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Review

The Hilbert–Schmidt Lagrangian Grassmannian in infinite dimension

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ABSTRACT

In this paper we study the action of the symplectic operators which are a perturbation of the identity by a Hilbert–Schmidt operator in the Lagrangian Grassmannian manifold. We prove several geometric properties using the quotient norm in the tangent spaces. We determinate the geodesic curves of this structure and we study the completeness of the corresponding geodesic distance.

Contents

1. Introduction

In finite dimension, the Lagrangian Grassmannian $\Lambda(n)$ of the Hilbert space $\mathcal{H} = \mathbb{R}^n \times \mathbb{R}^n$ with the canonical complex structure $J(x, y) = (-y, x)$ was introduced by V.I. Arnold in 1967 [1]. These notions have been generalized to infinite dimensional Hilbert spaces (see [2]) and have found several applications to Algebraic Topology, Differential Geometry and Physics.

In classical finite dimensional Riemannian theory it is well-known the fact that given two points there is a minimal geodesic curve that joins them and this is equivalent to the completeness of the metric space with the geodesic distance; this is the Hopf–Rinow theorem. In the infinite dimensional case this is no longer true. In [3] and [4], McAlpin and Atkin showed in two examples how this theorem can fail.

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In [5] E. Andruchow and G. Larotonda introduced a linear connection in the Lagrangian Grassmannian and focused on the geodesic structure of this manifold. There they proved that any two Lagrangian subspaces can be joined by a minimal geodesic.

In this paper we study a restricted version of the Lagrangian Grassmannian given by the action of the Hilbert–Schmidt symplectic group. We will focus on the geometric study and we will discuss which metric can be defined in each tangent space and which geometric properties it verifies. In particular we will find the geodesic curves of this structure and we will describe it in terms of exponentials of operators, moreover we will study the completeness of the geodesic distance. If we identify the Hilbert–Schmidt Lagrangian Grassmannian with a quotient of the Hilbert–Schmidt symplectic group it is natural to consider the quotient norm in the tangent spaces. The principal goal of this paper is to prove that the Hilbert–Schmidt Lagrangian Grassmannian is a complete metric space with the geodesic distance when we use this metric.

The case of the Fredholm Lagrangian Grassmannian of an infinite dimensional symplectic Hilbert space $\mathcal{H}$, modeled on the space of compact operators, was studied by J.C.C. Eidam and P. Piccione in [6]. See also the paper by A. Abbondandolo and P. Majer [7] for the general theory of infinite dimensional Grassmannians, and the book by G. Segal and A. Pressley for further references on the subject [8].

2. Background and definitions

In this paper we will focus in another Grassmannian, descripted using charts modeled on the algebra of Hilbert–Schmidt operators. We will consider the transitive action of the reduced (Hilbert–Schmidt) symplectic group on our Grassmannian. We will show that there exist smooth cross sections for this action. In [7] the authors built local cross sections using the polar decomposition of the sum of two symmetries. Here, we will use the polar decomposition of another auxiliary element. Since both formulas coincide the local sections are the same, but our formulation allows for an easier verification of the fact that the local sections fall into the unitary group $U_2(\mathcal{H}_J) = \{u \in U(\mathcal{H}) : uJ = Ju$ and $u - 1$ is a Hilbert–Schmidt operator$\}$. In this setting, it will be possible to study Riemannian metrics related to the infinite trace of $\mathcal{H}$.

To begin with, we will follow the notation and definitions of [9], so first we recall some of those. Let $\mathcal{H}$ be an infinite dimensional real Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the space of bounded operators. Denote by $\mathcal{B}_2(\mathcal{H})$ the Hilbert–Schmidt class

$$\mathcal{B}_2(\mathcal{H}) = \left\{a \in \mathcal{B}(\mathcal{H}) : Tr(a^*a) < \infty\right\}$$

where Tr is the usual trace in $\mathcal{B}(\mathcal{H})$. This space is a Hilbert space with the inner product

$$\langle a, b\rangle = Tr(b^*a).$$

The norm induced by this inner product is called the 2-norm and denoted by

$$\|a\|_2 = Tr(a^*a)^{1/2},$$

the usual operator norm will be denoted by $\| \;\|$.

If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is any subset of operators we use the subscript s (resp as) to denote the subset of symmetric (resp. anti-symmetric) operators of it, i.e. $\mathcal{A}_s = \{x \in \mathcal{A} : x^* = x\}$ and $\mathcal{A}_{as} = \{x \in \mathcal{A} : x^* = -x\}$.

We fix a complex structure; that is a linear isometry $J \in \mathcal{B}(\mathcal{H})$ such that,

$$J^2 = -1 \quad \text{and} \quad J^* = -J.$$

The symplectic form w is given by $w(\xi, \eta) = \langle J\xi, \eta\rangle$. We denote by $GL(\mathcal{H})$ the group of invertible operators and by $\mathrm{Sp}(\mathcal{H})$ the subgroup of invertible operators which preserve the symplectic form, that is $g \in \mathrm{Sp}(\mathcal{H})$ if $w(g\xi, g\eta) = w(\xi, \eta)$. Algebraically

$$\mathrm{Sp}(\mathcal{H}) = \left\{g \in GL(\mathcal{H}) : g^*Jg = J\right\}.$$

This group is a Banach–Lie group (see [9]) and its Banach–Lie algebra is given by

$$\mathfrak{sp}(\mathcal{H}) = \left\{x \in \mathcal{B}(\mathcal{H}) : xJ = -Jx^*\right\}.$$

Denote by $\mathcal{H}_J$ the Hilbert space $\mathcal{H}$ with the action of the complex field $\mathbb{C}$ given by J, that is; if $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}$ and $\xi \in \mathcal{H}$ we can define the action as $\lambda\xi := \lambda_1\xi + \lambda_2 J\xi$ and the complex inner product as $\langle \xi, \eta\rangle_{\mathbb{C}} = \langle \xi, \eta\rangle - iw(\xi, \eta)$.

Let $\mathcal{B}(\mathcal{H}_J)$ be the space of bounded complex linear operators in $\mathcal{H}_J$. A straightforward computation shows that $\mathcal{B}(\mathcal{H}_J)$ consists of the elements of $\mathcal{B}(\mathcal{H})$ which commute with J.

Following the notation of [9], we consider the Hilbert–Schmidt subgroup of $\mathrm{Sp}(\mathcal{H})$

$$\mathrm{Sp}_2(\mathcal{H}) = \{g \in \mathrm{Sp}(\mathcal{H}) : g - 1 \in \mathcal{B}_2(\mathcal{H})\}.$$

There it was proved that this group has a differentiable structure modeled on $\mathcal{B}_2(\mathcal{H})$. Some of the this facts have been well-known for general Schatten ideals, more precisely the Banach–Lie group structure was noted in the book [10]. The Lie algebra of $\mathrm{Sp}_2(\mathcal{H})$ is

$$\mathfrak{sp}_2(\mathcal{H}) = \left\{x \in \mathcal{B}_2(\mathcal{H}) : xJ = -Jx^*\right\}.$$

The Lagrangian Grassmannian $\Lambda(\mathcal{H})$ is the set of closed linear subspaces $L \subset \mathcal{H}$ such that $J(L) = L^{\perp}$. Clearly $\mathrm{Sp}(\mathcal{H})$ acts on $\Lambda(\mathcal{H})$ by means of $g.L = g(L)$. Since the action of the unitary group $U(\mathcal{H}_J)$ is transitive on $\Lambda(\mathcal{H})$ (see [11] Theorem 3.5), it is clear that the action of $\mathrm{Sp}(\mathcal{H})$ is also transitive on $\Lambda(\mathcal{H})$, so we can think of $\Lambda(\mathcal{H})$ as an orbit for a fixed $L_0 \in \Lambda(\mathcal{H})$, i.e.

$$\Lambda(\mathcal{H}) = \{g(L_0) : g \in \mathrm{Sp}(\mathcal{H})\}.$$

We denote by $P_L \in \mathcal{B}(\mathcal{H})$ the orthogonal projection onto L. It is customary to parametrize closed subspaces via orthogonal projections, $L \leftrightarrow P_L$, in order to carry on geometric or analytic computations. We shall also consider here an alternative description of the Lagrangian subspaces using projections and symmetries. That is, L is a Lagrangian subspace if and only if $P_L J + J P_L = J$, see [2] for a proof. Another description of this equation using symmetries is $\epsilon_L J = -J\epsilon_L$, where $\epsilon_L = 2P_L - 1$ is the symmetric orthogonal transformation which acts as the identity in L and minus the identity in $L^{\perp}$.

The isotropy subgroup at L is

$$\mathrm{Sp}(\mathcal{H})_L = \{g \in \mathrm{Sp}(\mathcal{H}) : g(L) = L\}.$$

It is obvious that this subgroup is a closed subgroup of $\mathrm{Sp}(\mathcal{H})$. In the infinite dimensional setting, this does not guarantee a nice submanifold structure; in Proposition 3.8 we will prove that $\mathrm{Sp}(\mathcal{H})_L$ is a Banach–Lie subgroup of $\mathrm{Sp}(\mathcal{H})$. We can restrict the natural action of the symplectic group in $\Lambda(\mathcal{H})$ to the Hilbert–Schmidt symplectic group and it will also be smooth. As before, we can consider the isotropy group at L

$$\mathrm{Sp}_2(\mathcal{H})_L = \{g \in \mathrm{Sp}_2(\mathcal{H}) : g(L) = L\}.$$

We will also prove in Proposition 3.8 that this subgroup is a Banach–Lie subgroup of $\mathrm{Sp}_2(\mathcal{H})$, with the topology induced by the metric $\|g_1 - g_2\|_2$.

If T is any operator we denote by Gr_T its graph, i.e. the subset $Gr_T = \{v + Tv : v \in \mathrm{Dom}(T)\} \subset \mathcal{H} \oplus \mathcal{H}$. Fix a Lagrangian subspace $L_0 \subset \mathcal{H}$, we consider the subset of $\Lambda(\mathcal{H})$

$$\mathcal{O}_{L_0} = \{g(L_0) : g \in \mathrm{Sp}_2(\mathcal{H})\} \subseteq \Lambda(\mathcal{H}).$$

We will see that this set is strictly contained in $\Lambda(\mathcal{H})$ and thus the action of $\mathrm{Sp}_2(\mathcal{H})$ on the Lagrangian Grassmannian is not transitive. Perhaps a more natural approach would be to consider the set of pairs (L_1, L_2) of Lagrangians such that $L_2 = g(L_1)$ for some $g \in \mathrm{Sp}_2(\mathcal{H})$. However the orbit approach makes the presentation of the metrics simple. The purpose of this paper is the geometric study of this orbit; its manifold structure and relevant metrics.

3. Manifold structure of $\mathcal{O}_{L_0}$

We start proving that the subset $\mathcal{O}_{L_0}$ is strictly contained in $\Lambda(\mathcal{H})$, to do it we need the following lemma.

Lemma 3.1. *Let $g \in \mathrm{Sp}_2(\mathcal{H})$ then $P_{g(L_0)} - P_{L_0} \in \mathcal{B}_2(\mathcal{H})$.*

Proof. To prove it, we use the formula of the orthogonal projector over the range of an operator Q given by

$$P_{R(Q)} = QQ^*(1 - (Q - Q^*)^2)^{1/2}. \tag{3.1}$$

This formula can be obtained using a block matrix representation. If we denote by Q the idempotent associated with $g(L_0)$, i.e. $Q := gP_{L_0}g^{-1}$ and if we suppose that $g = 1 + k$ and $g^{-1} = 1 + k'$ where $k, k' \in \mathcal{B}_2(\mathcal{H})$ we have

$$\begin{aligned} QQ^* &= (1+k)P_{L_0}(1+k')(1+k'^*)P_{L_0}(1+k^*) \\ &= \underbrace{(P_{L_0} + P_{L_0}k' + kP_{L_0} + kP_{L_0}k')}_{Q}\underbrace{(P_{L_0} + P_{L_0}k^* + k'^*P_{L_0} + k'^*P_{L_0}k^*)}_{Q^*} \\ &= P_{L_0} + \underbrace{P_{L_0}k^* + P_{L_0}k'^*P_{L_0} + \cdots}_{\in\, \mathcal{B}_2(\mathcal{H})} = P_{L_0} + T \ \in P_{L_0} + \mathcal{B}_2(\mathcal{H}). \end{aligned}$$

It is clear that $Q - Q^* \in \mathcal{B}_2(\mathcal{H})$, then $(Q - Q^*)^2 \in \mathcal{B}_1(\mathcal{H})$. From the spectral theorem we have,

$$1 - (Q - Q^*)^2 = 1 + \sum_i \lambda_i P_i = P_0 + \sum_i (\lambda_i + 1)P_i$$

where $(\lambda_i) \in \ell^1$ and P_0 is the projection to the kernel. Taking square roots, we have

$$\begin{aligned} (1 - (Q - Q^*)^2)^{1/2} &= P_0 + \sum_i (\lambda_i + 1)^{1/2} P_i \\ &= P_0 + \sum_i [(\lambda_i + 1)^{1/2} - 1]P_i + \sum_i 1P_i \\ &= 1 + \sum_i [(\lambda_i + 1)^{1/2} - 1]P_i = 1 + T' \ \in \ 1 + \mathcal{B}_2(\mathcal{H}) \end{aligned}$$

where $((\lambda_i+1)^{1/2}-1)\in\ell^2$, because $(\lambda_i)\in\ell^1$ and $\lim_{x\to 0}\frac{((x+1)^{1/2}-1)^2}{x}=0$. Then by the formula (3.1) we have

$$P_{g(L_0)}=(P_{L_0}+T)(1+T')\in P_{L_0}+\mathcal{B}_2(\mathcal{H}).\quad\square$$

Corollary 3.2. *The inclusion $\mathcal{O}_{L_0}\subset\Lambda(\mathcal{H})$ is strict.*

Proof. Suppose that $\Lambda(\mathcal{H})=\mathcal{O}_{L_0}$, since $L_0^\perp$ is Lagrangian, there exists $g\in\mathrm{Sp}_2(\mathcal{H})$ such that $L_0^\perp=g(L_0)$, then using its orthogonal projector and the above lemma we have,

$$1-P_{L_0}=P_{L_0^\perp}=P_{g(L_0)}=P_{L_0}+T$$

for some $T\in\mathcal{B}_2(\mathcal{H})$. Therefore, $2P_{L_0}-1=-T\in\mathcal{B}_2(\mathcal{H})$ and this is a contradiction because $2P_{L_0}-1$ is an unitary operator. $\square$

To build a manifold structure over $\mathcal{O}_{L_0}$, we will consider the charts of $\Lambda(\mathcal{H})$ given by the parametrization of Lagrangian subspaces as graphs of functions and we will adapt this charts to our set. This charts were used in [12] to describe the manifold structure of $\Lambda(\mathcal{H})$; in the followings steps we recall this charts and we fix the notation.

Given $L\in\Lambda(\mathcal{H})$, we have the Lagrangian decomposition $\mathcal{H}=L\oplus L^\perp$ and we denote by

$$\Omega(L^\perp)=\{W\in\Lambda(\mathcal{H}):\mathcal{H}=W\oplus L^\perp\}.$$

In [2] it was proved that these sets are open in $\Lambda(\mathcal{H})$. We consider the map $\phi_L:\Omega(L^\perp)\to\mathcal{B}(L)_s$ given by

$$W=Gr_T\longmapsto J|_{L^\perp}T$$

where $T:L\to L^\perp$ is the linear operator whose graph is W, more precisely

$$T=\pi_1|_W\circ(\pi_0|_W)^{-1}$$

where π_0,π_1 are the orthogonal projections to L and $L^\perp$.

Remark 3.3. The map ϕ_L is onto: Let $\psi\in\mathcal{B}(L)_s$, we consider the operator $T:=-J|_L\psi$ (T maps L into $L^\perp$) and $W:=Gr_T$. Since ψ is a symmetric operator, W is a Lagrangian subspace and $\mathcal{H}=Gr_T\oplus L^\perp$; for this $W\in\Omega(L^\perp)$ and it is a preimage of ψ.

The maps $\{\phi_L\}_{L\in\Lambda(\mathcal{H})}$ constitute a smooth atlas for $\Lambda(\mathcal{H})$, so that $\Lambda(\mathcal{H})$ becomes a smooth Banach manifold (see [13]). For every $W\in\Lambda(\mathcal{H})$ we can identify the tangent space $T_W\Lambda(\mathcal{H})$ with the Banach space $\mathcal{B}(W)_s$, this identification was used in [12] and [13]. For $W\in\Omega(L^\perp)$, the differential $d\phi_L$ of the chart at W is given by

$$d_W\phi_L(H)=\eta^*H\eta \tag{3.2}$$

for all $H\in\mathcal{B}(W)_s$, where $\eta:L\to W$ is the isomorphism given by the restriction to L of the projection $W\oplus L^\perp\to W$. It is easy to see that the inverse $d_\psi\phi_L^{-1}$ of this map at a point $\psi=\phi_L(W)$ is given by

$$\mathcal{B}(L)_s\xrightarrow{d_\psi\phi_L^{-1}}\mathcal{B}(W)_s$$
$$H\longmapsto(\eta^{-1})^*H\eta^{-1}.$$

Since the symplectic group acts smoothly we can consider for fixed $L\in\Lambda(\mathcal{H})$ the smooth map $\pi_L:\mathrm{Sp}(\mathcal{H})\to\Lambda(\mathcal{H})$ given by $g\mapsto g(L)$. Its differential map at a point $g\in\mathrm{Sp}(\mathcal{H})$ is given by

$$T_g\mathrm{Sp}(\mathcal{H})=\mathfrak{sp}(\mathcal{H})g\ni Xg\mapsto P_{g(L)}JX|_{g(L)}\in\mathcal{B}(g(L))_s,$$

see [12] and [13] for a proof. Throughout this paper, we will denote by $d_1\pi_L$ the differential at the identity. If $L\in\mathcal{O}_{L_0}$ we can restrict the map π_L to the subgroup $\mathrm{Sp}_2(\mathcal{H})$ obtaining a surjective map onto $\mathcal{O}_{L_0}$,

$$\pi_L|_{\mathrm{Sp}_2(\mathcal{H})}:\mathrm{Sp}_2(\mathcal{H})\to\mathcal{O}_{L_0}.$$

Theorem 3.4. *The set $\mathcal{O}_{L_0}$ is a submanifold of $\Lambda(\mathcal{H})$ and the natural map $i:\mathcal{O}_{L_0}\hookrightarrow\Lambda(\mathcal{H})$ is an embedding.*

Proof. We will adapt the above local chart ϕ_L to our set. Let $L=g(L_0)\in\mathcal{O}_{L_0}$, first we see that $\phi_L(\Omega(L^\perp)\cap\mathcal{O}_{L_0})\subset\mathcal{B}_2(L)_s$. Indeed, if W belongs to $\Omega(L^\perp)\cap\mathcal{O}_{L_0}$ then we can write $W=Gr_T=h(L_0)$ for some $h\in\mathrm{Sp}_2(\mathcal{H})$ and since $L_0=g^{-1}(L)$ we have that $W=hg^{-1}(L)$ and it is obvious that we can write now $W=\tilde g(L)$ with $\tilde g\in\mathrm{Sp}_2(\mathcal{H})$. If we write $\tilde g=1+k$ where $k\in\mathcal{B}_2(\mathcal{H})$ then the orthogonal projection π_1 restricted to W can be written as

$$\pi_1|_W(w)=\pi_1(\tilde g l)=\pi_1(l+kl)=\pi_1(k(l))=\pi_1(k(\tilde g^{-1}w))$$

where $W\ni w=\tilde g(l)$ and $l\in L$. Thus we have

$$\pi_1|_W=\pi_1\circ k\circ\tilde g^{-1}|_W\in\mathcal{B}_2(W,L^\perp).$$

Then it is clear that $\phi_L(W) = J|_{L^\perp}T \in \mathcal{B}_2(L)_s$. Now we have the restricted chart

$$\phi_L|_{\Omega(L^\perp)\cap\mathcal{O}_{L_0}} : \Omega(L^\perp) \cap \mathcal{O}_{L_0} \longrightarrow \mathcal{B}_2(L)_s.$$

To conclude we will see that this restricted map is also onto. Let $\psi \in \mathcal{B}_2(L)_s$ and as we did in Remark 3.3 we consider the operator $T := -J|_L\psi$, then the only fact to prove is that

$$Gr_T = \{v + (-J|_L\psi)v : v \in L\} \in \mathcal{O}_{L_0}.$$

To prove it we define $f := 1 - J|_L\psi P_L \in 1 + \mathcal{B}_2(\mathcal{H})$; it is invertible with inverse given by $1 + J|_L\psi P_L$ and it is clear that $Gr_T = f(L)$. Now we have to show that f is symplectic. Indeed, let $\xi, \eta \in \mathcal{H}$

$$w((1 - J|_L\psi P_L)\xi, (1 - J|_L\psi P_L)\eta) = w(\xi, \eta) + w(\xi, -J|_L\psi P_L\eta) + w(-J|_L\psi P_L\xi, \eta) + \underbrace{w(J|_L\psi P_L\xi, J|_L\psi P_L\eta)}_{=0}$$

and since J is an isometry we have

$$\begin{aligned} w(\xi, -J|_L\psi P_L\eta) + w(-J|_L\psi P_L\xi, \eta) &= \langle J\xi, -J|_L\psi P_L\eta\rangle + \langle J(-J|_L\psi P_L)\xi, \eta\rangle \\ &= -\langle \xi, \psi P_L\eta\rangle + \langle \psi P_L\xi, \eta\rangle. \end{aligned}$$

If $\xi = \xi_0 + \xi_0^\perp$ and $\eta = \eta_0 + \eta_0^\perp$ are the respective decompositions in $L \oplus L^\perp$, then by the symmetry of ψ the above equality results in

$$\begin{aligned} -\langle \xi, \psi P_L\eta\rangle + \langle \psi P_L\xi, \eta\rangle &= \langle \xi_0 + \xi_0^\perp, \psi\eta_0\rangle + \langle \psi\xi_0, \eta_0 + \eta_0^\perp\rangle \\ &= -\langle \xi_0, \psi\eta_0\rangle + \langle \psi\xi_0, \eta_0\rangle = 0. \end{aligned}$$

Then

$$w((1 - J|_L\psi P_L)\xi, (1 - J|_L\psi P_L)\eta) = w(\xi, \eta)$$

and $f \in \mathrm{Sp}_2(\mathcal{H})$. Since $L = g(L_0)$ we have

$$Gr_T = f(L) = fg(L_0) \in \mathcal{O}_{L_0}. \quad \square$$

As in the case of the full Lagrangian Grassmannian, for every $L \in \mathcal{O}_{L_0}$ we can identify the tangent space $T_L\mathcal{O}_{L_0}$ with the Hilbert space $\mathcal{B}_2(L)_s$.

Since the differential of the inclusion map is an inclusion map, it is clear that the differential of the adapted charts is the restriction of the differential of full charts given by Eq. (3.2). So, if $W \in \Omega(L^\perp) \cap \mathcal{O}_{L_0}$ then the differential of the adapted chart is given by $d_W\phi_L|_{\Omega(L^\perp)\cap\mathcal{O}_{L_0}}(H) = \eta^*H\eta$ where $H \in \mathcal{B}_2(W)_s$ and its inverse is

$$\begin{aligned} &\mathcal{B}_2(L)_s \overset{d_\psi\phi_L^{-1}|_{\Omega(L^\perp)\cap\mathcal{O}_{L_0}}}{\longrightarrow} \mathcal{B}_2(W)_s = T_W\mathcal{O}_{L_0} \\ &H \longmapsto (\eta^{-1})^*H\eta^{-1}. \end{aligned} \tag{3.3}$$

Remark 3.5. The differential of the map $\pi_L|_{\mathrm{Sp}_2(\mathcal{H})}$ at a point $g \in \mathrm{Sp}_2(\mathcal{H})$ is the restriction of the differential map $d_g\pi_L$ at $T_g\mathrm{Sp}_2(\mathcal{H})$ i.e.

$$d_g\pi_L|_{\mathrm{Sp}_2(\mathcal{H})} : T_g\mathrm{Sp}_2(\mathcal{H}) = \mathfrak{sp}_2(\mathcal{H})g \ni Xg \mapsto P_{g(L)}JX|_{g(L)} \in \mathcal{B}_2(g(L))_s.$$

Indeed, we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Sp}(\mathcal{H}) & \xrightarrow{\pi_L} & \Lambda(\mathcal{H}) \\ {\scriptstyle i_2}\uparrow & & \uparrow{\scriptstyle i_1} \\ \mathrm{Sp}_2(\mathcal{H}) & \xrightarrow[\pi_L|_{\mathrm{Sp}_2(\mathcal{H})}]{} & \mathcal{O}_{L_0} \end{array}$$

If we differentiate at a point $g \in \mathrm{Sp}_2(\mathcal{H})$ the equation $\pi_L \circ i_2 = i_1 \circ \pi_L|_{\mathrm{Sp}_2(\mathcal{H})}$ and use that the differential of the inclusion maps i_1 and i_2 at $h(L_0)$ and at h respectively is inclusions, we have $d_g\pi_L|_{\mathrm{Sp}_2(\mathcal{H})}(Xg) = d_g\pi_L(Xg)$ for every $X \in \mathfrak{sp}_2(\mathcal{H})$.

In the followings steps we will obtain the main result of this section, the Lie subgroup structure of the isotropy group. To do it we will use the above submanifold structure constructed over $\mathcal{O}_{L_0}$. If M and N are smooth Banach manifolds a smooth map $f : M \to N$ is a submersion if the tangent map d_xf is onto and its kernel is a complemented subspace of T_xM for all $x \in M$. This fact is equivalent to the existence of smooth local section (see [14]). The next proposition is essential for the proof.

Proposition 3.6. *The map $\pi_{L_0} : \mathrm{Sp}(\mathcal{H}) \to \Lambda(\mathcal{H})$ and its restriction $\pi_{L_0}|_{\mathrm{Sp}_2(\mathcal{H})} : \mathrm{Sp}_2(\mathcal{H}) \to \mathcal{O}_{L_0}$ are smooth submersions when we consider in $\Lambda(\mathcal{H})$ (resp. in $\mathcal{O}_{L_0}$) the above manifold structure.*

Proof. First we will prove that the map $\pi_{L_0}|_{\mathrm{Sp}_2(\mathcal{H})} : \mathrm{Sp}_2(\mathcal{H}) \to \mathcal{O}_{L_0}$ has local cross sections on a neighborhood of L_0, the proof is adapted from [5] and [7]. Using the symmetry over $R(Q)$ we have

$$\epsilon_{R(Q)} = 2P_{R(Q)} - 1 = 2(P_{L_0} + \mathcal{B}_2) - 1 = \epsilon_{L_0} + \mathcal{B}_2. \tag{3.4}$$

For $L \in \mathcal{O}_{L_0}$ close to L_0, we consider the element $g_L = 1/2(1 + \epsilon_L\epsilon_{L_0})$; it is invertible (in fact, it can be shown that it is invertible if $\|\epsilon_L - \epsilon_{L_0}\| < 2$) and it commutes with J, so it belongs to $GL(\mathcal{H}_J)$. From Eq. (3.4) we have

$$\epsilon_L\epsilon_{L_0} \in (\epsilon_{L_0} + \mathcal{B}_2(\mathcal{H}))\epsilon_{L_0} \in 1 + \mathcal{B}_2(\mathcal{H})$$

and then it is clear that $g_L \in 1 + \mathcal{B}_2(\mathcal{H}_J)$. Thus g_L is complex and invertible in a neighborhood of ϵ_{L_0}. Note that

$$g_L\epsilon_{L_0} = 1/2(\epsilon_{L_0} + \epsilon_L) = \epsilon_L g_L$$

and also that g^*g commutes with ϵ_{L_0}. If $|x| = (x^*x)^{1/2}$ denotes the modulus and $g_L = u_L\,|g_L|$ is the polar decomposition, then $u_L = g_L({g_L}^*g_L)^{-1/2} \in U(\mathcal{H}_J) \subset \mathrm{Sp}(\mathcal{H})$. We define the local cross section for L close to L_0 as

$$\sigma(L) = u_L.$$

Now we have to prove that $\pi_{L_0}|_{\mathrm{Sp}_2(\mathcal{H})}(\sigma(L)) = L$. If we identify the subspace with the symmetry this is equivalent to prove that $\epsilon_{\pi_{L_0}|_{\mathrm{Sp}_2(\mathcal{H})}(\sigma(L))} = \epsilon_L$. Indeed,

$$\epsilon_{\pi_{L_0}(u_L)} = u_L\epsilon_{L_0}u_L^* = g_L(g_L^*g_L)^{-1/2}\epsilon_{L_0}(g_L^*g_L)^{-1/2}g_L^* = g_L\epsilon_{L_0}g_L^{-1} = \epsilon_L.$$

Let us prove that it takes values in $\mathrm{Sp}_2(\mathcal{H})$. Since $\mathbb{C}1 + \mathcal{B}_2(\mathcal{H}_J)$ is a *-Banach algebra and $g_L \in GL_2(\mathcal{H}_J)$ by the Riesz functional calculus we have that $u_L = g_L\,|g_L|^{-1} \in \mathbb{C}1 + \mathcal{B}_2(\mathcal{H}_J)$. Thus $u_L = \beta 1 + b$ with $b \in \mathcal{B}_2(\mathcal{H}_J)$. On the other hand, note that ${g_L}^*g_L$ is a positive operator which lies in the C*-algebra $\mathbb{C}1 + \mathcal{K}(\mathcal{H}_J)$. Therefore its square root is of the form $r1 + k$ with $r \geq 0$ and k compact. Then

$${g_L}^*g_L = (r1 + k)^2 = r^2.1 + k'$$

and since ${g_L}^*g_L \in GL_2(\mathcal{H}_J)$ we have

$$r^2 1 + k' = 1 + b'$$

with $b' \in \mathcal{B}_2(\mathcal{H}_J)$. Since $\mathbb{C}1$ and $\mathcal{K}(\mathcal{H}_J)$ are linearly independent, it follows that $r = 1$. Then it is clear that $u_L \in U_2(\mathcal{H}_J) \subset \mathrm{Sp}_2(\mathcal{H})$ and σ is well defined. To conclude the proof we now show that the local section σ is smooth. If L lies in a small neighborhood of L_0 we have

$$L = \phi_{L_0}^{-1}(\psi) = Gr_{-J|_L\psi} = (1 - J|_{L_0}\psi P_{L_0})(L_0) = g(L_0) \;\in\; \Omega({L_0}^{\perp}) \cap \mathcal{O}_{L_0}.$$

The idempotent of range L is

$$Q := gP_{L_0}g^{-1} = (1 - J|_{L_0}\psi P_{L_0})P_{L_0}(1 + J|_{L_0}\psi P_{L_0}) = P_{L_0} - J|_{L_0}\psi P_{L_0}$$

and it is smooth as a function of ψ. Since the formula of the orthogonal projector (3.1) is smooth, the local expression of σ will also be smooth. Indeed, the symmetry in the chart will be

$$\epsilon_L = 2P_{R(gP_{L_0}g^{-1})} - 1 = 2QQ^*(1 - (Q - Q^*)^2)^{1/2} - 1$$

and it is clearly smooth as a function of ψ, because Q and the operations involved (product, involution, square root) are smooth. Then it is clear that the invertible element g_L and its unitary part u_L are smooth too. Finally the local expression $\sigma \circ \phi_{L_0}^{-1}$ is smooth as a function of ψ. Since the full Lagrangian Grassmannian can be expressed as an orbit for a fixed L_0, the proof of smoothness of the local section of π_{L_0} is analogous to that of the restricted map $\pi_{L_0}|_{\mathrm{Sp}_2(\mathcal{H})}$. □

Corollary 3.7. *If L is any subspace in the full Lagrangian Grassmannian or in $\mathcal{O}_{L_0}$ then the map $\pi_L : \mathrm{Sp}(\mathcal{H}) \to \Lambda(\mathcal{H})$ and its restriction $\pi_L|_{\mathrm{Sp}_2(\mathcal{H})} : \mathrm{Sp}_2(\mathcal{H}) \to \mathcal{O}_{L_0}$ have local cross sections on a neighborhood of L.*

Proof. The above map σ can be translated using the action to any $L = g(L_0)$. That is,

$$\sigma_L(h(L_0)) = g\sigma(g^{-1}h(L_0))g^{-1}$$

where $h(L_0)$ lies on a neighborhood of L. □

Proposition 3.8. *The isotropy groups* $\mathrm{Sp}(\mathcal{H})_L$ *and* $\mathrm{Sp}_2(\mathcal{H})_L$ *of the symplectic group and of the restricted symplectic group are Lie subgroups of them with their respective topology. Their Lie algebras are*

$$\mathfrak{sp}(\mathcal{H})_L = \{x \in \mathfrak{sp}(\mathcal{H}) : x(L) \subseteq L\}$$
$$\mathfrak{sp}_2(\mathcal{H})_L = \{x \in \mathfrak{sp}_2(\mathcal{H}) : x(L) \subseteq L\}.$$

Proof. Since the maps $d_1\pi_L$ and $d_1\pi_L|_{\mathrm{Sp}_2(\mathcal{H})}$ are submersions then by the inverse function theorem, we have that the isotropy groups are Lie subgroups and their Lie algebras are $\ker d_1\pi_L$ and $\ker d_1\pi_L|_{\mathrm{Sp}_2(\mathcal{H})}$ respectively. A short computation shows us that $\ker d_1\pi_L = \{x \in \mathfrak{sp}(\mathcal{H}) : x(L) \subseteq L\}$ and $\ker d_1\pi_L|_{\mathrm{Sp}_2(\mathcal{H})} = \{x \in \mathfrak{sp}_2(\mathcal{H}) : x(L) \subseteq L\}$. Indeed, if $P_L JX|_L = 0$ then $JX|_L \in L^\perp$ and thus $-X|_L \in J(L^\perp) = L$. □

Remark 3.9. The Lie algebra $\mathfrak{sp}_2(\mathcal{H})_L$ consists of all operators $x \in \mathfrak{sp}_2(\mathcal{H})$ that are L invariant, so we can give another characterization of this algebra using the orthogonal projection P_L. That is,

$$\mathfrak{sp}_2(\mathcal{H})_L = \{x \in \mathfrak{sp}_2(\mathcal{H}) : xP_L = P_L x P_L\}. \tag{3.5}$$

In block matrix form, this operators correspond to the upper triangular elements of $\mathfrak{sp}_2(\mathcal{H})$.

4. Metric structure in $\mathcal{O}_{L_0}$

In this section we will introduce a Riemannian structure on $\mathcal{O}_{L_0}$ using the Hilbert–Schmidt inner product. We will prove that this Riemannian structure coincides with the Riemannian structure given by the quotient norm. We also study the completeness of the geodesic distance and moreover we will find the corresponding geodesic curves.

4.1. The ambient metric

Given $v, w \in T_W\mathcal{O}_{L_0} = \mathcal{B}_2(W)_s$, we define the inner product

$$\langle v, w\rangle_W := tr_W(w^*v) = \sum_{i=1}^{\infty} \langle w^* v e_i, e_i\rangle$$

where $\{e_i\}$ is an orthonormal basis of the subspace W. The ambient metric for $v \in T_W\mathcal{O}_{L_0} = \mathcal{B}_2(W)_s$ is

$$\mathcal{A}(W, v) := tr_W(v^*v)^{1/2}. \tag{4.6}$$

Using the orthogonal projection over W, it can be expressed by $\|vP_W\|_2^2$. Indeed, if $\{e_i\}$ is an orthonormal basis for $\mathcal{H}$ then

$$\begin{aligned}\|vP_W\|_2^2 &= \sum_i \langle vP_W e_i, vP_W e_i\rangle = \sum_i \langle v^* v P_W e_i, P_W e_i\rangle \\ &= \sum_i \langle v^* v P_W e_i, e_i\rangle = tr(v^* v P_W) = tr_W(v^* v).\end{aligned} \tag{4.7}$$

To each point $W \in \mathcal{O}_{L_0}$, we associate the inner product $\langle\cdot,\cdot\rangle_W$ on the tangent space $T_W\mathcal{O}_{L_0}$. This correspondence allows us to introduce a Riemannian structure on the manifold $\mathcal{O}_{L_0}$. The fact to prove here is that the metric varies differentiably.

Proposition 4.1. *The Riemannian structure is well defined.*

Proof. Let $L \in \mathcal{O}_{L_0}$ and consider a neighborhood $U := \Omega(L^\perp) \cap \mathcal{O}_{L_0}$ of it. For any $W \in U$, we can write it in the local chart $W = \phi_L^{-1}\psi = Gr_{(-J|_L\psi)}$. Let $\eta_W : L \to W$ be the restriction of the orthogonal projection $W \oplus L^\perp \xrightarrow{\pi} W$, then its local expression;

$$\begin{aligned}\eta_W(v) &= \pi(v) = \pi((v - J|_L\psi(v)) + J|_L\psi(v)) \\ &= (1 - J|_L\psi)(v) \quad \text{for all } v \in L,\end{aligned}$$

and then it can be expressed by the compression of the operator $1 - J|_L\psi P_L$ into the subspace L i.e. $\eta_W = (1 - J|_L\psi P_L)|_L$. If we write the local expression of the metric using the classical differential structure of the tangent bundle with the differential of the chart ϕ_L^{-1} given in the formula (3.3), for every $v \in TU$ we have

$$\mathcal{A}(W, v) = \|d_\psi\phi_L^{-1}(H)P_W\|_2 = \|(\eta_W^{-1})^* H \eta_W^{-1} P_W\|_2, \tag{4.8}$$

where $\psi \in \phi_L(U)$ and $H \in \mathcal{B}_2(L)_s$ is the preimage of v. Since the projector $P_W = P_{Gr_{(-J|_L\psi)}}$ is smooth and the local expression of η_W is also smooth as a function of ψ and by smoothness of the operations involved (inverse, involution, product, trace) the formula (4.8) is smooth. □

4.2. The geodesic distance

The length of a smooth curve measured with the ambient metric will be denoted by

$$L_{\mathcal{A}}(\gamma) = \int_0^1 \mathcal{A}(\gamma(t), \dot\gamma(t))dt.$$

Given two Lagrangian subspaces S and T in $\mathcal{O}_{L_0}$, we denote by $d_{\mathcal{A}}$ the geodesic distance using the ambient metric,

$$d_{\mathcal{A}}(S,T) = \inf\{L_{\mathcal{A}}(\gamma) : \gamma \text{ joins } S \text{ and } T \text{ in } \mathcal{O}_{L_0}\}.$$

If $(L_n) \subset \mathcal{O}_{L_0}$ is any sequence we will denote by $L_n \overset{\mathcal{O}_{L_0}}{\to} L$ the convergence to some subspace $L \in \mathcal{O}_{L_0}$ in the topology given by the smooth structure of $\mathcal{O}_{L_0}$ (Theorem 3.4).

There is a naturally defined Hilbert space inner product on the tangent space at 1 of the group $\mathrm{Sp}_2(\mathcal{H})$, which is identified with the space of Hilbert–Schmidt operators on $\mathcal{H}$, and this inner product is employed to define a left-invariant and a right-invariant Riemannian structure on the group.

Given a smooth curve α in $\mathrm{Sp}_2(\mathcal{H})$ we can measure its length with the left or right invariant metric, depending on which identification of tangent spaces we use in the group. In [9] it was used the left one, hence they use the left invariant metric. The length of a curve using this metric is $L_{\mathcal{L}}(\alpha) = \int_0^1 \|\alpha^{-1}\dot\alpha\|_2$. In this paper we will use the right identification of the tangent spaces, so we have to introduce the right invariant metric. Although formally equivalent this choice will make some computations easier. Then the length of α is, $L_{\mathcal{R}}(\alpha) = \int_0^1 \|\dot\alpha\alpha^{-1}\|_2$.

Remark 4.2. Let G be a Banach–Lie group, if $d_{\mathcal{L}}$ and $d_{\mathcal{R}}$ denote the geodesic distance with the left and right invariant metrics respectively then,

$$d_{\mathcal{L}}(x^{-1}, y^{-1}) = d_{\mathcal{R}}(x,y) \quad \forall\, x, y \in G.$$

Indeed, since the geodesic distances are left and right invariant respectively, the only fact left to prove is the equality $d_{\mathcal{L}}(x^{-1}, 1) = d_{\mathcal{R}}(x, 1)$ for all $x \in G$. Then, if α is any curve that joins 1 to x^{-1}, the curve $\beta(t) = \alpha(t)^{-1}$ joins 1 to x; if we differentiate we have $\dot\beta(t)\beta(t)^{-1} = -\alpha(t)^{-1}\dot\alpha(t)$ and then the right length of β coincides with the left length of α.

If $\xi : [0,1] \to \mathcal{O}_{L_0}$ is a curve with $\xi(0) = L$ then a lifting of ξ is a map $\phi : [0,1] \to \mathrm{Sp}_2(\mathcal{H})$ with $\phi(0) = 1$ and $\phi(t)(L) = \xi(t)$, for all $t \in [0,1]$. The next lemma is an adaptation of Lemma 25 in [12].

Lemma 4.3. *Every smooth curve $\xi : [0,1] \to \mathcal{O}_{L_0}$ with $\xi(0) = L$ admits an isometric lifting, if we consider the right invariant metric in* $\mathrm{Sp}_2(\mathcal{H})$.

Proof. For each $t \in [0,1]$, set $X(t) = -J\dot\xi(t)P_{\xi(t)} \in \mathfrak{sp}_2(\mathcal{H})$ and consider the solution of the ODE

$$\begin{cases} \dot\phi(t) = X(t)\phi(t) \\ \phi(0) = 1. \end{cases} \tag{4.9}$$

A simple computation using Remark 3.5 shows that both $t \mapsto \phi(t)(L)$ and $\xi(t)$ are integral curves of the vector field $v(t)(L) = P_L JX(t)|_L \in T_L\mathcal{O}_{L_0} = \mathcal{B}_2(L)_s$ both starting at L, therefore the two curves coincide. Now, it is easy to see that the solution of the differential equations (4.9) is an isometric lifting of ξ. Indeed, if we take norms in the equation we have,

$$\|\dot\phi(t)\phi^{-1}(t)\|_2 = \|-J\dot\xi(t)P_{\xi(t)}\|_2 = \|\dot\xi(t)P_{\xi(t)}\|_2 = \mathcal{A}(\xi(t), \dot\xi(t)). \quad \square$$

The geodesic curves given by the left invariant metric in the group $\mathrm{Sp}_2(\mathcal{H})$ were calculated in [9]. There it was proved that if $g_0 \in \mathrm{Sp}_2(\mathcal{H})$ and $g_0 v_0 \in g_0.\mathfrak{sp}_2(\mathcal{H})$ are the initial position and the initial velocity then

$$\alpha(t) = g_0 e^{tv_0^*} e^{t(v_0 - v_0^*)} \subset \mathrm{Sp}_2(\mathcal{H})$$

is a geodesic of the Riemannian left invariant metric. This fact can be used to find the geodesic of the Riemannian connection induced by the ambient metric $\mathcal{A}$.

Theorem 4.4. *Let $\xi : [0,1] \to \mathcal{O}_{L_0}$ be a geodesic curve of the Riemannian connection induced by the ambient metric $\mathcal{A}$ with initial position $\xi(0) = L$ and initial velocity $\dot\xi(0) = w \in T_{\xi(0)}\mathcal{O}_{L_0} = \mathcal{B}_2(L)_s$. Then*

$$\xi(t) = e^{t(v^* - v)} e^{-tv^*}(L)$$

where $v \in \mathfrak{sp}_2(\mathcal{H})$ is a preimage of $-w$ by $d_1\pi_L$.

Proof. Since ξ is a geodesic curve, it is locally minimizing. Using Lemma 4.3 there exists an isometric lifting $\phi \subset \mathrm{Sp}_2(\mathcal{H})$ with initial condition $\phi(0) = 1$. By the isometric property ϕ results locally minimizing with the right invariant metric and then ϕ^{-1} results locally minimizing with the left invariant metric. Hence the curve $\phi^{-1} \subset \mathrm{Sp}_2(\mathcal{H})$ is a geodesic and it is $\phi^{-1}(t) = e^{tv^*}e^{t(v-v^*)}$ for some $v \in \mathfrak{sp}_2(\mathcal{H})$. Then it is clear that $\phi(t) = e^{t(v^*-v)}e^{-tv^*}$ and $\xi(t) = e^{t(v^*-v)}e^{-tv^*}(L)$. The only fact left to prove is that v is a lift of $-w$. Indeed, since $\dot\xi(t) = d_{e^{t(v^*-v)}e^{-tv^*}}\pi_L\big((v^*-v)e^{t(v^*-v)}e^{-tv^*} - e^{t(v^*-v)}e^{-tv^*}v^*\big)$, then $w = \dot\xi(0) = d_1\pi_L(-v) = -d_1\pi_L(v)$. $\square$

4.3. *The quotient metric*

Since the action of the Hilbert–Lie group $\mathrm{Sp}_2(\mathcal{H})$ on the Grassmannian $\mathcal{O}_{L_0}$ is smooth and transitive, we identify $\mathcal{O}_{L_0} \simeq \mathrm{Sp}_2(\mathcal{H})/\mathrm{Sp}(\mathcal{H})_{L_0}$ as manifolds. Then it is only natural to consider on our Grassmannian the quotient Riemannian metric. If $W \in \mathcal{O}_{L_0}$ and $v \in T_W\mathcal{O}_{L_0}$, we put

$$\mathcal{Q}(W, v) = \inf\{\|z\|_2 : z \in \mathfrak{sp}_2(\mathcal{H}),\ d_1\pi_W(z) = v\}.$$

This metric will be called the quotient metric of $\mathcal{O}_{L_0}$, because it is the quotient metric in the Banach space

$$T_W\mathcal{O}_{L_0} \simeq \mathfrak{sp}_2(\mathcal{H})/\mathfrak{sp}_2(\mathcal{H})_W.$$

Indeed, since $\mathfrak{sp}_2(\mathcal{H})_W = \ker d_1\pi_W$, if $z \in \mathfrak{sp}_2(\mathcal{H})$ with $d_1\pi_W(z) = v$ then

$$\mathcal{Q}(W, v) = \inf\{\|z - y\|_2 : y \in \mathfrak{sp}_2(\mathcal{H})_W\}.$$

If Q_L denotes the orthogonal projection onto $\mathfrak{sp}_2(\mathcal{H})_W$ then each $z \in \mathfrak{sp}_2(\mathcal{H})$ can be uniquely decomposed as

$$z = z - Q_L(z) + Q_L(z) = z_0 + Q_L(z)$$

hence

$$\|z - y\|_2^2 = \|z_0 + Q_L(z) - y\|_2^2 = \|z_0\|_2^2 + \|Q_L(z) - y\|_2^2 \ge \|z_0\|_2^2$$

for any $y \in \mathfrak{sp}_2(\mathcal{H})_W$ which shows that

$$\mathcal{Q}(W, v) = \|z_0\|_2 \tag{4.10}$$

where z_0 is the unique vector in $\mathfrak{sp}_2(\mathcal{H})_W^\perp$ such that $d_1\pi_W(z_0) = v$.

We denote the length for a piecewise smooth curve in $\mathcal{O}_{L_0}$, measured with the quotient norm introduced above as $L_{\mathcal{Q}}(\gamma)$.

Theorem 4.5. *The quotient metric and the ambient metric are equal.*

Proof. The proof is a straightforward computation using the definition of the metrics; indeed let $W \in \mathcal{O}_{L_0}$ and $v \in T_W\mathcal{O}_{L_0}$, by formula (4.10) we have $\mathcal{Q}(W, v) = \|z_0\|_2$ where z_0 is the unique vector in $\mathfrak{sp}_2(\mathcal{H})_W^\perp$ such that $d_1\pi_W(z_0) = v$. Since z_0 belongs to $\mathfrak{sp}_2(\mathcal{H})_W^\perp$, using the decomposition $W \oplus W^\perp$, we can write

$$z_0 = z_0P_W - P_Wz_0P_W = (1 - P_W)z_0P_W$$

and then since P_W is a Lagrangian projector we have $Jz_0 = (J - JP_W)z_0P_W = P_WJz_0P_W$. Therefore using the definition of the ambient metric (4.6) we have,

$$\begin{aligned}\mathcal{A}(W, v) = \|vP_W\|_2 &= \|d_1\pi_W(z_0)P_W\|_2 = \|P_WJz_0|_WP_W\|_2\\ &= \|Jz_0\|_2 = \|z_0\|_2 = \mathcal{Q}(W, v). \quad \square\end{aligned}$$

Now, it is obvious that the geometry of these Riemannian metrics is the same, in particular the geodesics and the geodesic distance.

To prove the main theorem in this paper we will use some facts that we obtained in [9]. The key is to use the completeness of the metric space $(\mathrm{Sp}_2(\mathcal{H}), \|.\|_2)$ and the lift property given in Lemma 4.3.

Theorem 4.6. *If (L_n) is a sequence in $\mathcal{O}_{L_0}$ and $L \in \mathcal{O}_{L_0}$ then*

1. $L_n \xrightarrow{\mathcal{O}_{L_0}} L \Longrightarrow L_n \xrightarrow{d_{\mathcal{Q}}} L$.
2. *The metric space $(\mathcal{O}_{L_0}, d_{\mathcal{Q}})$ is complete.*
3. *The distance $d_{\mathcal{Q}}$ defines the given topology on $\mathcal{O}_{L_0}$. Equivalently, $L_n \xrightarrow{\mathcal{O}_{L_0}} L \Longleftrightarrow L_n \xrightarrow{d_{\mathcal{Q}}} L$.*

Proof. Since $d_{\mathcal{A}}(S, T) = d_{\mathcal{Q}}(S, T)$ for all $S, T \in \mathcal{O}_{L_0}$, we can prove the three items with $d_{\mathcal{A}}$ to simplify the computations.

1. The map π_L has local continuous sections, let n_0 such that $L_n \in U \subset \mathcal{O}_{L_0}$ $\forall n \ge n_0$ (U a neighborhood of L) and such that $\sigma_L : U \to \mathrm{Sp}_2(\mathcal{H})$ is a section for π_L. By continuity we have $\sigma_L(L_n) \xrightarrow{\|.\|_2} \sigma_L(L) = 1$ if $n \ge n_0$. Since $\sigma_L(L_n)$ is close to 1, there is $z_n \in \mathfrak{sp}_2(\mathcal{H})$ such that $\sigma_L(L_n) = e^{z_n}$ and since $\|e^{z_n} - 1\|_2 = \|\sigma_L(L_n) - 1\|_2 \to 0$ we also have $\|z_n\|_2 \to 0$. Let $\gamma_n(t) = e^{tz_n}(L) \subset \mathcal{O}_{L_0}$ be a curve that joins L and L_n; using the equality (4.7) its length is $L_{\mathcal{A}}(\gamma_n) = \int_0^1 \mathcal{A}(\gamma_n(t), \dot\gamma_n(t))dt = \int_0^1 \|\dot\gamma_n(t)P_{\gamma_n(t)}\|_2$. Since $\gamma_n(t) = \pi_L \circ e^{tz_n}$ using the chain rule and Remark 3.5 we have

   $$\dot\gamma_n(t) = d_{e^{tz_n}}\pi_L(z_ne^{tz_n}) = P_{e^{tz_n}(L)}Jz_n|_{e^{tz_n}(L)},$$

   then taking norms and using the symmetric property of the 2-norm ($\|xyz\|_2 \le \|x\|\,\|y\|_2\|z\|$) we have

   $$\|\dot\gamma_n(t)P_{\gamma_n(t)}\|_2 = \|P_{e^{tz_n}(L)}Jz_nP_{e^{tz_n}(L)}\|_2 \le \|z_n\|_2.$$

   Then it is clear that $d_{\mathcal{A}}(L_n, L) \le L_{\mathcal{A}}(\gamma_n) \to 0$.

2. Let (L_n) be a $d_{\mathcal{A}}$-Cauchy sequence in $\mathcal{O}_{L_0}$ and fix $\varepsilon > 0$. Then there exists n_0 such that $d_{\mathcal{A}}(L_n, L_m) \leq \varepsilon$ if $n, m \geq n_0$. For the fixed Lagrangian L_{n_0}, we have the map
$$\pi = \pi_{L_{n_0}} : \mathrm{Sp}_2(\mathcal{H}) \to \mathcal{O}_{L_0}, \qquad \pi(g) = g(L_{n_0}).$$
If $n, m \geq n_0$ we can take a curve $\gamma_{n,m} \subset \mathcal{O}_{L_0}$ that joins L_n to L_m (for $t = 0$ and $t = 1$ respectively) such that
$$L_{\mathcal{A}}(\gamma_{n,m}) \leq d_{\mathcal{A}}(L_n, L_m) + \varepsilon.$$
Then by Lemma 4.3, the curves $\gamma_{n_0,m}$ are lifted, via π, to curves ϕ_m of $\mathrm{Sp}_2(\mathcal{H})$ with $\phi_m(0) = 1$ and $L_{\mathcal{R}}(\phi_m) = L_{\mathcal{A}}(\gamma_{n_0,m})$. Denote by $g_m = \phi_m(1) \subset \mathrm{Sp}_2(\mathcal{H})$ the end point. Then
$$\varepsilon + d_{\mathcal{A}}(L_{n_0}, L_m) \geq L_{\mathcal{A}}(\gamma_{n_0,m}) = L_{\mathcal{R}}(\phi_m) \geq d_{\mathcal{R}}(1, g_m).$$
For each $n, m \geq n_0$ we have,
$$d_{\mathcal{R}}(g_n, g_m) \leq d_{\mathcal{R}}(1, g_m) + d_{\mathcal{R}}(1, g_n) \leq 2\varepsilon + d_{\mathcal{A}}(L_{n_0}, L_m) + d_{\mathcal{A}}(L_{n_0}, L_n) \leq 4\varepsilon.$$
Thus the sequence $(g_m) \subset \mathrm{Sp}_2(\mathcal{H})$ is $d_{\mathcal{R}}$-Cauchy and then by Remark 4.2 we have that (g_m^{-1}) is $d_{\mathcal{L}}$-Cauchy. Using Lemma 7.1 of [9] we have that the sequence (g_m^{-1}) is a Cauchy sequence in $(\mathrm{Sp}_2(\mathcal{H}), \|.\|_2)$ and then since this metric space is closed, there exists $x \in \mathrm{Sp}_2(\mathcal{H})$ such that $g_m^{-1} \xrightarrow{\|.\|_2} x$. By continuity we have $\pi(g_m) \xrightarrow{\mathcal{O}_{L_0}} \pi(x^{-1})$ and since ϕ_m is a lift of $\gamma_{n_0,m}$ we also have $\pi(g_m) = g_m(L_{n_0}) = \phi_m(1)(L_{n_0}) = \gamma_{n_0,m}(1) = L_m$, so $L_m \xrightarrow{\mathcal{O}_{L_0}} \pi(x^{-1})$. Thus using the first item of this theorem we have $d_{\mathcal{A}}(L_m, \pi(x^{-1})) \to 0$.
3. Suppose that $L_n \xrightarrow{d_{\mathcal{A}}} L$, then it is a $d_{\mathcal{A}}$-Cauchy sequence. If we repeat the argument that we did above, there exists $x \in \mathrm{Sp}_2(\mathcal{H})$ such that $L_n \xrightarrow{\mathcal{O}_{L_0}} \pi(x^{-1})$. By the point first it is $d_{\mathcal{A}}$ convergent and therefore $L_n \xrightarrow{\mathcal{O}_{L_0}} L$. □

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