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A novel design approach for switched LPV controllers

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A novel design procedure for switched linear parameter-varying (LPV) controller is proposed. The new procedure, based on the Youla parameterisation ideas, decomposes the controller design into two steps. One focuses on ensuring global stability and the other on fulfilling the local performance specifications. This scheme allows the design of each local controller independently of each other, which may achieve higher performance without compromising the global stability and also simplifies the synthesis and the implementation of the local controllers. Any standard LPV synthesis procedure can be used to design these controllers. On the other hand, the stability during switching is ensured with convex constraints and no restrictions are imposed on the switching among controllers. The use of the proposed procedure is illustrated with an active magnetic bearing example.

Keywords: switched linear parameter-varying systems; linear parameter-varying systems; Youla parametrisation; linear matrix inequalities; active magnetic bearing systems

1. Introduction

Since the appearance of the first synthesis results (Becker and Packard 1994; Apkarian, Gahinet, and Becker 1995; Wu, Yang, Packard, and Becker 1996; Apkarian and Adams 1998), linear parameter-varying (LPV) tools have gradually replaced the traditional gain scheduling techniques. This new concept provides not only a formal framework ensuring stability and performance but also a systematic design procedure. To design the global control strategy, it is sufficient to solve only one optimisation problem with LMI constraints. The gain scheduling is now implemented without resorting to complex ad hoc algorithms because the synthesis procedure itself provides the interpolation formula. Nevertheless, the LPV formulation presents certain limitations. The optimisation problem in systems with large number of parameters demands a prohibitive computational effort with the current linear matrix inequality (LMI) algorithms (Lee 1997). On the other hand, a single LPV controller may not be effective in cases of plants with drastic dynamic changes or when highly demanding specifications must be fulfilled only in certain sectors of the parameter space. Usually, in these situations, the LPV synthesis focuses on the global behaviour at the expense of sacrificing the local performance.

Probably, the first attempt to overcome these limitations can be found in Lee (1997). The authors propose to divide the parameter space into overlapped subsets and to design one LPV controller for each subset. The global strategy is then constructed by interpolating the local controllers. A different approach is suggested in Wu (2001), Lu and Wu (2004) and Lu, Wu, and Kim (2006), putting the problem in the context of the recently introduced switched LPV systems (Lim and Chan 2003). In this context, the synthesis procedure differs from the traditional LPV techniques in the search for piecewise or multiple parameter-dependent Lyapunov functions. However, although these approaches put more emphasis on the local performance, the designs are still connected by a global stability condition that limits the maximum performance achieved.

In this article, we propose a different approach to the synthesis problem of switched LPV controllers partly inspired by the LTI works (Hespanha and Morse 2002; Blanchini, Miani, and Mesquine 2009). Based on the separation principle of the Youla parametrisation, the controller design is decomposed into two steps, one focused on ensuring global stability and the other on achieving the desirable performance in each subset. The appeal of this new scheme is that each local controller can be designed independently from the other subsets. This feature leads to higher performance designs without compromising the global stability, but it also makes the local controller synthesis

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more tractable and simplifies its implementation. The global stability is guaranteed by convex constraints with no restrictions imposed on the switching among controllers.

The article is organised as follows. Section 2 provides a brief introduction on switched LPV systems. Section 3 introduces the problem formulation and the novel design procedure for switched LPV controllers. Some implementation aspects are discussed at the end of this section. In Section 4, the application of the new procedure is illustrated with an active magnetic bearing (AMB) example. Finally, Section 5 summarises our conclusions.

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^{n\times m}$ the set of real matrices of $n\times m$. For a symmetric matrix $H\in\mathbb{R}^{n\times n}$, $H>0$ ($H<0$) denotes positive (negative) definite and $H\geq 0$ ($H\leq 0$) represents a positive (negative) semi-definite matrix. Given G a 2×2 block-partitioned matrix and a matrix Q such that $\det(I-G_{22}Q)\neq 0$, the lower linear fractional transformation is defined as $\mathcal{F}_l(G,Q)\triangleq G_{11}+G_{12}Q(I-G_{22}Q)^{-1}G_{21}$. For matrices $H_1, H_2, \ldots, H_n$,

$$\mathrm{diag}(H_1,H_2,\ldots,H_n)=\begin{bmatrix} H_1 & 0 & \cdots & 0\\ 0 & H_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & H_n\end{bmatrix}.$$

2. Switched LPV systems

In this section, we present a brief introduction on switched LPV systems for a better understanding of the contributions. A more complete discussion can be found in Lu (2004).

A switched LPV system depends on a scheduling parameter ρ, but the set $\mathcal{P}\subset\mathbb{R}^{n_\rho}$ where the parameter ranges is divided by means of a set of switching surfaces $\mathcal{S}_{ij}$ into closed subsets $\{\mathcal{P}_i\}_{i\in\mathbb{Z}_n}$ such that $\mathcal{P}=\bigcup\mathcal{P}_i$, where $\mathbb{Z}_n=\{1,2,\ldots,n\}$. The subsets can be either overlapped or not, and in this last case $\mathcal{S}_{ij}=\mathcal{S}_{ji}$.

The system dynamics in each subset is given by

$$G_{\mathrm{cl},i}(\rho):\begin{cases}\dot{x}_{\mathrm{cl}}=A_{\mathrm{cl},i}(\rho)x_{\mathrm{cl}}+B_{\mathrm{cl},i}(\rho)w\\ z=C_{\mathrm{cl},i}(\rho)x_{\mathrm{cl}}+D_{\mathrm{cl},i}(\rho)w\end{cases}\quad\forall\rho\in\mathcal{P}_i,\quad(1)$$

where $x_{\mathrm{cl}}\in\mathbb{R}^{n_x}$ is the state, $w\in\mathbb{R}^{n_w}$ is the disturbance and $z\in\mathbb{R}^{n_z}$ is the controlled signal. The evolution of the index i describes a piecewise constant function $\sigma(t)$ taking values in $\mathbb{Z}_n$. This switching signal indicates the active $G_{\mathrm{cl},i}(\rho)$ system at any time and permits us to express the dynamic behaviour as

$$G_{\mathrm{cl},\sigma}(\rho):\begin{cases}\dot{x}_{\mathrm{cl}}=A_{\mathrm{cl},\sigma}(\rho)x_{\mathrm{cl}}+B_{\mathrm{cl},\sigma}(\rho)w\\ z=C_{\mathrm{cl},\sigma}(\rho)x_{\mathrm{cl}}+D_{\mathrm{cl},\sigma}(\rho)w\end{cases}\quad\forall\rho\in\mathcal{P}.\quad(2)$$

The switching logic imposes the change of system depending on the parameter value. Therefore, it also states the set of switching signals and the stability characteristics of the switched system. Under arbitrary switching signals, the proof of exponential stability requires finding a common Lyapunov function $V_\sigma(x_{\mathrm{cl}},\rho)=x_{\mathrm{cl}}^TX_{\mathrm{cl}}(\rho)x_{\mathrm{cl}}$ such that

$$X_{\mathrm{cl}}(\rho)A_{\mathrm{cl},i}(\rho)+A_{\mathrm{cl},i}^T(\rho)X_{\mathrm{cl}}(\rho)+\dot{X}_{\mathrm{cl}}(\rho)<0$$
$$\forall\rho\in\mathcal{P}_i\quad\forall i\in\mathbb{Z}_n,\quad(3)$$

where $\dot{X}_{\mathrm{cl}}(\rho)=\mathrm{d}X_{\mathrm{cl}}(\rho)/\mathrm{d}t$ and $\underline{\nu}_l\leq\dot{\rho}_l\leq\overline{\nu}_l$ for $l=1,2,\ldots,n_\rho$.

This strong condition can be relaxed by limiting the set of switching signal with particular logics, such as hysteresis or average dwell time (Lu and Wu 2004). With a smaller set of switching signals, it is possible to employ piecewise or multiple Lyapunov functions. Nevertheless, these functions are not completely independent of each other. The functions must also satisfy additional constraints in the switching surfaces in order to guarantee global stability (Lu and Wu 2004; Lu et al. 2006).

3. Switched LPV control based on the Youla parametrisation

3.1 *Problem statement*

Consider an open-loop LPV system described by

$$G(\rho):\begin{cases}\dot{x}=A(\rho)x+B_1(\rho)w+B_2(\rho)u,\\ z=C_1(\rho)x+D_{11}(\rho)w+D_{12}(\rho)u,\\ y=C_2(\rho)x+D_{21}(\rho)w,\end{cases}\quad(4)$$

where $x\in\mathbb{R}^{n_s}$ is the state, $u\in\mathbb{R}^{n_u}$ is the control input and $y\in\mathbb{R}^{n_y}$ is the measured output. The system matrices are continuous and bounded functions of a parameter ρ measurable in real time. It is assumed that ρ takes values in a compact set $\mathcal{P}\subset\mathbb{R}^{n_\rho}$ and no bounds are imposed on the parameter rates. As usual, the pairs $(A(\rho),B_2(\rho))$ and $(A(\rho),C_2(\rho))$ are assumed quadratically stabilisable and detectable, respectively (Wu et al. 1996).

The parameter set $\mathcal{P}$ is divided into a finite number of closed subsets $\{\mathcal{P}_i\}_{i\in\mathbb{Z}_n}$ with $\mathcal{P}=\bigcup\mathcal{P}_i$. These subsets are considered non-overlapped, i.e. $S_{ij}=S_{ji}$. The objective is to formulate a methodology to design a family of n LPV controllers

$$K_i(\rho):\begin{cases}\dot{x}_K=A_{K,i}(\rho)x_K+B_{K,i}(\rho)\,y,\\ u=C_{K,i}(\rho)x_K+D_{K,i}(\rho)\,y,\end{cases}\quad i\in\mathbb{Z}_n\quad(5)$$

with $x_K\in\mathbb{R}^{n_k}$. Each controller must fulfil the performance specifications in the corresponding subset $\mathcal{P}_i$ whereas stability is guaranteed during the controller switching. Notice that the resulting closed-loop system fits the switched LPV system definition in (2).

The partition of the parameter range $\mathcal{P}$ provides additional flexibility during the controller design. For example, in the cases of LPV plants with substantial dynamic changes, a suitable partition of $\mathcal{P}$ may produce a higher performance controller and it may even be decisive for finding a solution. On the other hand, in situations with a considerable number of parameters, a clever subdivision of the parameter envelope can produce more tractable problems that would require less computational effort (Lee 1997).

3.2 *Switched LPV controller design*

Synthesis procedures for the previous problem have been proposed in Wu (2001), Lu and Wu (2004) and Lu et al. (2006). These procedures basically involve satisfying the set of LMI conditions used in the single controller case for each region plus a new set of constraints that the Lyapunov functions must fulfil in the switching surfaces. In general, these constraints are nonconvex due to the fact that these synthesis procedures address the stability and performance simultaneously. Furthermore, the n LMI sets must be solved simultaneously which may result in a highly demanding computational problem.

We propose a different approach based on the Youla parametrisation ideas. The new control scheme can be seen in Figure 1, where

$$M(\rho):\begin{cases}\dot{x}_M=(A(\rho)+B_2(\rho)F(\rho)\\ \qquad +L(\rho)C_2(\rho))x_M-L(\rho)y+B_2(\rho)v,\\ u=F(\rho)x_M+v,\\ h=-C_2(\rho)x_M+y,\end{cases}$$

and

$$Q_\sigma(\rho):\begin{cases}\dot{x}_Q=A_{Q,\sigma}(\rho)x_Q+B_{Q,\sigma}(\rho)h,\\ v=C_{Q,\sigma}(\rho)x_Q+D_{Q,\sigma}(\rho)h\end{cases} \tag{6}$$

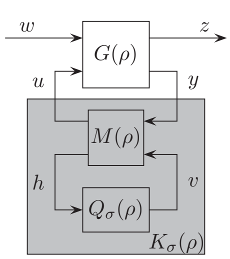

Figure 1. Proposed switched LPV control structure.

is any stable switched LPV system. As will be shown next, this control structure presents similar stability properties to other Youla parametrisations. That is, the stability of

$$J(\rho)=\mathcal{F}_l(G(\rho),M(\rho))$$

is not affected by the inclusion of any stable switched LPV system $Q_\sigma(\rho)$. This property allows the decomposition of the controller design into two steps. Firstly, a pre-compensator $M(\rho)$ is found in order to guarantee stability in the entire operating range $\mathcal{P}$. Then in the subsequent step, the parameters $Q_i(\rho)$ for achieving the desirable performance in each subset are designed. Observe that in this control scheme only $Q_\sigma(\rho)$ is a switched LPV system.

To formalise the previous ideas, the exponential stability of the switched closed-loop system

$$G_{\mathrm{cl},\sigma}(\rho)=\mathcal{F}_l(J(\rho),Q_\sigma(\rho))$$

needs to be proved. With this aim, a matrix function $X_{\mathrm{cl}}>0$ such that

$$X_{\mathrm{cl}}A_{\mathrm{cl},\sigma}(\rho)+A^T_{\mathrm{cl},\sigma}(\rho)X_{\mathrm{cl}}<0 \tag{7}$$

for all $\rho\in\mathcal{P}$ must be found (Lu and Wu 2004).[1]

After some system manipulations and a similarity transformation, it can be shown that

$$A_{\mathrm{cl},\sigma}(\rho)=\begin{bmatrix}A(\rho)+B_2(\rho)F(\rho) & B_2(\rho)C_{Q,\sigma}(\rho) & (B_2(\rho)F(\rho)-B_2(\rho)D_{Q,\sigma}(\rho)C_2(\rho))\\ 0 & A_{Q,\sigma}(\rho) & -B_{Q,\sigma}(\rho)C_2(\rho)\\ 0 & 0 & A(\rho)+L(\rho)C_2(\rho)\end{bmatrix}$$

(see e.g. Xie and Eisaka 2004). According to Lemma A.1 (see the Appendix), due to the triangular structure of $A_{cl,\sigma}(\rho)$, if there exist three positive definite matrices Y_1, X_Q and X_2 such that

$$Y_1(A(\rho)+B_2(\rho)F(\rho))+(A(\rho)+B_2(\rho)F(\rho))^TY_1<0, \tag{8}$$

$$X_2(A(\rho)+L(\rho)C_2(\rho))+(A(\rho)+L(\rho)C_2(\rho))^TX_2<0, \tag{9}$$

for all $\rho\in\mathcal{P}$ and

$$X_QA_{Q,i}(\rho)+A_{Q,i}(\rho)^TX_Q<0, \tag{10}$$

for all $\rho\in\mathcal{P}_i$ and $i\in\mathbb{Z}_n$, the constraint (7) is satisfied with $X_{\mathrm{cl}}=\mathrm{diag}(Y_1,X_Q,X_2)$.

The simple change of variables $Y_1=X_1^{-1}$, $V(\rho)=F(\rho)X_1$ and $W(\rho)=X_2L(\rho)$ turns (8) and (9) into the following convex constraints:

$$A(\rho)X_1+B_2(\rho)V(\rho)+(A(\rho)X_1+B_2(\rho)V(\rho))^T<0, \tag{11}$$

$$X_2A(\rho)+W(\rho)C_2(\rho)+(X_2A(\rho)+W(\rho)C_2(\rho))^T<0, \tag{12}$$

for all $\rho \in \mathcal{P}$. Notice that the plant (4) always satisfies these LMI constraints since by hypothesis it is quadratically stabilisable and detectable (Wu et al. 1996).

On the other hand, there is no need to compute the parameters $Q_i(\rho)$'s simultaneously to fulfil the condition (10). Actually, the LMI in (10) is held automatically if each $Q_i(\rho)$ is chosen quadratically stable $\forall \rho \in \mathcal{P}_i$, a condition that it must also fulfil to preserve stability in the subset $\mathcal{P}_i$. This fact is a consequence of Lemma A.2 (see the Appendix) which, using similar arguments to those introduced by Hespanha and Morse (2002), proves that it is always possible to find state transformations T_i such that the switched LPV system associated with

$$Q_i(\rho): \begin{cases} \dot{x}_Q = T_i A_{Q,i}(\rho) T_i^{-1} x_Q + T_i B_{Q,i}(\rho) h, \\ v = C_{Q,i}(\rho) T_i^{-1} x_Q + D_{Q,i}(\rho) h, \end{cases} \tag{13}$$

is exponentially stable. Therefore, each $Q_i(\rho)$ can be designed independently of the other ones and thus the performance achieved in each $\mathcal{P}_i$ is not affected by the $Q_i(\rho)$ corresponding to other subsets.

Each parameter $Q_i(\rho)$ can be designed by applying any standard LPV synthesis procedure to the plant

$$J(\rho): \begin{cases} \dot{x}_J = A_J(\rho) x_J + B_{J1}(\rho) w + B_{J2}(\rho) v, \\ z = C_{J1}(\rho) x_J + D_{J11}(\rho) w + D_{J12}(\rho) v, \\ h = C_{J2}(\rho) x_J + D_{J21}(\rho) w, \end{cases} \tag{14}$$

where the values of ρ are restricted to the subset $\mathcal{P}_i$ during the design of $Q_i(\rho)$ and $x_J^T = [x^T \ \ x_M^T]$,

$$\begin{aligned} A_J(\rho) &= \begin{bmatrix} A(\rho) + B_2(\rho)F(\rho) & B_2(\rho)F(\rho) \\ 0 & A(\rho) + L(\rho)C_2(\rho) \end{bmatrix}, \\ B_{J1}(\rho) &= \begin{bmatrix} B_1(\rho) \\ -(B_1(\rho) + L(\rho)D_{21}(\rho)) \end{bmatrix}, \\ B_{J2}(\rho) &= \begin{bmatrix} B_2(\rho) \\ 0 \end{bmatrix}, \\ C_{J1}(\rho) &= \begin{bmatrix} C_1(\rho) - D_{12}(\rho)F(\rho) & D_{12}(\rho)F(\rho) \end{bmatrix}, \\ C_{J2}(\rho) &= \begin{bmatrix} 0 & -C_2(\rho) \end{bmatrix}, \\ D_{J11}(\rho) &= D_{11}(\rho), \quad D_{J12}(\rho) = D_{12}(\rho), \\ D_{J21}(\rho) &= D_{21}(\rho). \end{aligned}$$

The Youla parameter $Q_i(\rho)$ is computed as a standard LPV controller valid in the subset $\mathcal{P}_i$. According to Theorem 1 in Xie and Eisaka (2004), the structure of $M(\rho)$ ensures that any $Q_i(\rho)$ quadratically stabilising $J(\rho)$ is also quadratically stable.

The particular synthesis procedure employed in the design of $Q_i(\rho)$ depends on the performance objectives. For example, measuring the performance as $\|z\|_2 < \gamma \|w\|_2$, the LPV system $Q_i(\rho)$ can be obtained from solving a convex optimisation problem with the following constraints:

$$\mathcal{N}_{R_i}^T \begin{bmatrix} R_i A_J(\rho)^T + A_J(\rho) R_i & R_i C_{J1}^T(\rho) & B_{J1}(\rho) \\ C_{J1}(\rho) R_i & -\gamma_i I_{n_z} & D_{J11}(\rho) \\ B_{J1}^T(\rho) & D_{J11}^T(\rho) & -\gamma_i I_{n_w} \end{bmatrix} \mathcal{N}_{R_i} < 0, \tag{15}$$

$$\mathcal{N}_{S_i}^T \begin{bmatrix} A_J(\rho)^T S_i + S_i A_J(\rho) & S_i B_{J1}(\rho) & C_{J1}^T(\rho) \\ B_{J1}^T(\rho) S_i & -\gamma_i I_{n_w} & D_{J11}^T(\rho) \\ C_{J1}(\rho) & D_{J11}(\rho) & -\gamma_i I_{n_z} \end{bmatrix} \mathcal{N}_{S_i} < 0, \tag{16}$$

$$\begin{bmatrix} R_i & I_{n_J} \\ I_{n_J} & S_i \end{bmatrix} \geq 0, \tag{17}$$

with $\mathcal{N}_{R_i} = \ker[B_{J2}^T(\rho) \ \ D_{J12}^T(\rho) \, 0]$ and $\mathcal{N}_{S_i} = \ker[C_{J2}(\rho) \ \ D_{J21}(\rho) \ \ 0]$, and the values of ρ restricted to the subset $\mathcal{P}_i$, $i \in \mathbb{Z}_n$. Then, the system matrices of $Q_i(\rho)$ are computed from R_i and S_i (see Wu et al. (1996) and Apkarian and Adams (1998) for more details). In order to obtain the n Youla parameters, the previous synthesis procedure must be repeated n times but there is no need to solve them simultaneously.

Once these Youla parameters are computed, it just remains to find the state transformations T_i to modify the realisations in order to guarantee the exponential stability of $Q_\sigma(\rho)$ and $G_{\mathrm{cl},\sigma}(\rho)$ (see Lemma A.2). These transformations do not depend on the parameter ρ and thus they do not affect the stability and performance characteristics achieved during the computation of the $Q_i(\rho)$'s.

To sum up, the proposed design procedure reduces to the following two steps:

(1) Find two positive definite matrices X_1, X_2 and matrices $V(\rho)$ and $W(\rho)$ such that LMIs (11) and (12) are satisfied. Then, compute $F(\rho) = V(\rho) X_1^{-1}$ and $L(\rho) = X_2^{-1} W(\rho)$ and construct the pre-compensator $M(\rho)$.
(2) Find one quadratically stable $Q_i(\rho)$ for each subset $\mathcal{P}_i$ such that the performance specifications are fulfilled. Each parameter $Q_i(\rho)$ can be designed by applying any standard LPV synthesis procedure to the plant $J(\rho)$ for all $\rho \in \mathcal{P}_i$. Finally, compute the state transformations T_i according to Lemma A.2.

With the parameters $Q_i(\rho)$ previously obtained, the system matrices of the controllers (5) are given by

$$A_{K,i}(\rho) = \begin{bmatrix} A(\rho) + B_2(\rho)F(\rho) + L(\rho)C_2(\rho) - B_2(\rho)D_{Q,i}(\rho)C_2(\rho) & B_2(\rho)C_{Q,i}(\rho)T_i^{-1}(\rho) \\ T_iB_{Q,i}(\rho)C_2(\rho) & T_iA_{Q,i}(\rho)T_i^{-1}(\rho) \end{bmatrix},$$
$$B_{K,i}(\rho) = \begin{bmatrix} B_2(\rho)D_{Q,i}(\rho) - L_i(\rho) \\ T_iB_{Q,i}(\rho) \end{bmatrix},$$
$$C_{K,i}(\rho) = \begin{bmatrix} F_i(\rho) - D_{Q,i}(\rho)C_2(\rho) & C_{Q,i}(\rho)T_i^{-1}(\rho) \end{bmatrix},$$
$$D_{K,i}(\rho) = D_{Q,i}(\rho),$$

with $\rho \in \mathcal{P}_i$, $i \in \mathbb{Z}_n$. These controllers can be arbitrarily switched without affecting the stability of $G_{cl,\sigma}(\rho)$.

It is worth emphasising that the particular choice of X_{cl} does not limit the existence of a stabilising $M(\rho)$ since (8)–(10) always hold by hypothesis. On the other hand, the local performance in each subset $\mathcal{P}_i$ is not affected by the particular choice of $F(\rho)$ and $L(\rho)$ because the parametrisation $K(\rho) = \mathcal{F}_l(M(\rho), Q(\rho))$ describes all quadratically stabilising controllers.

3.3 *Implementation aspects*

The most considerable difference with previous results is the decomposition of the design into two steps in order to reduce the solution to a couple of convex optimisation conditions. By separating the stability from the performance problem, a set of synthesis procedures that can be solved with available tools is obtained. Another positive point of this decomposition is the reduction of the number of variables and constraints in each optimisation problem, which allows its application to LPV plants with more parameters.

Furthermore, the new procedure does not force the use of parameter-dependent Lyapunov functions in each subset. Since $Q_i(\rho)$ must be quadratically stable, the parameterisation

$$K_i(\rho) = \mathcal{F}_l(M(\rho), Q_i(\rho))$$

describes only those controllers that quadratically stabilise the plant $G(\rho)$ for all $\rho \in \mathcal{P}_i$. Therefore, the existence of a constant Lyapunov matrix is guaranteed after finding each $Q_i(\rho)$. As a consequence, the online computations needed to obtain the control signal are substantially simpler than the parameter-dependent versions. Note that in previous results (Wu 2001; Lu and Wu 2004; Lu et al. 2006), the use of parameter-dependent Lyapunov functions is essential, otherwise the synthesis reduces to the single LPV controller design. Therefore, the online implementation in these cases becomes more complex than in the procedure presented here. On the other hand, the simplicity gained on the synthesis and on the implementation may produce a lower performance, in cases where a larger class of Lyapunov functions is considered. Recall that in LPV synthesis the achieved performance depends on the type of Lyapunov matrices employed (Wu et al. 1996).

The number of states of the resulting controller may become large in the cases of high order plants. Using standard LPV synthesis algorithms to design the $Q_i(\rho)$'s, the final order can reach $4n_s$. However, the order can be lower if the plant in (4) includes stable non-controllable or non-observable states, such as those added to consider performance specifications. Typically, the plant $G(\rho)$ includes weighting functions in order to translate the performance specifications into the LPV synthesis format. In this circumstance, only the controllable and observable states need to be considered during the computation of $F(\rho)$ and $L(\rho)$ and thus the order of $M(\rho)$ can be lower than n_s. In the extreme case where the plant is quadratically stable, the matrix gains $F(\rho)$ and $L(\rho)$ can be chosen equal to zero and the pre-compensator reduces to

$$M(\rho) : \begin{cases} \dot{x}_M = A(\rho)x_M + B_2(\rho)v, \\ u = v, \\ h = -C_2(\rho)x_M + y. \end{cases}$$

With this pre-compensator, in cases such as mixed sensitivity problems, it is possible to formulate an equivalent version of $J(\rho)$ of order n_s with the only aim of computing the $Q_i(\rho)$'s. Due to this reformulation, the order of the parameter $Q_i(\rho)$ is n_s and thus, the order of the resulting controller is $2n_s$.

4. AMB example

In order to illustrate the proposed methodology, we analyse the control of an AMB system. The example is borrowed from Lu and Wu (2004) and a detailed description of the system can be found in Mohamed and Busch-Vishniac (1995).

The AMB system consists of a rotor suspended by four pairs of electromagnets. The opposite electromagnetic forces maintain the rotor levitating in the centre line allowing high rotation speeds without mechanical contact and lubrication. The dynamic behaviour can be described, after some simplifications, by the following LPV model:

$$G(\rho): \begin{cases} \dot{x} = A(\rho)x + Bu, \\ y = Cx, \end{cases} \tag{18}$$

where the state-space matrices are

$$A(\rho) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{4c_2}{m} & 0 & 0 & -\frac{\rho J_a}{J_r} & \frac{2c_1}{m} & 0 \\ 0 & -\frac{4c_2}{m} & \frac{\rho J_a}{J_r} & 0 & 0 & \frac{2c_1}{m} \\ \frac{2d_2}{m} & 0 & 0 & 0 & -\frac{d_1}{m} & 0 \\ 0 & \frac{2d_2}{m} & 0 & 0 & 0 & -\frac{d_1}{m} \end{bmatrix},$$

$$B = \frac{1}{N}\begin{bmatrix} 0_{4\times 2} \\ I_2 \end{bmatrix},$$

$$C = \begin{bmatrix} I_2 & 0_{2\times 4} \end{bmatrix}.$$

The state vector is $x^T = \begin{bmatrix} l\theta & l\psi & l\dot{\theta} & l\dot{\psi} & \phi_\theta & \phi_\psi \end{bmatrix}$, the disturbance is $w^T = [f_{d\theta}\ f_{d\psi}]$, and the control action is $u^T = [e_\theta\ e_\psi]$. The angles θ and ψ indicate the orientation of the rotor centre line, and ϕ_θ and ϕ_ψ denote the differential fluxes produced by the electromagnetic pairs. The disturbance w is consequence of imbalances, modelling errors, etc. The orientation of the centre line can be controlled by means of the differential voltages e_θ and e_ψ applied to the electromagnet pairs. The symbol ρ represents the rotor speed, which ranges from 350 rad/s to 1100 rad/s and is assumed measurable in real time. A detailed explanation of the rest of the parameters can be found in Mohamed and Busch-Vishniac (1995).

The system is open-loop unstable; therefore, the first control objective is to stabilise it. The second objective is to minimise the gap displacements caused by the disturbances with a reasonable control effort. These control specifications are translated into the performance constraint $\|z\|_2 < \gamma\|w\|_2$ by augmenting the plant with weights as shown in Figure 2 where

$$W_y(s) = \frac{10(s+8)}{s+0.001}I_2, \quad W_u(s) = \frac{0.01(s+100)}{s+100000}I_2,$$
$$W_w(s) = 0.001 I_2.$$

The parameter set $\mathcal{P}$ has been divided into two sets $\mathcal{P}_1 = [315\ \ 720]$ and $\mathcal{P}_2 = [720\ \ 1100]$. Following the proposed procedure, firstly we found the matrix gains $F(\rho)$ and $L(\rho)$ by solving the LMI optimisation problem (11)–(12) and then the pre-compensator

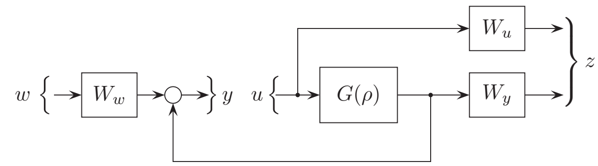

Figure 2. Control scheme for the controller design.

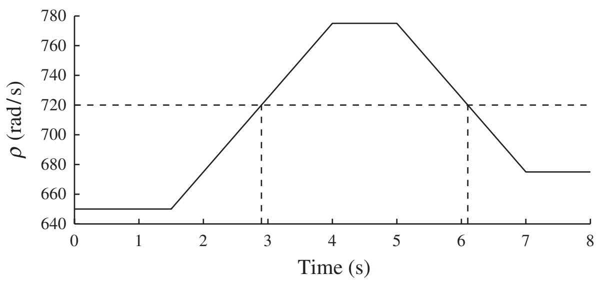

Figure 3. Parameter trajectory used in the simulations.

$M(\rho)$ is constructed. In this case, $F(\rho)$ and $L(\rho)$ cannot be zero because the plant is not quadratically stable. Once the stability is guaranteed, the parameter $Q_i(\rho)$'s are obtained by solving two independent convex optimisation problem given by (15)–(17). The achieved performance levels were $\gamma_1 = 3.31$ and $\gamma_2 = 3.32$, respectively. By comparison, the application of traditional LPV synthesis procedures for the whole parameter space ($\mathcal{P} = [315\ 1100]$) gives a performance level of 6.66, a worse result as compared with the switched option.

The simulation results in Figure 4 show the response to steps of 0.001 amplitude applied at the disturbance inputs, whereas the parameter trajectory follows the profile shown in Figure 3. It is worth noting the absence of glitches, even though the parameter trajectory crosses the switching surface at 2.9 s and 6.1 s. This is due to the fact that the parameters $Q_i(\rho)$'s are only active during the transient and in Figure 4 the system has reached the stationary state before the switching occurred. In this situation, only the pre-compensator focused on preserving stability is active, which is never switched. This is another difference with previous switching strategies where the whole controller is switched and then the glitches always arise.

In Figure 5 the closed-loop system has been excited with persistent signals of the form

$$w_1 = \tilde{w}\sin(\rho t + \tau_1),$$
$$w_2 = \tilde{w}\sin(\rho t + \tau_2),$$

where τ_i are initial phases and $\tilde{w} = 1.3 \times 10^{-6}$, which represent small imbalances. The parameter trajectory corresponds to the same signal shown in Figure 3.

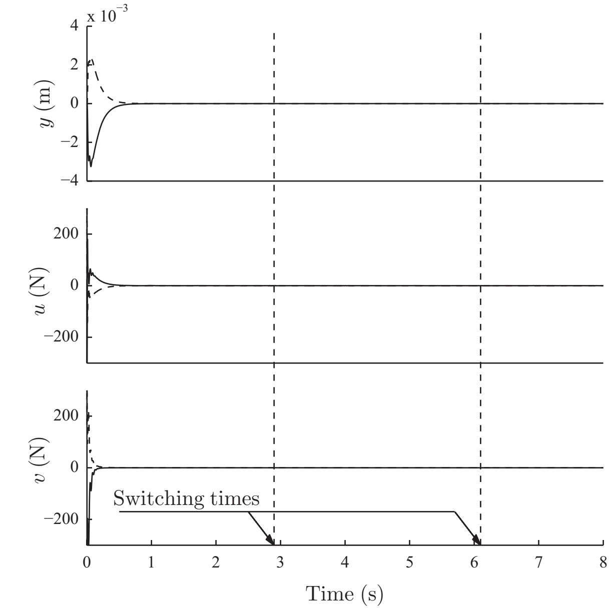

Figure 4. Step responses of the closed-loop system under the parameter trajectory in Figure 3.

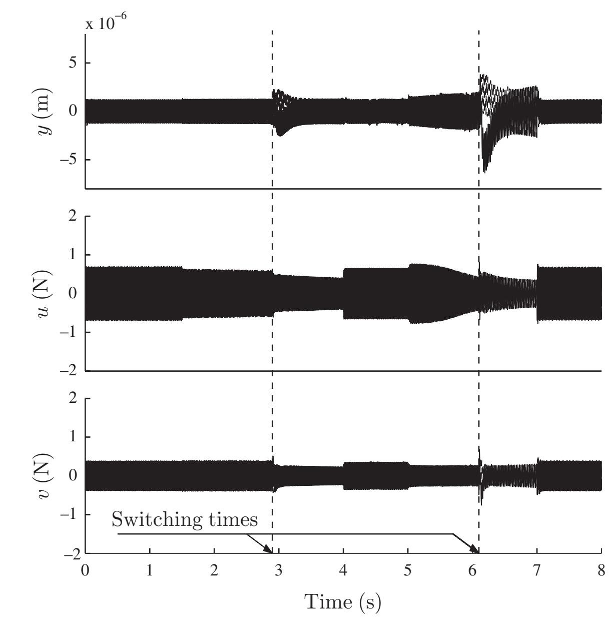

Figure 5. Responses of the closed-loop system under the parameter trajectory in Figure 3.

It can be seen that due to the persistency of the excitation the signal v is always non-zero and the effect of the controller switching is now visible on the output signal. However, the simulation shows that the stability is preserved and the transient due to the switching vanishes in a reasonable time.

5. Conclusions

A new synthesis procedure for switched LPV controllers has been discussed. Based on the separation principle provided by the Youla control structure, the design is divided into two steps. Firstly, the global stabilising pre-compensator is obtained and then a set of Youla parameters is designed to achieve the desirable performance in each subset in which the entire scheduling parameter envelope has been partitioned. Compared with previous results, the offline and online computational procedure in the proposed control structure is less demanding, but in certain situations may result to be more conservative. The controller computation is decomposed into several convex optimisation problems with a smaller number of variables and constraints, which makes the design more tractable for plants with large number of parameters. On the other hand, the use of constant Lyapunov matrices in the computation of the Youla parameter simplifies the controller online implementation. The new controller structure also exhibits a better behaviour during the switching. Due to the fact that only the parameter $Q_i(\rho)$ switches, the presence of glitches is now less visible.

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Note

1. The Lyapunov function is considered parameter independent because no bounds are assumed on the parameter rates.

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Appendix. Some useful results

Some technical results used through the article are presented in this section.

Lemma A.1: *If there exist $X_1>0$ and $X_2>0$ such that*

$$X_1A_{11}(\rho)+A_{11}^T(\rho)X_1<0$$
$$X_2A_{22}(\rho)+A_{22}^T(\rho)X_2<0$$

then there exists $X>0$ such that

$$XA(\rho)+A^T(\rho)X<0, \tag{A1}$$

where

$$A(\rho)=\begin{bmatrix} A_{11}(\rho) & A_{12}(\rho) \\ 0 & A_{22}(\rho)\end{bmatrix}$$

with $A_{12}(\rho)$ a bounded function of $\rho\in\mathcal{P}$.

Proof: With $X=\mathrm{diag}(\alpha X_1, X_2)$ and using Schur complements, the condition (A1) is equivalent to

$$\alpha M(\rho)=\alpha(X_1A_{11}(\rho)+A_{11}^T(\rho)X_1)<0, \tag{A2}$$

$$X_2A_{22}(\rho)+A_{22}^T(\rho)X_2-\alpha\underbrace{X_2A_{12}(\rho)M^{-1}(\rho)A_{12}^T(\rho)X_2}_{N(\rho)}<0. \tag{A3}$$

The $X_2A_{12}(\rho)$ is a possibly rectangular or singular matrix but a bounded function of ρ. Since $M^{-1}(\rho)<0$, it can be stated that $N(\rho)\leq 0$. Then, it is always possible to find a $\alpha>0$ such that (A2) and (A3) are satisfied. This proves the existence of a X such that (A1) holds. □

Lemma A.2: *Given the set of LPV systems*

$$G_i(\rho):\begin{cases}\dot{x}=A_i(\rho)x+B_i(\rho)w\\ z=C_i(\rho)x+D_i(\rho)w\end{cases}\quad\forall\rho_i\in\mathcal{P}_i \tag{A4}$$

If all $G_i(\rho)$ are quadratically stable, there always exist state-space transformations T_i's such that the switched LPV systems formed by the $G_i(\rho)$'s is exponential stable under arbitrary switching.

The proof is similar to the LTI case addressed in Hespanha and Morse (2002) and is sketched here in order to illustrate the computation of the state-space transformations. As $G_i(\rho)$ is quadratically stable there exist constant Lyapunov matrices $X_i>0$, $i\in\mathbb{Z}_n$ such that $X_i=S_i^TS_i$. Then, defining $X=S^TS>0$ and $\tilde{A}_i(\rho)=T_iA_i(\rho)T_i^{-1}$, with $T_i=S^{-1}S_i$, the quadratic stability of each $G_i(\rho)$ ensures that

$$X_iT_i^{-1}\tilde{A}_i(\rho)T_i+T_i^T\tilde{A}_i^T(\rho)T_i^{-1}X_i<0\quad\forall\rho\in\mathcal{P}_i\quad\forall i\in\mathbb{Z}_n,$$

which is equivalent to

$$X\tilde{A}_i(\rho)+\tilde{A}_i^T(\rho)X<0\quad\forall\rho\in\mathcal{P}_i\quad\forall i\in\mathbb{Z}_n$$

after applying the congruence transformation $S_i^{-T}S$. Hence, the existence of a common Lyapunov function $V(x)=x^TXx$ has been proved and thus the exponential stability of the switched system.

Unfortunately, this result cannot be extended to parameter-dependent quadratically stable systems. This would lead to parameter-dependent state transformations which would modify the input–output characteristics of the original LPV systems.