Rev Mat Complut
DOI 10.1007/s13163-010-0055-2

Orthogonally additive holomorphic functions on open subsets of $C(K)$

Jesús Ángel Jaramillo · Ángeles Prieto · Ignacio Zalduendo

Received: 16 July 2010 / Accepted: 3 November 2010

Abstract We introduce, study and characterize orthogonally additive holomorphic functions $f: U \to \mathbb{C}$ where U is an open subset of $C(K)$. We are led to consider orthogonal additivity at different points of U.

Keywords Infinite-dimensional holomorphy · Orthogonal additivity

Mathematics Subject Classification (2010) 46G20 · 46T25

1 Introduction

Orthogonally additive polynomials were first studied by Sundaresan [11] in the context of Banach lattices. He obtained a characterization of such functions for L^p-spaces. Interest in orthogonally additive polynomials was later renewed (see [2, 3, 5, 10]) and recently several papers have appeared dealing with various aspects of orthogonally additive functions [4, 9].

The third author is grateful for the hospitality of the Departamento de Análisis Matemático of the Universidad Complutense de Madrid. The authors have been partially supported by Ministerio de Ciencia e Innovación, grant MTM2009-07848.

J.Á. Jaramillo (✉) · Á. Prieto
Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain
e-mail: jaramil@mat.ucm.es

Á. Prieto
e-mail: angelin@mat.ucm.es

I. Zalduendo
Departamento de Matemática, Universidad Torcuato Di Tella, Miñones 2177 (C1428ATG), Buenos Aires, Argentina
e-mail: nacho@utdt.edu

A function $f : C(K) \to \mathbb{C}$ is said to be orthogonally additive if $f(x+y) = f(x) + f(y)$ whenever x and y are orthogonal (i.e. $xy = 0$ over K). As usual, we denote by $\mathcal{P}(^n C(K))$ the space of all continuous n-homogeneous polynomials $P : C(K) \to \mathbb{C}$. As was proved in [2, 3, 10], all orthogonally additive polynomials $P \in \mathcal{P}(^n C(K))$ are of the form

$$P(x) = \int_K x^n \, d\mu \qquad (*)$$

for some regular Borel measure μ. A holomorphic function $f : C(K) \to \mathbb{C}$ of bounded type (i.e. one which is bounded on bounded sets) has a Taylor series expansion $f = \sum_{k\geq 0} P_k$ at zero which is uniformly convergent on any bounded set, and it is easy to prove that f is orthogonally additive if and only if each P_k is orthogonally additive. Note that this does not hold for expansions around other points. Even so, as was shown in [4], a representation such as $(*)$ above cannot be obtained for such functions and in fact cannot be obtained even for non-homogeneous polynomials. However, a characterization of such functions was given in [4].

Holomorphic functions $f : C(K) \to \mathbb{C}$ which are not of bounded type cannot be expressed globally as Taylor series [6], and their study requires expansions around different points. More generally, in this paper we introduce an appropriate notion of orthogonally additive holomorphic functions $f : U \to \mathbb{C}$, where U is an arbitrary open subset of $C(K)$. We are therefore led to study some form of localization of the concept of orthogonal additivity. This is done in Sect. 2. In Sect. 3 we give our characterization of locally orthogonally additive holomorphic functions, more or less along the lines of [4], and we end with a concrete example of a locally orthogonally additive holomorphic function on an open subset of ℓ_∞.

2 Preliminary results

Throughout, K will be a compact topological space and U a non-empty open subset of $C(K)$. We say $u, v \in C(K)$ are orthogonal if $uv = 0$ over K (we denote this by $u \perp v$). If this happens, then $(u+v)^k = u^k + v^k$ for all $k \geq 1$. In order to begin our study of orthogonal additivity at different points of U, we require the following definition.

Definition 1 Given a function $f : U \to \mathbb{C}$, and $a \in U$, let f_a denote the function

$$f_a(x) = f(a+x) - f(a), \quad \text{where } x \in U_a = U - a.$$

Note that $f_0(x) = f(x) - f(0)$, so if $f(0) = 0$, then $f_0 = f$. This is the case for non-constant homogeneous polynomials. We have the following proposition.

Proposition 2.1 *Let $n \geq 1$ and $P \in \mathcal{P}(^n C(K))$. Then the following are equivalent.*

(i) *P is orthogonally additive.*
(ii) *For every $a \in C(K)$, P_a is orthogonally additive.*
(iii) *For some $a \in C(K)$ with $P(a) = 0$, P_a is orthogonally additive.*

Proof (i) implies (ii): Let $a \in C(K)$, and $u \perp v$. Since P is orthogonally additive, there is a regular Borel measure μ on K (see [2, 3, 10]) such that

$$P(x) = \int_K x^n \, d\mu.$$

Thus,

$$\begin{aligned}
P_a(u+v) &= P(a+u+v) - P(a) \\
&= \int_K (a+u+v)^n - a^n \, d\mu = \int_K \sum_{j=1}^n \binom{n}{j} a^{n-j}(u+v)^j \, d\mu \\
&= \int_K \sum_{j=1}^n \binom{n}{j} a^{n-j}(u^j + v^j) \, d\mu \\
&= \int_K \sum_{j=1}^n \binom{n}{j} a^{n-j} u^j \, d\mu + \int_K \sum_{j=1}^n \binom{n}{j} a^{n-j} v^j \, d\mu \\
&= \int_K (a+u)^n - a^n \, d\mu + \int_K (a+v)^n - a^n \, d\mu \\
&= P(a+u) - P(a) + P(a+v) - P(a) \\
&= P_a(u) + P_a(v).
\end{aligned}$$

(ii) implies (iii): P is zero at some point.

(iii) implies (i): Say P_a is orthogonally additive, $P(a) = 0$, and $u \perp v$. Applying [4, Theorem 2.3] to the non-homogeneous polynomial P_a, one sees that there is a regular Borel measure μ on K and functions $g_1, \ldots, g_n \in L_1(\mu)$ such that

$$P_a(x) = \int_K \sum_{j=1}^n g_j \cdot x^j \, d\mu.$$

Thus,

$$\begin{aligned}
P(u+v) &= P(a) + P_a(u+v-a) = P_a(u+v-a) \\
&= \int_K \sum_{j=1}^n g_j \cdot (u+v-a)^j \, d\mu \\
&= \int_K \sum_{j=1}^n g_j \cdot \sum_{i=0}^j \binom{j}{i} (-a)^{j-i}(u+v)^i \, d\mu \\
&= \int_K \sum_{j=1}^n g_j \cdot \sum_{i=0}^j \binom{j}{i} (-a)^{j-i}(u^i + v^i) \, d\mu
\end{aligned}$$

$$
\begin{aligned}
&= \int_K \sum_{j=1}^{n} g_j \cdot \sum_{i=0}^{j} \binom{j}{i} (-a)^{j-i} u^i \, d\mu + \int_K \sum_{j=1}^{n} g_j \cdot \sum_{i=0}^{j} \binom{j}{i} (-a)^{j-i} v^i \, d\mu \\
&= \int_K \sum_{j=1}^{n} g_j \cdot (u-a)^j \, d\mu + \int_K \sum_{j=1}^{n} g_j \cdot (v-a)^j \, d\mu \\
&= P_a(u-a) + P_a(v-a) = P(u) - P(a) + P(v) - P(a) \\
&= P(u) + P(v). \qquad \square
\end{aligned}
$$

We are going to apply the above result to the study of orthogonally additive holomorphic functions defined on an arbitrary open subset U of $C(K)$. We denote as usual by $\mathcal{H}(U)$ the space of all holomorphic functions $f: U \to \mathbb{C}$. We introduce the following definition.

Definition 2 Given a function $f \in \mathcal{H}(U)$, we will say f is *locally orthogonally additive on* U if for each $a \in U$ there exists a neighborhood V of a such that whenever $a+u, a+v, a+u+v \in V$ and $u \perp v$, then

$$f_a(u+v) = f_a(u) + f_a(v).$$

For short, we will refer to this local condition as *f is a-orthogonally additive on V*.

Proposition 2.2 *A function $f \in \mathcal{H}(U)$ is locally orthogonally additive on U if and only if for each $a \in U$ the Taylor series expansion of f at a is $f(a+x) = f(a) + \sum_{k\geq 1} Q_{k,a}(x)$, where all $Q_{k,a}$ are orthogonally additive k-homogeneous polynomials.*

Proof $\Rightarrow$: Take $a \in U$ and $u \perp v$. Note that for small λ we have $a+\lambda u, a+\lambda v, a+\lambda(u+v) \in V$, where V is a neighborhood of a on which f is a-orthogonally additive, and the Taylor series at a converges on them. Then for each $k \geq 1$

$$
\begin{aligned}
k! Q_{k,a}(u+v) &= \frac{d^k}{d\lambda^k}\Big|_{\lambda=0} f(a+\lambda(u+v)) \\
&= \frac{d^k}{d\lambda^k}\Big|_{\lambda=0} \big[f_a(\lambda u + \lambda v) + f(a)\big] \\
&= \frac{d^k}{d\lambda^k}\Big|_{\lambda=0} \big[f_a(\lambda u) + f_a(\lambda v) + f(a)\big] \\
&= \frac{d^k}{d\lambda^k}\Big|_{\lambda=0} f_a(\lambda u) + \frac{d^k}{d\lambda^k}\Big|_{\lambda=0} f_a(\lambda v) \\
&= \frac{d^k}{d\lambda^k}\Big|_{\lambda=0} (f(a+\lambda u) - f(a)) + \frac{d^k}{d\lambda^k}\Big|_{\lambda=0} (f(a+\lambda v) - f(a)) \\
&= k! Q_{k,a}(u) + k! Q_{k,a}(v).
\end{aligned}
$$

$\Leftarrow$: For each $a \in U$, consider $f(x+a) = f(a) + \sum_{k\geq 1} Q_{k,a}(x)$ the Taylor series expansion of f at a, which is convergent to f for $x+a$ in some neighborhood V of a. Being $Q_{k,a}$ orthogonally additive as a k-homogeneous polynomial, f is a-orthogonally additive on V. □

It follows from the proof above that the Taylor series expansion of f at a is composed of orthogonally additive polynomials, whenever f is an a-orthogonally additive holomorphic function on a neighborhood of a. Recall that a set $B \subset C(K)$, with $a \in B$, is said to be a-balanced if for each $x \in B$ and λ in the closed unit disc $\overline{\Delta}$, then $(1-\lambda)a + \lambda x \in B$.

Proposition 2.3 *Suppose $f \in \mathcal{H}(U)$, $a \in U$ and f is a-orthogonally additive on a neighborhood of a. Then f is a-orthogonally additive on any open a-balanced set $B \subset U$.*

Proof Consider $f(a+x) = f(a) + \sum_{k\geq 1} Q_{k,a}(x)$ the Taylor series of f at a, $B \subset U$ an open a-balanced set, and $a+u, a+v, a+u+v \in B$, where $u \perp v$. It is known [8, Theorem 7.11] that the Taylor series converges to f on any finite set of B. As shown in the proof of Proposition 2.2, each $Q_{k,a}$ is orthogonally additive. Then,

$$\begin{aligned} f_a(u+v) &= f(a+u+v) - f(a) \\ &= \sum_{k\geq 1} Q_{k,a}(u+v) \\ &= \sum_{k\geq 1} Q_{k,a}(u) + \sum_{k\geq 1} Q_{k,a}(v) \\ &= (f(a+u) - f(a)) + (f(a+v) - f(a)) \\ &= f_a(u) + f_a(v). \end{aligned}$$

□

Now we see that, for domains in $C(K)$, it suffices to verify orthogonal additivity for a single point.

Proposition 2.4 *Consider a connected open subset $U \subset C(K)$, $f \in \mathcal{H}(U)$, and $a \in U$. If f is a-orthogonally additive on a neighborhood of a, then f is locally orthogonally additive on U.*

Proof Let $A = \{x \in U : f \text{ is } x\text{-orthogonally additive on a neighborhood of } x\}$. Since $a \in A$, the set A is nonempty. For any $c \in A$, there exists $r > 0$, such that f is c-orthogonally additive on the open ball $B(c,r) \subset U$. If the Taylor series expansion of f at c is $f(c+x) = \sum_{k=0}^{\infty} Q_{k,c}(x)$, then $f_c(x) = f(c+x) - f(c) = \sum_{k=1}^{\infty} Q_{k,c}(x)$, for all $x \in B(0,r)$. We are going to see that for every $b \in B(c,r)$, if $s = r - \|b-c\|$, then f is b-orthogonally additive on $B(b,s)$. First note that the proof of Proposition 2.2 gives that each $Q_{k,c}$ is an orthogonally additive k-homogeneous polynomial. Therefore, if we denote $(Q_{k,c})_{b-c}(x) = Q_{k,c}(b-c+x) - Q_{k,c}(b-c)$, we obtain from Proposition 2.1 that each $(Q_{k,c})_{b-c}$ is also orthogonally additive. Now choose u, v with $u \perp v$, and such that $b+u, b+v, b+u+v \in B(b,s)$. Then,

$$\begin{aligned} f_b(u+v) &= f(b+u+v) - f(b) \\ &= \sum_{k=0}^{\infty} Q_{k,c}(b-c+u+v) - \sum_{k=0}^{\infty} Q_{k,c}(b-c) \\ &= \sum_{k=1}^{\infty}(Q_{k,c})_{b-c}(u+v) = \sum_{k=1}^{\infty}(Q_{k,c})_{b-c}(u) + \sum_{k=1}^{\infty}(Q_{k,c})_{b-c}(v) \\ &= \sum_{k=1}^{\infty} Q_{k,c}(b-c+u) - Q_{k,c}(b-c) + \sum_{k=1}^{\infty} Q_{k,c}(b-c+v) \\ &\quad - Q_{k,c}(b-c) \\ &= f_c(b-c+u) - f_c(b-c) + f_c(b-c+v) - f_c(b-c) \\ &= (f(b+u) - f(c)) - (f(b) - f(c)) + (f(b+v) - f(c)) \\ &\quad - (f(b) - f(c)) \\ &= f(b+u) - f(b) + f(b+v) - f(b) \\ &= f_b(u) + f_b(v). \end{aligned}$$

This shows that A is open. Now, let $\{a_n\}_n$ be a sequence in A converging to $b \in U$. Take $\delta > 0$ such that the open ball $B(b,\delta) \subset U$. Choose a_k with $\|b - a_k\| < \delta/4$. The ball $B(a_k, \delta/2)$ is still contained in U and b belongs to it. So, being f a_k-orthogonally additive on $B(a_k, \delta/2)$, f is b-orthogonally additive on a neighborhood of b, as we have just proved. Then, $b \in A$ and the set A is closed. Therefore, $A = U$ and the result follows. □

Note that the previous results show that our definition of "locally orthogonally additive" extends the classical notion of "orthogonally additive". More precisely, we have the following:

Corollary 2.5 *Let $f : C(K) \to \mathbb{C}$ be a holomorphic function. Then f is orthogonally additive if, and only if, $f(0) = 0$ and f is locally orthogonally additive on $C(K)$.*

We remark that the definition of locally orthogonally additive imposes no restriction on the value at any particular point and, in fact, if $f \in \mathcal{H}(U)$ is locally orthogonally additive on U, then so is $f + c$, for any $c \in \mathbb{C}$.

3 Characterization of locally orthogonally additive holomorphic functions on U

Given a Borel subset $A \subset K$, we denote the corresponding characteristic function by 1_A. Recall [3] that for any k-homogeneous polynomial Q on $C(K)$ we may define a k-homogeneous polynomial Q_A by setting

$$Q_A(u) = \overline{Q}(1_A u),$$

where $\overline{Q}$ denotes the Aron-Berner extension of Q to the bidual of $C(K)$ [1]. Note that when Q is orthogonally additive, there exists a regular Borel measure μ on K such that $Q(x) = \int_K x^k \, d\mu$, and the proof of [3, Corollary 2.1] shows that for any Borel subset $A \subset K$ and $u \in C(K)$, $Q_A(u) = \int_A u^k d\mu$. It is easily checked by using [7] that the norm of Q_A is bounded by the norm of Q.

Definition 3 Let $f \in \mathcal{H}(U)$ and μ be a Borel measure on K. We will say that $f \ll \mu$ at $a \in U$ if, given $\sum_{k=0}^{\infty} Q_{k,a}$ the Taylor series expansion of f at a, for every Borel set $A \subset K$ we have that if $\mu(A) = 0$, then $Q_{k,a_A} = 0$ for all $k \geq 1$.

Definition 4 If μ is a Borel measure on K, a $L_1(\mu)$-valued function h_a defined near a will be called a *power series function* if $h_a(u) = c + \sum_{k\geq 1} g_k \cdot (u-a)^k$, where $c \in \mathbb{C}$, we have that $g_k \in L_1(\mu)$ for all $k \geq 1$, and the series converges in $L_1(\mu)$.

Note that the radius of convergence of the series above can be computed in terms of $(\|g_k\|_{L_1})_k$, and is given by the Cauchy-Hadamard formula. A power series function is a special type of holomorphic function, for it uses the algebra structure of $L_1(\mu)$.

Theorem 3.1 *Given $f \in \mathcal{H}(U)$ locally orthogonally additive on U, and μ a probability measure on K, then $f \ll \mu$ at a if and only if there exists a power series function h_a defined near a, such that*

$$f(u) = \int_K h_a(u) \, d\mu \quad \textit{for } u \textit{ near } a.$$

Proof Suppose $f \ll \mu$ at a, and $\mu(A) = 0$. Thus for each $k \geq 1$, $Q_{k,a_A} = 0$. We have $Q_{k,a} \ll \mu$ at a for all k. By Proposition 2.2, $Q_{k,a}$ are all orthogonally additive so by [3] there are Borel measures $\mu_{k,a}$ such that

$$Q_{k,a}(x) = \int_K x^k \, d\mu_{k,a}.$$

Since $Q_{k,a} \ll \mu$, $\mu_{k,a} \ll \mu$: indeed, if $\mu(A) = 0$,

$$\mu_{k,a}(A) = \int_A 1 \, d\mu_{k,a} = Q_{k,a_A}(1) = 0.$$

Now by the Radon-Nikodým theorem, there are $g_{k,a} \in L_1(\mu)$ such that

$$Q_{k,a}(x) = \int_K x^k g_{k,a} \, d\mu.$$

We may see as in [4, Theorem 2.1] that $\|g_{k,a}\|_{L_1} \leq \|Q_{k,a}\|$. Now define h_a near a by

$$h_a(u) = f(a) + \sum_{k\geq 1} g_{k,a} \cdot (u-a)^k.$$

This series converges absolutely as in [4]. Now note that for u near $a \in U$,

$$\begin{aligned} f(u) &= f(a) + \sum_{k\geq 1} Q_{k,a}(u-a) \\ &= f(a) + \sum_{k\geq 1} \int_K (u-a)^k g_{k,a}\, d\mu \\ &= \int_K \Big(f(a) + \sum_{k\geq 1} g_{k,a} \cdot (u-a)^k \Big)\, d\mu \\ &= \int_K h_a(u)\, d\mu. \end{aligned}$$

Now suppose $a \in U$ is such that there exists a power series function h_a defined near a, with

$$f(u) = \int_K h_a(u)\, d\mu \quad \text{for } u \text{ near } a.$$

Then for u near a,

$$\begin{aligned} f(u) &= f(a) + \int_K \sum_{k\geq 1} g_{k,a} \cdot (u-a)^k\, d\mu \\ &= f(a) + \sum_{k\geq 1} \int_K g_{k,a} \cdot (u-a)^k\, d\mu. \end{aligned}$$

Thus for each k,

$$Q_{k,a}(x) = \int_K x^k g_{k,a}\, d\mu.$$

Now if A is such that $\mu(A) = 0$,

$$\begin{aligned} Q_{k,a_A}(x) &= \int_K x^k 1_A\, d\mu_{k,a} \\ &= \int_K x^k 1_A g_{k,a}\, d\mu \\ &= \int_A x^k g_{k,a}\, d\mu \\ &= 0, \end{aligned}$$

so $Q_{k,a} \ll \mu$ for all $k \geq 1$, and $f \ll \mu$ at a. □

Proposition 3.2 *Let $f \in \mathcal{H}(U)$ be locally orthogonally additive on U. Then for each $a \in U$ there is a probability measure μ_a such that $f \ll \mu_a$ at a.*

Proof Fix $a \in U$. If f is constant on a neighborhood of a, the result is obvious. Otherwise, let

$$f(u) = f(a) + \sum_{k\geq 1} Q_{k,a}(u-a)$$

be the Taylor series expansion of f at a. By Proposition 2.2, all $Q_{k,a}$ are orthogonally additive. As in [4, Proposition 2.2] there is a sequence of measures (μ_k), depending on a, such that

$$Q_{k,a}(x) = \int_K x^k \, d\mu_k,$$

and $|\mu_k| \leq \|Q_{k,a}\|$. There is then a positive number $r > 0$ such that $\sum_{k\geq 1} |\mu_k| r^k = 1$. We define

$$\mu_a(A) = \sum_{k\geq 1} r^k |\mu_k|(A).$$

Thus $\mu_k \ll \mu_a$ for each k, so $Q_k \ll \mu_a$ for each k and $f \ll \mu_a$ at a. □

We can now characterize holomorphic functions which are locally orthogonally additive on U.

Theorem 3.3 *$f \in \mathcal{H}(U)$ is locally orthogonally additive on U if and only if for each $a \in U$ there exists a probability measure μ_a on K and a power series function h_a defined near a such that*

$$f(u) = \int_K h_a(u)\, d\mu_a \quad \textit{for } u \textit{ near } a.$$

Proof ⇒: Follows immediately from the previous results.

⇐: For any $a \in U$ and small x,

$$\begin{aligned} f_a(x) &= f(x+a) - f(a) \\ &= \int_K (h_a(x+a) - f(a))\, d\mu_a \\ &= \sum_{k\geq 1} \int_K g_k \cdot x^k \, d\mu_a. \end{aligned}$$

Thus if $x \perp y$ (and both are small),

$$\begin{aligned} f_a(x+y) &= \sum_{k\geq 1} \int_K g_k \cdot (x+y)^k \, d\mu_a \\ &= \sum_{k\geq 1} \int_K g_k \cdot (x^k + y^k)\, d\mu_a \end{aligned}$$

$$= \sum_{k \geq 1} \int_K g_k \cdot x^k \, d\mu_a + \sum_{k \geq 1} \int_K g_k \cdot y^k \, d\mu_a$$
$$= f_a(x) + f_a(y).$$

Thus f is locally orthogonally additive on U. □

We conclude with a concrete example of a locally orthogonally additive holomorphic function which is not of bounded type, and its representation as in the previous theorem.

Example 1 We take $K = \beta\mathbb{N}$, the Stone-Čech compactification of the natural numbers, and $C(K) = \ell_\infty$. Consider the open subset of ℓ_∞

$$U = \{z \in \ell_\infty : d(z, c_0) < 1\},$$

and $f : U \to \mathbb{C}$ given by $f(z) = \sum_{n=1}^{\infty} z(n)^n$. This is a locally orthogonally additive holomorphic function on U. Nevertheless, f cannot be extended to all of ℓ_∞, for all entire functions on ℓ_∞ are of bounded type on c_0 (see [6, Corollary 6.17]), and f clearly is not.

For each $a \in U$, define the probability measure on $\beta\mathbb{N}$ with support in $\mathbb{N}$ and $\mu_a(\{n\}) = \frac{1}{2^n}$. Also, define—for z sufficiently near to a—the power series function $h_a(z)(n) = \sum_{k=0}^{\infty} g_k(n)(z(n) - a(n))^k$, where

$$g_k(n) = \begin{cases} 0, & \text{if } n < k \\ \binom{n}{k} a(n)^{n-k} \, 2^n, & \text{if } n \geq k \end{cases}$$

(here if $a(n) = 0$ and $n = k$ we understand $0^0 = 1$). Note that $h_a(z) \in L_1(\mu_a)$. Indeed, say $b \in c_0$ is at a distance $s < 1$ from a; and that for $n \geq n_s$

$$|a(n)| \leq |a(n) - b(n)| + |b(n)| < s + \frac{1-s}{2} = \frac{1+s}{2} = r < 1.$$

Then

$$\int_{\beta\mathbb{N}} |g_k| \, d\mu_a = \sum_{n=1}^{\infty} \binom{n}{k} |a(n)|^{n-k} \, 2^n \, \frac{1}{2^n}$$
$$< c + \sum_{n \geq n_s} \binom{n}{k} r^{n-k}$$
$$< c + \sum_{n \geq n_s} \frac{n^k \, r^n}{k! \, r^k}$$
$$= c + \frac{1}{k! \, r^k} \sum_{n \geq n_s} n^k \, r^n < \infty.$$

Moreover, integrating $h_a(z)$, we have

$$\begin{aligned}\int_{\beta\mathbb{N}} h_a(z)\,d\mu_a &= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} g_k(n)(z(n)-a(n))^k\,\frac{1}{2^n}\\ &= \sum_{n=1}^{\infty}\sum_{k=0}^{n}\binom{n}{k} a(n)^{n-k}\,2^n\,(z(n)-a(n))^k\,\frac{1}{2^n}\\ &= \sum_{n=1}^{\infty}(a(n)+(z(n)-a(n)))^n\\ &= \sum_{n=1}^{\infty} z(n)^n = f(z).\end{aligned}$$

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