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A differential bialgebra associated to a set theoretical solution of the Yang–Baxter equation ☆

Marco A. Farinati *, Juliana García Galofre

IMAS CONICET, Dep. de Matemática, Fac. de Cs. Exactas y Naturales, UBA, Argentina

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ABSTRACT

For a set theoretical solution of the Yang–Baxter equation (X, σ), we define a d.g. bialgebra $B = B(X, \sigma)$, containing the semigroup algebra $A = k\{X\}/\langle xy = zt : \sigma(x, y) = (z, t)\rangle$, such that $k \otimes_A B \otimes_A k$ and $\mathrm{Hom}_{A-A}(B, k)$ are respectively the homology and cohomology complexes computing biquandle homology and cohomology defined in [2,5] and other generalizations of cohomology of rack-quandle case (for example defined in [4]). This algebraic structure allows us to show the existence of an associative product in the cohomology of biquandles, and a comparison map with Hochschild (co)homology of the algebra A.

0. Introduction

A quandle is a set X together with a binary operation $*: X \times X \to X$ satisfying certain conditions (see definition in Example 1.1 below), it generalizes the operation of conjugation on a group, but also is an algebraic structure that behaves well with respect to Reidemeister moves, so it is very useful for defining knot/links invariants. Knot theorists have defined a cohomology theory for quandles (see [5] and [3]) in such a way that 2-cocycles give rise to knot invariants by means of the so-called state-sum procedure. Biquandles are generalizations of quandles in the sense that quandles give rise to solutions of the Yang–Baxter equation by setting $\sigma(x, y) := (y, x * y)$. For biquandles there is also a cohomology theory and state-sum procedure for producing knot/links invariants (see [4]).

The main tool of this work is, for any set theoretical solution of the Yang–Baxter equation (X, σ), to define a d.g. algebra $B = B(X, \sigma)$, containing the semigroup algebra $A = k\{X\}/\langle xy = zt : \sigma(x, y) = (z, t)\rangle$, in such a way that $k \otimes_A B \otimes_A k$ and $\mathrm{Hom}_{A-A}(B, k)$ canonically identify with the standard homology and cohomology complexes attached to general set theoretical solutions of the Yang–Baxter equation. As a product of this construction we have two main results: the first is Theorem 3.1 where we show the

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* Corresponding author.
E-mail addresses: mfarinat@dm.uba.ar (M.A. Farinati), jgarciag@dm.uba.ar (J. García Galofre).

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existence of an associative product in cohomology, already defined at the level of the complex. The second is Theorem 4.6 where we found an explicit comparison map between Yang–Baxter (co)homology of X and Hochschild (co)Homology of the semigroup algebra A.

The existence of an associative product on cohomology was known for rack cohomology (see [7]), but it was unknown for biquandles, or general solutions of the Yang–Baxter equation. Also, the proof in [7] was based on topological methods, our methods are purely algebraic.

The existence of a comparison map between Yang–Baxter (co)homology of X and Hochschild (co)homology of the semigroup algebra A was also unknown, and moreover, we prove that it factors through a complex of "size" $A \otimes \mathfrak{B} \otimes A$, where $\mathfrak{B}$ is the Nichols algebra associated to the solution $(X, -\sigma)$. This result leads to new questions, for instance when (X, σ) is involutive (that is $\sigma^2 = \mathrm{Id}$) and the characteristic is zero we show that this complex is acyclic (Proposition 3.10), we wonder if this is true in any other characteristic, and for non-necessarily involutive solutions.

Also, depending on properties of the solution (X, σ) (square-free, quandle type, biquandle, involutive, ...) this d.g. bialgebra B has natural d.g. bialgebra quotients, giving rise to the standard sub-complexes computing quandle cohomology (as sub-complex of rack homology), biquandle cohomology, etc.

This work is organized as follows: Section 1 contains the basic definitions and examples of solutions of the Yang–Baxter equation, in Section 2 we define the d.g. algebra B, we prove (Theorem 2.1) that it admits a structure of a d.g. bialgebra, and that after tensor product or Hom it canonically gives the standard complexes for Yang–Baxter (co)homology (Theorem 2.4). The maps we use are the natural ones, but the technical reason depends on the existence of "normal forms" for writing elements, this takes the rest of Section 2. In Section 3 we show the existence of the product in cohomology, and that this product is compatible with all types of natural quotients attached to special cases of solutions (e.g. square-free, quandles, biquandles, involutives). We also prove that the involutive quotient is acyclic in characteristic zero. Finally in Section 4 we derive from general reasons the existence of a comparison map, but introducing algebraic structure on the standard Hochschild resolution of A (e.g. braided shuffle product) we give (Theorem 4.6) an explicit chain map, that factors through a Nichols algebra.

1. Basic definitions

A set theoretical solution of the Yang–Baxter equation (YBeq) is a pair (X, σ) where $\sigma\colon X \times X \to X \times X$ is a bijection satisfying

$$(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma) = (\sigma \times \mathrm{Id})(\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})\colon X \times X \times X \to X \times X \times X$$

If $X = V$ is a k-vector space and $\sigma\colon V \otimes V \to V \otimes V$ is a linear bijective map satisfying $(\mathrm{Id} \otimes \sigma)(\sigma \otimes \mathrm{Id})(\mathrm{Id} \otimes \sigma) = (\sigma \otimes \mathrm{Id})(\mathrm{Id} \otimes \sigma)(\sigma \otimes \mathrm{Id})$ then σ is called a braiding on V.

Example 1.1. A set X with a binary operation $\triangleleft\colon X \times X \to X \times X$ is called a rack if

- $- \triangleleft x\colon X \to X$ is a bijection $\forall x \in X$ and
- $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z) \forall x, y, z \in X$.

If $(X, \triangleleft)$ is a rack, then

$$\sigma(x, y) := (y, x \triangleleft y)$$

is a set theoretical solution of the YBeq.

If $(X, \triangleleft)$ also satisfies that $x \triangleleft x = x$ for all $x \in X$ then it is called a *quandle*. An important example of rack, which is actually a quandle, is $X = G$ a group but with operation $x \triangleleft y = y^{-1}xy$. For X a rack, $x \triangleleft y$ is also usually denoted by x^y.

Example 1.2. If s and t are two commuting units in a ring and X is a module over that ring, then the formula $\sigma(x, y) = (sy, tx + (1 - st)y)$ is a solution of the YBeq, called the Alexander switch, or bi-Alexander.

Example 1.3 *(Wada solutions).* If G is a group, then $\sigma(x, y) := (xy^{-1}x^{-1}, xy^2)$ is a solution of the Yang–Baxter equation that is not of rack or quandle type, and in general non-linear.

Let $M = M_X$ be the monoid generated in X with relations

$$xy = zt$$

$\forall x, y, z, t$ such that $\sigma(x, y) = (z, t)$. Denote G_X the group with the same generators and relations. For example, when $\sigma = \text{flip}$ then $M = \mathbb{N}_0^{(X)}$ and $G_X = \mathbb{Z}_0^{(X)}$. If $\sigma = \text{Id}$ then M is the free (non-abelian) monoid in X. If σ comes from a rack $(X, \triangleleft)$ then M is the monoid with relation $xy \sim y(x \triangleleft y)$ and G_X is the group with relations $x \triangleleft y \sim y^{-1}xy$.

2. A d.g. bialgebra associated to (X, σ)

Let k be a commutative ring with 1. Fix X a set, and $\sigma\colon X \times X \to X \times X$ a solution of the YBeq. Denote $A_\sigma(X)$, or simply A if X and σ are understood, the quotient of the free k algebra on generators X modulo the ideal generated by elements of the form $xy - zt$ whenever $\sigma(x, y) = (z, t)$:

$$A := k\langle X\rangle / \langle xy - zt : x, y \in X,\ (z, t) = \sigma(x, y)\rangle = k[M]$$

It can be easily seen that A is a k-bialgebra declaring x to be group-like for any $x \in X$, since A agrees with the semigroup-algebra on M (the monoid generated by X with relations $xy \sim zt$). If one considers G_X, the group generated by X with relations $xy = zt$, then $k[G_X]$ is the (non-commutative) localization of A, where one has inverted the elements of X. An example of A-bimodule that will be used later, which is actually a $k[G_X]$-module, is k with A-action determined on generators by

$$x\lambda y = \lambda,\ \forall x, y \in X,\ \lambda \in k$$

We define $B(X, \sigma)$ (also denoted by B) the algebra generated by three copies of X, denoted x, e_x and x', with relations as follows: whenever $\sigma(x, y) = (z, t)$ we have

- $xy \sim zt$, $xy' \sim z't$, $x'y' \sim z't'$
- $xe_y \sim e_zt$, $e_xy' \sim z'e_t$

Since the relations are homogeneous, B is a graded algebra declaring

$$|x| = |x'| = 0, \quad |e_x| = 1$$

Theorem 2.1. *The algebra B admits the structure of a differential graded bialgebra, with d the unique superderivation satisfying*

$$d(x) = d(x') = 0, \quad d(e_x) = x - x'$$

and comultiplication determined by

$$\Delta(x) = x \otimes x,\ \Delta(x') = x' \otimes x',\ \Delta(e_x) = x' \otimes e_x + e_x \otimes x$$

By differential graded bialgebra we mean that the differential is both a derivation with respect to multiplication, and coderivation with respect to comultiplication.

Proof. In order to see that d is well-defined as super derivation, one must check that the relations are compatible with d. The first relations are easier since

$$d(xy - zt) = d(x)y + xd(y) - d(z)t - zd(t) = 0 + 0 - 0 - 0 = 0$$

and similar for the others (this implies that d is A-linear and A'-linear). For the rest of the relations:

$$\begin{aligned} d(xe_y - e_zt) &= xd(e_y) - d(e_z)t = x(y - y') - (z - z')t \\ &= xy - zt - (xy' - z't) = 0 \\ d(e_xy' - z'e_t) &= (x - x')y' - z'(t - t') = xy' - z't - (x'y' - z't') = 0 \end{aligned}$$

It is clear now that $d^2 = 0$ since d^2 vanishes on generators. In order to see that Δ is well defined, we compute

$$\begin{aligned} \Delta(xe_y - e_zt) &= (x \otimes x)(y' \otimes e_y + e_y \otimes y) - (z' \otimes e_z + e_z \otimes z)(t \otimes t) \\ &= xy' \otimes xe_y + xe_y \otimes xy - z't \otimes e_zt - e_zt \otimes zt \end{aligned}$$

and using the relations we get

$$= xy' \otimes xe_y + xe_y \otimes xy - xy' \otimes xe_y - xe_y \otimes xy = 0$$

similarly

$$\begin{aligned} \Delta(x'e_y - e_zt') &= (x' \otimes x')(y' \otimes e_y + e_y \otimes y) - (z' \otimes e_z + e_z \otimes z)(t' \otimes t') \\ &= x'y' \otimes x'e_y + x'e_y \otimes x'y - z't' \otimes e_zt' - e_zt' \otimes zt' \\ &= x'y' \otimes x'e_y + x'e_y \otimes x'y - x'y' \otimes x'e_y - x'e_y \otimes x'y = 0 \end{aligned}$$

This proves that B is a bialgebra, and d is (by construction) a derivation. Let us see that it is also a coderivation:

$$(d \otimes 1 + 1 \otimes d)(\Delta(x)) = (d \otimes 1 + 1 \otimes d)(x \otimes x) = 0 = \Delta(0) = \Delta(dx)$$

for x' is the same. For e_x:

$$\begin{aligned} (d \otimes 1 + 1 \otimes d)(\Delta(e_x)) &= (d \otimes 1 + 1 \otimes d)(x' \otimes e_x + e_x \otimes x) \\ &= x' \otimes (x - x') + (x - x') \otimes x = x' \otimes x - x' \otimes x' + x \otimes x - x' \otimes x \\ &= -x' \otimes x' + x \otimes x = \Delta(x - x') = \Delta(de_x) \quad \square \end{aligned}$$

Remark 2.2. Δ is coassociative.

For a particular element of the form $b = e_{x_1} \dots e_{x_n}$, the formula for $d(b)$ can be computed as follows:

$$d(e_{x_1} \dots e_{x_n}) = \sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \dots e_{x_{i-1}} d(e_{x_i}) e_{x_{i+1}} \dots e_{x_n}$$

$$= \sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \dots e_{x_{i-1}} (x_i - x'_i) e_{x_{i+1}} \dots e_{x_n}$$

$$= \overbrace{\sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \dots e_{x_{i-1}} x_i e_{x_{i+1}} \dots e_{x_n}}^{I} - \overbrace{\sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \dots e_{x_{i-1}} x'_i e_{x_{i+1}} \dots e_{x_n}}^{II}$$

If one wants to write it in a normal form (say, every x on the right, every x' on the left, and the e_x's in the middle), then one should use the relations in B: this might be a very complicated formula, depending on the braiding. We give examples in some particular cases. Lets denote $\sigma(x, y) = (\sigma^1(x, y), \sigma^2(x, y))$.

Example 2.3. In low degrees we have

- $d(e_x) = x - x'$
- $d(e_x e_y) = (e_z t - e_x y) - (x' e_y - z' e_t)$, where as usual $\sigma(x, y) = (z, t)$.
- $d(e_{x_1} e_{x_2} e_{x_3}) = A_I - A_{II}$ where

$$A_I = e_{\sigma^1(x_1,x_2)} e_{\sigma^1(\sigma^2(x_1,x_2),x_3)} \sigma^2(\sigma^2(x_1, x_2), x_3) - e_{x_1} e_{\sigma^1(x_2,x_3)} \sigma^2(x_2, x_3) + e_{x_1} e_{x_2} x_3$$

$$A_{II} = x'_1 e_{x_2} e_{x_3} - \sigma^1(x_1, x_2)' e_{\sigma^2(x_1,x_2)} e_{x_3} + \sigma^1(x_1, \sigma^1(x_2, x_3))' e_{\sigma^2(x_1,\sigma^1(x_2,x_3))} e_{\sigma^2(x_2,x_3)}$$

In particular, if $f : B \to k$ is an A–A' linear map, then

$$f(d(e_{x_1} e_{x_2} e_{x_3})) = f(e_{\sigma^1(x_1,x_2)} e_{\sigma^1(\sigma^2(x_1,x_2),x_3)}) - f(e_{x_1} e_{\sigma^1(x_2,x_3)}) + f(e_{x_1} e_{x_2})$$
$$-f(e_{x_2} e_{x_3}) + f(e_{\sigma^2(x_1,x_2)} e_{x_3}) - f(e_{\sigma^2(x_1,\sigma^1(x_2,x_3))} e_{\sigma^2(x_2,x_3)})$$

Erasing the e's we notice the relation with the cohomological complex given in [4], see Theorem 2.4 below.

If X is a rack and σ the braiding defined by $\sigma(x, y) = (y, x \triangleleft y) = (x, x^y)$, then:

- $d(e_x) = x - x'$
- $d(e_x e_y) = (e_y x^y - e_x y) - (x' e_y - y' e_{x^y})$
- $d(e_x e_y e_z) = e_x e_y z - e_x e_z y^z + e_y e_z x^{yz} - x' e_y e_z + y' e_{x^y} e_z - z' e_{x^z} e_{y^z}$.
- In general, expressions I and II are

$$I = \sum_{i=1}^{n} (-1)^{i+1} e_{x_1} \dots e_{x_{i-1}} e_{x_{i+1}} \dots e_{x_n} x_i^{x_{i+1} \dots x_n}$$

$$II = \sum_{i=1}^{n} (-1)^{i+1} x'_i e_{x_1^{x_i}} \dots e_{x_{i-1}^{x_i}} e_{x_{i+1}} \dots e_{x_n}$$

so

$$\partial f(x_1,\dots,x_n) = f(d(e_{x_1}\dots e_{x_n})) =$$
$$\sum_{i=1}^{n}(-1)^{i+1}\left(f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)x_i^{x_{i+1}\dots x_n} - x_i' f({x_1}^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n)\right)$$

Let us consider $k\otimes_{k[M']} B\otimes_{k[M]} k$ then d represents the canonical differential of rack homology and $\partial f(e_{x_1}\dots e_{x_n}) = f(d(e_{x_1}\dots e_{x_n}))$ gives the traditional rack cohomology structure.
In particular, taking trivial coefficients:

$$\partial f(x_1,\dots,x_n) = f(d(e_{x_1}\dots e_{x_n})) =$$
$$\sum_{i=1}^{n}(-1)^{i+1}\left(f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) - f({x_1}^{x_i},\dots,x_{i-1}^{x_i},x_{i+1}\dots,x_n)\right)$$

Theorem 2.4. *Taking in k the trivial A'–A-bimodule, the complexes associated to set theoretical Yang–Baxter solutions defined in [4] can be recovered as*

$$(C_\bullet(X,\sigma),\partial)\simeq(k\otimes_{A'} B_\bullet\otimes_A k, \partial = id_k\otimes_{A'} d\otimes_A id_k)$$
$$(C^\bullet(X,\sigma),\partial^*)\simeq(\mathrm{Hom}_{A'-A}(B,k),\partial^* = d^*)$$

In the proof of the theorem we will assume first Proposition 2.12 that says that one has a left A'-linear and right A-linear isomorphism:

$$B\cong A'\otimes TE\otimes A$$

where $A' = TX'/(x'y' = z't' : \sigma(x,y) = (z,t))$ and $A = TX/(xy = zt : \sigma(x,y) = (z,t))$. We will prove Proposition 2.12 later.

Proof. In this setting every expression in x, x', e_x, using the relations defining B, can be written as $x'_{i_1}\cdots x'_{i_n} e_{x_1}\cdots e_{x_k} x_{j_1}\cdots x_{j_l}$, tensorizing leaves the expression

$$1\otimes e_{x_1}\cdots e_{x_k}\otimes 1$$

This shows that $T = k\otimes_{k[M']} B\otimes_{k[M]} k\simeq T\{e_x\}_{x\in X}$, where $\simeq$ means isomorphism of k-modules. Now it is immediate to see that under these isomorphisms, the differentials correspond to each other, giving isomorphisms of complexes

$$(C_\bullet(X,\sigma),\partial)\simeq(k\otimes_{A'} B_\bullet\otimes_A k, \partial = id_k\otimes_{A'} d\otimes_A id_k)$$
$$(C^\bullet(X,\sigma),\partial^*)\simeq(\mathrm{Hom}_{A'-A}(B,k),d^*)\quad\square$$

Remark 2.5. This isomorphism gives an alternative proof of the fact that $\partial^2 = 0$, using that $d^2 = 0$ in B.

Now we will prove Proposition 2.12: Call $Y = \langle x, x', e_x\rangle_{x\in X}$ the free monoid in X with unit 1, $k\langle Y\rangle$ the k algebra associated to Y. Lets define $w_1 = xy'$, $w_2 = xe_y$ and $w_3 = e_x y'$. Let $S = \{r_1, r_2, r_3\}$ be the reduction system defined as follows: $r_i\colon k\langle Y\rangle\to k\langle Y\rangle$ the families of k-module endomorphisms such that r_i fix all elements except

$$r_1(xy') = z't,\quad r_2(xe_y) = e_z t \text{ and } r_3(e_x y') = z'e_t.$$

Note that S has more than 3 elements, each r_i is a family of reductions.

Definition 2.6. A reduction r_i *acts trivially* on an element a if w_i does not appear in a, i.e.: Aw_iB appears with coefficient 0.

Following [1], $a \in k\langle Y\rangle$ is called *irreducible* if Aw_iB does not appear for $i \in \{1,2,3\}$. Call $k_{irr}\langle Y\rangle$ the k submodule of irreducible elements of $k\langle Y\rangle$. A finite sequence of reductions is called *final* in a if $r_{i_n}\circ\cdots\circ r_{i_1}(a) \in k_{irr}(Y)$. An element $a \in k\langle Y\rangle$ is called *reduction-finite* if for every sequence of reductions r_{i_n} acts trivially on $r_{i_{n-1}}\circ\cdots\circ r_{i_1}(a)$ for sufficiently large n. If a is reduction-finite, then any maximal sequence of reductions, such that each r_{i_j} acts non-trivially on $r_{i_{(j-1)}}\ldots r_{i_1}(a)$, will be finite, and hence a final sequence. It follows that the reduction-finite elements form a k-submodule of $k\langle Y\rangle$. An element $a \in k\langle Y\rangle$ is called *reduction-unique* if it is reduction finite and it's image under every finite sequence of reductions is the same. This common value will be denoted $r_s(a)$.

Definition 2.7. Given a monomial $a \in k\langle Y\rangle$ we define the disorder degree of a, $\mathrm{disdeg}(a) = \sum_{i=1}^{n_x} rp_i + \sum_{i=1}^{n_{x'}} lp_j$, where rp_i is the position of the i-th letter "x" counting from right to left, and lp_i is the position of the i-th letter "x'" counting from left to right.

If $a = \sum_{i=1}^{n} k_i a_i$ where a_i are monomials in letters of X, X', e_X and $k_i \in K - \{0\}$,

$$\mathrm{disdeg}(a) := \sum_{i=1}^{n} \mathrm{disdeg}(a_i)$$

Example 2.8.

- $\mathrm{disdeg}(x_1 e_{y_1} x_2 z'_1 x_3 z'_2) = (2+4+6) + (4+6) = 22$
- $\mathrm{disdeg}(x e_y z') = 3+3 = 6$ and $\mathrm{disdeg}(x' e_y z) = 1+1$
- $\mathrm{disdeg}(\prod_{i=1}^{n} x'_i \prod_{i=1}^{m} e_{y_i} \prod_{i=1}^{k} z_i) = \frac{n(n+1)}{2} + \frac{k(k+1)}{2}$

The reduction r_1 lowers disorder degree in two and reductions r_2 and r_3 lower disorder degree in one.

Remark 2.9.

- $k_{irr}(Y) = \{\sum A' e_B C : A'$ word in X', e_B word in $e_x s, C$ word in $X\}$.
- $k_{irr} \simeq TX' \otimes TE \otimes TX$.

Take for example $a = x e_y z'$, there are two possible sequences of final reductions: $r_3\circ r_1\circ r_2$ or $r_2\circ r_1\circ r_3$. The result will be $a = A' e_B C$ and $a = D' e_E F$ respectively, where

$$A = \sigma^{(1)}\left(\sigma^{(1)}(x,y), \sigma^{(1)}(\sigma^{(2)}(x,y),z)\right)$$

$$B = \sigma^{(2)}\left(\sigma^{(1)}(x,y), \sigma^{(1)}(\sigma^{(2)}(x,y),z)\right)$$

$$C = \sigma^{(2)}\left(\sigma^{(2)}(x,y),z\right)$$

$$D = \sigma^{(1)}\left(x, \sigma^{(1)}(y,z)\right)$$

$$E = \sigma^{(1)}\left(\sigma^{(2)}(x, \sigma^{(1)}(y,z), \sigma^{(2)}(y,z))\right)$$

$$F = \sigma^{(2)}\left(\sigma^{(2)}(x, \sigma^{(1)}(y,z), \sigma^{(2)}(y,z))\right)$$

We have $A = D$, $B = E$ and $C = F$ as σ is a solution of YBeq, hence $r_3\circ r_1\circ r_2(x e_y z') = r_2\circ r_1\circ r_3(x e_y z')$.

A monomial a in $k\langle Y\rangle$ is said to have an *overlap ambiguity* of S if $a = ABCDE$ such that $w_i = BC$ and $w_j = CD$. We shall say the overlap ambiguity is *solvable* if there exist compositions of reductions, r, r' such that $r(Ar_i(BC)DE) = r'(ABr_j(CD)E)$. Notice that it is enough to take $r = r_s$ and $r' = r_s$.

Remark 2.10. In our case, there is only one type of overlap ambiguity and is the one we solved previously.

Proof. There is no rule with x' on the left nor rule with x on the right, so there will be no overlap ambiguity including the family r_1. There is only one type of ambiguity involving reductions r_2 and r_3. □

Notice that r_s is a projector and $I = \langle xy' - z't, xe_y - e_zt, e_xy' - z'e_t\rangle$ is trivially included in the kernel. We claim that it is actually equal:

Proof. As r_s is a projector, an element $a \in \ker$ must be $a = b - r_s(b)$ where $b \in k\langle Y\rangle$. It is enough to prove it for monomials b.

- If $a = 0$ the result follows trivially.
- If not, then take a monomial b where at least one of the products xy', xe_y or e_xy' appear. Lets suppose b has a factor xy' (the rest of the cases are analogous).
  $b = Axy'B$ where A or B may be empty words. $r_1(b) = Ar_1(xy')B = Az'tB$. Now we can rewrite: $b - r_s(b) = \underbrace{Axy'B - Az'tB}_{\in I} + Az'tB - r_s(b)$. As r_1 lowers disdeg in two, we have $\mathrm{disdeg}(Az'tB - r_s(b)) < \mathrm{disdeg}(b - r_s(b))$ then in a finite number of steps we get $b = \sum_{k=1}^{N} i_k$ where $i_k \in I$. It follows that $b \in I$. □

Corollary 2.11. *r_s induces a k-linear isomorphism:*

$$k\langle Y\rangle/\langle xy' - z't, xe_y - e_zt, e_xy' - z'e_t\rangle \to TX' \otimes TE \otimes TX$$

Returning to our bialgebra, taking quotients we obtain the following:

Proposition 2.12. $B \simeq (TX'/(x'y' = z't')) \otimes TE \otimes (TX/(xy = zt))$

Notice that $\overline{x_1 \dots x_n} = \overline{\prod \beta_m \circ \dots \circ \beta_1(x_1, \dots, x_n)}$ where $\beta_i = \sigma_{j_i}^{\pm 1}$, analogously with $\overline{x'_1 \dots x'_n}$. This ends the proof of Theorem 2.4.

Example 2.13. If the coefficients are trivial, $f \in C^1(X, k)$ and we identify $C^1(X, k) = k^X$, then

$$(\partial f)(x, y) = f(d(e_xe_y)) = -f(x) - f(y) + f(z) + f(t)$$

where as usual $\sigma(x, y) = (z, t)$ (if instead of considering $\mathrm{Hom}_{A'-A}$, we consider $\mathrm{Hom}_{A-A'}$ then $(\partial f)(x, y) = f(d(e_xe_y)) = f(x) + f(y) - f(z) - f(t)$ but with $\sigma(z, t) = (x, y)$).

Again with trivial coefficients, and $\Phi \in C^2(X, k) \cong k^{X^2}$, then

$$(\partial\Phi)(x, y, z) = \Phi(d(e_xe_ye_z)) = \Phi\left(\overbrace{xe_ye_z}^{I} - \overbrace{x'e_ye_z}^{II} - \overbrace{e_xye_z}^{III} + \overbrace{e_xy'e_z}^{IV} + \overbrace{e_xe_yz}^{V} - \overbrace{e_xe_yz'}^{VI}\right)$$

If considering $\mathrm{Hom}_{A'-A}$ then, using the relations defining B, the terms I, III, IV and VI change leaving

$$\partial\Phi(x,y,z) = \Phi(\sigma^1(x,y),\sigma^1(\sigma^2(x,y),z)) - \Phi(y,z) - \Phi(x,\sigma^1(y,z)) +$$
$$\Phi(\sigma^2(x,y),z) + \Phi(x,y) - \Phi(\sigma^2(x,\sigma^1(y,z)),\sigma^2(y,z))$$

If M is a $k[T]$-module (notice that T need not to be invertible as in [3]) then M can be viewed as an $A'-A$-bimodule via

$$x'\cdot m = m, \quad m\cdot x = Tm$$

The actions are compatible with the relations defining B:

$$(m\cdot x)\cdot y = T^2 m\ , \quad (m\cdot z)\cdot t = T^2 m$$

and

$$x'\cdot(y'\cdot m) = m\ , \quad z'\cdot(t'\cdot m) = m$$

Using these coefficients we get twisted cohomology as in [3] but for general YB solutions.

If one takes the special case of (X,σ) being a rack, namely $\sigma(x,y) = (y, x\triangleleft y)$, then the general formula gives

$$\partial f(x_1,\dots,x_n) = f(d(e_{x_1}\dots e_{x_n})) =$$
$$\sum_{i=1}^{n}(-1)^{i+1}\left(Tf(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n) - f({x_1}^{x_i},\dots,x_{i-1}^{x_i},x_{i+1},\dots,x_n)\right)$$

that agree with the differential of the twisted cohomology defined in [3].

Remark 2.14. If $f\colon X\times X\to k^\times$, where $k^\times$ denotes the units of k viewed as (multiplicative) abelian group, then $c_f(x\otimes y) := f(x,y)\sigma^1(x,y)\otimes\sigma^2(x,y)$, is a solution of YBeq if and only if f is a 2-cocycle with trivial coefficients.

Proof. Direct checking. □

3. First application: multiplicative structure on cohomology

Theorem 3.1. $\Delta = \Delta_B$ *induces an associative product in* $\mathrm{Hom}_{A'-A}(B,k)$ *(the graded Hom) and hence in* $(C^\bullet(X,\sigma),\partial^*)$ *and* $H^\bullet_{YB}(X,\sigma,k)$.

Proof. It is clear that Δ induces an associative product on $\mathrm{Hom}_k(B,k)$ (the graded Hom), and $\mathrm{Hom}_{A'-A}(B,k)\subset\mathrm{Hom}_k(B,k)$ is a k-submodule. We will show that it is in fact a subalgebra. Notice that Δ is a coderivation, so d^* will be a derivation with respect to that product.

Consider the A'–A diagonal structure on $B\otimes B$ (i.e. $x'_1.(b\otimes b').x_2 = x'_1bx_2\otimes x'_1b'x_2$ for $x_1,x_2\in X$) and denote $B\otimes^D B$ the k-module $B\otimes B$ considered as $A'-A$-bimodule in this diagonal way. We claim that $\Delta\colon B\to B\otimes^D B$ is a morphism of $A'-A$-modules:

$$\Delta(x'_1yx_2) = x'_1yx_2\otimes x'_1yx_2 = x'_1(y\otimes y)x_2$$

same with y', and with e_x:

$$\Delta(x'_1e_yx_2) = (x'_1\otimes x'_1)(y'\otimes e_y + e_y\otimes y)(x_2\otimes x_2) = x'_1\Delta(e_y)x_2$$

Dualizing Δ one gets:

$$\Delta^*\colon \mathrm{Hom}_{A'-A}(B\otimes^D B,k)\to \mathrm{Hom}_{A'-A}(B,k)$$

consider the natural map

$$\iota\colon \mathrm{Hom}_k(B,k)\otimes \mathrm{Hom}_k(B,k)\to \mathrm{Hom}_k(B\otimes B,k)$$
$$\iota(f\otimes g)(b_1\otimes b_2)=f(b_1)g(b_2)$$

and denote $\iota|$ by $\iota| = \iota|_{\mathrm{Hom}_{A'-A}(B,k)\otimes \mathrm{Hom}_{A'-A}(B,k)}$. Let us see that

$$Im(\iota|)\subset \mathrm{Hom}_{A'-A}(B\otimes B,k)\subset \mathrm{Hom}_k(B\otimes B,k)$$

If $f,g\colon B\to k$ are two $A'-A$-module morphisms (recall k has trivial actions, i.e. $x'\lambda=\lambda$ and $\lambda x=x$), then

$$\begin{aligned}\iota(f\otimes g)(x'(b_1\otimes b_2)) &= f(x'b_1)g(x'b_2)=(x'f(b_1))(x'g(b_2))\\ &= f(b_1)g(b_2)=x'\iota(f\otimes g)(b_1\otimes b_2)\\ \iota(f\otimes g)((b_1\otimes b_2)x) &= f(b_1x)g(b_2x)=(f(b_1)x)(g(b_2)x)\\ &= (f(b_1)g(b_2))x=\iota(f\otimes g)(b_1\otimes b_2)x\end{aligned}$$

So, it is possible to compose $\iota|$ and Δ, and obtain in this way an associative multiplication in $\mathrm{Hom}_{A'-A}(B,k)$. □

Now we will describe several natural quotients of B, each of them give rise to a subcomplex of the cohomological complex of X with trivial coefficients that are not only subcomplexes but also subalgebras; in particular they are associative algebras.

3.1. Square-free case

A very interesting type of solutions are the so-called square-free ones. A solution (X,σ) of YBeq is called *square-free* if $\sigma(x,x)=(x,x)$ for all $x\in X$. For instance, if X is a rack, then this condition is equivalent to X being a quandle, but the property $\sigma(x,x)=(x,x)$ makes sense for general solutions of the YBeq. The name comes from the fact that for each $x\in X$ such that $\sigma(x,x)\neq(x,x)$ one has a nontrivial condition for x^2 in the semigroup M_X. The square-free type was first considered by T. Gateva-Ivanova and M. Van den Bergh for involutive solutions in [10], in the study of what they call semigroups of I-type and semigroups of skew-polynomial-type. Later on, numerous works on that direction were published, for instance Gateva-Ivanova proved in [9] that all these three notions introduced in [9] were equivalent, and conjecture that every involutive square-free solution is retractable (in the sense of Etingof, Schedler and Soloviev [8]). This conjecture was proved to be true in some cases by Cedó, Jespers and Okniński [6] but developing the theory of extensions for involutive solutions L. Vendramin [15] found a family of counter-examples, so even in the involutive case, this family of solutions is far from being understood.

In the square-free situation, but not necessarily involutive, namely when X is such that $\sigma(x,x)=(x,x)$ for all x, one may add to the d.g. bialgebra B the relation $e_xe_x\sim 0$. If (X,σ) is a square-free solution of the YBeq, let us denote sf the two sided ideal of B generated by $\{e_xe_x\}_{x\in X}$.

Proposition 3.2. *sf is a differential Hopf ideal. More precisely,*

$$d(e_xe_x)=0 \textit{ and } \Delta(e_xe_x)=x'x'\otimes e_xe_x+e_xe_x\otimes xx.$$

In particular B/sf is a differential graded bialgebra. We may identify $\mathrm{Hom}_{A'A}(B/sf,k) \subset \mathrm{Hom}_{A'A}(B,k)$ as the elements f such that $f(\dots,x,x,\dots)=0$. If X is a quandle, this construction leads to the quandle-complex. We have $\mathrm{Hom}_{A'A}(B/sf,k) \subset \mathrm{Hom}_{A'A}(B,k)$ is not only a subcomplex, but also a subalgebra.

3.2. Biquandles

In [11], a generalization of quandles is proposed (we recall it with different notation), a solution (X,σ) is called non-degenerated, or *birack* if in addition,

1. for any $x,z \in X$ there exists a unique y such that $\sigma^1(x,y)=z$, (if this is the case, σ^1 is called *left invertible*),
2. for any $y,t \in X$ there exists a unique x such that $\sigma^2(x,y)=t$, (if this is the case, σ^2 is called *right invertible*).

A birack is called *biquandle* if, given $x_0 \in X$, there exists a unique $y_0 \in X$ such that $\sigma(x_0,y_0)=(x_0,y_0)$. In other words, if there exists a bijective map $s\colon X \to X$ such that

$$\{(x,y):\sigma(x,y)=(x,y)\}=\{(x,s(x)):x\in X\}$$

Remark 3.3. Every quandle solution is a biquandle, moreover, given a rack $(X,\triangleleft)$, then $\sigma(x,y)=(y,x\triangleleft y)$ is a biquandle if and only if $(X,\triangleleft)$ is a quandle.

If (X,σ) is a biquandle, for all $x\in X$ we add in B the relation $e_x e_{s(x)} \sim 0$. Let us denote bQ the two sided ideal of B generated by $\{e_x e_{sx}\}_{x\in X}$.

Proposition 3.4. *bQ is a differential Hopf ideal. More precisely, $d(e_x e_{sx})=0$ and $\Delta(e_x e_{sx}) = x's(x)'\otimes e_x e_{sx} + e_x e_{sx} \otimes xs(x)$.*

In particular B/bQ is a differential graded bialgebra. We may identify

$$\mathrm{Hom}_{A'A}(B/bQ,k) \cong \{f\in \mathrm{Hom}_{A'A}(B,k) : f(\dots,x,s(x),\dots)=0\} \subset \mathrm{Hom}_{A'A}(B,k)$$

In [4], the condition $f(\dots,x_0,s(x_0),\dots)=0$ is called the *type I condition*, because of its relation with Reidemeister move of type I, they prove that they form a subcomplex and they define biquandle cohomology as the homology of this subcomplex. A consequence of the above proposition is that the (cohomological) biquandle subcomplex is not only a subcomplex, but also a subalgebra. Before proving this proposition we will review some other similar constructions.

3.3. Identity case

The two cases above may be generalized in the following way:

Consider $S \subseteq X\times X$ a subset of elements satisfying $\sigma(x,y)=(x,y)$ for all $(x,y)\in S$. Define idS the two sided ideal of B given by $idS=\langle e_x e_y/(x,y)\in S\rangle$.

Proposition 3.5. *idS is a differential Hopf ideal. More precisely, $d(e_x e_y)=0$ for all $(x,y)\in S$ and $\Delta(e_x e_y) = x'y'\otimes e_x e_y + e_x e_y \otimes xy$.*

In particular B/idS is a differential graded bialgebra.

If one identifies $\mathrm{Hom}_{A'A}(B/sf,k) \subset \mathrm{Hom}_{A'A}(B,k)$ as the elements f such that

$$f(\dots,x,y,\dots) = 0 \ \forall (x,y) \in S$$

We have that $\mathrm{Hom}_{A'A}(B/idS,k) \subset \mathrm{Hom}_{A'A}(B,k)$ is not only a subcomplex, but also a subalgebra.

3.4. Flip case

Consider the condition $e_x e_y + e_y e_x \sim 0$ for all pairs such that $\sigma(x,y) = (y,x)$. For such a pair (x,y) we have the equations $xy = yx$, $xy' = y'x$, $x'y' = y'x'$ and $xe_y = e_y x$. Note that there is no equation for $e_x e_y$. The two sided ideal $D = \langle e_x e_y + e_y e_x : \sigma(x,y) = (y,x)\rangle$ is a differential and Hopf ideal. Moreover, the following generalization is still valid.

3.5. Involutive case

Assume $\sigma(x,y)^2 = (x,y)$. This case is called *involutive* in [8]. Define *Invo* the two sided ideal of B given by $Invo = \langle e_x e_y + e_z e_t : (x,y) \in X, \sigma(x,y) = (z,t)\rangle$.

Proposition 3.6. *Invo is a differential Hopf ideal. More precisely, $d(e_x e_y + e_z e_t) = 0$ for all $(x,y) \in X$ (with $(z,t) = \sigma(x,y)$) and if $\omega = e_x e_y + e_z e_t$ then $\Delta(\omega) = x'y' \otimes \omega + \omega \otimes xy$.*

In particular $B/Invo$ is a differential graded bialgebra. If one identifies $\mathrm{Hom}_{A'A}(B/Invo,k) \subset \mathrm{Hom}_{A'A}(B,k)$ then $\mathrm{Hom}_{A'A}(B/Invo,k) \subset \mathrm{Hom}_{A'A}(B,k)$ is not only a subcomplex, but a subalgebra.

Conjecture 3.7. *$B/Invo$ is acyclic in positive degrees.*

Example 3.8. If $\sigma = \mathit{flip}$ and $X = \{x_1,\dots,x_n\}$ then $A = k[x_1,\dots,x_n] = SV$, the symmetric algebra on $V = \oplus_{x\in X} kx$. In this case $(B/Invo, d) \cong (S(V) \otimes \Lambda V \otimes S(V), d)$ gives the Koszul resolution of $S(V)$ as $S(V)$-bimodule.

Example 3.9. If $\sigma = Id$, $X = \{x_1,\dots,x_n\}$ and $V = \oplus_{x\in X} kx$, then $A = TV$ the tensor algebra. If $\frac{1}{2} \in k$, then $(B/invo, d) \cong TV \otimes (k \oplus V) \otimes TV$ gives the Koszul resolution of TV as TV-bimodule. Notice that we don't really need $\frac{1}{2} \in k$, one could replace $invo = \langle e_x e_y + e_x e_y : (x,y) \in X \times X\rangle$ by $idX \times X = \langle e_x e_y : (x,y) \in X \times X\rangle$.

The conjecture above, besides these examples, is supported by next result:

Proposition 3.10. *If $\mathbb{Q} \subseteq k$, then $B/Invo$ is acyclic in positive degrees.*

Proof. In $B/Invo$ it can be defined h as the unique (super)derivation such that:

$$h(e_x) = 0; h(x) = e_x, h(x') = -e_x$$

Let us see that h is well defined:

$$h(xy - zt) = e_x y + x e_y - e_z t - z e_t = 0$$
$$h(xy' - z't) = e_x y' - x e_y + e_z t - z' e_t = 0$$
$$h(x'y' - z't') = -e_x y' - x' e_y + e_z t' + z' e_t = 0$$
$$h(x e_y - e_z t) = e_x e_y + e_z e_t = 0$$

Notice that in particular next equation shows that h is not well-defined in B.

$$h(e_x y' - z' e_t) = e_x e_y + e_z e_t = 0$$
$$h(zt' - x'y) = e_z t' - z e_t + e_x y - x' e_y = 0$$
$$h(z e_t - e_x y) = e_z e_t + e_x e_y = 0$$
$$h(e_z t' - x' e_y) = e_z e_t + e_x e_y = 0$$
$$h(e_x e_y + e_z e_t) = 0$$

Since (super)commutator of (super)derivations is again a derivation, we have that $[h, d] = hd + dh$ is also a derivation. Computations on generators:

$$h(e_x) = 2e_x,\ h(x) = x - x',\ h(x') = x' - x$$

or equivalently

$$h(e_x) = 2e_x,\ h(x + x') = 0,\ h(x - x') = 2(x - x')$$

One can also easily see that $B/Invo$ is generated by e_x, $x_\pm$, where $x_\pm = x \pm x'$, and that their relations are homogeneous. We see that $hd + dh$ is nothing but the Euler derivation with respect to the grading defined by

$$\deg e_x = 2,\ \deg x_+ = 0,\ \deg x_- = 2.$$

We conclude automatically that the homology vanish for positive degrees of the e_x's (and similarly for the x_-'s). □

Next, we generalize Propositions 3.2, 3.4, 3.5 and 3.6.

3.6. Braids of order N

Let $(x_0, y_0) \in X \times X$ such that $\sigma^N(x_0, y_0) = (x_0, y_0)$ for some $N \geq 1$. If $N = 1$ we have the "identity case" and all subcases, if $N = 2$ we have the "involutive case". Denote

$$(x_i, y_i) := \sigma^i(x_0, y_0)\ 1 \leq i \leq N - 1$$

Notice that the following relations hold in B:

- ⋆ $x_{N-1}y_{N-1} \sim x_0 y_0,\ \ x_{N-1}y'_{N-1} \sim x'_0 y_0,\ \ x'_{N-1}y'_{N-1} = x'_0 y'_0$
- ⋆ $x_{N-1}e_{y_{N-1}} \sim e_{x_0} y_0,\ \ e_{x_{N-1}} y'_{N-1} \sim x'_0 e_{y_0}$

and for $1 \leq i \leq N - 1$:

- ⋆ $x_{i-1}y_{i-1} \sim x_i y_i,\ \ x_{i-1}y'_{i-1} \sim x'_i y_i,\ \ x'_{i-1}y'_{i-1} = x'_i y'_i$
- ⋆ $x_{i-1}e_{y_{i-1}} \sim e_{x_i} y_i,\ \ e_{x_{i-1}} y'_{i-1} \sim x'_i e_{y_i}$

Take $\omega = \sum_{i=0}^{N-1} e_{x_i} e_{y_i}$, then we claim that

$$d\omega = 0$$

and

$$\Delta\omega = x_0 y_0 \otimes \omega + \omega \otimes x'_0 y'_0$$

For that, we compute

$$d(\omega) = \sum_{i=0}^{N-1} (x_i - x'_i)e_{y_i} - e_{x_i}(y_i - y'_i) =$$

$$\sum_{i=0}^{N-1} (x_i e_{y_i} - e_{x_i} y_i) - \sum_{i=0}^{N-1} (x'_i e_{y_i} - e_{x_i} y'_i) = 0$$

For the comultiplication, we recall that

$$\Delta(ab) = \Delta(a)\Delta(b)$$

where the product on the right hand side is defined using the Koszul sign rule:

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 b_1 \otimes a_2 b_2$$

So, in this case we have

$$\Delta(\omega) = \sum_{i=0}^{N-1} \Delta(e_{x_i} e_{y_i}) =$$

$$\sum_{i=0}^{N-1} (x'_i y'_i \otimes e_{x_i} e_{y_i} - x'_i e_{y_i} \otimes e_{x_i} y_i + e_{x_i} y'_i \otimes x_i e_{y_i} + e_{x_i} e_{y_i} \otimes x_i y_i)$$

the middle terms cancel telescopically, giving

$$= \sum_{i=0}^{N-1} (x'_i y'_i \otimes e_{x_i} e_{y_i} + e_{x_i} e_{y_i} \otimes x_i y_i)$$

and the relation $x_i y_i \sim x_{i+1} y_{i+1}$ gives

$$= x'_0 y'_0 \otimes (\sum_{i=0}^{N-1} e_{x_i} e_{y_i}) + (\sum_{i=0}^{n-1} e_{x_i} e_{y_i}) \otimes x_0 y_0$$

$$= x'_0 y'_0 \otimes \omega + \omega \otimes x_0 y_0$$

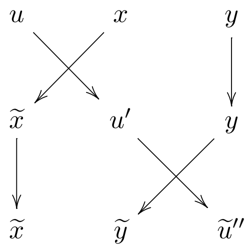
Then the two-sided ideal of B generated by ω is a Hopf ideal. If instead of a single ω we have several $\omega_1, \dots \omega_n$, we simply remark that the sum of differential Hopf ideals is also a differential Hopf ideal.

Remark 3.11. If X is finite, then for every (x_0, y_0) there exists $N > 0$ such that $\sigma^N(x_0, y_0) = (x_0, y_0)$.

Remark 3.12. Let us suppose $(x_0, y_0) \in X \times X$ is such that $\sigma^N(x_0, y_0) = (x_0, y_0)$ and $u \in X$ an arbitrary element. Consider the element

$$((\mathrm{Id} \times \sigma)(\sigma \times \mathrm{Id})(u, x_0, y_0) = (\widetilde{x}_0, \widetilde{y}_0, u'')$$

graphically



then $\sigma^N(\widetilde{x}_0, \widetilde{y}_0) = (\widetilde{x}_0, \widetilde{y}_0)$.

Proof.

$$(\sigma^N \times id)(\widetilde{x}_0, \widetilde{y}_0, u'') = (\sigma^N \times id)(id \times \sigma)(\sigma \times id)(u, x_0, y_0) =$$
$$(\sigma^{N-1} \times id)(\sigma \times id)(id \times \sigma)(\sigma \times id)(u, x_0, y_0) =$$

using YBeq

$$(\sigma^{N-1} \times id)(id \times \sigma)(\sigma \times id)(id \times \sigma)(u, x_0, y_0) =$$

repeating the procedure $N-1$ times leaves

$$(id \times \sigma)(\sigma \times id)(id \times \sigma^N)(u, x_0, y_0) = (id \times \sigma)(\sigma \times id)(u, x_0, y_0) = (\widetilde{x}_0, \widetilde{y}_0, u'') \quad \square$$

4. Second application: comparison with Hochschild cohomology

B is a differential graded algebra, and on each degree n it is isomorphic to $A \otimes (TV)_n \otimes A$, where $V = \oplus_{x \in X} ke_x$. In particular B_n is free as A^e-module. We have *for free* the existence of a comparison map

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & B_n & \longrightarrow \cdots \longrightarrow & B_2 & \xrightarrow{d} & B_1 & \xrightarrow{d} & B_0 & & & \\
& \| & & \| & & \| & & \| & & & \\
\cdots \longrightarrow & A'(TX)_nA & \longrightarrow \cdots \longrightarrow & \oplus_{x,y\in X} A'e_xe_yA & \xrightarrow{d} & \oplus_{x\in X} A'e_xA & \xrightarrow{d} & A'A & & & \\
& \|\cong & & \|\cong & & \| & & \|\cong & & & \\
\cdots \longrightarrow & A \otimes V^{\otimes n} \otimes A & \longrightarrow \cdots \longrightarrow & A \otimes V^{\otimes 2} \otimes A & \xrightarrow{d_2} & A \otimes V \otimes A & \xrightarrow{d_1} & A \otimes A & \xrightarrow{m} & A & \longrightarrow 0 \\
& \downarrow \widetilde{\mathrm{Id}} & & \downarrow \widetilde{\mathrm{Id}} & & \downarrow \widetilde{\mathrm{Id}} & & \downarrow \mathrm{Id} & & \downarrow \mathrm{Id} & \\
\cdots \longrightarrow & A \otimes A^{\otimes n} \otimes A & \longrightarrow \cdots \longrightarrow & A \otimes A^{\otimes 2} \otimes A & \xrightarrow{b'} & A \otimes A \otimes A & \xrightarrow{b'} & A \otimes A & \xrightarrow{m} & A & \longrightarrow 0
\end{array}$$

Corollary 4.1. *For all A-bimodule M, there exists natural maps*

$$\widetilde{\mathrm{Id}}_* \colon H^{YB}_\bullet(X, M) \to H_\bullet(A, M)$$
$$\widetilde{\mathrm{Id}}^* \colon H^\bullet(A, M) \to H^\bullet_{YB}(X, M)$$

that are the identity in degrees zero and one.

Moreover, one can choose an explicit map with extra properties. For that we recall some definitions: there is a set theoretical section to the canonical projection from the Braid group to the symmetric group

$$\mathbb{B}_n \twoheadrightarrow \mathbb{S}_n$$

$$T_s := \sigma_{i_1} \ldots \sigma_{i_k} \longleftarrow\!\dashv s = \tau_{i_1} \ldots \tau_{i_k}$$

where

- $\tau \in S_n$ are transpositions of neighboring elements i and $i+1$, so-called simple transpositions,
- σ_i are the corresponding generators of $\mathbb{B}_n$,
- $\tau_{i_1} \ldots \tau_{i_k}$ is one of the shortest words representing s.

This inclusion factorizes trough

$$\mathbb{S}_n \hookrightarrow \mathbb{B}_n^+ \hookrightarrow \mathbb{B}_n$$

It is a set inclusion not preserving the monoid structure.

Definition 4.2. The permutation sets

$$\mathrm{Sh}_{p_1,\ldots,p_k} := \{s \in \mathbb{S}_{p_1+\cdots+p_k} / s(1) < \cdots < s(p_1), \cdots, s(p+1) < \cdots < s(p+p_k)\},$$

where $p = p_1 + \cdots + p_{k-1}$, are called *shuffle sets*.

Remark 4.3. It is well known that a braiding σ gives an action of the positive braid monoid B_n^+ on $V^{\otimes n}$, i.e. a monoid morphism

$$\rho \colon B_n^+ \to End_{\mathbb{K}}(V^{\otimes n})$$

defined on generators σ_i of B_n^+ by

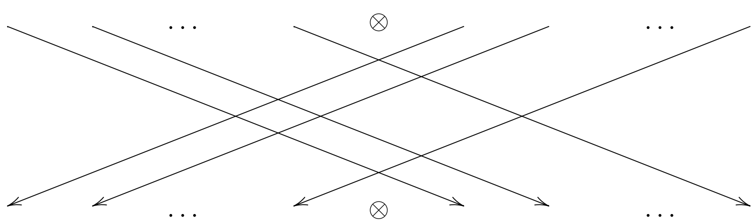
$$\sigma_i \mapsto \mathrm{Id}_V^{\otimes(i-1)} \otimes \sigma \otimes \mathrm{Id}_V^{\otimes(n-i+1)}$$

Then there exists a natural extension of a braiding in V to a braiding in $T(V)$.

$$\sigma(v \otimes w) = (\sigma_k \ldots \sigma_1) \circ \cdots \circ (\sigma_{n+k-2} \ldots \sigma_{n-1}) \circ (\sigma_{n+k-1} \ldots \sigma_n)(vw) \in V^k \otimes V^n$$

for $v \in V^{\otimes n}$, $w \in V^k$ and vw being the concatenation.

Graphically



Definition 4.4. The quantum shuffle multiplication on the tensor space $T(V)$ of a braided vector space (V, σ) is the k-linear extension of the map

$$\sqcup\!\sqcup_\sigma = \sqcup\!\sqcup_\sigma^{p,q} \colon V^{\otimes p} \otimes V^{\otimes q} \to V^{\otimes(p+q)}$$
$$\overline{v} \otimes \overline{w} \mapsto \overline{v} \sqcup\!\sqcup_\sigma \overline{w} := \sum_{s \in Sh_{p,q}} T_s^\sigma(\overline{vw})$$

Notation: T_s^σ stands for the lift $T_s \in \mathbb{B}_n^+$ acting on $V^{\otimes n}$ via the braiding σ. The algebra $Sh_\sigma(V) := (TV, \sqcup\!\sqcup_\sigma)$ is called the quantum shuffle algebra on (V, σ)

It is well-known that $\sqcup\!\sqcup_\sigma$ is an associative product on TV (see for example [12] for details) that makes it a Hopf algebra with deconcatenation coproduct.

Definition 4.5. Let V be a braided vector space, then the quantum symmetrizer map $\sqcup\!\sqcup_\sigma \colon V^{\otimes n} \to V^{\otimes n}$ defined by

$$QS_\sigma(v_1 \otimes \cdots \otimes v_n) = \sum_{\tau \in \mathbb{S}_n} T_\tau^\sigma(v_1 \otimes \cdots \otimes v_n)$$

where T_τ^σ is the lift $T_\tau^\sigma \in \mathbb{B}_n^+$ of τ, acting on $V^{\otimes n}$ via the braiding σ.

In terms of shuffle products the quantum symmetrizer can be computed as

$$\omega \sqcup\!\sqcup_\sigma \eta := \sum_{\tau \in \mathrm{Sh}_{p,q}} T_\tau^\sigma(\omega \otimes \eta)$$

The quantum symmetrizer map can also be defined as

$$QS_\sigma(v_1 \otimes \cdots \otimes v_n) = v_1 \sqcup\!\sqcup_\sigma \cdots \sqcup\!\sqcup_\sigma v_n$$

With this notation, next result reads as follows:

Theorem 4.6. *The A'–A-linear quantum symmetrizer map*

$$A'V^{\otimes n}A \xrightarrow{\widetilde{\mathrm{Id}}} A \otimes A^{\otimes n} \otimes A$$

$$a_1' e_{x_1} \cdots e_{x_n} a_2 \longmapsto a_1 \otimes (x_1 \sqcup\!\sqcup_{-\sigma} \cdots \sqcup\!\sqcup_{-\sigma} x_n) \otimes a_2$$

is a chain map lifting the identify. Moreover, $\widetilde{\mathrm{Id}} \colon B \to (A \otimes TA \otimes A, b')$ is a differential graded algebra map, where in TA the product is $\sqcup\!\sqcup_{-\sigma}$, and in $A \otimes TA \otimes A$ the multiplicative structure is not the usual tensor product algebra, but the braided one. In particular, this map factors through $A \otimes \mathfrak{B} \otimes A$, where $\mathfrak{B}$ is the Nichols algebra associated to the braiding $\sigma'(x \otimes y) = -z \otimes t$, where $x, y \in X$ and $\sigma(x, y) = (z, t)$.

Remark 4.7. The Nichols algebra $\mathfrak{B}$ is the quotient of TV by the ideal generated by (skew)primitives that are not in V, so the result above explains the good behavior of the ideals *invo*, *idS*, or in general the ideal generated by elements of the form $\omega = \sum_{i=0}^{N-1} e_{x_i} e_{y_i}$ where $\sigma(x_i, y_i) = (x_{i+1}, y_{i+1})$ and $\sigma^N(x_0, y_0) = (x_0, y_0)$. It would be interesting to know the properties of $A \otimes \mathfrak{B} \otimes A$ as a differential object, since it appears to be a candidate of Koszul-type resolution for the semigroup algebra A (or similarly the group algebra $k[G_X]$).

The rest of the paper is devoted to the proof of Theorem 4.6. Most of the Lemmas are “folklore” but we include them for completeness. The interested reader can look at [13] and references therein.

Lemma 4.8. *Let σ be a braid in the braided (sub)category that contains two associative algebras A and C, meaning there exists bijective functions*

$$\sigma_A \colon A \otimes A \to A \otimes A,\ \sigma_C \colon C \otimes C \to C \otimes C,\ \sigma_{C,A} \colon C \otimes A \to A \otimes C$$

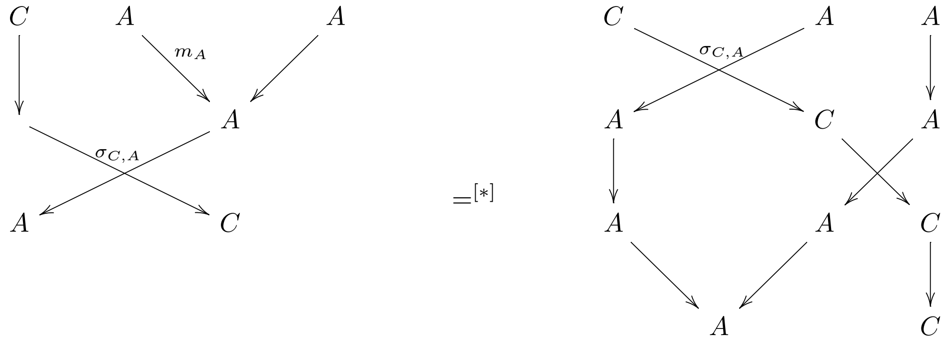
such that

$$\sigma_*(1,-) = (-,1) \text{ and } \sigma_*(-,1) = (1,-) \quad \text{for } * \in \{A,C;C,A\}$$
$$\sigma_{C,A} \circ (1 \otimes m_A) = (m_A \otimes 1)(1 \otimes \sigma_{C,A})(\sigma_{C,A} \otimes 1)$$

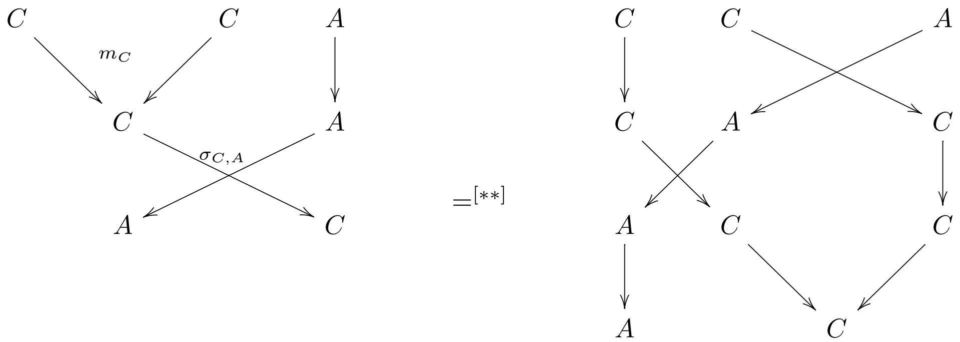
and

$$\sigma_{C,A} \circ (m_C \otimes 1) = (1 \otimes m_C)(\sigma_{C,A} \otimes 1)(1 \otimes \sigma_{C,A})$$

Diagrammatically



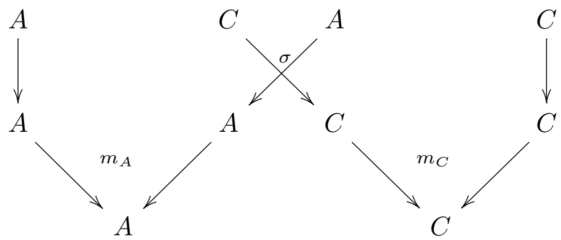
and



Assume that they satisfy the braid equation with any combination of σ_A, σ_C *or* $\sigma_{A,C}$*. Then,* $A \otimes_\sigma C = A \otimes C$ *with product defined by*

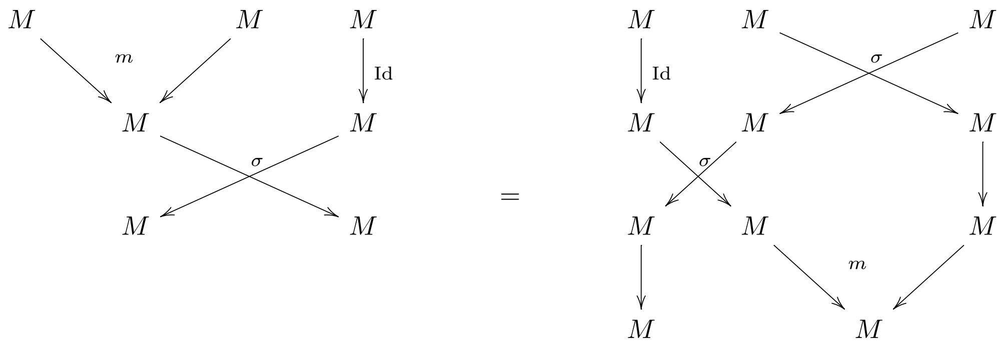
$$(m_A \otimes m_C) \circ (\mathrm{Id}_A \otimes \sigma_{C,A} \otimes \mathrm{Id}_C) \colon (A \otimes C) \otimes (A \otimes C) \to A \otimes C$$

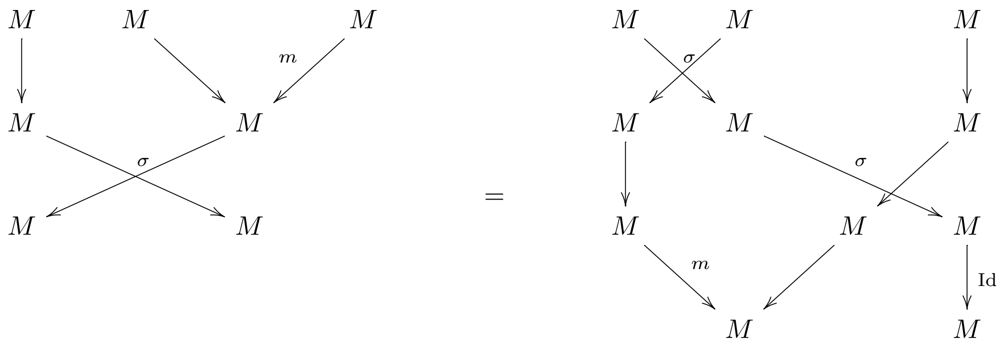
is an associative algebra. In diagram:



Proof. Take $m \circ (1 \otimes m)((a_1 \otimes c_2) \otimes ((a_2 \otimes c_2) \otimes (a_3 \otimes c_3))$ use [$*$], associativity in A, associativity in C then [$**$] and the result follows. □

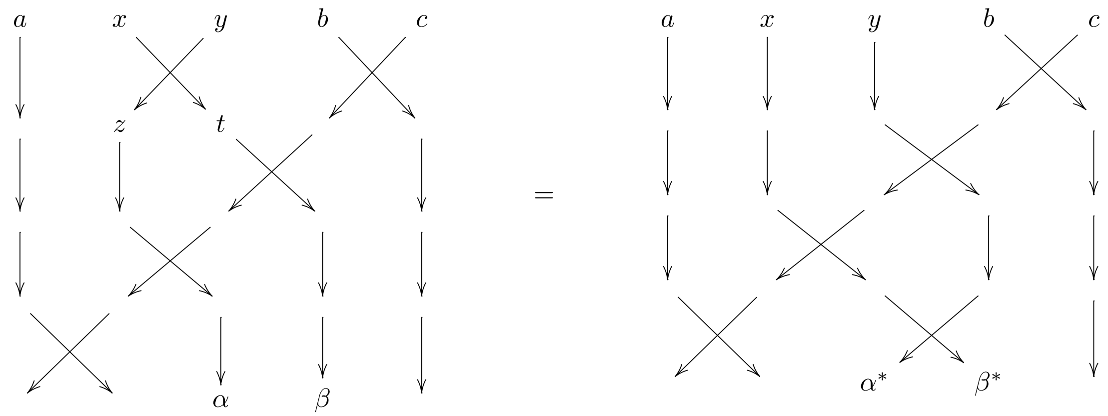
Lemma 4.9. *Let M be the monoid generated by X module the relation $xy = zt$ where $\sigma(x, y) = (z, t)$, then, $\sigma \colon X \times X \to X \times X$ naturally extends to a braiding in M and satisfies*





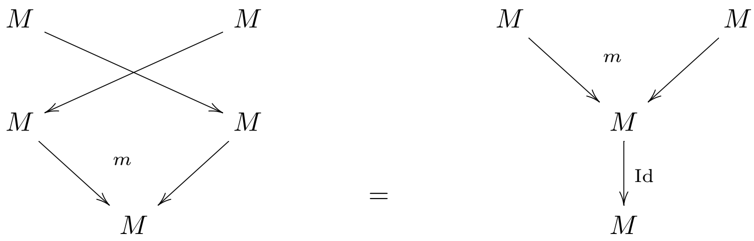
Proof. It is enough to prove that the extension mentioned before is well defined in the quotient. Inductively, it will be enough to see that $\sigma(axyb, c) = \sigma(aztb, c)$ and $\sigma(c, axyb) = \sigma(c, aztb)$ where $\sigma(x, y) = (z, t)$, and this follows immediately from the braid equation:

A diagram for the first equation is the following:



As $\alpha\beta = \alpha^*\beta^*$ the result follows. □

Lemma 4.10. *$m \circ \sigma = m$, diagrammatically:*



Proof. Using successively that $m \circ \sigma_i = m$, we have:

$$\begin{aligned} m \circ \sigma(x_1 \dots x_n, y_1 \dots y_k) &= m\left((\sigma_k \dots \sigma_1) \dots (\sigma_{n+k-1} \dots \sigma_n)_{(x_1 \dots x_n y_1 \dots y_k)}\right) \\ &= m\left((\sigma_{k-1} \dots \sigma_1) \dots (\sigma_{n+k-1} \dots \sigma_n)_{(x_1 \dots x_n y_1 \dots y_k)}\right) = \dots \\ &= m(x_1 \dots x_n, y_1 \dots y_k) \quad \square \end{aligned}$$

Corollary 4.11. *If one considers $A = k[M]$, then the algebra A satisfies all diagrams in previous lemmas.*

Lemma 4.12. *If $T = (TA, ⧢_\sigma)$ there are bijective functions*

$$\begin{aligned} \sigma_{T,A} &:= \sigma|_{T \otimes A} \colon T \otimes A \to A \otimes T \\ \sigma_{A,T} &:= \sigma|_{A \otimes T} \colon A \otimes T \to T \otimes A \end{aligned}$$

that satisfies the hypothesis of Lemma 4.8, and the same for $(TA, ⧢_{-\sigma})$.

Corollary 4.13. *$A \otimes (TA, ⧢_{-\sigma}) \otimes A$ is an algebra.*

Proof. Use Lemma 4.8 twice and the result follows. $\square$

Corollary 4.14. *Taking $A = k[M]$, then the standard resolution of A as A-bimodule has a natural algebra structure defining the braided tensorial product as follows:*

$$A \otimes TA \otimes A = A \otimes_\sigma (T^c A, ⧢_{-\sigma}) \otimes_\sigma A$$

Recall the differential of the standard resolution is defined as $b' \colon A^{\otimes n+1} \to A^{\otimes n}$

$$b'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

for all $n \geq 2$. If A is a commutative algebra then the Hochschild resolution is an algebra viewed as $\oplus_{n \geq 2} A^{\otimes n} = A \otimes TA \otimes A$, with right and left A-bilinear extension of the shuffle product on TA, and b' is a (super) derivation with respect to that product (see for instance Prop. 4.2.2 [14]). In the braided-commutative case we have the analogous result:

Lemma 4.15. *b' is a derivation with respect to the product mentioned in Corollary 4.14.*

Proof. Recall the commutative proof as in Prop. 4.2.2 [14]. Denote $*$ the product

$$(a_0 \otimes \cdots \otimes a_{p+1}) * (b_0 \otimes \cdots \otimes b_{q+1}) = a_0 b_0 \otimes ((a_1 \cdots \otimes a_p) ⧢ (b_1 \otimes \cdots \otimes b_q)) \otimes a_{p+1} b_{q+1}$$

Since $\oplus_{n\geq 2}A^{\otimes n} = A\otimes TA\otimes A$ is generated by $A\otimes A$ and $1\otimes TA\otimes 1$, we check on generators. For $a\otimes b \in A\otimes A$, $b'(a\otimes b) = 0$, in particular, it satisfies Leibnitz rule for elements in $A\otimes A$. Also, b' is A-linear on the left, and right-linear on the right, so

$$\begin{aligned} b'\big((a_0\otimes a_{n+1}) * (1\otimes a_1\otimes\cdots\otimes a_n\otimes 1)\big) &= b'(a_0\otimes a_1\otimes\cdots\otimes a_n\otimes a_{n+1}) \\ = a_0 b'(1\otimes a_1\otimes\cdots\otimes a_n\otimes 1)a_{n+1} &= (a_0\otimes a_{n+1}) * b'(1\otimes a_1\otimes\cdots\otimes a_n\otimes 1) \\ &= 0 + (a_0\otimes a_{n+1}) * b'(1\otimes a_1\otimes\cdots\otimes a_n\otimes 1) \\ = b'(a_0\otimes a_{n+1}) * (1\otimes a_1\otimes\cdots\otimes a_n\otimes 1) &+ (a_0\otimes a_{n+1}) * b'(1\otimes a_1\otimes\cdots\otimes a_n\otimes 1) \end{aligned}$$

Now consider $(1\otimes a_1\otimes\cdots\otimes a_p\otimes 1)*(1\otimes b_1\otimes\cdots\otimes b_q\otimes 1)$, it is a sum of terms where two consecutive tensor terms can be of the form (a_i, a_{i+1}), or (b_j, b_{j+1}), or (a_i, b_j) or (b_j, a_i). When one computes b', multiplication of two consecutive tensor factors will give, respectively, terms of the form

$$\cdots\otimes a_i a_{i+1}\otimes\cdots\ ,\ \cdots\otimes b_j b_{j+1}\otimes\cdots\ ,\ \cdots\otimes a_i b_j\otimes\cdots\ ,\cdots\otimes b_j a_i\otimes\cdots$$

The first type of terms will recover $b'((1\otimes a_1\otimes\cdots\otimes a_n\otimes 1)) * (1\otimes b_1\otimes\cdots\otimes b_q\otimes 1)$ and the second type of terms will recover $\pm(1\otimes a_1\otimes\cdots\otimes a_n\otimes 1) * b'((1\otimes b_1\otimes\cdots\otimes b_q\otimes 1))$. On the other hand, the difference between the third and forth type of terms is just a single transposition so they have different signs, while $a_i b_j = b_j a_i$ because the algebra is commutative, if one take the *signed* shuffle then they cancel each other.

In the *braided* shuffle product, the summands are indexed by the same set of shuffles, so we have the same type of terms, that is, when computing b' of a (signed) shuffle product, one may do the product of two elements in coming form the first factor, two elements of the second factor, or a mixed term. For the mixed terms, they will have the form

$$\cdots\otimes A_i B_j\otimes\cdots,\ \text{or}\ \cdots\otimes\sigma^1(A_i,B_j)\sigma^2(A_i,B_j)\otimes\cdots$$

As in the algebra A we have $A_i B_j = \sigma^1(A_i,B_j)\sigma^2(A_i,B_j)$ then these terms will cancel leaving only the terms corresponding to $b'(1\otimes a_1\otimes\cdots\otimes a_p\otimes 1)\sqcup_{-\sigma}(1\otimes b_1\otimes\cdots\otimes b_q\otimes)$ and $\pm(1\otimes a_1\otimes\cdots\otimes a_p\otimes 1)\sqcup_{-\sigma}$ $b'(1\otimes b_1\otimes\cdots\otimes b_q\otimes 1)$ respectively. □

Corollary 4.16. *There exists a comparison morphism* $f\colon (B,d)\to(A\otimes TA\otimes A, b')$ *which is a differential graded algebra morphism,* $f(d) = b'(f)$*, simply defining it on* e_x *(*$x\in X$*) and satisfying* $f(x'-x) = b'(f(e_x))$*.*

Proof. Define f on e_x, extend k-linearly to V, multiplicatively to TV, and A'–A linearly to $A'\otimes TV\otimes A = B$. In order to see that f commutes with the differential, by A'–A-linearity it suffices to check on TV, but since f is multiplicative on TV it is enough to check on V, and by k-linearity we check on basis, that is, we only need $f(de_x) = b'f(e_x)$. □

Corollary 4.17. $f|_{TX}$ *is the quantum symmetrizer map, and therefore* $\mathrm{Ker}(f)\cap TX\subset B$ *defines the Nichol's ideal associated to* $-\sigma$*.*

Proof.

$$f(e_{x_1}\cdots e_{x_n}) = f(e_{x_1}) * \cdots * f(e_{x_n}) = (1\otimes x_1\otimes 1) * \cdots * (1\otimes x_n\otimes 1) = 1\otimes(x_1\sqcup\cdots\sqcup x_n)\otimes 1 \quad \square$$

The previous corollary explains why $\mathrm{Ker}(\mathrm{Id}-\sigma)\subset B_2$ gives a Hopf ideal and also ends the proof of Theorem 4.6.

Question 4.18. *Im*$(f) = A\otimes\mathfrak{B}\otimes A$ is a resolution of A as a A-bimodule? Namely, is $(A\otimes\mathfrak{B}\otimes A, d)$ acyclic?

This is the case for involutive solutions in characteristic zero, but also for $\sigma =$ flip in any characteristic, and $\sigma = \mathrm{Id}$ (notice this Id-case gives the Koszul resolution for the tensor algebra). If the answer to that question is yes, and $\mathfrak{B}$ is finite dimensional then A have necessarily finite global dimension. Another interesting question is how to relate generators for the relations defining $\mathfrak{B}$ and cohomology classes for X.

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