On a Ermakov-Painlevé II reduction in three-ion electrodiffusion. A Dirichlet boundary value problem

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Abstract

Two-point boundary value problems of Dirichlet type are investigated for a Ermakov-Painlevé II equation which arises out of a reduction of a three-ion electrodiffusion Nernst-Planck model system. In addition, it is shown how Ermakov invariants may be employed to solve a hybrid Ermakov-Painlevé II triad in terms of a solution of the single component integrable Ermakov-Painlevé II reduction. The latter is related to the classical Painlevé II equation.

1. Introduction

The theory of multi-ion electrodiffusion has its origin in the liquid-junction theory of Nernst [1] and Planck [2]. It provides a macroscopic description of the migration of charged particles through material barriers and has applications notably in the modelling of biological membranes [3–8] and in electrochemistry [9]. In this multi-ion transmission context, Schlögl [10] observed that it is convenient to partition the ions into m classes characterized by the same electric charge $q_j = q_0\nu_j$, where q_0 is the unit of change and ν_j is a nonzero integral signed valency. The m-ion electrodiffusion model in steady régimes then reduces to a nonlinear $m+1$-component coupled system, namely (Leuchtag [11])

$$\begin{aligned} \frac{dn_i}{dx} &= \nu_i n_i p - c_i \ , \quad i = 1, \cdots m \\ \frac{dp}{dx} &= \sum_{i=1}^{m} \nu_i n_i \ , \quad (\nu_i - \nu_j)\nu_i n_i \neq 0 \qquad i \neq j \ , \end{aligned} \tag{1.1}$$

where x is a coordinate normal to the planar boundaries, p is the electric field and n_j is the number of ions with the same charge $q_j = q_0\nu_j$. Here, the c_i are constants each proportional to $\sum J_{ij}/u_{ij}$ where the J_{ij} are the current densities and the u_{ij} are the ion mobilities (see Leuchtag [11])

The two-ion case when $\nu_1 + \nu_2 = 0$ was originally investigated by Grafov and Chernenko [12] and independently by Bass [13]. An analogous system was subsequently derived, again independently, in the context of semi-conductor theory by Kudryashov [14]. In both cases, reduction to the integrable Painlevé II equation was obtained. This integrable connection has been subsequently exploited to apply Bäcklund transformations iteratively to generate sequences of solutions of the two-ion system (Rogers *et al* [15], Bass *et al* [16]). Such sequences have been shown by Bracken *et al* [17] to be characterized by quantized fluxes of the two ionic species with associated quantization of the electric current density.

Boundary value problems for the classical Painlevé II equation and certain extensions have been previously investigated in such diverse physical contexts as two-ion electrodiffusion, the solitonic Korteweg-de Vries equation and superconductivity (see e.g. [18]– [30]).

In three-ion electrodiffusion, existence and solvability aspects of classes of two-point boundary value problems have been investigated by Amster *et al* [31] via topological and upper and lower solution methods. In [32], Conte *et al* undertook an *ab initio* Painlevé analysis of the general m-ion system (1.1). Interestingly, it was established that in a pair of three-ion cases with valency ratios $\nu_1 : \nu_2 : \nu_3 = 1 : -2 : -1$, reduction may be made to a well-known integrable Painlevé-Gambier equation, namely (see e.g. Gromak [33])

$$v_{xx} - \frac{v_x^2}{2v} + xv + 2\epsilon v^2 = -\frac{(\alpha - \epsilon/2)^2}{2v} \quad , \qquad (\epsilon^2 = 1) \tag{1.2}$$

where v is connected to the electric field p via a relation of the type

$$p \sim v_x / v \quad . \tag{1.3}$$

Remark 1.

In Conte *et al* [32], the three-ion case $\nu_1 : \nu_2 : \nu_3 = 1 : 1 : 1$ was not isolated as leading to underlying integrable Painlevé structure in the Nernst-Planck system and is outside the scope of the present investigation.

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On setting $v = \mu^2$, it is seen that (1.2) adopts a hybrid Ermakov-Painlevé II form in μ, namely

$$\mu_{xx} + (x/2)\mu + \epsilon\mu^3 = -\frac{(\alpha - \epsilon/2)^2}{4\mu^3} \quad . \tag{1.4}$$

Here, we shall be concerned with existence and uniqueness properties of solutions to Dirichlet boundary value problems for this kind of equation which also has recently been shown to arise in the nonlinear elastodynamics of transverse wave propagation in generalised Mooney-Rivlin [34].

It is recalled that (Gromak [33])

$$\omega = \frac{\epsilon}{2v} \left(\alpha - \frac{\epsilon}{2} - \frac{dv}{dx}\right) \tag{1.5}$$

with v governed by (1.2), obeys the canonical Painlevé II equation

$$\omega_{xx} = 2\omega^3 + x\omega + \alpha \quad . \tag{1.6}$$

However, the avatar (1.4) of (1.6) has the advantage, that it makes connection with the hitherto unrelated subject of Ermakov-type systems (see e.g. [35–41] and works cited therein). Thus, in particular, characteristic invariants of such systems have recently been exploited in [34] to isolate integrable Ermakov-Painlevé II reductions of coupled systems of resonant nonlinear Schrödinger type [42, 43].

Thus, motivated by the genesis of the Ermakov-Painlevé II equation in a three-ion electrodiffusion system, we here investigate a class of two-point boundary value problems involving the Ermakov-Painlevé II type equation

$$\mu_{xx} = a\mu^3 + bx\mu + \frac{c}{\mu^3} \ , \quad 0 < x < 1 \tag{1.7}$$

under Dirichlet boundary conditions

$$\mu(0) = \mu_0, \qquad \mu(1) = \mu_1 \tag{1.8}$$

with $\mu_0, \mu_1 > 0$. An existence and uniqueness result is established for the class of boundary value problems with $c < 0$ when $a > 0$. Moreover, a sufficient condition for the existence of solutions is obtained for the case with $c < 0$ and $a \leq 0$. Interestingly, in the repulsive case with $c > 0$ and $a < 0$ the results differ markedly from the attractive case $c < 0$ wherein the Ermakov-Painlevé II equation may be linked via a relation of the type (1.5) to the classical integrable Painlevé II equation. Thus, in the repulsive case with $a < 0$ a winding number argument is adduced to establish a non-uniqueness result.

In the Appendix, a novel coupled nonlinear integrable triad is introduced which is shown, via its admitted Ermakov invariants, to possess a key Ermakov-Painlevé II component. The present results concerning Dirichlet boundary value problems apply *mutatis mutandis* and have potential application to the overlying triad with appropriate side conditions.

2 The Dirichlet problem

In this section, we shall prove various existence and uniqueness/multiplicity results for the boundary value problem encapsulated in (1.7)−(1.8). The non-singular case $c = 0$ has been intensively studied; thus, in the sequel we shall focus only on the case $c \neq 0$. According to the standard terminology in singular problems, the case $c > 0$ shall be called *repulsive* and the case $c < 0$ shall be called *attractive*. Note that (1.4) corresponds to this latter case, with $a = -\epsilon$, $b = -1/2$ and $c = -(\alpha - \epsilon/2)^2/4$. Moreover, in the search of classical solutions which are smooth and non-vanishing in $(0, 1)$, we may assume without loss of generality, that $\mu_0, \mu_1 \geq 0$. This is due to the fact that the right-hand side of (1.7) is an odd function of μ.

2.1 Non-vanishing solutions

In this section we shall assume $\mu_0, \mu_1 > 0$, and study the existence of strictly positive solutions. For convenience, the repulsive and attractive cases shall be analyzed separately.

2.1.1 Repulsive case

Throughout this section, we shall assume that $c > 0$.

Let us firstly consider the case $a > 0$ and define, for convenience, for $A > w_* > 0$:

$$J(w_*, A) := \int_{w_*}^{A} \frac{dw}{\sqrt{\left(\frac{a}{2}(w + w_*)w + b^+w + \frac{c}{w_*}\right)(w - w_*)}}$$

where, as usual, $b^+ := \max\{b, 0\}$. Observe that

$$J(w^*, A) \leq \frac{2\sqrt{A - w_*}}{\sqrt{aw_*^2 + b^+w_* + \frac{c}{w_*}}};$$

thus, we may extend J continuously by setting $J(w_*, w_*) := 0$. for $w_* > 0$.

Theorem 1 *Let $a, c > 0$. If there exists a positive $w_* \leq \mu_0^2, \mu_1^2$ such that*

$$J(w_*, \mu_0^2) + J(w_*, \mu_1^2) \geq 2,$$

then (1.7)-(1.8) *has at least one positive solution.*

Proof. In the first place, observe that if v is a large positive constant then $v''(x) = 0 \leq av^3 + bxv + cv^{-3}$ for all $x \in [0, 1]$. If moreover $v \geq \mu_0, \mu_1$, it follows that v is an upper solution of the problem. Thus, it suffices to prove that the problem

$$u''(x) = au(x)^3 + b^+u(x) + \frac{c}{u(x)^3} \tag{2.9}$$

$$u(0) = u_0, \quad u(1) = u_1 \tag{2.10}$$

admits a positive solution u for some $u_j \in (0, \mu_j]$, $j = 0, 1$. Indeed, in this case

$$u''(x) \geq au(x)^3 + bxu(x) + \frac{c}{u(x)^3}, \qquad u(0) \leq \mu_0, \quad u(1) \leq \mu_1$$

and hence u serves as a lower solution of the original problem. By standard results (see e.g. [44, II.4]), existence of a solution μ with $u \leq \mu \leq v$ is deduced.

We shall now prove that the assumption of Theorem 1, in fact, is necessary and sufficient for the existence of such a lower solution. For simplicity, from now on we may assume, without loss of generality, that $b \geq 0$. Multiply (2.9) by $u'(x)$ and integrate to obtain

$$u'(x)^2 = \frac{a}{2}u(x)^4 + bu(x)^2 - \frac{c}{u(x)^2} + d$$

for some constant d. Setting $w := u^2$,

$$w'(x)^2 = 2aw(x)^3 + 4bw(x)^2 - 4c + 4dw(x).$$

Remark 2.

It is observed that the latter may be reduced to the canonical Weierstrass elliptic function $\wp$ equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

under a Möbius transformation

$$w = \frac{r_1\wp + r_3}{r_2\wp + r_4} \qquad r_1r_4 - r_2r_3 \neq 0$$

for appropriate constants r_i, $i = 1, \ldots, 4$. In the above, g_2 and g_3 are the classical Weierstrass invariants.

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Suppose that u is a classical solution of (2.9)-(2.10). Since the right-hand side of the equation is positive, it follows that u (and consequently w) is strictly convex. Thus, w achieves its (unique) absolute minimum $w_* > 0$ at some $x_* \in [0, 1]$.

Assume firstly that $x^* \in (0, 1)$, then using the fact that $w'(x_*) = 0$ we may write the constant d as

$$d = d(w_*) := -\frac{a}{2}w_*^2 - bw_* + \frac{c}{w_*}$$

and hence

$$w'(x)^2 = \left(2aw(x)(w(x) + w_*) + 4bw(x) + \frac{4c}{w_*}\right)(w(x) - w_*),$$

that is

$$w'(x) = -2S(x) \qquad \text{for } x \leq x_*$$

and

$$w'(x) = 2S(x) \qquad \text{for } x \geq x_*,$$

where

$$S(x) := \sqrt{\left(\frac{a}{2}(w(x) + w_*)w(x) + bw(x) + \frac{c}{w_*}\right)(w(x) - w_*)}.$$

This implies that

$$J(w_*, u_0^2) = -\int_0^{x_*} \frac{w'(x)}{S(x)}\, dx = 2x_*,$$

$$J(w_*, u_1^2) = \int_{x_*}^1 \frac{w'(x)}{S(x)}\, dx = 2(1 - x_*)$$

and hence

$$J(w_*, u_0^2) + J(w_*, u_1^2) = 2.$$

Conversely, if the latter equality holds for some $w_* < u_0^2, u_1^2$, then let $x_* := \frac{J(w_*,u_0^2)}{2} \in (0, 1)$, and let a function w be implicitly defined by the equations

$$J(w_*, w(x)) = 2x \qquad x \leq x_*$$

$$J(w_*, w(x)) = 2(1-x) \qquad x \geq x_*.$$

Observe that w is non-increasing in $[0, x_*]$ and non-decreasing in $[x_*, 1]$; thus, by standard arguments it is deduced that w is smooth and $u(x) := \sqrt{w(x)}$ is a solution of (2.9)-(2.10). Finally, observe that

$$J(w_*, A) \leq 2\sqrt{\frac{w_*}{c}}\sqrt{A - w_*} \to 0 \qquad \text{as } w_* \to 0$$

and, moreover, for each fixed A the function $J(\cdot, A)$ is continuous. Consequently, if the hypothesis of the theorem holds for some $w_* < \mu_0^2, \mu_1^2$, then making w_* smaller if necessary we obtain

$$J(w_*, \mu_0^2) + J(w_*, \mu_1^2) = 2$$

and the result follows.

Now assume that $x_* = 0$ or $x_* = 1$: in other words, w is monotone and $w_* = \min\{u_0^2, u_1^2\}$. For convenience, let us denote $M := \max\{u_0^2, u_1^2\} = \max_{x\in[0,1]} w(x)$. As before, from the equality

$$w'(x)^2 = 2aw(x)^3 + 4bw(x)^2 - 4c + 4dw(x)$$

we deduce that

$$\int_{w_*}^{w(x)} \frac{dw}{\sqrt{\frac{a}{2}w^3 + bw^2 + dw - c}} = 2x$$

and, conversely, if w is smooth and satisfies this latter equality for all x then $u := \sqrt{w}$ is a solution of the problem. Set

$$\tilde{J}(d) := \int_{w_*}^{M} \frac{dw}{\sqrt{\frac{a}{2}w^3 + bw^2 + dw - c}},$$

then:

- $\tilde{J}$ is nonincreasing.
- $\tilde{J}(d) \to 0$ as $d \to +\infty$.

We deduce that the problem has a solution if $\tilde{J}(d_*) \geq 2$, where d_* is the smallest possible value of d verifying

$$\frac{a}{2}w^3 + bw^2 + dw - c \geq 0$$

for all $w \in [w_*, M]$, that is:

$$d_* := \frac{c}{w_*} - \frac{a}{2}w_*^2 - bw_*.$$

Thus, the result follows since

$$\tilde{J}(d_*) = J(w_*, M) = J(w_*, u_0^2) + J(w_*, u_1^2).$$

∎

Remark 3.

1. Observe that

$$J(w_*, A) \geq \int_{w_*}^{A} \frac{dw}{\sqrt{\left(\frac{a}{2}(A+w_*)A + bA + \frac{c}{w_*}\right)(w-w_*)}}$$

$$= 2\sqrt{\frac{A-w_*}{\frac{a}{2}(A+w_*)A + bA + \frac{c}{w_*}}}\ .$$

   Thus, a sufficient condition for the existence of solutions is

$$\sqrt{\frac{\mu_0^2 - w_*}{\frac{a}{2}(\mu_0^2+w_*)\mu_0^2 + b\mu_0^2 + \frac{c}{w_*}}} + \sqrt{\frac{\mu_1^2 - w_*}{\frac{a}{2}(\mu_1^2+w_*)\mu_1^2 + b\mu_1^2 + \frac{c}{w_*}}} \geq 1$$

   for some $w_* \leq \mu_0^2, \mu_1^2$.

2. The previous result is still valid when $a = 0$, provided that $b > 0$. In this case, an explicit expression for $J(w_*, A)$ is readily obtained.

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Next, we shall introduce a different approach for the case $a < 0$.

Theorem 2 *Let $a < 0 < c$. Then problem* (1.7)-(1.8) *has infinitely many positive solutions. More precisely, for each $N \in \mathbb{N}$ large enough there exist at least two positive solutions that take the values μ_0 and μ_1 exactly N or $N+1$ times.*

Proof. We shall apply a shooting type argument. Consider problem (1.7) with the initial condition

$$\mu(0) = \mu_0, \qquad \mu'(0) = \lambda \tag{2.11}$$

and integrate to obtain

$$\mu'(x)^2 - \frac{a}{2}\mu(x)^4 + \frac{c}{\mu(x)^2} = d + bx\mu(x)^2 - b\int_0^x \mu(s)^2\, ds \tag{2.12}$$

where

$$d = \lambda^2 + \frac{c}{\mu_0^2} - \frac{a}{2}\mu_0^4.$$

It is seen from (2.12) that μ cannot take arbitrarily large values and, consequently, μ' and μ^{-1} are also bounded. In other words, there exist $0 < \delta(\lambda) < R(\lambda)$ such that

$$\delta \leq \mu(x) \leq R, \qquad |\mu'(x)| \leq R$$

for all x. In particular, μ is defined on $[0,1]$, so it suffices to search for those values of λ such that $\mu(1) = \mu_1$. Also from (2.12), observe that if $M := \max_{x\in[0,1]} \mu(x)$ and $|\lambda| \gg 0$, then M cannot be larger than $O(|\lambda|^{1/2})$. Moreover, if $x_c \in (0,1)$ is a critical point of μ, then

$$-\frac{a}{2}\mu(x_c)^4 + \frac{c}{\mu(x_c)^2} = d + bx_c\mu(x_c)^2 - b\int_0^{x_c} \mu(s)^2\, ds = O(\lambda^2).$$

This implies, when $|\lambda|$ is large, that either $\mu(x_c) = O(|\lambda|^{1/2})$ and μ achieves a maximum at x_c or $\mu(x_c) = O(|\lambda|^{-1/2})$ and μ achieves a minimum at x_c. In particular we deduce that between two consecutive critical points the solution takes the values μ_0 and μ_1 exactly once.

Next, we shall prove that μ cannot be monotone over large intervals. To this end, observe that if $|\lambda|$ is large enough then

$$\mu'(x)^2 - \frac{a}{2}\mu(x)^4 \geq \frac{\lambda^2}{2} \qquad \text{for all } x.$$

Now suppose that μ increases over an interval $[x_1, x_2]$, then set $\theta := \sup \mathcal{A}$, where

$$\mathcal{A} := \{x \in [x_1, x_2] : \lambda^2 + a\mu(x)^4 \geq 0\}.$$

Since $\mu'(x) \geq \sqrt{\frac{\lambda^2 + a\mu(x)^4}{2}}$ for $x \in \mathcal{A}$, it is readily seen that

$$\int_{\mu(x_1)}^{\mu(\theta)} \frac{du}{\sqrt{\lambda^2 + au^4}} \geq \frac{\sqrt{2}}{2}(\theta - x_1)$$

and hence $\theta - x_1 = O(|\lambda|^{-1/2})$. Moreover, on $[\theta, x_2]$ we have that $\mu(x) \geq \left(\frac{\lambda^2}{-a}\right)^{1/4}$ and hence

$$\mu'(\theta) = \mu'(x_2) - \int_\theta^{x_2} \mu''(s)\, ds \geq \frac{(-a)^{1/4}}{2}|\lambda|^{3/2}(x_2 - \theta)$$

if $|\lambda|$ is large. As $\|\mu'\|_\infty = O(|\lambda|)$, we conclude that $x_2 - \theta = O(|\lambda|^{-1/2})$.

Next observe that, when $|\lambda|$ is large,

$$\mu'(x)^2 + \frac{c}{\mu(x)^2} \geq \frac{\lambda^2}{2} \qquad \text{for all } x$$

and assume now that μ decreases over $[x_1, x_2]$. Set $\theta := \sup \mathcal{A}$, where

$$\mathcal{A} := \{x \in [x_1, x_2] : \lambda^2\mu(x)^2 \geq 2c\}.$$

In this case, the fact that $-\sqrt{2}\mu(x)\mu'(x) \geq \sqrt{\lambda^2\mu(x)^2 - 2c}$ implies, by substitution,

$$\int_{\mu^2(\theta)}^{\mu^2(x_1)} \frac{dw}{\sqrt{\lambda^2 w - 2c}} \geq \sqrt{2}(\theta - x_1)$$

and hence $\theta - x_1 = O(|\lambda|^{-1/2})$. On the other hand, enlarging $|\lambda|$ if necessary we may write

$$\mu'(\theta) = \mu'(x_2) - \int_\theta^{x_2} \mu''(s)\, ds \leq -(x_2 - \theta)\frac{\lambda^2}{4}$$

and consequently $x_2 - \theta = O(|\lambda|^{-1})$.

As a final step, consider the curve $\gamma : [0,1] \to \mathbb{C}$ given by $\gamma(x) := \mu'(x) + i\mu(x)$. The number

$$I = I(\lambda) := Re\left(\frac{1}{2\pi i}\int_\gamma \frac{dz}{z - i\mu_1}\right)$$

counts the (net) number of turns that γ performs around the point $i\mu_1 \in \mathbb{C}$. Recall that $\gamma(0) = \lambda + i\mu_0$; thus, $\mu(1) = \mu_1$ if and only if $I(\lambda) + \frac{r(\lambda)}{2\pi} \in \frac{1}{2}\mathbb{Z}$, where $r(\lambda) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ denotes the argument of the complex number $\lambda + i(\mu_0 - \mu_1)$. Clearly, $r(\lambda) \to 0$ or π as $\lambda \to +\infty$ or $-\infty$ respectively. A simple computation shows that

$$I(\lambda) = \frac{1}{2\pi}\int_0^1 \frac{\mu'(x)^2 - \mu(x)\mu''(x)}{\mu'(x)^2 + (\mu(x) - \mu_1)^2}\,dx.$$

From the previous analysis we know that if x_1 and x_2 are two consecutive critical points of μ, then

$$\int_{x_1}^{x_2} \frac{\mu'(x)^2 - \mu(x)\mu''(x)}{\mu'(x)^2 + (\mu(x) - \mu_1)^2}\,dx = \pi$$

and, moreover, the number of critical points of μ tends to $+\infty$ as $|\lambda| \to \infty$. We conclude that $I(\lambda) \to +\infty$ as $|\lambda| \to \infty$. Finally, observe that I is continuous, so for each $n \in \mathbb{N}$ large enough there exist a positive and a negative values of λ such that $I(\lambda) + \frac{r(\lambda)}{2\pi} = \frac{n}{2}$. Observe that before the first critical point, μ takes the value μ_0 (and possibly the value μ_1) exactly once. An analogous situation occurs after the last critical point of μ, with the roles of μ_0 and μ_1 reversed and so the proof is complete. ∎

2.1.2 Attractive case

Throughout this section, we assume $c < 0$.

Theorem 3 *If $a > 0 > c$, then problem* (1.7)-(1.8) *admits at least one positive solution. If moreover $b \geq -\pi^2$, then the solution is unique. In particular,* (1.4)-(1.8) *with $\epsilon = -1$ has a unique solution.*

Proof. Since $a > 0$, any large enough constant v is an upper solution of the problem. On the other hand, $au^3 + bxu + cu^{-3} < 0$ for any small enough positive constant u. If moreover $u \leq \mu_0, \mu_1$, then u is a lower solution and existence of a solution μ such that $u \leq \mu \leq v$ follows.

In order to prove uniqueness, assume that $b \geq -\pi^2$ and suppose that μ and ν are solutions. Let $y := \mu - \nu$, then

$$y''(x) = \theta(x)y(x), \qquad y(0) = y(1) = 0$$

with $\theta(x) > bx$ for all x. If $y \not\equiv 0$, then

$$\int_0^1 y'(x)^2\,dx = -\int_0^1 y''(x)y(x)\,dx = -\int_0^1 \theta(x)y(x)^2\,dx < -b\int_0^1 xy^2(x)\,dx.$$

Thus, a contradiction is obtained since

$$\int_0^1 xy^2(x)\,dx < \int_0^1 y^2(x)\,dx \le \frac{1}{\pi^2}\int_0^1 y'(x)^2\,dx.$$

∎

Before establishing the next result, we introduce the following notation for $a \le 0$ and $M > v > 0$:

$$I(v, M) := \int_v^M \frac{dw}{\sqrt{\left(-\frac{a}{2}(w+M)w + b^-w - \frac{c}{M}\right)(M-w)}}$$

with $b^- := \max\{-b, 0\}$. As before, I can be extended continuously by $I(v, v) := 0$ for $v > 0$.

Theorem 4 *Let $a \le 0$ and $c < 0$. If there exists $M \ge \mu_0^2, \mu_1^2$ such that*

$$I(\mu_0^2, M) + I(\mu_1^2, M) \ge 2, \tag{2.13}$$

then (1.7)-(1.8) *has at least one solution.*

Proof. The proof follows the general outline of that in the preceding section. Since $c < 0$, any small enough constant $u > 0$ is a lower solution of the problem. Moreover, if $v(x) > 0$ and $x \in [0, 1]$ then $-b^-v(x) \le bxv(x)$; hence, it suffices to prove that the problem

$$v''(x) = av(x)^3 - b^-v(x) + \frac{c}{v(x)^3}$$

has a solution such that $v(0) = v_0 \ge \mu_0$, $v(1) = v_1 \ge \mu_1$. Set $w = v^2$ as before, then

$$w'(x)^2 = 2aw(x)^3 - 4b^-w(x)^2 + 4dw(x) - 4c.$$

There are two possible cases:

1. w increases up to some $x^* \in (0, 1)$ and then decreases. In this case, one gets

$$d = -\frac{a}{2}M^3 + b^-M^2 + \frac{c}{M}$$

   and

$$I(v_0^2, M) = 2x^*, \qquad I(v_1^2, M) = 2(1 - x^*).$$

   In a similar manner to that of the previous section, it is seen that the condition $I(v_0^2, M) + I(v_1^2, M) = 2$ for some $M > v_0^2, v_1^2$ allows to define implicitly a solution of the problem, and the conclusion follows from the properties of I.

2. w is monotone. In this case, set $M := \max\{v_0^2, v_1^2\}$ and $w_* := \min\{v_0^2, v_1^2\}$. A sufficient condition is that $\tilde{I}(d) = 2$ for some d, where

$$\tilde{I}(d) := \int_{w_*}^{M} \frac{dw}{\sqrt{\frac{a}{2}w^3 - b^- w^2 + dw - c}}.$$

As before, the function $\tilde{I}$ is nonincreasing and $I(d) \to 0$ as $d \to +\infty$. Thus, a sufficient condition is $\tilde{I}(d_*) \geq 2$, where d_* is the minimum possible value of d, namely $d_* := -\frac{a}{2}M^2 + b^- M + \frac{c}{M}$. A simple computation shows that this latter condition is equivalent to $2 \leq I(w_*, M) = I(v_0^2, M) + I(v_1^2, M)$ and the result follows from the properties of I.

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Remark 4. An alternative proof can be obtained by the shooting method. Indeed, it is verified that the problem has a solution if and only if the initial value problem for (1.7) with initial data (2.11) has, for some λ, a solution μ defined on $[0,1]$ such that $\mu(1) \geq \mu_1$. In fact, this is a somewhat stronger result, although the latter condition is not explicit and hence not readily verified. Instead, one may seek an upper solution as previously: in this case, a shooting argument shows that the existence of such an upper solution is equivalent to (2.13).

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2.2 Solutions vanishing at the boundary

In this section, we investigate the existence of solutions $\mu > 0$ in $(0,1)$ such that $\mu(0) = \mu(1) = 0$. The analysis of other situations with solutions vanishing only at one end, or with non-classical solutions vanishing inside the interval may be similarly undertaken.

To begin, let us observe that vanishing solutions can exist only for the attractive case $c < 0$. Indeed, if $c > 0$ and μ is a solution, then

$$\mu'(x)^2 = \frac{a}{2}\mu(x)^4 + b\left(x\mu(x)^2 - \frac{\mu(1/2)^2}{2}\right) - b\int_{1/2}^{x} \mu(s)^2\, ds - \frac{c}{\mu(x)^2} + d$$

for some constant d. The right-hand side term tends to $-\infty$ as $\mu(x) \to 0$, a contradiction.

From now on, we shall assume $c < 0$. For simplicity, only the case $a > 0$ and $b \geq 0$ shall be considered, although the argument may be traced in order to cover the more general situation.

Remark 5.

In view of Theorem 3, it might be interesting to investigate whether it is possible or not to obtain a solution as a limit of a sequence of solutions of (1.7)-(1.8) with $\mu_0 = \mu_1 \to 0$.

However, there are no *a priori* bounds for μ' in the attractive case, so the existence of a convergent sequence cannot be ensured.

∎

Under the previous conditions $a > 0 > c$ and $b \geq 0$, the integral

$$T(M) := \int_0^M \frac{dw}{\sqrt{\left(-\frac{a}{2}(w+M)w - bw - \frac{c}{M}\right)(M-w)}}$$

is defined for any positive $M \leq \tilde{M}$, where $\tilde{M} > 0$ is the unique root of the polynomial $P(M) := aM^3 + bM^2 + c$. Furthermore, a simple computation shows that $T(M) \to 0$ as $M \to 0$, and that T is strictly increasing in $(0, \tilde{M}]$.

Theorem 5 *Let $a > 0 > c$ and $b \geq 0$. If $T(\tilde{M}) \geq 1$, then* (1.7)-(1.8) *with $\mu_0 = \mu_1 = 0$ has at least one solution μ such that $\mu(x) > 0$ for all $x \in (0,1)$. In particular, the problem admits a solution if*

$$\frac{a\sqrt{-c}}{2} + b \leq 4.$$

Proof. As before, we shall firstly prove that the autonomous problem (2.9) has a positive solution u such that $u(0) = u(1) = 0$, which will serve as a lower solution of the original problem. With this aim, observe that if u is such a solution and $w := u^2$ achieves its absolute maximum M at some $x^* \in (0,1)$, then u also achieves its maximum at x^*, and hence $u''(x^*) \leq 0$. This implies

$$aM^{3/2} + bM^{1/2} + cM^{-3/2} \leq 0$$

or, equivalently, $P(M) \leq 0$.

Furthermore, the right-hand side of the equation is an increasing function of u, so $u''(x) < 0$ for $u(x) < M$ and hence the solution cannot achieve any local minima in $(0,1)$. We deduce that w is strictly increasing in $(0, x^*)$ and strictly decreasing in $(x^*, 1)$. Integration yields

$$w'(x)^2 = 2aw(x)^3 + 4bw(x)^2 + 4dw(x) - 4c,$$

with

$$d = -\frac{a}{2}M^2 - bM + \frac{c}{M}.$$

Hence,

$$w'(x) = 2\sqrt{(M - w(x))C(w(x))} \qquad 0 < x < x^*$$

and

$$w'(x) = -2\sqrt{(M - w(x))C(w(x))} \qquad x^* < x < 1$$

where $C(w) := -\left[\frac{c}{M} + bw + \frac{a}{2}w(w+M)\right]$. It follows that

$$\int_0^{w(x)} \frac{dw}{\sqrt{(M-w)C(w)}} = 2x \qquad x \leq x^*$$

and

$$\int_0^{w(x)} \frac{dw}{\sqrt{(M-w)C(w)}} = 2(1-x) \qquad x \geq x^*.$$

In particular, it is seen that $x^* = \frac{1}{2}$ and w is symmetric, namely $w(x) = w(1-x)$. Conversely, if

$$T(M) = \int_0^M \frac{dw}{\sqrt{(M-w)C(w)}} = 1$$

for some positive M such that $P(M) \leq 0$, then a solution $u := \sqrt{w}$ of (2.9) with $u(0) = u(1) = 0$ is implicitly defined. Since P and T are increasing functions and $T(M) \to 0$ for $M \to 0$, it is seen that the inequality $T(\tilde{M}) \geq 1$ implies that $T(M) = 1$ for some $M \in (0, \tilde{M}]$ satisfying $P(M) \leq 0$.

For the particular case $\frac{a\sqrt{-c}}{2} + b \leq 4$, observing that $C(w) \leq C(0)$ for $0 \leq w \leq M$ one then has

$$T(M) \geq \int_0^M \frac{dw}{\sqrt{(M-w)C(0)}} = \frac{2M}{\sqrt{-c}}.$$

Thus, the existence of solutions is guaranteed if $\frac{2\tilde{M}}{\sqrt{-c}} \geq 1$, that is, $\tilde{M} \geq \frac{\sqrt{-c}}{2}$. Equivalently,

$$a\left(\frac{\sqrt{-c}}{2}\right)^3 + b\left(\frac{\sqrt{-c}}{2}\right)^2 + c \leq 0.$$

The latter inequality is fulfilled if and only if $\frac{a\sqrt{-c}}{2} + b \leq 4$.

For the moment, we have proven that the assumptions of the theorem yield the existence of a lower solution u of (1.7)-(1.8) with $\mu_0 = \mu_1 = 0$. On the other hand, we know as before that any large enough constant v is an upper solution of the problem. Thus, the result follows from Lemma 1 below.

∎

To conclude the proof of Theorem 5, we need to adapt the method of upper and lower solutions to this singular case. The following result does not hold for arbitrary singular problems, but can be extended to a considerable number of situations.

Lemma 1 *Let $u \leq v$ satisfy*

$$u'' \geq au^3 + bu + \frac{c}{u^3}, \qquad v'' \leq av^3 + bv + \frac{c}{v^3}$$

$$u(0) = u(1) = 0, \quad v(0), v(1) \geq 0.$$

Then (1.7)-(1.8) *with $\mu_0 = \mu_1 = 0$ has at least one solution μ such that $u \leq \mu \leq v$.*

Proof.

Let
$$\mathcal{C} := \{y \in C([0,1]) : u(x) \le y(x) \le v(x)\,\text{for all }\ x\}$$
and observe that, if $y \in \mathcal{C}$, then the function φ_y defined by
$$\varphi_y(x) := \int_{1/2}^x ay(s)^3 + bsy(s) + \frac{c}{y(s)^3}\,ds$$
belongs to $L^1(0,1)$, since $\frac{-c}{y(x)^3} \le \frac{-c}{u(x)^3} = -u''(x) + au(x)^3 + bu(x)$ for all $x \in (0,1)$. Thus, the claim follows from the fact that $\int_{1/2}^x u''(s)\,ds = u'(x)$, which is an integrable function. Next, define
$$P(x,y) := \begin{cases} y & \text{if } u(x) \le y \le v(x) \\ u(x) & \text{if } y < u(x) \\ v(x) & \text{if } y > v(x) \end{cases}$$
and consider the problem
$$\mu''(x) = aP(x,\mu(x))^3 + bxP(x,\mu(x)) + \frac{c}{P(x,\mu(x))^3} \tag{2.14}$$
under the Dirichlet condition (1.8) with $\mu_0 = \mu_1 = 0$. If μ is a solution, then $u \le \mu \le v$ and hence μ solves the original problem. Indeed, if for example $\mu \not\ge u$ then we may fix θ such that $\mu(\theta) - u(\theta) = \min_{x\in[0,1]} \mu(x) - u(x) < 0$. It is seen that $\theta \in (0,1)$ and
$$0 \le \mu''(\theta) - u''(\theta) = au(\theta)^3 + b\theta u(\theta) + \frac{c}{u(\theta)^3} - u''(\theta) = b(\theta - 1)u(\theta) < 0$$
a contradiction. The remaining inequality is proven in an analogous manner. Thus, it suffices to show that (2.14)-(1.8) with $\mu_0 = \mu_1 = 0$ has at least one solution. To this end, for $y \in C([0,1])$ let $Ty := \mu$ be defined as the unique solution of the problem
$$\mu''(x) = aP(x,y(x))^3 + bxP(x,y(x)) + \frac{c}{P(x,y(x))^3}$$
satisfying the Dirichlet condition. It is easily verified that
$$Ty(x) = \int_0^x \varphi_{\hat{y}}(s)\,ds - x\int_0^1 \varphi_{\hat{y}}(s)\,ds,$$
where $\hat{y} := P(\cdot, y)$. Thus, $Ty \in W^{1,1}(0,1) \hookrightarrow C([0,1])$ and, moreover,
$$\|Ty\|_\infty \le \|(Ty)'\|_{L^1} \le 2\|\varphi_{\hat{y}}\|_{L^1} \le a\|v\|_\infty^3 + b\|v\|_\infty + 4v(1/2).$$
This proves that $T : C([0,1]) \to C([0,1])$ is a compact operator and, furthermore, that its range is bounded; hence, the conclusion follows from Schauder's Theorem. ∎

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Appendix
A Multi-Component Ermakov-Painlevé II System

Here, the importance of the Ermakov-Painlevé II equation of the type (1.4) as a canonical form is emphasied by establishing its key role in the construction of solutions of an over-arching coupled Ermakov-Painlevé II system. Importantly, systems of the latter type have recently been obtained as symmetry reductions of N+1-dimensional coupled nonlinear Schrödinger systems incorporating de Broglie-Bohm quantum potential terms (Rogers [34]).

Thus, a hybrid coupled Ermakov-Painlevé II triad is introduced, namely

$$\begin{aligned} u_{xx} + \frac{1}{2}\, xu + \epsilon[\, u^2 + v^2 + w^2 \,]u &= \frac{\mathcal{I}}{u^3} \ , \\ v_{xx} + \frac{1}{2}\, xv + \epsilon[\, u^2 + v^2 + w^2 \,]v &= \frac{\mathcal{J}}{v^3} \ , \\ w_{xx} + \frac{1}{2}\, xw + \epsilon[\, u^2 + v^2 + w^2 \,]w &= \frac{\mathcal{K}}{w^3} \ . \end{aligned} \tag{A.1}$$

$$(\epsilon^2 = 1)$$

This constitutes a particular three-component Ermakov-Ray-Reid system and accordingly admits associated characteristic Ermakov invariants (see e.g. [39]). Thus, $(\text{A.1})_{1,2}$ combine to produce the Ermakov invariant

$$(u_x v - v_x u)^2 + \mathcal{J}\left(\frac{u}{v}\right)^2 + \mathcal{I}\left(\frac{v}{u}\right)^2 = \mathcal{E}_{\text{I}} \ , \tag{A.2}$$

while, similarly, $(\text{A.1})_{2,3}$ and $(\text{A.1})_{3,1}$ lead, in turn, to the Ermakov invariants

$$(v_x w - w_x v)^2 + \mathcal{K}\left(\frac{v}{w}\right)^2 + \mathcal{J}\left(\frac{w}{v}\right)^2 = \mathcal{E}_{\text{II}} \ , \tag{A.3}$$

and

$$(w_x u - u_x w)^2 + \mathcal{I}\left(\frac{w}{u}\right)^2 + \ \mathcal{K}\left(\frac{u}{w}\right)^2 = \mathcal{E}_{\text{III}} \ . \tag{A.4}$$

We now seek to determine the quantity

$$\Sigma = u^2 + v^2 + w^2 \tag{A.5}$$

so that the constituent equations in the Ermakov triad (A.1) become determinate. In this connection, the system is seen to admit the (non-local) Hamiltonian

$$u_x^2 + v_x^2 + w_x^2 + \frac{1}{2}\int x d\Sigma + \frac{\epsilon}{2}\ \Sigma^2 + \frac{\mathcal{I}}{u^2} + \frac{\mathcal{J}}{v^2} + \frac{\mathcal{K}}{w^2} = 2\mathcal{H} \ . \tag{A.6}$$

On use of the identity

$$\begin{aligned} (u^2+v^2+w^2)(u_x^2+v_x^2+w_x^2) - [\ (u_x v - v_x u)^2 + (v_x w - w_x v)^2 + (w_x u - u_x w)^2\ ] \\ \equiv (uu_x + vv_x + ww_x)^2 \end{aligned} \tag{A.7}$$

the Ermakov invariant relations (A.2)$-$(A.4) together with the 'Hamiltonian' (A.6) show that

$$\begin{aligned}&\Sigma[\ 2\mathcal{H}-\frac{1}{2}\int x d\Sigma-\frac{\epsilon}{2}\ \Sigma^2-(\frac{\mathcal{I}}{u^2}+\frac{\mathcal{J}}{v^2}+\frac{\mathcal{K}}{w^2})\ ]\\ &-[\ \mathcal{E}_{\rm I}+\mathcal{E}_{\rm II}+\mathcal{E}_{\rm III}-\{\mathcal{J}\left(\frac{u}{v}\right)^2+\mathcal{I}\left(\frac{v}{u}\right)^2+\mathcal{K}\left(\frac{v}{w}\right)^2+\mathcal{J}\left(\frac{w}{v}\right)^2+\mathcal{I}\left(\frac{w}{u}\right)^2+\mathcal{K}\left(\frac{u}{w}\right)^2\}\ ]\\ &=\frac{1}{4}\ \Sigma_x^2\end{aligned}\qquad\text{(A.8)}$$

which reduces to

$$\Sigma[\ 2\mathcal{H}-\frac{1}{2}\int x d\Sigma-\frac{\epsilon}{2}\ \Sigma^2\ ]-(\ \mathcal{E}_{\rm I}+\mathcal{E}_{\rm II}+\mathcal{E}_{\rm III}+\mathcal{I}+\mathcal{J}+\mathcal{K}\ )=\frac{1}{4}\ \Sigma_x^2\ \ .\qquad\text{(A.9)}$$

The latter, in turn, shows that

$$\Sigma_{xx}-\frac{1}{2\Sigma}\ (\Sigma_x)^2+x\Sigma+2\epsilon\Sigma^2=\frac{2\mathcal{E}}{\Sigma}\qquad\text{(A.10)}$$

where

$$\mathcal{E}=(\ \mathcal{E}_{\rm I}+\mathcal{E}_{\rm II}+\mathcal{E}_{\rm III}+\mathcal{I}+\mathcal{J}+\mathcal{K}\ )/2\ \ .\qquad\text{(A.11)}$$

Thus, Σ as given by (A.10) is governed by a Painlevé-Gambier type equation (c.f. (1.2)) while $\Omega=\sqrt{\Sigma}$ is governed by a Ermakov-Painlevé II equation

$$\Omega_{xx}+\frac{x}{2}\ \Omega+\epsilon\ \Omega^3=\frac{\mathcal{E}}{\Omega^3}\qquad\text{(A.12)}$$

which, in turn, may be connected to the classical integrable Painlevé II equation (1.6) if $\mathcal{E}=-(\alpha-\epsilon/2)^2/4$. Thus, here it is required that $\mathcal{E}<0$.

The nonlinear pair consisting of (A.12) together with $(A.1)_1$, namely

$$u_{xx}+\frac{x}{2}\ u+\epsilon\ \Omega^2 u=\frac{\mathcal{I}}{u^3}\qquad\text{(A.13)}$$

constitutes a particular Ermakov-Ray-Reid system with characteristic invariant

$$(u_x\Omega-u\ \Omega_x)^2+\mathcal{E}\left(\frac{u}{\Omega}\right)^2+\mathcal{I}\left(\frac{\Omega}{u}\right)^2=\mathbb{R}_{\rm I}\ \ ,\qquad\text{(A.14)}$$

so that

$$[\ \Omega^2\frac{d}{dx}\ \left(\frac{u}{\Omega}\right)^2\ ]+\mathcal{E}\left(\frac{u}{\Omega}\right)^2+\mathcal{I}\left(\frac{\Omega}{u}\right)^2=\mathbb{R}_{\rm I}\qquad\text{(A.15)}$$

where

$$\Omega^2=\omega_x-\epsilon\ \omega^2+\epsilon\ x\qquad\text{(A.16)}$$

with ω being a solution of the canonical Painlevé II equation (1.6).

On introduction of the new variables $\bar{\xi}$, U according to

$$d\bar{x}=\Omega^{-2}dx\ \ ,\qquad\text{(A.17)}$$

and

$$U = \left(\frac{u}{\Omega}\right)^2 \tag{A.18}$$

the invariant relation (A.15), on integration yields

$$U = \frac{1}{2\mathcal{E}} \; [\; \mathbb{R}_{\rm I} - \sqrt{\mathbb{R}_{\rm I}^2 - 4\mathcal{E}\mathcal{I}}\cosh(2\bar{x}(-\mathcal{E})^{1/2} + \mathbb{K}_{\rm I}) \;] \tag{A.19}$$

where, in addition to $\mathcal{E} < 0$, here we require that $\mathcal{I} > 0$ whence the condition $U > 0$ is met: in the above, $\mathbb{K}_{\rm I}$ is an arbitrary constant of integration. In a similar manner, the nonlinear pair consisting of (A.12) together with $(A.1)_2$ admits the Ermakov invariant

$$(\; v_x\Omega - v\Omega_x)^2 + \mathcal{E}\left(\frac{v}{\Omega}\right)^2 + \mathcal{J}\left(\frac{\Omega}{v}\right)^2 = \mathbb{R}_{\rm II} \tag{A.20}$$

so that if

$$V = \left(\frac{v}{\Omega}\right)^2 \tag{A.21}$$

then, in analogy with (A.19),

$$V = \frac{1}{2\mathcal{E}} \; [\; \mathbb{R}_{\rm II} - \sqrt{\mathbb{R}_{\rm II}^2 - 4\mathcal{E}\mathcal{J}}\cosh(2\bar{x}(-\mathcal{E})^{1/2} + \mathbb{K}_{\rm II}) \;] \tag{A.22}$$

where, it is assumed that $\mathcal{J} > 0$, while $\mathbb{K}_{\rm II}$ is an arbitrary constant of integration. Lastly, the pair (A.12), $(A.1)_3$ admit a Ermakov invariant

$$(\; w_x\Omega - w\Omega_x)^2 + \mathcal{E}\left(\frac{w}{\Omega}\right)^2 + \mathcal{K}\left(\frac{\Omega}{w}\right)^2 = \mathbb{R}_{\rm III} \;\;, \tag{A.23}$$

where it is assumed that $\mathcal{K} > 0$. However, the invariant relation (A.23) is not necessary in the calculation of $W = \left(\frac{w}{\Omega}\right)^2$ since the latter is determined via the relation

$$U^2 + V^2 + W^2 = 1 \tag{A.24}$$

in terms of U, V given by (A.19) and (A.22) respectively. In conclusion, it is noted that application of the identity (A.7) together with the Ermakov-type constants of motion establishes an interesting relation between invariants, namely

$$\mathcal{E} = \mathbb{R}_{\rm I} + \mathbb{R}_{\rm II} + \mathbb{R}_{\rm III} - [\; \mathcal{E}_{\rm I} + \mathcal{E}_{\rm II} + \mathcal{E}_{\rm III} + \mathcal{I} + \mathcal{J} + \mathcal{K} \;] = \frac{1}{3}\; [\; \mathbb{R}_{\rm I} + \mathbb{R}_{\rm II} + \mathbb{R}_{\rm III} \;] \tag{A.25}$$

on use of (A.11).

Thus, in summary, solution of the original Ermakov-Painlevé II triad (A.1) is given in terms of a seed solution Ω of the canonical Ermakov-Painlevé II equation (A.12) via the relations

$$u = \pm\Omega\; U^{1/2} \;\;, \quad v = \pm\Omega\; V^{1/2} \;\;, \quad w = \pm\Omega\; W^{1/2} \;\;. \tag{A.26}$$

where $\bar{x}$ is determined by integration of the relation (A.17).

It is emphasised, that the above procedure may be readily extended to N-component Ermakov-Painlevé II systems which admit Ermakov invariants.