathrm{A}(g)\right)_{\mathfrak{su}_2} - \frac{1}{2}\left(\mathrm{A}(g), \mathrm{A}(g)\right)_{\mathfrak{su}_2}$$

that describe the dynamics of a particle moving on the group manifold $S^3$ under the action of non-Abelian potential vector potential A, which confines its movement to the $S^2$ sphere determined by the constraint $\arg\beta = cte$.

## 8. Conclusions

We have shown how the theory of integrable systems, in particular AKS theory, can be used in its full scope to solve effectively the systems involved in a Poisson–Lie $T$-duality scheme. In doing so, we have also introduced a variant for the Hamiltonian Poisson–Lie $T$-duality scheme, by using as a central object of the scheme a coadjoint orbit of one of the Iwasawa factors. This fact enhances the applicability of the PL duality, including a wider class of systems, finite or infinite dimensional, on which the techniques of integrable systems can be used.

The explicit finite dimensional example $\mathrm{SL}(2,\mathbb{C}) = \mathrm{SU}(2)\times B$ exhibits a detailed description of the way in which this duality works, constructing explicitly the solutions for all the involved systems from the factorization of the solution curve of an almost trivial system on $\mathfrak{sl}_2^*$. This curve $\tilde{b}(t) \subset B$ gives rise to the solution curves in each case through the corresponding actions. As an alternative way for using the scheme, the solutions would be obtained retrieving the curve $\tilde{b}(t) \subset B$ from the well known solution of the Toda system on $\mathbb{R}^2$. It is worth to remark that the election of the symmetry group defines the master integrable system ruling the dynamics and it is realized in this example by the choice of $B$ as the main symmetry, putting the Toda system in the center of the scheme, or in the loop group case of Refs. [2,3], where the WZNW model appears on the double Lie group $LD = LG \times LG^*$. The compatible dynamics was obtained from collective Hamiltonian functions after fixing a Hamiltonian on the selected coadjoint orbit.

The systems in the equivalence class includes a kind of generalized top on the group $B$ and a dressing invariant system on the group manifold $S^3$ which suffers a reduction to the $S^2$ submanifold characterized by arg $\beta$. Dressing symmetry becomes relevant for the so called *Poisson sigma models*, so our example may serve as a laboratory for understanding issues related to the reduced space of systems with this kind of symmetry.

From the Lagrangian point of view, the PL $T$-duality transformation relates a constrained systems on the compact configuration space SU(2) with a system on the noncompact space $B$, by a rather nontrivial transformation. A remarkable fact is that these nonlinear systems arise from kinetic Lagrangians, that means, bilinear forms on the corresponding tangent bundles. In the SU(2) case the bilinear form amounts to be metric, while in $B$ case, because a solvable Lie group lacks of an Ad-invariant bilinear form on it, the bilinear form is inherited from $\mathfrak{sl}_2$ through a linear operator $\mathbb{K}$. This relation between two different *target geometries* relies on the dynamical equivalence of the reduced Hamiltonian systems and the coadjoint orbit. In both cases, the structure of the reduced phase spaces were explicitly determined, and the PL $T$-duality equivalence between the Lagrangian system (34), (36) and the (38) was established.

Most of the theory of integrable systems applied in this work can be used, with some cares, in the infinite dimensional case (loop groups). In fact, the Refs. [4,10,11] deal with Kac–Moody algebras and infinite dimensional integrable systems like KdV and others, so we expect they can be applied in the natural setting of Poisson–Lie $T$-duality, namely the loop groups case and $T$-dualizable sigma models.

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