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Gap probabilities for the cardinal sine

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ABSTRACT

We study the zero sets of random analytic functions generated by a sum of the cardinal sine functions which form an orthonormal basis for the Paley–Wiener space. As a model case, we consider real-valued Gaussian coefficients. It is shown that the asymptotic probability that there is no zero in a bounded interval decays exponentially as a function of the length.

1. Introduction

We study the asymptotic behavior of the probability that a particular type of simple point process does not have any point in an interval of increasing length (gap probabilities). The simple point process we consider is given by the real zeros of the random function

$$f(z) = \sum_{n\in\mathbb{Z}} a_n \frac{\sin \pi(z-n)}{\pi(z-n)},$$

where a_n are i.i.d. random variables with zero mean and unit variance. Kolmogorov's inequality shows that this sum is almost surely pointwise convergent. In fact, since

$$\sum_{n\in\mathbb{Z}} \left| \frac{\sin \pi(z-n)}{\pi(z-n)} \right|^2$$

converges uniformly on compact subsets of the plane, this series almost surely defines an entire function. If we take a_n to be Gaussian random variables then f is a Gaussian analytic function (GAF). See [1, Lemma 2.2.3] for details.

We are chiefly concerned with the functions given by taking a_n to be real Gaussian random variables. These functions are an example of a stationary symmetric GAF. We denote by n_f the counting measure on the set $Z(f)$ of zeros of f. As a counterpart of the Kac–Rice formula [2,3], Feldheim [4] has shown that the density of zeros is given by

$$\mathbb{E}[n_f(z)] = S(y)m(x,y) + \frac{1}{2\sqrt{3}}\mu(x), \tag{1}$$

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where $z = x + iy$, m denotes the planar Lebesgue measure, μ is the singular measure with respect to m supported on $\mathbb{R}$ and identical to Lebesgue measure there, and

$$S\left(\frac{y}{2\pi}\right) = \pi \left| \frac{d}{dy} \left(\frac{\cosh y - \frac{\sinh y}{y}}{\sqrt{\sinh^2 y - y^2}} \right) \right|.$$

(Here S is defined only for $y \neq 0$, in fact the atom appearing in (1) is the distributional derivative at 0.) We observe that since $S(y) = O(y)$ as y approaches zero there are almost surely zeros on the real line, but that they are sparse close by. Moreover the zero set is on average uniformly distributed on the real line. We are interested in the 'gap probability', that is the probability that there are no zeros in a large interval on the line.

Our result is the following asymptotic estimate.

Theorem 1. *Let f be the symmetric GAF given by the almost surely convergent series*

$$\sum_{n\in\mathbb{Z}} a_n \frac{\sin \pi (z-n)}{\pi (z-n)},$$

where a_n are i.i.d. real Gaussian variables with mean 0 and variance 1. Then, there exist constants $c, C > 0$ such that for all $r \geq 1$,

$$e^{-cr} \leq \mathbb{P}\left(\#(Z(f) \cap (-r, r)) = 0\right) \leq e^{-Cr}.$$

Remark 1. If instead of considering intervals we consider the rectangle $D_r = (-r, r) \times (-a, a)$ for some fixed $a > 0$, then we obtain a similar exponential decay for $\mathbb{P}(\#(Z(f) \cap D_r) = 0)$.

Remark 2. Consider the case when the a_n are i.i.d. Rademacher distributed. I.e., each a_n is equal to either -1 or 1 with equal probability. Since $f(n) = a_n$ for $n \in \mathbb{N}$, it follows that if not all a_n for $|n| \leq N$ are of equal sign, then by the mean value theorem, f has to have a zero in $(-N, N)$. Following the proof of Theorem 1, the remaining two choices of the a_n for $|n| \leq N$ each yield an f without zeros in $(-N, N)$, whence the desired probability is exactly $e^{-2N \log 2}$ for $r = N$.

Remark 3. Whereas the Rademacher distribution is in some sense a simplified Gaussian, the Cauchy distribution, given by the density

$$p(x) = \frac{1}{\pi} \frac{1}{x^2 + 1},$$

is in some sense its opposite: It has neither an expectation, nor a standard deviation. If we suppose that the a_n are i.i.d. Cauchy distributed, it is not hard to see that with probability one the sum $\sum a_n/n$ diverges, whence the related random function diverges everywhere. For a study of random zeros in the polynomial case, see [5].

We now give a short description of the main motivation for our work which comes from the 'hole theorems' proved by Sodin and Tsirelson [6] for point processes uniformly distributed in the plane, and by Peres and Virág [7] for the, so called, hyperbolic GAF in the disk.

The GAF considered by Sodin and Tsirelson [6] is given by

$$F(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}},$$

where a_n are i.i.d. standard complex Gaussian variables. For this function, the density of zeros is proportional to the planar Lebesgue measure and they show that the probability that there are no zeros in a disk of radius r asymptotically behaves as e^{-cr}, where $c > 0$.

It was later shown by Nishry [8] that $c = 3e^2/4 + o(1)$ as $r \to \infty$. Observe that $\left\{ \frac{z^n}{\sqrt{n!}} \right\}_{n=0}^{\infty}$ constitutes an orthonormal basis for the Bargmann–Fock space

$$\mathcal{F} = \left\{ f \in H(\mathbb{C}) : \|f\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-2|z|^2} \frac{dm(z)}{\pi} < +\infty \right\},$$

where m is the planar Lebesgue measure. For a generalization to several variables, see [9].

The hyperbolic GAF considered in [7] is the determinantal process defined in $\mathbb{D}$ by the zeros of

$$F(z) = \sum_{n=0}^{\infty} a_n z^n,$$

where a_n are i.i.d. standard complex Gaussian variables. In this case, the asymptotic probability that there are no zeros in a disk of radius $r < 1$ centered at zero is $e^{-ca(r)}$, where $c > 0$ and $a(r)$ stands for the area of the disk in the hyperbolic metric. The set $\{z^n\}_{n=0}^{\infty}$ is an orthonormal basis for the Hardy space $H^2(\mathbb{D})$.

In analogy with these two previous cases, we consider the Paley–Wiener space

$$PW = \left\{ f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(t) e^{izt} dt : \phi \in L^2[-\pi, \pi] \right\},$$

of all entire functions of exponential type at most π that are square integrable on the real axis. It is well known that the integer translations of the cardinal sine function $\{\sin \pi(z-n)/\pi(z-n)\}_{n\in\mathbb{Z}}$ constitute an orthonormal basis for the Paley–Wiener space. The GAF we consider

$$\sum_{n\in\mathbb{Z}} a_n \frac{\sin \pi(z-n)}{\pi(z-n)}$$

is a stationary Gaussian process with mean zero and continuous covariance function

$$\mathbb{E}F(\tau)F(\tau+t) = \frac{\sin \pi t}{\pi t}.$$

The main difficulty in studying the gap probabilities for this Gaussian process is to deal with the oscillations of the covariance function. Considering just the decay is not enough to get the right asymptotics, see [10,11].

An important caveat is that, though the GAFs above are constructed from orthonormal bases, almost surely they do not belong to their respective spaces, since the sequence of coefficients a_n is almost surely not in $\ell^2(\mathbb{Z})$. In our case, however, it is not hard to see that f belongs almost surely to the Cartwright class of entire functions of exponential type such that

$$\int_{-\infty}^{\infty} \frac{\log^+ |f(x)|}{1+x^2} dx < \infty.$$

Finally, we mention that a generalization of the Paley–Wiener space is given by the class of de Branges spaces

$$H(E) = \left\{ f \text{ entire } : \int_{\mathbb{R}} \left| \frac{f(x)}{E(x)} \right|^2 dx < \infty, \text{and } f/E, f/E^* \in H^2(\mathbb{C}_+) \right\},$$

where $H^2(\mathbb{C}_+)$ is the Hardy space of the upper-half plane, $f^*(z) = \overline{f(\overline{z})}$ and E is an entire function such that $|E(z)| > |E^*(z)|$ for whenever $\operatorname{Im} z > 0$. Analogous to the Paley–Wiener space, these spaces admit natural orthonormal bases (consisting of reproducing kernels). An important measure of the behavior of these spaces is the phase function $\phi(x) = -\operatorname{Arg} E(x)$. As was shown by Lyubarskii and Seip [12], when $a \le \phi'(x) \le b$, for $a, b > 0$ these spaces are in some sense similar to the Paley–Wiener space in terms of sampling and interpolating sequences. This analogy continues to hold under the weaker assumption that $\phi'(x)dx$ is a doubling measure, as was shown in [13]. Under the former assumption, it is not hard to show that our results continue to hold. We include our results in this direction in a forthcoming paper.

2. Proof of Theorem 1

2.1. Upper bound

We want to compute the probability of an event that contains the event of not having any zeros on $(-N, N)$, for $N \in \mathbb{N}$. One such event is that the values $f(n)$ have the same sign for $|n| \le N$. The probability of this event is

$$\mathbb{P}\left(a_n > 0 \text{ for } |n| \le N \text{ or } a_n < 0 \text{ for } |n| \le N\right) = 2(1/2)^{2N+1} = e^{-CN},$$

for some constant $C > 0$.

Remark 4. The same upper bound holds when a_n are i.i.d. random variables with $0 < \mathbb{P}(a_n > 0) < 1$ for which the random function $\sum_{n\in\mathbb{Z}} a_n \sin \pi(x-n)/\pi(x-n)$ converges.

2.2. Lower bound

To compute the lower hole probability, we use the following scheme. First, we introduce the deterministic function

$$f_0(x) = \sum_{n=-2N}^{2N} \frac{\sin \pi(x-n)}{\pi(x-n)}.$$

We show in Lemma 1 that it has no zeros on $(-N, N)$, and we find an explicit lower bound on $(-N, N)$ for it. This lower bound does not depend on N. Second, we consider the functions

$$f_1(x) = \sum_{n=-2N}^{2N} (a_n - 1) \frac{\sin \pi(x-n)}{\pi(x-n)} \quad \text{and} \quad f_2(x) = \sum_{|n|>2N} a_n \frac{\sin \pi(x-n)}{\pi(x-n)},$$

which induce the splitting

$$f = f_0 + f_1 + f_2.$$

We show that for all $x \in [-N, N]$ we have $|f_1(x)| \le \epsilon$ with probability at least e^{-cN} for large N and some constant $c > 0$. Moreover, we show that

$$\mathbb{P}\left(\sup_{x\in[-N,N]} |f_2(x)| \le \epsilon \right)$$

is larger than, say, $1/2$ for big enough N. As the events on f_1 and f_2 are clearly independent, the lower bound now follows by choosing ϵ small enough.

We turn to the first part of the proof.

Lemma 1. *Given $N \in \mathbb{N}$ and*

$$f_0(x) = \sum_{n=-2N}^{2N} \frac{\sin \pi(x-n)}{\pi(x-n)} = \sin \pi x \sum_{n=-2N}^{2N} \frac{(-1)^n}{\pi(x-n)}. \tag{2}$$

Then, there exists a constant $C > 0$ such that, for N big enough,

$$1 - \frac{C}{N} \le \inf_{|x|\le N} f_0(x) \le \sup_{|x|\le N} f_0(x) \le 1 + \frac{C}{N}.$$

Proof. Let $R = R(N)$ be the boundary of the square of length $4N + 1$, centered at the origin. By the residue theorem, it holds that for $-N \le x \le N$ not an integer

$$\frac{1}{2\pi i} \oint_R \frac{d\xi}{(\xi - x)\sin \pi\xi} = \frac{1}{\pi} \sum_{n=-2N}^{2N} \frac{(-1)^n}{n-x} + \frac{1}{\sin \pi x}.$$

Observe that if we shift around the terms, this yields

$$\frac{\sin \pi x}{\pi} \sum_{n=-2N}^{2N} \frac{(-1)^n}{x-n} = 1 + \frac{\sin \pi x}{2\pi i} \oint_R \frac{d\xi}{(\xi - x)\sin \pi\xi}. \tag{3}$$

It is easy now to bound this last integral by C/N. □

Remark 5. The same bound holds for all points z in a strip with fixed height $[-N, N] \times [-C, C]$ for some $C > 0$ and N big enough depending on C. We observe though, that the function $f_0(z)$ in (2) does not yield the desired estimate on rectangles. Indeed, let R_0 be the square centered at $x = 1/2$ with side lengths $4N$. By considering horizontal and vertical contributions to the contour integral corresponding to (3) separately, then on the line $z = 1/2 + iy$ we get

$$\frac{1}{\pi} \sum_{n=-2N}^{2N} \frac{(-1)^n}{z-n} = \frac{1}{\cos \pi iy} - \frac{1}{\pi} \int_{-2N}^{2N} \frac{2N}{(y-v)^2 + 4N^2} \frac{dv}{\cos \pi iv} + O(e^{-cN}).$$

By considering the signs of the expression of the right-hand side, it follows that for large and fixed N, then for y around log N the main terms will cancel each other out, leaving only the exponentially small error term. Therefore the estimate proved in the lemma above does not work for z on squares.

2.2.1. The middle terms

Let $\epsilon > 0$ be given, and consider $N \in \mathbb{N}$ to be fixed. We look at the function

$$f_1(x) = \sum_{n=-2N}^{2N} (a_n - 1) \frac{\sin \pi(x-n)}{\pi(x-n)} = \frac{\sin \pi x}{\pi} \sum_{n=-2N}^{2N} (a_n - 1) \frac{(-1)^n}{x-n}.$$

To simplify the expression, we set $b_n = (a_n - 1)(-1)^n$. We want to compute a lower bound for the probability that, for $x \in [-N, N]$,

$$|f_1(x)| \lesssim \epsilon.$$

Define $B_n = b_{-2N} + \cdots + b_n$ for $|n| \le 2N$, and suppose that $x \notin \mathbb{Z}$. With this, summation by parts yields

$$\sum_{-2N}^{2N} \frac{b_n}{x-n} = -\sum_{-2N}^{2N-1} \frac{B_n}{(x-n)(x-n-1)} + \frac{B_{2N}}{x-2N-1}. \tag{4}$$

We now claim that under the event

$$E = \{|B_n| \le \epsilon \text{ for } |n| \le 2N\}, \tag{5}$$

we have $|f_1(x)| \lesssim \epsilon$ for $|x| \le N$, and the proportionality constant involved does not depend on N. Indeed, the second summand at the right hand side of (4) converges to zero on E uniformly for $|x| \le N$, because

$$\left|\frac{B_{2N}}{x-2N-1}\right| \le \frac{\epsilon}{N}.$$

Suppose that $x \in (k, k+1)$ and split the first sum in (4) as

$$\sum_{\substack{n=-2N\\ n\ne k-1,k,k+1}}^{2N-1} \frac{B_n}{(x-n)(x-n-1)} + \sum_{n=k-1}^{k+1} \frac{B_n}{(x-n)(x-n-1)}.$$

Then

$$\left|\sum_{\substack{n=-2N\\ n\ne k-1,k,k+1}}^{2N-1} \frac{B_n}{(x-n)(x-n-1)}\right| \le \sum_{n\ge k+2} \frac{\epsilon}{(k+1-n)^2} + \sum_{n\le k-2} \frac{\epsilon}{(k-n)^2} \lesssim \epsilon.$$

For the remaining terms, the function $\sin \pi x$ comes into play. E.g., suppose that $|x-k| \le 1/2$, then

$$\left|\sin \pi x \frac{B_k}{(x-k)(x-k-1)}\right| \lesssim \frac{\epsilon}{|x-k-1|}\left|\frac{\sin \pi(x-k)}{\pi(x-k)}\right| \lesssim \epsilon.$$

The remaining terms are treated in exactly the same way. So, we have obtained that $|f_1(x)| \lesssim \epsilon$ for $x \notin \mathbb{Z}$ on E. By continuity, this bound also holds for $x \in \mathbb{Z}$.

What remains is to compute the probability of the event E defined by (5). We recall that the b_n were all defined in terms of the real and independent Gaussian variables a_n. So the event E above defines a set

$$V = \left\{(t_{-2N}, \dots, t_{2N}) \in \mathbb{R}^{4N+1} : \left|\sum_{-2N}^{n} t_n\right| \le \epsilon, |n| \le 2N\right\}$$

in terms of the values of the b_n. Hence,

$$\mathbb{P}(E) = c^{4N+1} \int_V e^{-((t_{-2N}-1)^2+\dots+(t_{2N}-1)^2)/2} dt_{-2N} \cdots dt_{2N}.$$

Here, c is the normalizing constant of the one dimensional Gaussian. It follows that

$$\mathbb{P}(E) \ge c^{4N+1} e^{-(4N+1)(1+C\epsilon)^2/2} \int_V dt_{-2N} \cdots dt_{2N} = c^{4N+1} e^{-(4N+1)(1+C\epsilon)^2/2} \mathrm{Vol}(V).$$

We now seek a lower bound for this euclidean $(4N+1)$-volume. To simplify notation, we pose this problem as follows. For real variables $x_1, \dots, x_N$, we wish to compute the euclidean volume of the solid V_N defined by

$$|x_1| \le \epsilon,$$
$$|x_1 + x_2| \le \epsilon,$$
$$\vdots$$
$$|x_1 + x_2 + \dots + x_N| \le \epsilon.$$

One way to do this is as follows. Write $y_N = x_1 + \dots + x_{N-1}$, then

$$\mathrm{Vol}(V_N) = \int_{V_{N-1}} \left(\int_{-\epsilon-y_N}^{\epsilon-y_N} dx_N\right) dx_1 \cdots dx_{N-1}.$$

This is illustrated in Fig. 1. Clearly, if $y_N < 0$, $\epsilon - y_N \ge \epsilon$ and $-\epsilon - y_N \le 0$, while if $y_N > 0$ then $\epsilon - y_N \ge 0$ and $-\epsilon - y_N \le -\epsilon$.

Therefore,

$$\mathrm{Vol}(V_N) \ge \int_{V_{N-1}\cap\{y_N<0\}} \left(\int_0^{\epsilon} dx_N\right) dx_1 \cdots dx_{N-1} + \int_{V_{N-1}\cap\{y_N>0\}} \left(\int_{-\epsilon}^{0} dx_N\right) dx_1 \cdots dx_{N-1} = \epsilon \mathrm{Vol}(V_{N-1}).$$

Iterating this, we get

$$\mathrm{Vol}(V_N) \ge \epsilon^N.$$

In conclusion,

$$\mathbb{P}(E) \ge e^{-cN},$$

which concludes this part of the proof.

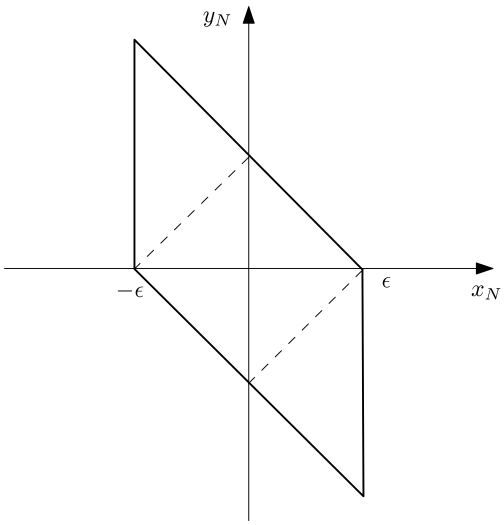

Fig. 1. Illustration of the solid V_N.

2.2.2. The tail

We now turn to the tail term

$$f_2(x) = \sin \pi x \sum_{|n|>2N} \frac{a_n(-1)^n}{\pi(x-n)}.$$

Clearly, we need only consider the terms for which n is positive. Set $c_n = (-1)^n a_n$. We apply summation by parts, to get

$$\sum_{n>2N}^{L} \frac{c_n}{x-n} = \sum_{2N+1}^{L} C_n \frac{-1}{(x-n)(x-n-1)} + \frac{C_L}{x-L-1} \tag{6}$$

where

$$C_n = c_{2N+1} + \cdots + c_n, \qquad C_{2N} = 0.$$

We want to take the limit as $L \to \infty$. It is easy to see that the last term almost surely converges to zero uniformly for $|x| \leq N$. Indeed, C_L is a sum of independent Gaussian variables with mean 0 and variance 1, and therefore is itself Gaussian with mean 0 and variance $L - 2N$. Moreover, since for all $|x| \leq N$

$$\left| \frac{C_L}{x-L-1} \right| \lesssim \frac{C_L}{L-2N},$$

and the random variable inside of the absolute values on the right-hand side has variance $L - 2N$, it follows by the law of large numbers that the limit is almost surely equal to 0, whence we are allowed to let $L \to \infty$ in (6).

We prove the following. With a positive probability, we have for $|x| \leq N$

$$\left| \sum_{2N+1}^{L} C_n \frac{1}{(x-n)(x-n-1)} \right| \leq \epsilon.$$

As $n^2 \lesssim |(x-n)(x-n-1)|$ for $|x| \leq N$ and $n > 2N$, it is enough to consider the expression

$$\sum_{2N+1}^{\infty} \frac{|C_n|}{n^2}.$$

The absolute value of a Gaussian random variable has the folded-Gaussian distribution. In particular, if $X \sim N(0, \sigma^2)$, then

$$\mathbb{E}(|X|) = \sigma \sqrt{\frac{2}{\pi}}.$$

Since, in our case, $\sigma^2 = n - 2N$, this yields

$$\mathbb{E}\left(\sum_{2N+1}^{\infty} \frac{|C_n|}{n^2} \right) \lesssim \sum_{2N+1}^{\infty} \frac{\sqrt{n-2N}}{n^2} \lesssim \sum_{1}^{\infty} \frac{1}{(n-2N)^{3/2}} \lesssim \frac{1}{\sqrt{N}}.$$

Finally, by Chebyshev's inequality,

$$\mathbb{P}\left(\sum_{2N+1}^{\infty}\frac{|C_n|}{n^2}\leq\epsilon\right)\geq 1-\frac{1}{\epsilon}\mathbb{E}\left(\sum_{2N+1}^{\infty}\frac{|C_n|}{n^2}\right)\geq 1-\frac{C}{\epsilon\sqrt{N}}.$$

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