ON A GENERALIZATION OF THE SEATING COUPLES PROBLEM

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Abstract. We prove a conjecture of Adamaszek generalizing the seating couples problem to the case of $2n$ seats. Concretely, we prove that given a positive integer n and $d_1, \dots, d_n \in (\mathbb{Z}/2n)^\times$ we can partition $\mathbb{Z}/2n$ into n pairs with differences $d_1, \dots, d_n$.

1. Introduction

Preissmann and Mischler [6] proved the following result, confirming a conjecture of R. Bacher.

Theorem 1.1. *Let $p = 2n + 1$ be an odd prime. Suppose we are given n elements $d_1, \dots, d_n \in (\mathbb{Z}/p)^\times$. Then there exists a partition of $\mathbb{Z}/p - \{0\}$ into pairs with differences $d_1, \dots, d_n$.*

We gave a simpler proof of this theorem using the Combinatorial Nullstellensatz [4]. Karasev and Petrov, independently, gave a proof of Theorem 1.1 along the same lines and provided further generalizations [3]. In this work, they also conjectured two generalizations of Theorem 1.1, replacing p by an arbitrary integer N. The conjecture in the case that N is even is originally due to Adamaszek.

Conjecture 1.2 ([3, Conjecture 1])**.** *Let $N = 2n + 1$ be a positive integer. Suppose we are given n elements $d_1, \dots, d_n \in (\mathbb{Z}/N)^\times$. Then there exists a partition of $\mathbb{Z}/N - \{0\}$ into pairs with differences $d_1, \dots, d_n$.*

We will prove the conjecture when N is even:

Theorem 2.5 ([3, Conjecture 2])**.** *Let $N = 2n$ be a positive integer. Suppose we are given n elements $d_1, \dots, d_n \in (\mathbb{Z}/N)^\times$. Then there exists a partition of $\mathbb{Z}/N$ into pairs with differences $d_1, \dots, d_n$.*

While finishing this paper we found out that, in his master's thesis [5], T.R. Mezei suggests a possible way to solve the conjecture that is similar to ours. Furthermore, he shows that Theorem 2.5 holds whenever $N = 2p$ for p a prime number.

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2. The even case

We recall the following version of the Cauchy-Davenport theorem.

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Theorem 2.1 ([1, Theorem 1])**.** *If A and B are nonempty subsets of $\mathbb{Z}/N$ where $0 \in B$, and $\gcd(b, N) = 1$ for all $b \in B \setminus \{0\}$, then*

$$|A + B| \geq \min\{N, |A| + |B| - 1\}.$$

Suppose that we have a partition as in Theorem 2.5. Since the d_i are odd numbers, each pair contains exactly one even number. Therefore, if Theorem 2.5 holds there must exist signs s_i such that

$$s_1 d_1 + \ldots + s_n d_n = 1 - 2 + 3 - \ldots + (2n-1) - 2n = n \bmod N.$$

Theorem 2.2. *Let $N = 2n$ and let $d_1, \ldots, d_n \in (\mathbb{Z}/N)^\times$. Then there exists $s_1, \ldots, s_n \in \{1, -1\}$ such that*

$$s_1 d_1 + \ldots + s_n d_n = n \bmod 2n.$$

Proof. It is enough to prove that there exists $I \subset \{1, \ldots, n\}$ such that

$$\sum_{i \in I} 2d_i = d_1 + \ldots + d_n + n \bmod 2n$$

Since d_i is odd for every i, $d_1 + \ldots + d_n + n$ is even and therefore our task is equivalent to finding I such that

$$\sum_{i \in I} d_i = \frac{d_1 + \ldots + d_n + n}{2} \bmod n.$$

Let $A_i = \{d_i, 0\}$. Applying Theorem 2.1 inductively, we see that

$$\#(A_1 + \ldots + A_n) \geq \min\left\{n, \sum \#A_i - (n-1)\right\} = n,$$

concluding the proof. □

Remark 2.3. Theorem 2.1 was stated in full strength for the benefit of the reader. However, in the previous proof we only needed to use this result in the case $|B| = 2$, which follows from the fact that $A + b = A$ and $\gcd(b, N) = 1$ imply that $A = \mathbb{Z}/N$.

The last ingredient is the following theorem due to Hall.

Theorem 2.4 ([2])**.** *Let A be an abelian group of order n and $a_1, \ldots, a_n$ be a numbering of the elements of A. Let $d_1, \ldots, d_n \in A$ be elements such that $d_1 + \ldots + d_n = 0$. Then there are permutations $\sigma, \tau \in S_n$ such that*

$$a_i - a_{\sigma(i)} = d_{\tau(i)}$$

Theorem 2.5. *Let $N = 2n$ be a positive integer. Suppose we are given n elements $d_1, \ldots, d_n \in (\mathbb{Z}/N)^\times$. Then there exists a partition of $\mathbb{Z}/N$ into pairs with differences $d_1, \ldots, d_n$.*

Proof. First, using Theorem 2.2, we may assume that $d_1 + \ldots + d_n = n \bmod 2n$. Now it is enough to find a numbering $a_1, \ldots, a_n$ of the odd numbers in $\mathbb{Z}/N$ and $\sigma \in S_n$ such that $2i - a_i = d_{\sigma(i)} \bmod 2n$ for every $i \in \{1, \ldots, n\}$, for then the partition in pairs $\{2, a_1\}, \{4, a_2\}, \ldots, \{2n, a_n\}$ works.

Equivalently, we need to find a numbering $b_1, \ldots, b_n$ of the even numbers in $\mathbb{Z}/N$ such that $2i - b_i = d_{\sigma(i)} + 1 \bmod N$ for some $\sigma \in S_n$. Now since $d_i + 1$ is even for all i, this is the same as finding a permutation $c_1, \ldots, c_n$ of $\{1, \ldots, n\}$ such that

$$i - c_i = \frac{d_{\sigma(i)} + 1}{2} \bmod n,$$

for some $\sigma \in S_n$. If we verify that

$$\frac{d_1+1}{2} + \ldots + \frac{d_n+1}{2} = 0 \bmod n$$

this will follow from Theorem 2.4. But this holds, since $d_1 + \ldots + d_n = n \bmod 2n$ and therefore $(d_1+1) + \ldots + (d_n+1) = 0 \bmod 2n$, proving that

$$\frac{d_1+1}{2} + \ldots + \frac{d_n+1}{2} = 0 \bmod n.$$

□

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