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On commuting matrices in max algebra and in classical nonnegative algebra☆

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ABSTRACT

This paper studies commuting matrices in max algebra and nonnegative linear algebra. Our starting point is the existence of a common eigenvector which directly leads to max analogues and nonnegative analogues of some classical results for complex matrices. We also investigate Frobenius normal forms of commuting matrices, particularly when the Perron roots of the components are distinct. For the case of max algebra, we show how the intersection of eigencones of commuting matrices can be described and we consider connections with Boolean algebra which enables us to prove that two commuting irreducible matrices in max algebra have a common eigennode.

1. Introduction

The study of commuting complex matrices has a long history. As observed in [13], Cayley considers what appears to be a generic case of commuting matrices in his famous memoir [7]. Frobenius [15,

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16] showed that if A_i, $i = 1, \ldots, r$, are pairwise commuting matrices, then the eigenvalues α_i^j, $j = 1, \ldots, n$, of the matrices A_i may be ordered so that the eigenvalues of any polynomial $p(A_1, \ldots, A_r)$ are $p(\alpha_1^j, \ldots, \alpha_r^j)$, $j = 1, \ldots, n$. Another proof may be found in Schur [35]. Surprisingly, none of these proofs mention eigenvectors. Frobenius [15] also showed that if for given matrices A, B the equation $AX = XB$ has a nonzero solution, then A and B have a common eigenvalue. Another well-known result is that pairwise commuting matrices have a common eigenvector. We have found no reference for the first explicit appearance of this property, though it easily follows from, e.g., the canonical form derived by Weyr [37] and his discussion of commuting matrices. Many generalizations and applications of this result exist, see [14], [32] or [28]. Several books on matrix theory, such as [20], contain proofs of the results stated above.

It is the purpose of this paper to prove analogs of these results for matrices over two semirings:

1. The semiring of nonnegative reals under the usual addition, here called (classical) nonnegative algebra.
2. The semiring of nonnegative reals with the operation of maximum playing the role of addition, here called max algebra.

Spectral theory of matrices in nonnegative algebra is usually called Perron–Frobenius theory after the founders of this topic, see [29,30,17–19]. The basic results are again found in many books on matrix theory, such as [20,24,33]. Commuting nonnegative matrices can be found in [3], see Section 7, and in [31]. For further information relevant to the present article, see [34].

Spectral theory for matrices in max algebra was developed by Cuninghame-Green [9] and Gaubert [21], see [5,4] for recent expositions.

See [11,12] for studies of commuting matrices in more general semirings.

We devote the main sections to properties of commuting matrices in max algebra. At the end of this introduction we give a formal definition of max algebra and make some remarks on the relation between the two theories. We then review basic max algebra spectral theory in Section 2 (for those who are unfamiliar with this topic). We provide a proof that pairwise commuting matrices have a common eigenvector in Section 3. We also derive some immediate consequences of this theorem, concerning inequalities for Perron roots and matrix polynomials, and describe the intersection of principal eigencones by means of the product of spectral projectors. In Section 4, we investigate Frobenius normal forms of commuting matrices, showing that in the important special case where the Perron roots of the components are distinct, the transitive closures of the associated reduced digraphs coincide. In Section 5, we consider the eigenvector scaling, which leads us to study commuting matrices in Boolean algebra. As a result of this study we show that the critical digraphs of two commuting irreducible matrices in max algebra share a common node. In Section 7, it is indicated that most results in Sections 3 and 4 also hold in nonnegative matrix algebra. Section 6 is devoted to numerical examples.

By *max algebra* we understand the analog of linear algebra developed over the max-times semiring $(\mathbb{R}_+, \oplus, \times)$, which is the set of nonnegative numbers $\mathbb{R}_+$ equipped with the operations of "addition" $a \oplus b := \max(a, b)$ and ordinary multiplication $a \times b$. The zero and unity of this semiring coincide with the usual 0 and 1. The operations of the semiring are extended to nonnegative matrices and vectors in the same way as in conventional linear algebra. That is, if $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ are matrices of compatible sizes with entries in $\mathbb{R}_+$, we write $C = A \oplus B$ if $c_{ij} = a_{ij} \oplus b_{ij}$ for all i, j and $C = A \otimes B$ if $c_{ij} = \oplus_k a_{ik} b_{kj} = \max_k (a_{ik} b_{kj})$ for all i, j. If A is a square matrix over $\mathbb{R}_+$, then the iterated product $A \otimes A \otimes \cdots \otimes A$ in which the symbol A appears k times will be denoted by A^k.

It is significant that max algebra can be obtained from nonnegative linear algebra by means of a limit passage called Maslov dequantization [27]:

$$a \oplus b = \lim_{p \to \infty} a \oplus_p b, \tag{1}$$

where $a \oplus_p b := (a^p + b^p)^{1/p}$. Note that $(\mathbb{R}_+, \oplus_p, \times)$ forms a semiring which is isomorphic to the semiring $(\mathbb{R}_+, +, \times)$ of nonnegative numbers equipped with the usual addition and multiplication. Thus one may expect, and this is indeed the case, that max algebra and nonnegative linear algebra have many interesting properties in common.[1] For example, the Frobenius (normal) from of a reducible matrix plays an important role in the study of reducible matrices in both theories. In view of the above discussion, it is not surprising that a comparison of spectral properties of reducible matrices shows that one needs to replace strict inequalities in classical nonnegative spectral theory by weak inequalities in max algebraic spectral theory, see [5] for a remark along these lines concerning eigenvectors.

The above notation employing $\oplus$ and $\otimes$ is standard in max algebra. However, as many results of the present paper are true both in max algebra and in nonnegative linear algebra, it will be convenient to write $a + b$ for $\max(a, b)$ when the argument works in both theories. On the other hand, we emphasize by using the specific max algebraic notation when this is not the case.

2. The spectral problem in max algebra

We recall here some notation and basic facts about the spectral problem in max algebra, which we use further in this paper. See [1,2,4,9,25] for general reference and more information.

The *max algebraic spectral problem* for $A \in \mathbb{R}_+^{n\times n}$ consists in finding *eigenvalues* $\alpha \in \mathbb{R}_+$ and nonzero *eigenvectors* $v \in \mathbb{R}_+^n$ such that $Av = \alpha v$ is satisfied. Observe that the set $V(A, \alpha) := \{v | Av = \alpha v\}$ is a (max) *cone* of $\mathbb{R}_+^n$, that is, a subset of $\mathbb{R}_+^n$ closed under (max) addition and (nonnegative) scalar multiplication. This cone will be called the *eigencone* of A associated with α. The set of eigenvalues, which is nonempty like in the usual Perron–Frobenius theory, is called the *spectrum* of A and denoted $\Lambda(A)$. The largest eigenvalue of A will be denoted $\lambda(A)$ and called the *Perron root* of A (since we want the same terminology for max algebra and nonnegative matrix algebra), and the associated eigencone will be called the *principal eigencone* of A.

Unlike in the case of classical algebra, there is an explicit formula for the max algebraic Perron root of $A = (a_{ij}) \in \mathbb{R}_+^{n\times n}$:

$$\lambda(A) = \bigoplus_{k=1}^{n} \bigoplus_{i_1,\cdots,i_k} (a_{i_1 i_2} \ldots a_{i_k i_1})^{1/k}. \tag{2}$$

This is also known as the maximal cycle (geometric) mean of A.

For $A \in \mathbb{R}_+^{n\times n}$ we construct the associated digraph $\mathcal{G} = (N, E)$ by setting $N = \{1, \ldots, n\}$ and letting $(i, j) \in E$ whenever $a_{ij} > 0$. When this digraph contains at least one cycle, one distinguishes *critical cycles*, where the maximum in (2) is attained. Further, one constructs the *critical digraph* $\mathcal{C}(A) = (N_c^A, E_c^A)$, which consists of all the nodes N_c^A and edges E_c^A of $\mathcal{G}$ on critical cycles. The nodes in N_c^A will be called *critical nodes* or *eigennodes*.

The critical digraph is closely related to the series

$$A^* = I \oplus A \oplus A^2 \oplus \cdots, \tag{3}$$

where I is the unit matrix. This series is known to converge if, and only if, $\lambda(A) \leqslant 1$, in which case it is called the *Kleene star* of A. If $\lambda(A) \leqslant 1$, then this series can be truncated: $A^* = I \oplus A \oplus A^2 \oplus \cdots \oplus A^{n-1}$. Note that generally, any series like (3) can be defined as supremum, which is finite if the sequence of matrices is bounded.

For $A \in \mathbb{R}_+^{n\times n}$ such that $\lambda(A) = 1$, the principal eigencone is the set of max-linear combinations of all columns of A^* with indices in N_c^A:

$$V(A, 1) = \left\{\oplus_{i\in N_c^A} \beta_i A^*_{\cdot i} | \beta_i \in \mathbb{R}_+\right\}. \tag{4}$$

In particular, we have

$$AA^*_{\cdot i} = A^*_{\cdot i}, \quad A^*_{i\cdot} A = A^*_{i\cdot}, \quad \forall i \in N_c^A. \tag{5}$$

Thus, unlike in the usual Perron–Frobenius theory, even if $A \in \mathbb{R}_+^{n\times n}$ is irreducible (that is, the associated digraph is strongly connected), the principal eigencone in max algebra may contain more than

[1] Referring to max algebra spectral theory, Gaubert [23] remarks "The theory is extremely similar to the well-known Perron–Frobenius theory".

just one ray. However, for irreducible A, $\lambda(A)$ given by (2) is the only eigenvalue and every eigenvector is positive, see Theorem 4.1.

By standard optimal path algorithms, the critical digraph and the columns of A^* can be computed in $O(n^3)$ operations. For further details we refer the reader to [4,25,2].

A (max) cone $K \subset \mathbb{R}^n_+$ is said to be finitely generated if it is the set of max-linear combinations of a finite subset of vectors of $\mathbb{R}^n_+$. Equivalently, a cone $K \subset \mathbb{R}^n_+$ is finitely generated if there exists a matrix $X \in \mathbb{R}^{n\times r}_+$, for some $r \in \mathbb{N}$, such that $K = \mathrm{Im}(X)$, where as usual $\mathrm{Im}(X) := \{Xu | u \in \mathbb{R}^r_+\}$. Observe that if K is not trivial, we may assume that X does not have a null column. By (4), it follows that the principal eigencone is finitely generated. Indeed, this property holds for any eigencone of A, see, e.g., [5, Theorem 4.1]. Therefore, in what follows, for $\alpha \in \Lambda(A)$ we shall denote by X^A_α any matrix with nonzero columns satisfying $V(A, \alpha) = \mathrm{Im}(X^A_\alpha)$.

Let us finally mention that like in classical algebra, any finite intersection of finitely generated (max) cones is also finitely generated (this property follows from [6], see e.g. [22, Theorem 1]).

We summarize the main properties that will be used in this paper in the next proposition.

Proposition 2.1. *In max algebra the following statements hold*:

(i) *Every matrix has an eigenvalue with a corresponding eigenvector.*
(ii) *Eigencones are finitely generated.*
(iii) *The intersection of two finitely generated (max) cones is finitely generated.*

Further information on max algebra spectral theory will be given in Section 4.

3. Existence of common eigenvectors

3.1. Common eigenvector of two matrices

In this section on max algebra we prove that two commuting matrices have a common eigenvector. With this aim, we shall need the following lemma.

Lemma 3.1. *If $A, B \in \mathbb{R}^{n\times n}_+$ commute, then any eigencone $V(A, \alpha)$ of A is invariant under B and any eigencone $V(B, \alpha)$ of B is invariant under A.*

Proof. Let $v \in V(A, \alpha)$. For $u = Bv$, we have

$$Au = ABv = BAv = \alpha Bv = \alpha u. \tag{6}$$

Therefore, $B(V(A, \alpha)) \subset V(A, \alpha)$. $\square$

Now it is possible to prove the following key result, which relates the eigencones of two commuting matrices.

Theorem 3.2. *If $A, B \in \mathbb{R}^{n\times n}_+$ commute, then for any eigencone $V(A, \alpha)$ of A there exists an eigencone $V(B, \mu)$ of B such that $V(A, \alpha) \cap V(B, \mu)$ contains a nonzero vector*.

Proof. Let $V(A, \alpha) = \mathrm{Im}(X^A_\alpha)$ be an eigencone of A. Then,

$$AX^A_\alpha = \alpha X^A_\alpha, \tag{7}$$

and since by Lemma 3.1 we have $B(\mathrm{Im}(X^A_\alpha)) \subset \mathrm{Im}(X^A_\alpha)$, there exists a (nonnegative square) matrix C such that $BX^A_\alpha = X^A_\alpha C$. Let z be any eigenvector of C, so that $Cz = \mu z$ and $z \neq 0$, and consider $u = X^A_\alpha z$. Then, $u \neq 0$ (recall that all the columns of X^A_α are nonzero) and we obtain

$$Au = AX^A_\alpha z = \alpha X^A_\alpha z = \alpha u$$

and

$$Bu = BX_\alpha^A z = X_\alpha^A Cz = \mu X_\alpha^A z = \mu u.$$

Thus, $u \in V(A, \alpha) \cap V(B, \mu)$. $\square$

As an immediate consequence, we obtain:

Corollary 3.3. *If $A, B \in \mathbb{R}_+^{n\times n}$ commute, then they have a common eigenvector.*

We remark that our proof of Theorem 3.2 also shows the following result:

Proposition 3.4. *Let $A \in \mathbb{R}_+^{n\times n}$ and let K be a (nontrivial) finitely generated cone of $\mathbb{R}_+^n$. If $AK \subseteq K$, then A has an eigenvector in K.*

3.2. Common eigenvector of several matrices

The results above can be generalized to several pairwise commuting matrices.

Theorem 3.5. *Assume the matrices $A_1, \ldots, A_r \in \mathbb{R}_+^{n\times n}$ commute in pairs. Then, given any eigenvalue $\alpha_i \in \Lambda(A_i)$, where $i \in \{1, \ldots, r\}$, there exist $\alpha_j \in \Lambda(A_j)$ for all $j \neq i$ such that $V(A_1, \alpha_1) \cap \cdots \cap V(A_r, \alpha_r)$ contains a nonzero vector.*

Proof. The case $r = 2$ is precisely Theorem 3.2. So assume that the statement of the theorem holds for $r = k$ and let $A_1, \ldots, A_k, A_{k+1}$ be $k + 1$ matrices which commute in pairs.

Without loss of generality, assume $\alpha_1 \in \Lambda(A_1)$ is given. By the induction hypothesis, there exist $\alpha_j \in \Lambda(A_j)$, for $j = 2, \ldots, k$, such that $V(A_1, \alpha_1) \cap \cdots \cap V(A_k, \alpha_k)$ contains a nonzero vector. Moreover, since by Proposition 2.1 any eigencone is finitely generated and any finite intersection of finitely generated max cones is also finitely generated, there exists a (nonnegative) matrix X such that $V(A_1, \alpha_1) \cap \cdots \cap V(A_k, \alpha_k) = \mathrm{Im}(X)$. Note that we may assume, without loss of generality, that all the columns of X are nonzero because $\mathrm{Im}(X)$ contains nonzero vectors.

Since A_i and A_{k+1} commute for $i = 1, \ldots, k$, by Lemma 3.1 it follows that $A_{k+1}(V(A_i, \alpha_i)) \subseteq V(A_i, \alpha_i)$ for $i = 1, \ldots, k$. Therefore, $A_{k+1}(\mathrm{Im}(X)) = A_{k+1}(\cap_{i=1}^k V(A_i, \alpha_i)) \subseteq \cap_{i=1}^k V(A_i, \alpha_i) = \mathrm{Im}(X)$ and thus there exists a (nonnegative square) matrix C such that $A_{k+1}X = XC$.

Like in the proof of Theorem 3.2, let z be any eigenvector of C so that $Cz = \mu z$, for some $\mu \in \Lambda(C)$, and define $u = Xz$. Since $z \neq 0$ and the columns of X are nonzero, we have $u \neq 0$ and

$$A_{k+1}u = A_{k+1}Xz = XCz = \mu Xz = \mu u.$$

Thus, $u \in \mathrm{Im}(X) \cap V(A_{k+1}, \mu) = V(A_1, \alpha_1) \cap \cdots \cap V(A_k, \alpha_k) \cap V(A_{k+1}, \mu)$. $\square$

Next we investigate the eigenvalues of polynomials of commuting matrices, assuming that their coefficients are nonnegative. For the polynomials in max algebra (also known as *max-polynomials*), replace the usual addition of monomials by maximum.

Theorem 3.6. *Let $A_1, \ldots, A_r \in \mathbb{R}_+^{n\times n}$ commute in pairs and let $p(x_1, \ldots, x_r)$ be a polynomial. Then,*

(i) *For each $i \in \{1, \ldots, r\}$ and $\alpha_i \in \Lambda(A_i)$, there exist $\alpha_j \in \Lambda(A_j)$ for all $j \neq i$ such that $p(\alpha_1, \ldots, \alpha_r) \in \Lambda(p(A_1, \ldots, A_r))$.*

(ii) *For each $\lambda \in \Lambda(p(A_1, \ldots, A_r))$ there exist $\alpha_i \in \Lambda(A_i)$ for all $i = 1, \ldots, r$ such that $\lambda = p(\alpha_1, \ldots, \alpha_r)$.*

Proof. (i) Let $i \in \{1, \ldots, r\}$ and $\alpha_i \in \Lambda(A_i)$. By Theorem 3.5, there exist $\alpha_j \in \Lambda(A_j)$ for all $j \neq i$ and a nonzero vector $v \in \mathbb{R}_+^n$ such that $A_i v = \alpha_i v$ for all $i = 1, \ldots, r$. But then we also have $p(A_1, \ldots, A_r)v = p(\alpha_1, \ldots, \alpha_r)v$, and so $p(\alpha_1, \ldots, \alpha_r) \in \Lambda(p(A_1, \ldots, A_r))$.

(ii) Let $\lambda \in \Lambda(p(A_1, \ldots, A_r))$. Since $A_1, \ldots, A_r$ and $p(A_1, \ldots, A_r)$ commute in pairs, by Theorem 3.5 there is an eigenvector $v \in V(p(A_1, \ldots, A_r), \lambda)$ which is also an eigenvector of A_i associated with some eigenvalue $\alpha_i \in \Lambda(A_i)$, for all $i = 1, \ldots, r$. But then $\lambda v = p(A_1, \ldots, A_r)v = p(\alpha_1, \ldots, \alpha_r)v$ and it follows that $\lambda = p(\alpha_1, \ldots, \alpha_r)$. $\square$

Corollary 3.7. *Let $A_1 \cdots A_r \in \mathbb{R}_+^{n\times n}$ commute in pairs and let $p(x_1, \ldots, x_r)$ be a polynomial. Then,*

(i) $\lambda(p(A_1 \cdots A_r)) \leqslant p(\lambda(A_1) \cdots \lambda(A_r))$.
(ii) $\lambda(A_1 + \cdots + A_r) \leqslant \lambda(A_1) + \cdots + \lambda(A_r)$.
(iii) $\lambda(A_1 \cdots A_r) \leqslant \lambda(A_1) \cdots \lambda(A_r)$.

Moreover, equality holds in all the above relations if the matrices $A_1 \cdots A_r$ are irreducible.

Proof. Part (i) follows from Theorem 3.6 and the monotonicity of polynomials, and parts (ii) and (iii) are special cases. If the matrices are irreducible, then each of them has unique eigenvalue, and we have the equalities. $\square$

In the case of max algebra we also have $\lambda(A_1 \oplus \cdots \oplus A_r) \geqslant \lambda(A_i)$ for all $i = 1, \ldots, r$, as the Perron root expressed by (2) is monotonic. Hence (ii) always holds with equality in max algebra.

3.3. Intersection of principal eigencones

A matrix Q is called a *projector* on a cone $K \subset \mathbb{R}_+^n$ if $\mathrm{Im}(Q) = K$ and $Q^2 = Q$. This implies that $Qx = x$ if, and only if, $x \in K$. In general, there are many projectors on the same cone, but if two such projectors P, Q commute, then they are identical because we have $Px = QPx = PQx = Qx$ for all $x \in \mathbb{R}_+^n$.

We recall that the eigencone $V(A, \lambda(A))$ associated with $\lambda(A)$ (assumed to be nonzero) is called the principal eigencone of A and a projector on $V(A, \lambda(A))$ which commutes with A is called a *spectral projector* for A. Since $V(A, \lambda(A)) = V(A/\lambda(A), 1)$, there is no loss of generality in assuming that $\lambda(A) = 1$. In max algebra, one can explicitly define such projector. There are two definitions in the literature:

$$\widetilde{Q}(A) = \bigoplus_{i \in N_c^A} A^*_{\cdot i} A^*_{i \cdot}, \tag{8}$$

and

$$Q(A) = \lim_{p\to\infty} \bigoplus_{m \geqslant p} A^m. \tag{9}$$

The first of these is found in Baccelli et al. [2, Section 3.7.3], see also [8], and the second one is found in a more general context in Kolokoltsov and Maslov [26, Section 2.4].

We shall need the following proposition, which shows that these projectors are indeed identical. See [26, Theorem 2.11] for a closely related result.

Proposition 3.8. *Let $A \in \mathbb{R}_+^{n\times n}$ with $\lambda(A) = 1$. Then, there is a unique spectral projector on $V(A, 1)$ which is given, equivalently, by (8) or (9).*

Proof. In the first place, note that in the matrix case, (9) may be replaced by

$$Q(A) = \lim_{p\to\infty} A^p A^*, \tag{10}$$

see the remarks on A^* in Section 2. Since using (3) and the continuity of operations we have $A^{p+1}A^* \leqslant A^p A^*$, it follows that the limit in (10) exists.

By the continuity of operations, $\lim_{p\to\infty}(C_pB) = (\lim_{p\to\infty} C_p)B$ for any converging sequence of matrices C_p and any matrix B. Using this, we observe that if B is any matrix which commutes with A, then B also commutes with $Q(A)$. Since as shown above any two commuting projectors on the same cone are identical, we conclude that any spectral projector for A is equal to $Q(A)$. Therefore, in particular we have $\widetilde{Q}(A) = Q(A)$. $\square$

Next we state two lemmas. The first one exploits (9) and follows from the continuity of multiplication. The second lemma is standard and its proof is recalled for the convenience of the reader.

Lemma 3.9. *If $A, B \in \mathbb{R}_+^{n\times n}$ commute, then $Q(A)$ and $Q(B)$ commute.*

Lemma 3.10. *Let Q_i, $i = 1, \ldots, r$, be pairwise commuting projectors. Then,*

$$\mathrm{Im}(Q_1) \cap \cdots \cap \mathrm{Im}(Q_r) = \mathrm{Im}(Q_1 \cdots Q_r). \tag{11}$$

Proof. If $x \in \mathrm{Im}(Q_1) \cap \cdots \cap \mathrm{Im}(Q_r)$, then $Q_ix = x$ for $i = 1, \ldots, r$, and hence $(Q_1 \cdots Q_r)x = x$. Thus, $x \in \mathrm{Im}(Q_1 \cdots Q_r)$. Conversely, if $x \in \mathrm{Im}(Q_1 \cdots Q_r)$, then $(Q_1 \cdots Q_r)y = x$ for some vector y. Multiplying this equation by Q_i, for $i = 1, \ldots, r$, using the idempotency of Q_i and commutativity, it follows that $Q_ix = x$. $\square$

Note that Lemma 3.10 also holds if we require that any permutation of $Q_1, \ldots, Q_r$ yields the same product, which is a weaker commutativity condition.

Lemma 3.10 implies that we can express the intersection of the principal eigencones of commuting matrices as follows:

$$V(A_1, 1) \cap \cdots \cap V(A_r, 1) = \mathrm{Im}(Q(A_1)) \cap \cdots \cap \mathrm{Im}(Q(A_r)) = V(Q(A_1) \cdots Q(A_r), 1). \tag{12}$$

In the general (reducible) case, this intersection may reduce to the zero vector. Since by (iii) of Corollary 3.7 we have $\lambda(Q(A_1) \cdots Q(A_r)) \leqslant 1$, it follows that (12) is not trivial if, and only if, the Perron root of $Q(A_1) \cdots Q(A_r)$ is 1, in which case this intersection is given by the principal eigencone of $Q(A_1) \cdots Q(A_r)$. Using definition (8), we can compute this product in $O(rn^3)$ operations, and then it requires no more than $O(n^3)$ operations to compute its Perron root and describe its principal eigencone when the Perron root is 1.

4. Frobenius normal forms

Let $\mathcal{G} = (N, E)$ be the associated digraph of $A \in \mathbb{R}_+^{n\times n}$ and $\mathcal{G}^\mu = (N^\mu, E^\mu)$, for $\mu = 1, \ldots, t$, be the connected components of $\mathcal{G}$. We construct the *reduced digraph* $\mathcal{R}$ with set of nodes $\{1, \ldots, t\}$ setting an edge (μ, ν) whenever there exist $i \in N^\mu$ and $j \in N^\nu$ such that $(i, j) \in E$. We shall call a connected component (or the corresponding set of nodes) of $\mathcal{G}$ a *class* of A and also use that term for the nodes of $\mathcal{R}$. Further, we also identify subsets S of nodes of $\mathcal{R}$ with the union of the corresponding classes of A, that is S may denote $\cup_{\nu\in S} N^\nu$.

Each class μ is labeled by the corresponding maximal cycle (geometric) mean α_μ, which will be also called the *Perron root* of the class. We write $\mu \to \nu$ if $\mu = \nu$ or if there exists a path in $\mathcal{R}$ connecting μ to ν (in other words, if μ has access to ν). A set I of classes is an *initial segment* of $\mathcal{R}$ if $\nu \in I$ and $\mu \to \nu$ imply that $\mu \in I$. The set of all classes μ such that $\mu \to \nu$ will be denoted by $\mathrm{Intl}(\nu)$ and called the *initial segment* generated by ν in $\mathcal{R}$. If S is a set of classes, then a class $\nu \in S$ is said to be *initial* in S if $\mu \to \nu$ and $\mu \in S$ imply that $\mu = \nu$. Similarly, a class $\nu \in S$ is called *final* in S if $\nu \to \mu$ and $\mu \in S$ imply that $\mu = \nu$. An initial (respectively, final) class in $\{1, \ldots, t\}$ is simply called initial (respectively, final). A class ν is said to be *spectral* if ν is initial, or if $\alpha_\nu > 0$ and $\mu \to \nu$ imply that $\alpha_\mu \leqslant \alpha_\nu$. A spectral class ν is called *premier spectral* if $\mu \to \nu$ and $\mu \neq \nu$ imply that $\alpha_\mu < \alpha_\nu$.

Access relations for $\mathcal{G}$ and $\mathcal{R}$ are normally visualized in terms of a *Frobenius form*. There exists a simultaneous permutation of rows and columns of A such that

$$A = \begin{pmatrix} A_{11} & 0 & \cdots & 0 & 0 \\ A_{21} & A_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{(t-1)1} & A_{(t-1)2} & \cdots & A_{(t-1)(t-1)} & 0 \\ A_{t1} & A_{t2} & \cdots & A_{t(t-1)} & A_{tt} \end{pmatrix}$$

with irreducible diagonal blocks $A_{\mu\mu}$ for $\mu = 1, \ldots, t$.

A Frobenius (normal) form of A arises from each total ordering of the classes of $\mathcal{R}$ that is anti-compatible with the partial order given by the access relations, viz. $\mu \to \nu$ implies $\mu \geqslant \nu$. In particular, given any initial segment I of $\mathcal{R}$ there is a Frobenius form of A for which the classes of I are $s, s+1, \ldots, t$ for some $s \in \{1, \ldots, t\}$.

We now state the *fundamental spectral theorem of max algebra*. Recall that the *support* of a vector $x \in \mathbb{R}_+^n$ consists of all $i \in N$ such that $x_i > 0$.

Theorem 4.1. *Let $A \in \mathbb{R}_+^{n \times n}$ and $\lambda \in \mathbb{R}_+$. Then, a subset U of N is the support of an eigenvector associated with λ if, and only if,*

(i) *There is an initial segment I of $\mathcal{R}$ such that $U = \cup_{\nu \in I} N^\nu$.*
(ii) *All final classes ν in I are spectral and satisfy $\alpha_\nu = \lambda$.*

This theorem has a long history and has been stated in different ways, see, e.g., [23,21,9,5,4,2]. The statement in Theorem 4.1 is essentially the same as the one that appeared in [23].

The following corollary is immediate.

Corollary 4.2. *Let $A \in \mathbb{R}_+^{n \times n}$. Then,*

(i) *λ is an eigenvalue if, and only if, there is a spectral class ν such that $\alpha_\nu = \lambda$.*
(ii) *ν is a spectral class if, and only if, there exists an eigenvector with support* $\mathrm{Intl}(\nu)$.
(iii) *A spectral class ν is premier spectral if, and only if, any eigenvector associated with α_ν whose support is contained in* $\mathrm{Intl}(\nu)$ *has its support equal to* $\mathrm{Intl}(\nu)$.
(iv) *If the reduced digraph of A has a unique spectral class ν with Perron root α_ν, then any eigenvector associated with α_ν has support* $\mathrm{Intl}(\nu)$.
(v) *If the Perron roots of all classes are distinct, then all spectral classes are premier spectral and all eigenvectors have support* $\mathrm{Intl}(\nu)$ *for some spectral class ν.*

The following well-known corollary also follows easily from Theorem 4.1.

Corollary 4.3. *For any $A \in \mathbb{R}_+^{n \times n}$ with $\lambda(A) > 0$ the following statements are equivalent*:

(i) *A has a positive eigenvector*.
(ii) *The Perron root of any final class is $\lambda(A)$* (*and so, in particular, all final classes are spectral*).

If either condition holds, then any positive eigenvector is associated with the eigenvalue $\lambda(A)$.

The proof of our next lemma essentially repeats arguments used to prove Corollary 3.3 and Theorem 3.5.

Lemma 4.4. *Let $A \in \mathbb{R}_+^{n \times n}$ and $C \in \mathbb{R}_+^{m \times m}$. If $AX = XC$, where $X \in \mathbb{R}_+^{n \times m}$ and every column of X is nonzero, then any eigenvalue of C is also an eigenvalue of A.*

Proof. Suppose that $\lambda \in \Lambda(C)$ and let $z \in \mathbb{R}_+^m$ be an eigenvector of C associated with λ. Then, $AXz = XCz = \lambda Xz$. Since every column of X is nonzero, we have $Xz \neq 0$ and thus $\lambda \in \Lambda(A)$. $\square$

If A and C are irreducible, then in Lemma 4.4 it is enough to assume that X is nonzero because the vector z in the proof above is positive. Thus, we obtain:

Lemma 4.5. *Let $A \in \mathbb{R}_+^{n\times n}$ and $C \in \mathbb{R}_+^{m\times m}$ be irreducible matrices. If $AX = XC$, where $X \in \mathbb{R}_+^{n\times m}$ is nonzero, then $\lambda(A) = \lambda(C)$.*

The following important lemma indicates what happens if a matrix commutes with an irreducible matrix.

Lemma 4.6. *If $A, B \in \mathbb{R}_+^{n\times n}$ commute and B is irreducible, then*

(i) *The Perron root of every final class and every initial class of A is $\lambda(A)$ (and so, in particular, all final classes are spectral).*
(ii) *A has the unique eigenvalue $\lambda(A)$.*
(iii) *If A is reducible, then at least two distinct classes of A have Perron root $\lambda(A)$.*

Proof. In the first place, note that the lemma is obvious if $\lambda(A) = 0$, so we may assume that $\lambda(A) > 0$.

(i) From Corollary 3.3, we know that A and B have a common eigenvector. Since B is irreducible, all its eigenvectors are positive. It follows by (ii) of Corollary 4.3 that all final classes of A have Perron root $\lambda(A)$ and are therefore spectral. Similarly, the transpose A^T commutes with the irreducible matrix B^T and therefore all final classes of A^T have Perron root $\lambda(A)$. But the final classes of A^T are precisely the initial classes of A.
(ii) This follows easily from (i), Theorem 4.1 and the definition of spectral class.
(iii) If A is reducible, either it has two initial classes or an initial class and a distinct final class. □

Remark 4.7. In Corollary 3.7, the irreducibility assumption can be relaxed. We need there that just *one* of the matrices is irreducible, for then by (ii) of Lemma 4.6 each matrix has a unique eigenvalue.

The *transitive closure* of $\mathcal{R}$ is the digraph $\mathcal{R}^*$ which has the edge (μ, ν) if, and only if, $\mu \to \nu$ in $\mathcal{R}$. We shall say that μ *covers* ν in $\mathcal{R}^*$ if $\mu \neq \nu$, $\mu \to \nu$ and the following property is satisfied: $\mu \to \delta \to \nu$ implies that either $\delta = \mu$ or $\delta = \nu$.

The main result of this section is the following theorem.

Theorem 4.8. *Suppose that $A_1, \ldots, A_r \in \mathbb{R}_+^{n\times n}$ pairwise commute and that all classes of A_i, for each $i \in \{1, \ldots, r\}$, have distinct Perron roots. Then,*

(i) *All classes of $A_1, \ldots, A_r$ and $A_1 + \cdots + A_r$ coincide.*
(ii) *The transitive closures of the reduced digraphs of $A_1, \ldots, A_r$ and $A_1 + \cdots + A_r$ coincide.*
(iii) *The spectral classes of the reduced digraphs of $A_1, \ldots, A_r$ and $A_1 + \cdots + A_r$ coincide. In particular, $A_1, \ldots, A_r$ have the same number of distinct eigenvalues.*
(iv) *Let $\mu_1, \ldots, \mu_m$ be the common spectral classes of $A_1, \ldots, A_r$ and denote the Perron root of the μ_jth class of A_i by α_i^j. Then, for any polynomial $p(x_1, \ldots, x_r)$, the eigenvalues of $p(A_1, \ldots, A_r)$ are precisely $p(\alpha_1^j, \ldots, \alpha_r^j)$ for $j = 1, \ldots, m$ (possibly with repetitions).*

Proof. (i) Suppose that $C := A_1 + \cdots + A_r$ is in Frobenius form and partition A_i, for $i = 1, \ldots, r$, correspondingly. Evidently, a Frobenius form of $B := A_i$, for $i = 1, \ldots, r$, is a refinement of the Frobenius form of C. Since $B_{\mu\mu}$ and $C_{\mu\mu}$ commute and $C_{\mu\mu}$ is irreducible, by (iii) of Lemma 4.6 and our assumption, it follows that $B_{\mu\mu}$ is also irreducible. Therefore, $B = A_i$ is also in Frobenius form. This proves (i).

(ii) Now suppose that ν covers μ in the reduced digraph associated with C. Then, for $B := A_i$ the matrices

$$\begin{pmatrix} B_{\mu\mu} & 0 \\ B_{\nu\mu} & B_{\nu\nu} \end{pmatrix} \text{ and } \begin{pmatrix} C_{\mu\mu} & 0 \\ C_{\nu\mu} & C_{\nu\nu} \end{pmatrix}$$

commute and, by assumption, $C_{\nu\mu} \neq 0$. Suppose that $B_{\nu\mu} = 0$. Examining the $(2, 1)$ block of the products of these matrices we obtain

$$B_{\nu\nu}C_{\nu\mu} = C_{\nu\mu}B_{\mu\mu}. \tag{13}$$

Since $B_{\mu\mu}$ and $B_{\nu\nu}$ are irreducible, it follows from Lemma 4.5 that the Perron roots of $B_{\mu\mu}$ and $B_{\nu\nu}$ are equal. This contradicts our assumption and hence $B_{\nu\mu} \neq 0$. But two transitive digraphs coincide if the cover relations are identical. This proves (ii).

(iii) In the first place, observe that any initial segment $\mathrm{Intl}(\nu)$ generated by a class ν in the reduced digraph associated with one of the matrices $A_1, \dots, A_r$ or $A_1 + \cdots + A_r$ is independent of the choice of the matrix because the transitive closures of their reduced digraphs coincide. For this reason, in what follows we shall denote by $\mathrm{Intl}(\nu)$ this common initial segment and we shall not specify the matrix it corresponds to.

Let μ_j be a spectral class of A_i. Since all classes of A_i have distinct Perron roots, from (v) of Corollary 4.2 it follows that every spectral class is premier spectral and that every eigenvector of A_i associated with α_i^j has support $\mathrm{Intl}(\mu_j)$. But, by Theorem 3.5, there are eigenvalues of A_k for $k \neq i$ that share an eigenvector with the eigenvalue α_i^j of A_i. Since this eigenvector has support $\mathrm{Intl}(\mu_j)$, by (ii) of Corollary 4.2 it follows that μ_j is a spectral class for all A_k.

Note that the above argument shows that any spectral class of A_i is also a spectral class of $A_1 + \cdots + A_r$. To prove the converse in max algebra, suppose that μ is a spectral class of $A_1 + \cdots + A_r$. Using the additivity of Perron roots (see Corollary 3.7), we obtain

$$\oplus_{i=1}^r \lambda((A_i)_{\nu\nu}) = \lambda((\oplus_{i=1}^r A_i)_{\nu\nu}) \leqslant \lambda((\oplus_{i=1}^r A_i)_{\mu\mu}) = \oplus_{i=1}^r \lambda((A_i)_{\mu\mu}), \tag{14}$$

for all $\nu \in \mathrm{Intl}(\mu)$. Without loss of generality, assume that $\oplus_{i=1}^r \lambda((A_i)_{\mu\mu}) = \lambda((A_1)_{\mu\mu})$. Then, from (14) it follows that $\lambda((A_1)_{\nu\nu}) \leqslant \lambda((A_1)_{\mu\mu})$ for all $\nu \in \mathrm{Intl}(\mu)$, implying that μ is a spectral class of A_1, and hence of all A_i.

(iv) By Theorem 3.5, for each common spectral class μ_j of $A_1, \dots, A_r$ there exists a common eigenvector v^j which has support $\mathrm{Intl}(\mu_j)$. Since $A_i v^j = \alpha_i^j v^j$ for $i = 1, \dots, r$, it follows that $p(A_1, \dots, A_r)v^j = p(\alpha_1^j, \dots, \alpha_r^j)v^j$ and thus $p(\alpha_1^j, \dots, \alpha_r^j)$ is an eigenvalue of $p(A_1, \dots, A_r)$. Let now λbe an eigenvalue of $p(A_1, \dots, A_r)$. As $p(A_1, \dots, A_r)$ commutes with A_i for all $i = 1, \dots, r$, by Theorem 3.5 there exists an eigenvector v of $p(A_1, \dots, A_r)$ associated with λwhich is also an eigenvector of A_i for all i. Then, by (v) of Corollary 4.2 there exists a common spectral class μ_j of $A_1, \dots, A_r$ such that the support of v is equal to $\mathrm{Intl}(\mu_j)$. Therefore, we have $A_i v = \alpha_i^j v$ for all $i = 1, \dots, r$, implying that $\lambda = p(\alpha_1^j, \dots, \alpha_r^j)$ because $\lambda v = p(A_1, \dots, A_r)v = p(\alpha_1^j, \dots, \alpha_r^j)v$. □

As it was already observed, under the assumptions of Theorem 4.8, the eigenvalues α_i^j, $i = 1, \dots, r$, of the matrices $A_1, \dots, A_r$ are associated with some common spectral class μ_j of their reduced digraphs. We next show how to compute the intersection of the corresponding eigencones. Let I be the initial segment generated by the spectral class μ_j in any of the reduced digraphs associated with the matrices A_i (recall that this initial segment is independent of the choice of the matrix because the transitive closures of their reduced digraphs coincide). We write uniquely each vector $x \in \mathbb{R}_+^n$ as $x[I] + x[I']$, where I' is the complement of I in $\{1, \dots, n\}$. Since I is an initial segment of the reduced digraphs associated with all the matrices A_i, there is a Frobenius form of all these matrices such that $I = \{s, s+1, \dots, t\}$ for some $s \in \{1, \dots, t\}$. If we denote the submatrix of A_i based on the set of classes I by $A_i[I, I]$, then as the matrices $A_1, \dots, A_r$ commute in pairs, it follows that also the matrices $A_1[I, I], \dots, A_r[I, I]$ commute in pairs. Therefore, we can apply the method described in Section 3.3 to compute the intersection of their principal eigencones. Moreover, by Corollary 4.2 we know that $x \in V(A_1, \alpha_1^j) \cap \cdots \cap V(A_r, \alpha_r^j)$ if, and only if, $x[I'] = 0$ and $x[I] \in V(A_1[I, I], \alpha_1^j) \cap \cdots \cap V(A_r[I, I], \alpha_r^j)$, where the latter is the intersection of the principal eigencones of $A_i[I, I]$, because by the definition of these matrices we have $\lambda(A_i[I, I]) = \alpha_i^j$ for all $i = 1, \dots, r$.

5. Common scaling and application of Boolean algebra

5.1. Common scaling and saturation digraphs

The whole of this section is in max algebra only. It is inspired by the works of Cuninghame-Green and Butkovič [10,4], where commuting matrices are studied in the context of two-sided systems and generalized eigenproblem. In these works, commuting irreducible matrices are *assumed* to have a common eigennode. We are going to show that it is always the case.

If $A = (a_{ij})$ and $B = (b_{ij})$ are irreducible and $AB = BA$, then they have a common positive eigenvector u, and using $U = \operatorname{diag}(u)$ they can be simultaneously scaled to $\widetilde{A} := U^{-1}AU$ and $\widetilde{B} := U^{-1}BU$. Assumed that $\lambda(A) = \lambda(B) = 1$, for $\widetilde{A} = (\tilde{a}_{ij})$ and $\widetilde{B} = (\tilde{b}_{ij})$ we obtain

$$\begin{aligned} Au = u \Rightarrow\ & \forall i \exists j\colon a_{ij}u_j = u_i \Leftrightarrow \tilde{a}_{ij} = 1, \\ & \forall i, j\colon a_{ij}u_j \leqslant u_i \Leftrightarrow \tilde{a}_{ij} \leqslant 1. \\ Bu = u \Rightarrow\ & \forall i \exists j\colon b_{ij}u_j = u_i \Leftrightarrow \tilde{b}_{ij} = 1, \\ & \forall i, j\colon b_{ij}u_j \leqslant u_i \Leftrightarrow \tilde{b}_{ij} \leqslant 1. \end{aligned} \tag{15}$$

Defining $\widetilde{A}^{[1]} = (\tilde{a}_{ij}^{[1]})$ and $\widetilde{B}^{[1]} = (\tilde{b}_{ij}^{[1]})$ by:

$$\tilde{a}_{ij}^{[1]} = \begin{cases} 1, & \tilde{a}_{ij} = 1, \\ 0, & \text{otherwise.} \end{cases} \quad \tilde{b}_{ij}^{[1]} = \begin{cases} 1, & \tilde{b}_{ij} = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

it follows that

$$\forall i \exists j\colon \tilde{a}_{ij}^{[1]} = 1, \quad \forall i \exists k\colon \tilde{b}_{ik}^{[1]} = 1. \tag{17}$$

Defining digraphs $\mathcal{G}_1 = (N, E_1)$ and $\mathcal{G}_2 = (N, E_2)$ so that $(i, j) \in E_1$, respectively, $(i, k) \in E_2$, if and only if $\tilde{a}_{ij}^{[1]} = 1$, respectively, $\tilde{b}_{ik}^{[1]} = 1$, we see that by (17) each node in these digraphs has an outgoing edge. The matrices $\widetilde{A}^{[1]}$ and $\widetilde{B}^{[1]}$ are the *adjacency matrices* of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. These digraphs are also the *saturation digraphs* of u with respect to A and B [2], meaning that $(i, j) \in E_1$ (respectively, $(i, j) \in E_2$) if, and only if, $a_{ij}u_j = u_i$ (respectively, $b_{ij}u_j = u_i$). We recall the following well-known result, providing a proof for the reader's convenience.

Proposition 5.1 (Baccelli et al. [2]). *Let $A \in \mathbb{R}_+^{n\times n}$ be irreducible and let $v \in \mathbb{R}_+^n$ be an eigenvector of A. Then, the strongly connected components of the saturation digraph of v with respect to A are the same as those of the critical digraph $\mathcal{C}(A)$.*

Proof. We need to show that any edge in a strongly connected component of a saturation digraph is critical, and the other way around, that any critical edge is present in any saturation digraph. Recalling that every edge in a strongly connected component of a digraph belongs to a cycle, it suffices to show that any *cycle* in a saturation digraph is critical and the other way around, that any critical cycle is present in any saturation digraph. Assume w.l.o.g. that $\lambda(A) = 1$. The first part: if $(i_1, \dots, i_k)$ is a cycle of a saturation digraph w.r.t. an eigenvector v, then $a_{i_1i_2}v_{i_2} = v_{i_1}, \dots, a_{i_ki_1}v_{i_1} = v_{i_k}$. Multiplying all these equalities and canceling the product $v_{i_1}, \dots, v_{i_k}$, we obtain $a_{i_1i_2} \dots a_{i_ki_1} = 1$, thus $(i_1, \dots, i_k)$ is critical. The second part: assume that $(i_1, \dots, i_k)$ is critical but it is not a cycle of the saturation digraph. Then, we have $a_{i_1i_2}v_{i_2} \leqslant v_{i_1}, \dots, a_{i_ki_1}v_{i_1} \leqslant v_{i_k}$ where one of the inequalities is strict. Multiplication and cancelation of $v_{i_1}, \dots, v_{i_k}$ now yield $a_{i_1i_2} \dots a_{i_ki_1} < 1$, a contradiction. The proof is complete. □

This proposition tells us that the strongly connected components of $\mathcal{G}_1$ and $\mathcal{G}_2$ are those of $\mathcal{C}(A)$ and $\mathcal{C}(B)$.

5.2. Commuting Boolean matrices

Now we study in more detail the case of Boolean matrices, to show that two irreducible commuting matrices in max algebra always have a common eigennode. In a similar way, the graph of commuting Boolean matrices is studied in [11, Proposition 10] to make an observation about the general case.

We need a couple of simple facts. Combined with Proposition 5.1, they will provide the connection between max algebra and the Boolean case.

Lemma 5.2. *If matrices $A, B \in \mathbb{R}_+^{n\times n}$ are such that $a_{ij} \leqslant 1$ and $b_{ij} \leqslant 1$ for all $i, j \in N$, then $(AB)^{[1]} = A^{[1]}B^{[1]}$.*

Proof. For all $i, k \in N$, we may have two cases:

$$\bigoplus_{j=1}^{n} a_{ij}b_{jk} = 1 \text{ or } \bigoplus_{j=1}^{n} a_{ij}b_{jk} < 1. \tag{18}$$

In the first case of (18), there exists h such that $a_{ih}b_{hk} = 1$, which implies $a_{ih} = b_{hk} = 1$, since $a_{ij} \leqslant 1$ and $b_{ij} \leqslant 1$ for all i and j. Passing to $A^{[1]}$ and $B^{[1]}$ we have $a_{ih}^{[1]} = b_{hk}^{[1]} = 1$ and thus $a_{ih}^{[1]}b_{hk}^{[1]} = 1$. Using this we obtain

$$\bigoplus_{j=1}^{n} a_{ij}^{[1]}b_{jk}^{[1]} = 1, \tag{19}$$

In the second case of (18), there are no such h as above, and we obtain

$$\bigoplus_{j=1}^{n} a_{ij}^{[1]}b_{jk}^{[1]} = 0. \tag{20}$$

It follows that $(AB)^{[1]} = A^{[1]}B^{[1]}$. □

We immediately deduce the following observation.

Lemma 5.3. *If the matrices $A, B \in \mathbb{R}_+^{n\times n}$ are such that $AB = BA$, and $a_{ij} \leqslant 1$, $b_{ij} \leqslant 1$ for all $i, j \in N$, then $A^{[1]}B^{[1]} = B^{[1]}A^{[1]}$.*

This motivates us to study the Boolean case in more detail.

Theorem 5.4. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two commuting digraphs (meaning that their adjacency matrices commute) with nonzero out-degree of each node, and let $\mathcal{G}_1^{\mu} = (N_1^{\mu}, E_1^{\mu})$ for $\mu = 1, \ldots, m_1$ and $\mathcal{G}_2^{\nu} = (N_2^{\nu}, E_2^{\nu})$ for $\nu = 1, \ldots, m_2$ be the nontrivial strongly connected components of $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively. Then, there exists a cycle $c_1 \in \mathcal{G}_1$ such that all nodes on this cycle belong to $\bigcup_{\nu=1}^{m_2} N_2^{\nu}$, and a cycle $c_2 \in \mathcal{G}_2$ such that all nodes on this cycle belong to $\bigcup_{\mu=1}^{m_1} N_1^{\mu}$.*

Proof. We show the first part of the claim, i.e., that there exists a cycle $c_1 \in \mathcal{G}_1$ such that all nodes on this cycle belong to $\bigcup_{\nu=1}^{m_2} N_2^{\nu}$.

Pick $\nu_1 \in \{1, \ldots, m_2\}$ and consider the subdigraph $\mathcal{G}_1[N_2^{\nu_1}]$ of $\mathcal{G}_1$ induced by the nodes in $N_2^{\nu_1}$ (informally, the part of $\mathcal{G}_1$ which penetrates the component ν_1 of $\mathcal{G}_2$). Either $\mathcal{G}_1[N_2^{\nu_1}]$ has a cycle and then there is nothing to prove, or it is acyclic. In the latter case, let $i_1 \in N_2^{\nu_1}$ be a leaf in $\mathcal{G}_1[N_2^{\nu_1}]$ (a node with no arcs back into $N_2^{\nu_1}$). Denote $M = \{j\colon (i_1, j) \in E_1\}$. As i_1 is a leaf in $\mathcal{G}_1[N_2^{\nu_1}]$, we have $M \cap N_2^{\nu_1} = \emptyset$. There is a cycle $c \in \mathcal{G}_2$, which goes through i_1 (this cycle is unrelated to c_1 which we are going to construct). Select $j \in M$ and consider the path $c \circ (i_1, j)$ (first turn around along c in $\mathcal{G}_2$ then move $i_1 \to j$ in $\mathcal{G}_1$). As the digraphs commute, there is a path $P = (i_1, k) \circ P'$, where $(i_1, k) \in E_1$

and the path $P' \in \mathcal{G}_2$ ending at j is of the same length as c. Hence, for each node $j \in M$ there exists a node $k \in M$ such that k has access to j in $\mathcal{G}_2$. This implies that some nodes in M lie on a cycle in $\mathcal{G}_2$, and hence M intersects with $N_2^{\nu_2}$ for some $\nu_2 \in \{1, \ldots, m_2\}$. Taking $j_2 \in M \cap N_2^{\nu_2}$ we obtain the edge $i_1 \to j_2$ in $\mathcal{G}_1$ such that $i_1 \in N_2^{\nu_1}$ and $j_2 \in N_2^{\nu_2}$.

Consider the digraph $\mathcal{G}_1[N_2^{\nu_2}]$. If it is not acyclic then there is nothing to prove, otherwise we proceed to a leaf i_2 accessed by j_2 in $\mathcal{G}_1[N_2^{\nu_2}]$. We have obtained the path $i_1 \to j_2 \to \cdots \to i_2$ in $\mathcal{G}_1$, whose nodes lie in $\bigcup_{\nu=1}^{m_2} N_2^{\nu}$. Arguing as above we can continue this path until we obtain a cycle c_1 in $\mathcal{G}_1$ which has all nodes in $\bigcup_{\nu=1}^{m_2} N_2^{\nu}$. This shows the first part of the claim.

The second part of the claim, i.e., the cycle c_2 in $\mathcal{G}_2$ which has all nodes in $\bigcup_{\nu=1}^{m_1} N_1^{\nu}$, is obtained analogously. $\square$

Theorem 5.4 implies notable facts about the critical digraphs of two commuting matrices in max algebra.

Theorem 5.5. *If two irreducible matrices $A, B \in \mathbb{R}_+^{n\times n}$ commute, then the claim of Theorem 5.4 holds for the strongly connected components of $\mathcal{C}(A)$ and $\mathcal{C}(B)$. In particular, A and B have a common eigennode.*

Proof. If $A, B \in \mathbb{R}_+^{n\times n}$ commute, then they have a common eigenvector u by Corollary 3.3. If the matrices are irreducible, then u is positive, and $U := \operatorname{diag}(u)$ can be used to make a simultaneous diagonal similarity scaling: $\widetilde{A} := U^{-1}AU$ and $\widetilde{B} = U^{-1}BU$. Evidently $\widetilde{A}\widetilde{B} = \widetilde{B}\widetilde{A}$. Also we have $\mathcal{C}(\widetilde{A}) = \mathcal{C}(A)$ and $\mathcal{C}(\widetilde{B}) = \mathcal{C}(B)$. Notice that $\widetilde{A}^{[1]}$, respectively, $\widetilde{B}^{[1]}$, is the adjacency matrix of the saturation digraph of u with respect to A, respectively, to B. These saturation digraphs will be denoted by $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively (with the intention to use Theorem 5.4). By Lemma 5.3, we have $\widetilde{A}^{[1]}\widetilde{B}^{[1]} = \widetilde{B}^{[1]}\widetilde{A}^{[1]}$. As $\mathcal{G}_1$ and $\mathcal{G}_2$ are saturation digraphs, each node in these digraphs has an outgoing edge. Applying Theorem 5.4 we obtain that the claim of Theorem 5.4 holds for the strongly connected components of $\mathcal{G}_1$ and $\mathcal{G}_2$. By Proposition 5.1, these components are precisely the strongly connected components of $\mathcal{C}(A)$ and $\mathcal{C}(B)$. Now, Theorem 5.4 also implies that A and B have a common eigennode. $\square$

Let us consider a special case, which presumably appears if the commuting A and B are taken at random.

Corollary 5.6. *Let two irreducible matrices $A, B \in \mathbb{R}_+^{n\times n}$ commute. If $\mathcal{C}(A) = (N_c^A, E_c^A)$ and $\mathcal{C}(B) = (N_c^B, E_c^B)$ both consist of just one cycle, then $N_c^A = N_c^B$.*

6. Examples of commuting matrices in max algebra

In this section, we give several examples in max algebra, which will appear now as the semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$, i.e. the set $\mathbb{R} \cup \{-\infty\}$ equipped with max as "addition" and the usual sum as "multiplication". This semiring is isomorphic to $(\mathbb{R}_+, \max, \times)$ via the logarithmic transform.

Consider the irreducible commuting matrices

$$A_1 = \begin{pmatrix} -2 & 1 & -\infty \\ -1 & -1 & -2 \\ -1 & -\infty & -2 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & -1 & -1 \\ -\infty & 0 & -4 \\ -3 & -\infty & 0 \end{pmatrix}.$$

Then, it is straightforward to check that $\lambda(A_1) = \lambda(A_2) = 0$, $N_c^{A_1} = \{1, 2\}$ and $N_c^{A_2} = \{1, 2, 3\}$. Therefore, as claimed in Theorem 5.5, A_1 and A_2 have a common eigennode.

In order to compute their common eigenvectors, we apply the method described in Section 3.3. Since

$$Q(A_1) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ -1 & 0 & -2 \end{pmatrix} \quad \text{and} \quad Q(A_2) = \begin{pmatrix} 0 & -1 & -1 \\ -7 & 0 & -4 \\ -3 & -4 & 0 \end{pmatrix},$$

it follows that

$$Q(A_1)Q(A_2) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -2 \\ -1 & 0 & -2 \end{pmatrix}.$$

Then, by (12) we have

$$V(A_1, 0) \cap V(A_2, 0) = V(Q(A_1)Q(A_2), 0) = \left\{\lambda(1, 0, 0)^T | \lambda \in \mathbb{R} \cup \{-\infty\}\right\}.$$

The following example of commuting matrices illustrates Lemma 4.6. Let

$$A = \begin{pmatrix} 1 & -\infty & -\infty \\ 1 & 0 & -\infty \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, A and B commute, B is irreducible and A satisfies the conditions of Lemma 4.6.

As an example of reducible commuting matrices, consider

$$A_1 = \begin{pmatrix} 0 & -\infty & -\infty & -\infty \\ 1 & 3 & -\infty & -\infty \\ 2 & -\infty & -1 & -\infty \\ -\infty & -\infty & 0 & 2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 6 & -\infty & -\infty & -\infty \\ 5 & 7 & -\infty & -\infty \\ 8 & -\infty & 5 & -\infty \\ 5 & -\infty & 6 & 8 \end{pmatrix}.$$

The classes of these matrices are their diagonal elements. Since the Perron roots of the classes (i.e., the diagonal entries in this case) of each of these matrices are distinct, we know by Theorem 4.8 that the transitive closure of the reduced digraph associated with these matrices are the same, even if these digraphs are different, as can be easily checked. By the same theorem, we know that the spectral classes of the associated reduced digraph coincide. In this case, for both matrices the spectral classes are 2 and 4. Each of these matrices has two different eigenvalues corresponding to their spectral classes. The eigenvalues of A_1 are 3 and 2 and the ones of A_2 are 7 and 8.

7. Classical nonnegative matrices

In this section, we assume knowledge of some basic results on nonnegative matrices found in, e.g., [3] or [20]. Most of the results and arguments of Section 3 (except for the last subsection) and Section 4 were meant to be true also for nonnegative matrices in classical matrix algebra. In this section, we explicitly state the most important of such results and show where the classical nonnegative theory is different.

Let $A \in \mathbb{R}_+^{n\times n}$. Following standard terminology, we call an eigenvalue λ of A a *distinguished eigenvalue* of A if $\lambda \geqslant 0$ and there is a nonnegative eigenvector corresponding to it. In this section, $\Lambda(A)$ will be the set of distinguished eigenvalues of A and $V(A, \lambda)$ the convex cone of nonnegative eigenvectors (and the 0 vector) associated with a distinguished eigenvalue λ. By the Perron–Frobenius theorem, $\Lambda(A)$ is nonempty and the largest element in $\Lambda(A)$ is called the *Perron root* of A. Moreover, any eigencone $V(A, \lambda)$ is finitely generated, and the intersection of finitely generated convex cones is again finitely generated. Matrices leaving a cone invariant in $\mathbb{R}_+^n$ (indeed in $\mathbb{R}^n$) have been much studied, see, e.g., [36]. Proposition 3.4 is well known in this context.

Lemma 3.1, Theorems 3.2, 3.5, Corollary 3.3 and their proofs go through without further change to the classical nonnegative case, except that we need to insert the adjective "nonnegative" in Corollary 3.3.

Corollary 3.3A. *If $A, B \in \mathbb{R}_+^{n\times n}$ commute, then they have a common nonnegative eigenvector.*

It follows that if A and B are commuting nonnegative matrices and one of them is irreducible, then they have a common Perron vector.

Theorem 3.6 and Corollary 3.7 are also valid in the classical nonnegative case under the following assumptions: the matrices $A_1, \ldots, A_r \in \mathbb{R}_+^{n\times n}$ commute in pairs and $p(x_1, \ldots, x_r)$ is a real polynomial such that $p(A_1, \ldots, A_r)$ is nonnegative and, in the case of Corollary 3.7, all the coefficients of

$p(x_1, \dots, x_r)$ are nonnegative. In the latter, by the analog of Remark 4.7, we need to assume only one of the A_i is irreducible.

Turning to Section 4, we again construct the reduced digraph of $A \in \mathbb{R}_+^{n\times n}$ and we now label each class μ with its classical Perron root α_μ. By a theorem of Frobenius [19], we replace Theorem 4.1 by:

Theorem 4.1A. *Let $A \in \mathbb{R}_+^{n\times n}$ and $\lambda \in \mathbb{R}_+$. Then, a subset U of N is the support of a nonnegative eigenvector associated with λ if, and only if,*

(i) *There is an initial segment I such that $U = \cup_{\nu\in I} N^\nu$.*
(ii) *All final classes ν in I are premier spectral and satisfy $\alpha_\nu = \lambda$.*

See, e.g., [34]. We observe that the supports of nonnegative eigenvectors of $A \in \mathbb{R}_+^{n\times n}$ are completely determined in Theorem 4.1A by (i) the classes (i.e., the strongly connected components) of $\mathcal{G}$, (ii) the Perron roots of these classes and (iii) the access relations of $\mathcal{R}$ (equivalently the arcs of $\mathcal{R}^*$). A similar remark holds for Theorem 4.1 and other results in Sections 3 and 4.

We restate Corollary 4.2 as:

Corollary 4.2A. *Let $A \in \mathbb{R}_+^{n\times n}$. Then,*

(i) *λ is a distinguished eigenvalue if, and only if, there is a premier spectral class ν such that $\alpha_\nu = \lambda$.*
(ii) *ν is a premier spectral class if, and only if, there exists a nonnegative eigenvector with support* $\mathrm{Intl}(\nu)$.
(iii) *If ν is a premier spectral class, then any nonnegative eigenvector associated with α_ν whose support is contained in* $\mathrm{Intl}(\nu)$ *has its support equal to* $\mathrm{Intl}(\nu)$.
(iv) *If the reduced digraph of A has a unique premier spectral class ν with Perron root α_ν, then any nonnegative eigenvector associated with α_ν has support* $\mathrm{Intl}(\nu)$.
(v) *If the Perron roots of all classes are distinct, then all nonnegative eigenvectors have support* $\mathrm{Intl}(\nu)$ *for some premier spectral class ν.*

The analog of Corollary 4.3 in nonnegative linear algebra is well known, but we need to replace (ii) of Corollary 4.3 by: "The Perron root of any final class is $\lambda(A)$ and all final classes are premier spectral". Lemma 4.4 goes through without change except that we need to replace "eigenvalue" with "distinguished eigenvalue" and Lemma 4.5 also holds in nonnegative linear algebra.

In the classical nonnegative case we obtain the following known stronger form of Lemma 4.6, which may be found on [3, p. 53]. We give a short proof along the lines of the proof of Lemma 4.6.

Lemma 4.6A. *If $A, B \in \mathbb{R}_+^{n\times n}$ commute and B is irreducible, then the Perron root of A is its unique distinguished eigenvalue. Moreover, if A is reducible, it is completely reducible (viz, the direct sum of irreducible matrices after a permutation similarity).*

Proof. We repeat the proof of (i) of Lemma 4.6 to show that both A and A^T have positive eigenvectors. This implies that all initial and final classes in the reduced digraph of A are premier spectral with Perron root $\lambda(A)$. But a premier spectral class cannot have access to another premier spectral class with the same Perron root. It follows that all initial classes are final and vice versa. This means that a class has access only to itself, which proves the lemma. □

Theorem 4.8 also holds in nonnegative algebra, with exception of the last part of (iii) whose proof is specific to max algebra. Thus we obtain the following main theorem of this section.

Theorem 4.8A. *Suppose that $A_1, \dots, A_r \in \mathbb{R}_+^{n\times n}$ pairwise commute and that all classes of A_i, for each $i \in \{1, \dots, r\}$, have distinct Perron roots. Then,*

(i) *All classes of $A_1, \dots, A_r$ and $A_1 + \cdots + A_r$ coincide.*

(ii) *The transitive closures of the reduced digraphs of* $A_1, \ldots, A_r$ *and* $A_1 + \cdots + A_r$ *coincide*.
(iii) *The reduced digraphs of* $A_1, \ldots, A_r$ *have the same premier spectral classes, which are premier spectral classes of* $A_1 + \cdots + A_r$. *In particular,* $A_1, \ldots, A_r$ *have the same number of distinct distinguished eigenvalues*.
(iv) *Let* $\mu_1, \ldots, \mu_m$ *be the common premier spectral classes of* $A_1, \ldots, A_r$ *and denote the Perron root of the* μ_j*-th class of* A_i *by* α_i^j. *Then, for any real polynomial* $p(x_1, \ldots, x_r)$ *such that* $p(A_1, \ldots, A_r)$ *is nonnegative, the distinguished eigenvalues of* $p(A_1, \ldots, A_r)$ *are precisely* $p(\alpha_1^j, \ldots, \alpha_r^j)$ *for* $j = 1, \ldots, m$ (*possibly with repetitions*).

We end this section with an example to illustrate Theorem 4.8A.

Example 7.1. *Let*

$$A = \begin{pmatrix} 10 & 0 & 0 \\ 5 & 0 & 0 \\ 2 & 3 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Then, $AB = BA$. *The classes of A and B are their diagonal elements, and the skeleton of their reduced digraphs (meaning the diagram of cover relations) is*

$$1 \leftarrow 2 \leftarrow 3 \ .$$

The premier spectral classes of both matrices are 1 *and* 3 *and the distinguished eigenvalues are the corresponding entries. Their common (nonnegative) eigenvectors are* $(2, 1, 1)^T$ *and* $(0, 0, 1)^T$, *respectively.*

Of course, A^T *and* B^T *also commute. Note that the skeleton of their reduced digraphs is obtained by reversing the arrows in the diagram above. The only spectral class of* A^T *or* B^T *is 1 and their common eigenvector is* $(1, 0, 0)^T$.

To illustrate (iv) of Theorem 4.8A, consider polynomial $p(x, y) = x^2y - xy$. *We have*

$$p(A, B) = A^2B - AB = \begin{pmatrix} 270 & 0 & 0 \\ 135 & 0 & 0 \\ 123 & 12 & 12 \end{pmatrix}. \tag{21}$$

Thus $p(A, B)$ *is nonnegative and, as predicted by (iv) of Theorem 4.8A, the eigenvalues are* $p(10, 3) = 270$ *and* $p(3, 2) = 12$.

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