THE FIXED POINT PROPERTY IN EVERY WEAK HOMOTOPY TYPE

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Abstract. We prove that for any connected compact CW-complex K there exists a space X weak homotopy equivalent to K which has the fixed point property, that is, every continuous map $X \to X$ has a fixed point. The result is known to be false if we require X to be a polyhedron. The space X we construct is a non-Hausdorff space with finitely many points.

A topological space X has the fixed point property if every continuous map $f : X \to X$ has a fixed point. We will prove the following

Theorem 1. *Let K be a connected compact CW-complex. Then there exists a topological space X weak homotopy equivalent to K with the fixed point property.*

If we require X to be a polyhedron, the result is known to be false. Though the fixed point property is not a homotopy invariant, every polyhedron homotopy equivalent to a sphere lacks the fixed point property (see [6, Theorem 7.1], [8] or the proof of [11, Theorem]). The space X we find has finitely many points. Therefore, we are also proving the following result. The homotopy type of any connected compact CW-complex can be realized by the order complex of a finite partially ordered set with the fixed point property.

A simplicial complex K has the *fixed simplex property* if for every simplicial map $f : K \to K$ there exists a simplex $\sigma \in K$ such that $f(\sigma) = \sigma$ or, equivalently, if every simplicial endomorphism of K fixes a point of the realization of K. The spheres S^n do not have the fixed point property, but they do have triangulations with the fixed simplex property provided that $n \geq 2$ (see Proposition 3). We will show that for every simply connected compact polyhedron K there exists a finite simplicial complex L homotopy equivalent to K with the fixed simplex property. Then the finite topological space $\mathcal{X}(L)$ associated to L has the fixed point property. This will prove Theorem 1 for simply connected complexes. If K is not simply connected we will be able to modify the construction above to obtain a finite model of K with the fixed point property but it will not be the poset of faces of a complex.

We sketch in a few lines the idea of the construction of L from K and the main parts of the proof. We first consider integer homology classes $\alpha_{k,l} \in H_k(K)$ which are a basis of the rational k-homology of K and then realize each $\alpha_{k,l}$ as the image of the fundamental class $[M_{k,l}]$ of a k-dimensional oriented pseudomanifold through a map $M_{k,l} \to K$. We construct L as follows. We find a sufficiently fine (j-th barycentric) subdivision K^j of K and attach to K^j mapping cylinders of the maps $M^j_{k,l} \to K^j$ and of approximations to the

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identities $M^j_{k,l} \to M^r_{k,l}$ $(r < j)$. In this way we manage to concentrate the homology classes $\alpha_{k,l}$ in "few" simplices, those of $M^r_{k,l}$. The complex L satisfies this singular property: if $c = \sum t_i\sigma_i \in C_k(L)$ is a cycle such that $\sum |t_i|$ is less than or equal to the number of k-simplices in $M^r_{k,l}$ and the $(\alpha_{k,l}\otimes 1_{\mathbb{Q}})$-coordinate of $[c] \in H_k(L;\mathbb{Q})$ is different from zero, then c is, up to sign, the fundamental cycle of $M^r_{k,l}$. Therefore if f is a simplicial endomorphism of L, one of the following holds: (1) the matrices of $f_* : H_k(L;\mathbb{Q}) \to H_k(L;\mathbb{Q})$ in the basis $\{\alpha_{k,l} \otimes 1_{\mathbb{Q}}\}_l$ have all the diagonal zero for each $k \geq 2$, in which case the Lefschetz fixed point theorem gives us a fixed simplex, or (2) the map f maps one of the $M^r_{k,l}$ into itself and $f|_{M^r_{k,l}} : M^r_{k,l} \to M^r_{k,l}$ is a simplicial automorphism. The argument is then complete if we show that the pseudomanifolds $M^r_{k,l}$ can be assumed to be *asymmetric*, in the sense that every automorphism fixes a vertex.

The following notions are a rigid version of Gromov's simplicial volume and the ℓ^1-norm. Let C be a finitely generated free $\mathbb{Z}$-module with a fixed basis $\{b_1, b_2, \dots, b_r\}$. The *norm* $\|c\|$ of an element $c = \sum t_ib_i \in C$ is $\sum |t_i|$. If $f : C \to C'$ is a morphism between finitely generated free $\mathbb{Z}$-modules (each of them with a chosen basis), the *norm* of f is $\|f\| = \max\limits_{c\neq 0} \frac{\|f(c)\|}{\|c\|}$. Note that $\|f\|$ is well-defined since $\|f(c)\| \leq \|c\| \max\limits_i \|f(b_i)\|$, where $\{b_i\}_i$ is the chosen basis of C. If $C = 0$ define $\|f\| = 0$. For a composition fg we have $\|fg\| \leq \|f\|\|g\|$.

Let C_* be a finitely generated free chain complex with a given basis for each C_k. The *norm* of $\alpha \in H_k(C)$ is $\|\alpha\| = \min\{\|c\| \mid [c] = \alpha\}$. Here $[c]$ denotes the class of a cycle c in homology.

When K is a finite simplicial complex we will always consider the chain complex $C_*(K)$ with the usual basis for $C_k(K)$ given by one oriented k-simplex $[v_0, v_1, \dots, v_k]$ for each k-simplex $\{v_0, v_1, \dots, v_k\} \in K$. We denote by $H_*(K)$ the simplicial homology of K with integer coefficients.

If M is a closed n-dimensional oriented pseudomanifold, the norm $\|[M]\| \in H_n(M)$ of its fundamental class is the number of n-simplices in M.

If $\varphi : K \to L$ is a simplicial map between finite simplicial complexes, $\varphi_* : C_*(K) \to C_*(L)$ maps an oriented k-simplex $[v_0, v_1, \dots, v_k]$ to $[\varphi(v_0), \varphi(v_1), \dots, \varphi(v_k)]$ if $\varphi(v_i) \neq \varphi(v_j)$ for $i \neq j$ and to 0 otherwise. Therefore $\varphi_k : C_k(K) \to C_k(L)$ has norm at most 1.

If L is a finite simplicial complex and K is a subdivision of L, the subdivision operator $\lambda : C_*(L) \to C_*(K)$ is a homotopy inverse to the chain map induced by any simplicial approximation to the identity and maps a k-simplex $\sigma \in L$ into a signed sum of all the k-simplices of K contained in σ. Therefore, the norm of $\lambda_k : C_k(L) \to C_k(K)$ is the maximum number of k-simplices in which a k-simplex of L is subdivided. In particular, when K is the first barycentric subdivision L' of L, $\|\lambda_k\| = (k+1)!$ if $\dim L \geq k$. In this case, if $\alpha \in H_k(L)$, $\|\lambda_*(\alpha)\| \leq \|\lambda_k\|\|\alpha\| \leq (k+1)!\|\alpha\|$. In general the equality $\|\lambda_*(\alpha)\| = (k+1)!\|\alpha\|$ does not hold if $k < \dim(L)$.

Let K be a finite simplicial complex. The barycenter of a simplex $\sigma \in K$ will be denoted by $b(\sigma)$ or $\hat{\sigma}$. The simplices of K' are then the sets $\{\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_k\}$ such that $\sigma_i \subsetneq \sigma_{i+1}$ for every i. Given a simplicial map $\varphi : K \to L$, we denote by $\varphi' : K' \to L'$ the map $b(\sigma) \mapsto b(\varphi(\sigma))$ and in general $\varphi^j : K^j \to L^j$ is the map induced in the j-th barycentric subdivisions.

In [1, Example 2.4], Baclawski and Björner construct a finite model of S^2 with the fixed point property. The following definition is inspired by their example.

Definition 2. We will say that a complex K is *asymmetric* if there exists a vertex $v \in K$ which is fixed by every simplicial automorphism of K.

Proposition 3. *Let M be an n-dimensional pseudomanifold with $n \geq 2$. Then there exists a subdivision L of M such that the j-th barycentric subdivision L^j of L is asymmetric for every $j \geq 0$.*

Proof. Given a complex K and a simplex $\sigma \in K$, we denote by $\deg_K(\sigma)$ the number of maximal simplices of K containing σ. Then $\deg_K(\sigma)$ is the number of maximal simplices in the link $\mathrm{lk}_K(\sigma)$ if σ is not maximal, and 1 if σ is maximal in K. Define $d(K) = \max\limits_{\sigma \in K} \deg_K(\sigma) = \max\limits_{v \in K} \deg_K(v)$ where the second maximum is taken over all the vertices v of K. It is not hard to see that there exists a subdivision L of M which contains a vertex v_0 such that $\deg_L(v) < \deg_L(v_0)$ for any other $v \in L$. Then clearly L is asymmetric since the degree deg is preserved by automorphisms of L and thus v_0 is fixed by any such an automorphism. Moreover, we will show that $\deg_{L'}(v) < \deg_{L'}(v_0)$ for every vertex $v_0 \neq v \in L'$. It follows by induction that L^j is asymmetric for each $j \geq 0$.

Let $v \in L'$, $v = b(\sigma)$ where σ is a simplex of L. Let $k = \dim \sigma$. A maximal simplex of $\mathrm{lk}_{L'}(b(\sigma))$ is obtained by choosing a chain $\sigma^0 < \sigma^1 < \ldots < \sigma^{k-1}$ of proper faces of σ and a chain $\sigma^{k+1} < \sigma^{k+2} < \ldots < \sigma^n$ of simplices containing σ. There are $(k+1)$ possible choices for σ^{k-1}, k for σ^{k-2}, ..., 2 for σ^0. On the other hand there are $\deg_L(\sigma)$ choices for σ^n, $(n-k)$ for σ^{n-1}, $(n-k-1)$ for σ^{n-2}, ..., 2 for σ^{k+1}. Therefore

$$\deg_{L'}(b(\sigma)) = (k+1)!(n-k)! \deg_L(\sigma).$$

-If $k = n$, $\deg_{L'}(b(\sigma)) = (n+1)! < n!d(L)$ since $d(L) = n+1$ only when the pseudomanifold is isomorphic to the boundary of an $(n+1)$-simplex, which is not the case since L is asymmetric.
-If $k = n-1$, $\deg_{L'}(b(\sigma)) = n! \deg_L(\sigma) = 2n! < n!d(L)$.
-If $1 \leq k \leq n-2$, $\binom{n}{k} \geq n$, so $\deg_{L'}(b(\sigma)) \leq (n-1)!(k+1) \deg_L(\sigma) < n!d(L)$.
-If $k = 0$, $\sigma = v$ is a vertex of L and $\deg_{L'}(v) = n! \deg_L(v)$, which is strictly smaller than $n!d(L)$ if $v \neq v_0$ and which is $n! \deg_L(v_0) = n!d(L)$ if $v = v_0$. Then $d(L') = n!d(L)$ and this degree is only achieved by v_0.

□

Lemma 4. *Let τ be a simplex. Let $\sigma_0, \sigma_1, \ldots, \sigma_k$ be faces of τ such that for every $0 \leq i, j \leq k$ one of the following holds:*

(1) $\sigma_i \subseteq \sigma_j$.
(2) $\sigma_j \subseteq \sigma_i$.
(3) $\sigma_i \cap \sigma_j = \emptyset$.

Then the convex hull of $\{\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_k\}$ is a subcomplex of τ'.

Proof. We proceed by induction in the number of pairs i, j satisfying (3). If (1) or (2) holds for every i, j then the convex hull S of $\{\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_k\}$ is a simplex of τ'. Otherwise take i, j such that $\sigma = \sigma_i \cup \sigma_j$ has maximum cardinality among all the pairs satisfying (3). Since $\hat{\sigma}$ is a convex combination of $\hat{\sigma}_i$ and $\hat{\sigma}_j$, by induction it suffices to prove that $\{\sigma, \sigma_0, \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_k\}$ and $\{\sigma, \sigma_0, \sigma_1, \ldots, \sigma_{j-1}, \sigma_{j+1}, \ldots, \sigma_k\}$ satisfy the hypothesis of the lemma. Let $0 \leq l \leq k$, $i \neq l \neq j$. We have to verify that σ and σ_l are comparable or disjoint. If (i, l) and (j, l) satisfy (3), then σ and σ_l are disjoint. Suppose (i, l) satisfies (3) and σ_j is comparable with σ_l. By the choice of i and j, σ_j cannot be a proper face of σ_l. Then $\sigma_l \subseteq \sigma_j \subseteq \sigma$ so σ_l and σ are comparable. By symmetry it only remains to analyze the case that σ_l is comparable with both σ_i and σ_j. If σ_l is a face of any of them, then it is a face of σ. If σ_i and σ_j are faces of σ_l, then so is σ. □

Remark 5. In the conditions of Lemma 4, note that if the convex hull S of $\{\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_k\}$ is a k-dimensional subcomplex of τ', then it contains at most $(k+1)!$ many k-simplices of τ'. Moreover, if the equality holds then all the pairs i, j satisfy condition (3). The proof of Lemma 4 shows that the vertices of $S \leq \tau'$ are barycenters of unions of simplices σ_i. Therefore, the k-simplices of τ' contained in S are of the form $\{\hat{\tau}_0, \hat{\tau}_1, \dots, \hat{\tau}_k\}$ where $\tau_l = \bigcup_{i \in A_l} \sigma_i$, $A_l \subseteq [0,k]$, and $\tau_l \subsetneq \tau_{l+1}$. Moreover, we can assume that $A_l \subsetneq A_{l+1}$. Since there are only $(k+1)!$ sequences $\emptyset \neq A_0 \subsetneq A_1 \subsetneq \dots \subsetneq A_k \subseteq [0,k]$, S is decomposed in at most $(k+1)!$ many k-simplices. If $\sigma_i \subseteq \sigma_j$ for some $i \neq j$ then half of these sequences do not give a k-simplex, so S is decomposed in at most $\frac{(k+1)!}{2}$ many k-simplices.

Let $\varphi : K \to L$ be a simplicial map between finite simplicial complexes. We will work with the following version of the simplicial mapping cylinder Z_φ of φ. First we choose a total ordering in the set of vertices of K. The vertex set of Z_φ is the disjoint union of the vertex set of K and of L. The simplices of the cylinder are the simplices of L together with sets of the form $\{v_0, v_1, \dots, v_l, \varphi(v_{l+1}), \varphi(v_{l+2}), \dots, \varphi(v_m)\}$ where $\{v_0, v_1, \dots, v_m\}$ is a simplex of K and $v_i \leq v_{i+1}$ for every i (v_l could be equal to v_{l+1}).

There is a simplicial retraction $p : Z_\varphi \to L$ of the canonical inclusion $j : L \to Z_\varphi$ defined by $p(v) = \varphi(v)$ if $v \in K$. Therefore $pi = \varphi$ where i denotes the canonical inclusion of K into the cylinder. The composition jp lies in the same contiguity class as the identity 1_{Z_φ}, so $p_* : C_*(Z_\varphi) \to C_*(L)$ is a homotopy equivalence [10, p.151].

Suppose K is a subdivision of a complex L and that $\psi : K \to L$ is a simplicial approximation to the identity. In other words, ψ is a vertex map which maps each vertex $v \in K$ to any vertex $w \in L$ of the unique open simplex of L containing v. In this case $\psi_* : C_*(K) \to C_*(L)$ is a homotopy equivalence. Since $p_* : C_*(Z_\psi) \to C_*(L)$ is a homotopy equivalence and $\psi = pi$, $i_* : C_*(K) \to C_*(Z_\psi)$ is a homotopy equivalence. Since $C_*(K)$ is a subcomplex of $C_*(Z_\psi)$, it is known that there exists a retraction $r : C_*(Z_\psi) \to C_*(K)$. However, we need to control the norm $\|r_k\|$ of each $r_k : C_k(Z_\psi) \to C_k(K)$. We will prove that for barycentric subdivisions $K = L'$ there is a retraction r such that $\|r_k\| \leq (k+1)!$. It is not true that this inequality holds for any retraction r.

Lemma 6. *Let K be a finite simplicial complex. Then there exists an ordering of the vertices of K', a simplicial approximation to the identity $\psi : K' \to K$ and a retraction $r : C_*(Z_\psi) \to C_*(K')$ satisfying the following*

1. *If S is a k-simplex of Z_ψ, then $\|r_k(S)\| \leq (k+1)!$.*
2. *If S is a k-simplex of Z_ψ with $k \geq 1$ such that $\|r_k(S)\| = (k+1)!$, then $S \in K$.*

Proof. Order the vertices $\hat{\sigma}$ of K' in such a way that $\hat{\sigma} < \hat{\tau}$ implies $\dim(\sigma) \geq \dim(\tau)$. Let $\psi : K' \to K$ be any approximation to the identity. In other words, if $\hat{\sigma}$ is a vertex of K', then $\psi(\hat{\sigma}) \in \sigma$. A k-simplex of Z_ψ is of the form $S = \{\hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_l, \psi(\hat{\sigma}_{l+1}), \psi(\hat{\sigma}_{l+2}), \dots, \psi(\hat{\sigma}_m)\}$ where $l \geq -1$, $m \geq l$ and $\sigma_{i+1} \subseteq \sigma_i$ for all $0 \leq i \leq m-1$ (the simplices of K are included in these). We can consider S as a set of vertices of K' identifying $v_i = \psi(\hat{\sigma}_i)$ with the barycenter of $\{v_i\}$. The hypothesis of Lemma 4 is satisfied. For any $0 \leq i, j \leq l$, σ_i and σ_j are comparable; if $l+1 \leq i, j \leq m$, $\{v_i\}$ and $\{v_j\}$ are disjoint or equal; if $0 \leq i \leq l$ and $l+1 \leq j \leq m$, then $\sigma_i \supseteq \sigma_j \supseteq \{v_j\}$. Thus, by Lemma 4, the convex hull of S is a subcomplex $\Phi(S)$ of K'. The application Φ defines an acyclic carrier from Z_ψ to K'. Let

$r : C_*(Z_\psi) \to C_*(K')$ be a chain map carried by Φ. Note that r is uniquely determined by Φ since for a k-simplex $S \in Z_\psi$, $\Phi(S)$ is j-dimensional with $j \leq k$. Thus, any homotopy $F : C_*(Z_\psi) \to C_{*+1}(K')$ carried by Φ, given by the Acyclic carrier theorem, must be trivial.

If $S \in Z_\psi$ is a k-simplex such that $\dim \Phi(S) < k$, then $r(S) \in C_k(\Phi(S)) = 0$ is trivial. Otherwise $\dim \Phi(S) = k$ and then $\Phi(S)$ is a subdivision of S (considered as a set of $k+1$ affinely independent vertices of K'). One has then the subdivision operator $\lambda : C_*(S) \to C_*(\Phi(S))$. Since for each j-face $\widetilde{S}$ of S, $\Phi(\widetilde{S})$ is a j-dimensional subcomplex, the acyclic carrier Φ when restricted to $C_*(S)$ is the usual subdivision carrier $\Phi(\widetilde{S}) = K'(\widetilde{S})$. Thus $\lambda, r|_{C_*(S)} : C_*(S) \to C_*(\Phi(S))$ are carried by the same acyclic carrier Φ and by the same argument as before, they coincide. Hence $\|r_k(S)\| = \|\lambda_k(S)\|$ is the number of k-simplices in $\Phi(S)$ which is at most $(k+1)!$ by Remark 5.

Cearly $r : C_*(Z_\psi) \to C_*(K)$ is a retraction since for $S \in K'$ we have $\Phi(S) = S$ and $\lambda(S) = S$.

Finally, suppose $S = \{\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_l, \psi(\hat{\sigma}_{l+1}), \psi(\hat{\sigma}_{l+2}), \ldots, \psi(\hat{\sigma}_m)\}$ is a k-simplex of Z_ψ with $k \geq 1$. If $l \geq 0$, σ_0 is comparable with each σ_j and each $\{v_j\}$. By Remark 5, S is subdivided in less than $(k+1)!$ k-simplices of K' so $\|r_k(S)\| < (k+1)!$. This proves the second assertion of the lemma. □

Remark 7. It is well-known that every singular k-homology class α of a space X can be realized by a disjoint union $\sqcup M_i$ of closed k-dimensional oriented pseudomanifolds, meaning that there is a continuous map $f : \sqcup M_i \to X$ such that $\sum f_*([M_i]) = \alpha$ (see [5, p.108] for example). Then for every simplicial homology class $\alpha \in H_k(K)$ of a simplicial complex K there exists a simplicial map $\varphi : \sqcup M_i \to K$ from a disjoint union of closed oriented pseudomanifolds such that $\sum \varphi_*([M_i]) = \alpha$.

Theorem 8. *Let K be a finite simplicial complex which is simply connected or, more generally, such that $H_1(K) = 0$. Then there exists a finite simplicial complex L homotopy equivalent to K with the fixed simplex property.*

Proof. **First part: Construction of L**

Let $n = \dim(K)$. For each $2 \leq k \leq n$ let d_k be the rank of $H_k(K;\mathbb{Q})$. Take for each $2 \leq k \leq n$ homology classes $\alpha_{k,1}, \alpha_{k,2}, \ldots, \alpha_{k,d_k} \in H_k(K)$ such that $\{\alpha_{k,1} \otimes 1_\mathbb{Q}, \alpha_{k,2} \otimes 1_\mathbb{Q}, \ldots, \alpha_{k,d_k} \otimes 1_\mathbb{Q}\}$ is a basis of $H_k(K;\mathbb{Q})$. By Remark 7 each $\alpha_{k,l}$ can be realized by a disjoint union of closed oriented pseudomanifolds. Moreover, by changing the $\alpha_{k,l}$'s if needed we can assume that each of them is realized by a single pseudomanifold. For each $k \geq 2$ and $1 \leq l \leq d_k$ let $M_{k,l}$ be a k-dimensional oriented pseudomanifold and let $\varphi_{k,l} : M_{k,l} \to K$ be a simplicial map such that $(\varphi_{k,l})_*([M_{k,l}]) = \alpha_{k,l}$. By Proposition 3 we can assume that $M_{k,l}$ is asymmetric for each k, l as well as all their iterated barycentric subdivisions.

We define an increasing sequence $s_1, s_2, s_3, \ldots, s_n$ of non-negative integers as follows. Let $s_1 = 0$. Let $N_{2,l} = \|[M_{2,l}]\|$ be the number of 2-simplices in $M_{2,l}$ for each $1 \leq l \leq d_2$ and let $N_2 = \max\limits_l N_{2,l}$. If P is any finite simplicial complex, the cover $\mathcal{U}$ of P given by the open stars $\operatorname{st}_P(v)$ of the vertices of P has a Lebesgue number $\delta > 0$. Therefore, there exists a positive integer s_2 such that for each $s \geq s_2$, every connected subcomplex of P^s generated by at most N_2 many simplices is contained in an element of $\mathcal{U}$, and in particular in a contractible subcomplex of P. We take s_2 in such a way that the assertion above holds for P when P is any $M_{k,l}$ with $k \geq 3$ and $1 \leq l \leq d_k$.

Now let $N_{3,l} = \|[M_{3,l}^{s_2}]\|$ for each $1 \leq l \leq d_3$ and let $N_3 = \max_l N_{3,l}$. Take $s_3 \geq s_2$ such that for each $s \geq s_3$, $k \geq 4$ and $1 \leq l \leq d_k$, every connected subcomplex of $M_{k,l}^s$ generated by at most N_3 many simplices is contained in a contractible subcomplex.

In general, suppose $s_2, s_3, \dots, s_m$ are defined, with $m \leq n-2$. Then define $N_{m+1,l} = \|[M_{m+1,l}^{s_m}]\|$ for each $1 \leq l \leq d_{m+1}$ and let $N_{m+1} = \max_l N_{m+1,l}$. Take $s_{m+1} \geq s_m$ such that for each $s \geq s_{m+1}$, every connected subcomplex of $M_{k,l}^s$ generated by at most N_{m+1} many simplices is contained in a contractible subcomplex for each $k \geq m+2$ and $1 \leq l \leq d_k$.

Finally define $N_{n,l} = \|[M_{n,l}^{s_{n-1}}]\|$, $N_n = \max_l N_{n,l}$, $N = \max_{2 \leq k \leq n} N_k$ and take $s_n \geq s_{n-1}$ such that for each $s \geq s_n$, any connected subcomplex of K^s generated by at most N simplices is contained in a contractible subcomplex.

We now define for each $k \geq 2$ and $1 \leq l \leq d_k$ a cylinder $C_{k,l}$ which will be attached to K^{s_n}. Each $C_{k,l}$ consists of three parts. The first one is $C_{k,l}^a = Z_{\varphi_{k,l}^{s_n}}$, the cylinder of $\varphi_{k,l}^{s_n} : M_{k,l}^{s_n} \to K^{s_n}$ (see Figure 1). The second part $C_{k,l}^b$ is constructed as follows. We glue N cylinders $Z_{1_{M_{k,l}^{s_n}}}$ of the identity $1_{M_{k,l}^{s_n}} : M_{k,l}^{s_n} \to M_{k,l}^{s_n}$, the second base of one with the first base of the following, to build a long cylinder $C_{k,l}^b$ with both bases equal to $M_{k,l}^{s_n}$. The last part $C_{k,l}^c$ is the union of $s_n - s_{k-1}$ mapping cylinders. For each $s_{k-1} < m \leq s_n$ there is a simplicial approximation to the identity $\psi_{k,l,m} : M_{k,l}^m \to M_{k,l}^{m-1}$ and a retraction $R_m = R_{k,l,m} : C_*(Z_{\psi_{k,l,m}}) \to C_*(M_{k,l}^m)$ satisfying properties (1) and (2) in the statement of Lemma 6. When we glue the cylinders $Z_{\psi_{k,l,m}}$, identifying a base of one with a base of the following, we obtain a cylinder $C_{k,l}^c$ with one base equal to $M_{k,l}^{s_n}$ and the other equal to $M_{k,l}^{s_{k-1}}$. Finally we glue $C_{k,l}^a$ with one extreme of $C_{k,l}^b$ and $C_{k,l}^c$ with the other extreme of $C_{k,l}^b$. This is $C_{k,l}$.

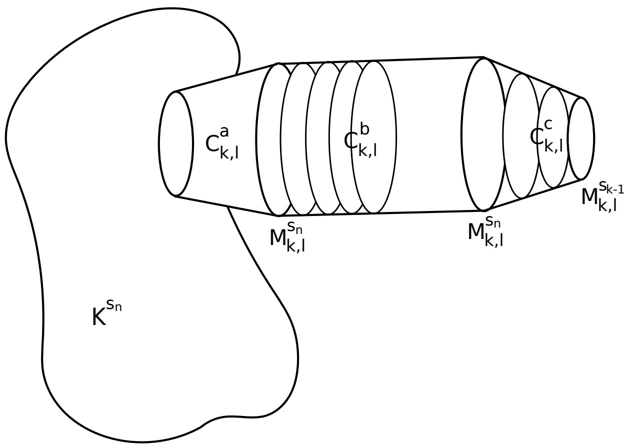

FIGURE 1. The cylinder $C_{k,l}$, union of $C_{k,l}^a$, $C_{k,l}^b$ and $C_{k,l}^c$.

Let L be the union of all the cylinders $C_{k,l}$ for $k \geq 2$ and $1 \leq l \leq d_k$, all intersecting in K^{s_n}. Each $C_{k,l}^c$ deformation retracts to $M_{k,l}^{s_n}$, $C_{k,l}^b$ deformation retracts to $M_{k,l}^{s_n}$ and $C_{k,l}^a$ deformation retracts to K^{s_n}. Therefore L is homotopy equivalent to K. We will show that L has the fixed simplex property.

Second part: L has the fixed simplex property

Let $i_{k,l} : M_{k,l}^{s_{k-1}} \hookrightarrow L$ be the inclusion map in the free extreme of $C_{k,l}^c$. Note that $(i_{k,l})_*[M_{k,l}^{s_{k-1}}] = i_*(\varphi_{k,l}^{s_n})_*[M_{k,l}^{s_n}] = i_*\lambda_*(\alpha_{k,l})$, where $i : K^{s_n} \hookrightarrow L$ is the inclusion. Hence, $\mathcal{B}_k = \{(i_{k,l})_*([M_{k,l}^{s_{k-1}}]) \otimes 1_{\mathbb{Q}}\}_l$ is a basis of $H_k(L;\mathbb{Q})$.

Let $f : L \to L$ be a simplicial map. We study for each k the matrix of $f_* : H_k(L;\mathbb{Q}) \to H_k(L;\mathbb{Q})$ in the basis $\mathcal{B}_k$. Since $M_{k,l}^{s_{k-1}}$ has $N_{k,l} \leq N_k$ many k-simplices, $f(M_{k,l}^{s_{k-1}})$ lies in a connected subcomplex of L generated by at most N_k simplices. Since each $C_{k',l'}^b$ is constructed gluing $N \geq N_k$ cylinders then one of the following holds:

(1) $f(M_{k,l}^{s_{k-1}}) \subseteq \bigcup\limits_{k',l'} (C_{k',l'}^a \cup C_{k',l'}^b)$ or

(2) $f(M_{k,l}^{s_{k-1}}) \subseteq C_{k',l'}^b \cup C_{k',l'}^c$ for some k', l'.

In the first case, call $C^{ab} = \bigcup\limits_{k',l'} (C_{k',l'}^a \cup C_{k',l'}^b)$. Just as L, the complex C^{ab} deformation retracts to K^{s_n}, but in contrast to L, for C^{ab} the retraction $r^{ab} : C^{ab} \to K^{s_n}$ may be taken simplicial. Since we are assuming $fi_{k,l} : M_{k,l}^{s_{k-1}} \to C^{ab}$, $r^{ab}fi_{k,l}(M_{k,l}^{s_{k-1}})$ is contained in a connected subcomplex of K^{s_n} generated by $N_{k,l} \leq N_k$ simplices. By the choice of s_n, this connected subcomplex lies in a contractible subcomplex of K^{s_n}, and then $r_*^{ab}(fi_{k,l})_*[M_{k,l}^{s_{k-1}}] = 0 \in H_k(K^{s_n})$. Since r^{ab} is a homotopy equivalence $(fi_{k,l})_*[M_{k,l}^{s_{k-1}}] = 0 \in H_k(C^{ab})$, and then $f_*(i_{k,l})_*[M_{k,l}^{s_{k-1}}] = 0 \in H_k(L)$. In this case the l-th column of the matrix of f_* is zero.

Assume then that (2) holds. Call $C_{k',l'}^{bc} = C_{k',l'}^b \cup C_{k',l'}^c$ and suppose that $fi_{k,l}(M_{k,l}^{s_{k-1}}) \subseteq C_{k',l'}^{bc}$. The complex $C_{k',l'}^{bc}$ deformation retracts to $M_{k',l'}^{s_{k'-1}}$ by a simplicial retraction $r = r_{k',l'}^{bc} : C_{k',l'}^{bc} \to M_{k',l'}^{s_{k'-1}}$. Since $rfi_{k,l}(M_{k,l}^{s_{k-1}})$ is contained in a connected subcomplex of $M_{k',l'}^{s_{k'-1}}$ generated by $N_{k,l} \leq N_k$ simplices, $r_*(fi_{k,l})_*[M_{k,l}^{s_{k-1}}] = 0 \in H_k(M_{k',l'}^{s_{k'-1}})$ if $k' > k$, by the choices of s_k and $s_{k'-1}$. If $k' < k$, then $H_k(M_{k',l'}^{s_{k'-1}}) = 0$ since $\dim M_{k',l'} = k'$. Therefore in this case we also have $r_*(fi_{k,l})_*[M_{k,l}^{s_{k-1}}] = 0 \in H_k(M_{k',l'}^{s_{k'-1}})$. Since r is a homotopy equivalence we conclude that if $k' \neq k$, then $(fi_{k,l})_*[M_{k,l}^{s_{k-1}}] = 0 \in H_k(C_{k',l'}^{bc})$ and therefore $f_*(i_{k,l})_*[M_{k,l}^{s_{k-1}}] = 0 \in H_k(L)$. Hence the l-th column of f_* is also zero. It remains to analyze the case $k' = k$. In that case $r_*(fi_{k,l})_*[M_{k,l}^{s_{k-1}}] \in H_k(M_{k,l'}^{s_{k-1}})$ is an integer multiple of the fundamental class $[M_{k,l'}^{s_{k-1}}]$, and then $f_*(i_{k,l})_*[M_{k,l}^{s_{k-1}}] = (i_{k,l'})_*r_*(fi_{k,l})_*[M_{k,l}^{s_{k-1}}]$ is an integer multiple of $(i_{k,l'})_*[M_{k,l'}^{s_{k-1}}]$. If $l' \neq l$, the l-th column of f_* has a zero in the l-th entry. The last and most important case is $l' = l$.

If for each k, l the case (1) occurs or the case (2) for $(k', l') \neq (k, l)$, then the trace of the matrix of f_* in each positive degree is zero and by the Lefschetz fixed point theorem, f must fix a simplex. We can assume then that there exists one pair k, l such that (2) holds for $(k', l') = (k, l)$ and that $(fi_{k,l})_*[M_{k,l}^{s_{k-1}}] \in H_k(C_{k,l}^{bc})$ is non-trivial.

The simplicial projection $C_{k,l}^b \to M_{k,l}^{s_n}$ of the cylinder into the extreme in contact with $C_{k,l}^c$ extends to a simplicial retraction $p : C_{k,l}^{bc} \to C_{k,l}^c$. On the other hand Lemma 6 provides retractions $R_m : C_*(Z_{\psi_{k,l,m}}) \to C_*(M_{k,l}^m)$ for each $s_{k-1} < m \leq s_n$. Each of them extends to a retraction

$$\widetilde{R}_m : C_*(\bigcup_{q=m}^{s_n} Z_{\psi_{k,l,q}}) \to C_*(\bigcup_{q=m+1}^{s_n} Z_{\psi_{k,l,q}})$$

When $m = s_n$, $\widetilde{R}_m$ is just another notation for R_{s_n}.

By Lemma 6, the norm of the map R_m in degree k is $\|R_m\| \leq (k+1)!$ for each m, so $\|\widetilde{R}_m\| \leq (k+1)!$.

Let $c \in Z_k(M_{k,l}^{s_{k-1}})$ be the fundamental cycle of $M_{k,l}^{s_{k-1}}$. Then $\|c\|$ is the number of k-simplices of $M_{k,l}^{s_{k-1}}$ and $p_*(fi_{k,l})_*(c) \in Z_k(C_{k,l}^c)$ is a k-cycle in $C_{k,l}^c$. Thus

$$\widetilde{c} = \widetilde{R}_{s_n}\widetilde{R}_{s_n-1}\ldots\widetilde{R}_{s_{k-1}+2}\widetilde{R}_{s_{k-1}+1}p_*(fi_{k,l})_*(c) \in Z_k(M_{k,l}^{s_n})$$

is a cycle with norm at most $((k+1)!)^{s_n-s_{k-1}}\|c\|$. But this number is exactly the number of k-simplices in the pseudomanifold $M_{k,l}^{s_n}$. If $\|\widetilde{c}\| < ((k+1)!)^{s_n-s_{k-1}}\|c\|$, the cycle $\widetilde{c}$ is carried by a proper subcomplex of $M_{k,l}^{s_n}$ and then it is trivial in homology. Since each $\widetilde{R}_m$ induces an isomorphism in homology, $f_*(i_{k,l})_*[M_{k,l}^{s_{k-1}}] = 0 \in H_k(L)$. This contradicts the assumption. Therefore, $\|\widetilde{c}\| = ((k+1)!)^{s_n-s_{k-1}}\|c\|$.

Since the equality holds, we have in particular $\|\widetilde{R}_{s_{k-1}+1}p_*(fi_{k,l})_*(c)\| = (k+1)!\|c\|$. Then for every k-simplex $\sigma \in M_{k,l}^{s_{k-1}}$, $S = pfi_{k,l}(\sigma)$ is a k-simplex of $C_{k,l}^c$ and $\|\widetilde{R}_{s_{k-1}+1}(S)\|$ $= (k+1)!$. By Lemma 6, $S \in M_{k,l}^{s_{k-1}}$. We conclude then that $pfi_{k,l}(M_{k,l}^{s_{k-1}}) \subseteq M_{k,l}^{s_{k-1}}$ and therefore $fi_{k,l}(M_{k,l}^{s_{k-1}}) \subseteq M_{k,l}^{s_{k-1}}$. If $fi_{k,l}(M_{k,l}^{s_{k-1}})$ is contained in a proper subcomplex of $M_{k,l}^{s_{k-1}}$, $(fi_{k,l})_*[M_{k,l}^{s_{k-1}}] = 0$ and we have a contradiction. Then $f|_{M_{k,l}^{s_{k-1}}} : M_{k,l}^{s_{k-1}} \to M_{k,l}^{s_{k-1}}$ is an automorphism and the asymmetry of $M_{k,l}^{s_{k-1}}$ gives the desired fixed simplex. $\square$

The poset of simplices of a finite simplicial complex K is denoted by $\mathcal{X}(K)$. Recall that a finite poset X can be regarded as a topological space with finitely many points in which open sets are those subsets $U \subseteq X$ such that any $x \in X$ which is smaller than or equal to an element of U is itself in U. This space satisfies the T_0 separation axiom and in fact any finite T_0-space is a poset in this sense. Order preserving maps correspond to continuous maps and comparable maps are homotopic ([2]). For every finite simplicial complex K there is a weak homotopy equivalence $K \to \mathcal{X}(K)$ (see [9]). The simplicial complex of chains of a poset X is denoted by $\mathcal{K}(X)$. There is a weak homotopy equivalence $\mathcal{K}(X) \to X$. A simplicial map $\varphi : K \to L$ and a continuous map $f : X \to Y$ between finite T_0-spaces induce maps $\mathcal{X}(\varphi)$ and $\mathcal{K}(f)$ in the obvious way and one has the following commutative diagrams up to homotopy where the vertical maps are the weak homotopy equivalences mentioned above

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & L \\ \downarrow & & \downarrow \\ \mathcal{X}(K) & \xrightarrow{\mathcal{X}(\varphi)} & \mathcal{X}(L) \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \\ \mathcal{K}(X) & \xrightarrow{\mathcal{K}(f)} & \mathcal{K}(Y). \end{array}$$

Corollary 9. *Let K be a simply connected compact CW-complex. Then there exists a topological space X weak homotopy equivalent to K which has the fixed point property.*

Proof. By the previous theorem there is a finite simplicial complex L homotopy equivalent to K with the fixed simplex property. We claim that the associated finite space $\mathcal{X}(L)$ has the fixed point property. Let $f : \mathcal{X}(L) \to \mathcal{X}(L)$ be a continuous map. For every $v \in L$ choose a vertex $g(v)$ of L such that $g(v) \leq f(v)$. The vertex map $g : L \to L$ is simplicial since f maps a bounded set of minimal points into a bounded set. Then g fixes some simplex $\sigma \in L$, so $\mathcal{X}(g) : \mathcal{X}(L) \to \mathcal{X}(L)$ fixes σ. Since $f \geq \mathcal{X}(g)$, $f(\sigma) \geq \sigma$ and then $f^i(\sigma) = f \circ f \circ \ldots \circ f(\sigma)$ is a fixed point of f for i large enough. □

In order to extend the last corollary to non-simply connected complexes, we need to modify the construction of the space $\mathcal{X}(L)$. The idea we used in the proof of Theorem 8 fails if $H_1(K) \neq 0$ since no 1-dimensional pseudomanifold is asymmetric. We will adapt the proofs of Theorem 8 and Corollary 9 to the general case using the rigidity of finite spaces. Recall from [2, 3] that the non-Hausdorff mapping cylinder B_f of an order preserving map $f : X \to Y$ between finite T_0-spaces is the set $X \sqcup Y$ keeping the given ordering within X and Y and setting $x < y$ for $x \in X$ and $y \in Y$ if $f(x) \leq y$. The cylinder B_f deformation retracts to Y so $\mathcal{K}(B_f)$ deformation retracts to $\mathcal{K}(Y)$ by a simplicial retraction. If f is a weak homotopy equivalence, $\mathcal{K}(B_f)$ deformation retracts to $\mathcal{K}(X)$. If X is a finite T_0-space, a point $x \in X$ such that $X_{<x}$ or $X_{>x}$ is contractible is called a *weak point*. In this case the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence. In other words $\mathcal{K}(X)$ deformation retracts to $\mathcal{K}(X \smallsetminus \{x\})$ (see [2, 3] for more details).

Lemma 10. *There exists a topological space weak homotopy equivalent to S^1 with the fixed point property.*

Proof. Consider the space $\mathfrak{K}$ of 14 points in Figure 2. This space is the *core* of a space considered by G. Kun in [7, Remark 38] in a different context. It is constructed by gluing two non-Hausdorff mapping cylinders of 1 and 2-degree maps from an 8-point model of S^1 to a 4-point model.

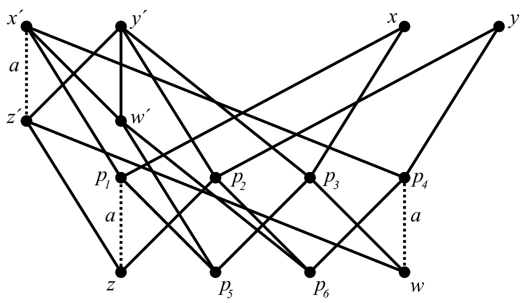

FIGURE 2. The space $\mathfrak{K}$, a finite model of S^1 with the fixed point property.

Since x and y are weak points of $\mathfrak{K}$, $\mathfrak{K} \smallsetminus \{x, y\} \hookrightarrow \mathfrak{K}$ is a weak homotopy equivalence. The space $\mathfrak{K} \smallsetminus \{x, y\}$ is a non-Hausdorff mapping cylinder and then it deformation retracts to $\{x', y', z', w'\}$. Therefore $\mathfrak{K}$ is weak homotopy equivalent to S^1. We show that $\mathfrak{K}$ has the fixed point property.

We will prove the following **assertion**: there is, up to sign, a unique 1-cycle of norm at most 4 in the order complex $\mathcal{K}(\mathfrak{K})$ which represents the double of a generator of $H_1(\mathcal{K}(\mathfrak{K})) \simeq \mathbb{Z}$.

An easy way to prove this assertion is by using *colorings* (see [4]). The $\mathbb{Z}$-coloring of $\mathfrak{K}$ which colors the solid edges with the identity and the dotted edges with a generator a of $\mathbb{Z}$ is connected and admissible, so it is the standard coloring of $\mathfrak{K}$. To each directed edge vw of the complex $\mathcal{K}(\mathfrak{K})$ we assign a weight $\omega(vw)$ which is the sum of the colors of the edges in any increasing path from v to w if $v < w$. If $v > w$, $\omega(vw) = -\omega(wv)$. For example

$\omega(zx')=a$, $\omega(w'x')=0$, $\omega(p_4w)=-a$. The map $H_1(\mathcal{K}(\mathfrak{K}))\to\mathbb{Z}$ which maps the class of a 1-cycle $\sum v_iw_i$ to $\sum\omega(v_iw_i)$ is a well defined isomorphism (see [10, p.208] and [4]). It is now easy to check that $c=zx+xw+wy+yz$ is the unique cycle of $\mathcal{K}(\mathfrak{K})$ with norm at most 4 which corresponds to $2a\in\mathbb{Z}$.

Alternatively, in order to prove the assertion, the reader not familiar with colorings may consider the order complex $\mathcal{K}(X)$ of the poset X given by the solid edges of $\mathfrak{K}$. This complex is contractible and $\mathcal{K}(\mathfrak{K})$ is obtained from $\mathcal{K}(X)$ by adding seven 1-simplices and six 2-simplices. Moreover, $\mathcal{K}(\mathfrak{K})$ collapses to $\mathcal{K}(X)\cup\{x'z'\}$ and then the homology of the 1-cycles of $\mathcal{K}(\mathfrak{K})$ of norm 4 is easy to understand.

We use now the assertion to prove the fixed point property. Suppose $f:\mathfrak{K}\to\mathfrak{K}$ is a fixed point free map. Then $\mathcal{K}(f)$ has no fixed point and by the Lefschetz fixed point theorem $\mathcal{K}(f)_*:H_1(\mathcal{K}(\mathfrak{K}))\to H_1(\mathcal{K}(\mathfrak{K}))$ is the identity. Thus $\mathcal{K}(f)_*(c)=c$ and then f maps $\{x,y,z,w\}$ into itself, so $f(x)=y, f(y)=x$, $f(z)=w$ and $f(w)=z$. In particular the set of points greater than w and z is mapped to itself, so $f(\{x',y',z'\})\subseteq\{x',y',z'\}\sqcup\{x\}\sqcup\{y\}$. If the connected subspace $\{x',y',z'\}$ is mapped into the point x or into y, then the generating cycle $z'x'+x'w'+w'y'+y'z'$ is mapped to 0. Therefore $\{x',y',z'\}$ is mapped into itself and then f has a fixed point, a contradiction. □

Proof of Theorem 1. We may suppose that K is a finite simplicial complex. We begin with the construction of L performed in the proof of Theorem 8, except that this time we consider also integer homology classes $\alpha_{1,1},\alpha_{1,2},\dots,\alpha_{1,d_1}$ which are a basis for $H_1(K;\mathbb{Q})$ and 1-pseudomanifolds $M_{1,l}$ along with simplicial maps $\varphi_{1,l}:M_{1,l}\to K$ which map $[M_{1,l}]$ to $\alpha_{1,l}$. The pseudomanifolds $M_{k,l}$ will be assumed to be asymmetric for $k\geq 2$, but of course this is not possible for $k=1$.

The numbers s_1, $N_{k,l}$, N_k, s_k for $k\geq 2$ are defined as before. Let $s_0=0, N_1=1$ and $N=\max\limits_{1\leq k\leq n}N_k$. The cylinders $C_{k,l}$ are built just as before, except that $C^b_{k,l}$ will be constructed by gluing not N cylinders but $(n+1)!N$ cylinders of the identity. Also, we include now the $C_{1,l}$'s. The complex L is the union of all these cylinders and it is homotopy equivalent to K. The unique difference was the incorporation of the 1-dimensional manifolds with their cylinders and that we increased the length of the cylinders $C^b_{k,l}$.

The space X will contain $\mathcal{X}(L)$ as a subspace. For each $1\leq l\leq d_1$ consider a weak homotopy equivalence $\mathcal{X}(M_{1,l})\to\{x',y',z',w'\}$ where the codomain is the subspace of $\mathfrak{K}$ defined in Lemma 10. Since $\{x',y',z',w'\}\hookrightarrow\mathfrak{K}$ is a weak equivalence, the composition $h_l:\mathcal{X}(M_{l,1})\to\mathfrak{K}$ is a weak equivalence. We take a different copy of $\mathfrak{K}$ for each $1\leq l\leq d_1$, so the non-Hausdorff mapping cylinders B_{h_l} are disjoint. Let $X=\mathcal{X}(L)\cup\bigcup\limits_{l=1}^{d_1}B_{h_l}$. Since h_l is a weak equivalence, $\mathcal{K}(B_{h_l})$ deformation retracts to $\mathcal{K}(\mathcal{X}(M_{1,l}))=M'_{1,l}$ and then $\mathcal{K}(X)$ deformation retracts to L'. Therefore X is weak homotopy equivalent to K. We prove that X has the fixed point property.

Let $f:X\to X$ be a continuous map. Then $\mathcal{K}(f):\mathcal{K}(X)\to\mathcal{K}(X)$ is a simplicial map. If $\mathcal{K}(f)$ fixes a simplex, then f fixes a chain, so f fixes all the elements of the chain. For each $k\geq 1$ we consider the basis $\mathcal{B}_k=\{(i'_{k,l})_*([M^{s_{k-1}+1}_{k,l}])\otimes 1_{\mathbb{Q}}\}_l$ of $H_k(\mathcal{K}(X);\mathbb{Q})$. Let $k\geq 2$ and $1\leq l\leq d_k$. Then $\mathcal{K}(f)(M^{s_{k-1}+1}_{k,l})$ lies in a complex generated by $(k+1)!\|[M^{s_{k-1}}_{k,l}]\|\leq(k+1)!N_k\leq(n+1)!N$ simplices. We consider now three cases in order to take into account the 1-dimensional part:

(1) $\mathcal{K}(f)(M_{k,l}^{s_{k-1}+1}) \subseteq (C^{ab})'$ or

(2) $\mathcal{K}(f)(M_{k,l}^{s_{k-1}+1}) \subseteq (C^{bc}_{k',l'})'$ for some $k' \geq 2$ and some l' or

(3) $\mathcal{K}(f)(M_{k,l}^{s_{k-1}+1}) \subseteq \mathcal{K}(\mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}})$ for some l'.

In the third case, since $\mathcal{K}(\mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}})$ deformation retracts to $(C^{bc}_{1,l'})'$ which in turn deformation retracts into the 1-dimensional complex $M'_{1,l'}$, then $\mathcal{K}(f)_*(i'_{k,l})_*[M_{k,l}^{s_{k-1}+1}] = 0 \in H_k(\mathcal{K}(X))$.

If (1) holds, $f(\mathcal{X}(M_{k,l}^{s_{k-1}})) \subseteq \mathcal{X}(C^{ab})$. As in the proof of Corollary 9, $f : \mathcal{X}(M_{k,l}^{s_{k-1}}) \to \mathcal{X}(C^{ab})$ induces a simplicial map $g : M_{k,l}^{s_{k-1}} \to C^{ab}$ which is zero in H_k by the proof of Theorem 8. Then $\mathcal{X}(g) : \mathcal{X}(M_{k,l}^{s_{k-1}}) \to \mathcal{X}(C^{ab})$ is zero in k-homology and the same is true for $f : \mathcal{X}(M_{k,l}^{s_{k-1}}) \to \mathcal{X}(C^{ab})$ since this restriction of f is homotopic to $\mathcal{X}(g)$. Then $\mathcal{K}(f) : M_{k,l}^{s_{k-1}+1} \to (C^{ab})'$ induces the trivial map so $\mathcal{K}(f)_*(i'_{k,l})_*[M_{k,l}^{s_{k-1}+1}] = 0$. This idea works also to adapt the proof of Theorem 8 to the cases (2) for $k' < k$, (2) for $k' > k$, (2) for $k' = k$ and $l' \neq l$. Also, if we are in the case (2) with $k' = k, l' = l$ then $f : \mathcal{X}(M_{k,l}^{s_{k-1}}) \to \mathcal{X}(C^{bc}_{k,l})$ induces $g : M_{k,l}^{s_{k-1}} \to C^{bc}_{k,l}$. If $\mathcal{K}(f)_*(i'_{k,l})_*[M_{k,l}^{s_{k-1}+1}] \neq 0$, $g_*[M_{k,l}^{s_{k-1}}] \neq 0$ and then by the proof of Theorem 8 g fixes a simplex, so $\mathcal{X}(g)$ fixes a point $\sigma \in \mathcal{X}(M_{k,l}^{s_{k-1}})$. Then $f(\sigma) \geq \mathcal{X}(g)(\sigma) = \sigma$ and $f : X \to X$ has a fixed point.

We can then assume that the trace of $\mathcal{K}(f)_* : H_k(\mathcal{K}(X); \mathbb{Q}) \to H_k(\mathcal{K}(X); \mathbb{Q})$ is zero for each $k \geq 2$.

Now we study the 1-dimensional component. Let $1 \leq l \leq d_1$. Then $\{x, y, z, w\} \subseteq B_{h_l}$ and if $j_l : \{x, y, z, w\} \hookrightarrow X$ denotes the inclusion which factors through B_{h_l} and $\beta_l = [\mathcal{K}(\{x, y, z, w\})]$ denotes the fundamental class of $\mathcal{K}(\{x, y, z, w\})$, then $\mathcal{K}(j_l)_*(\beta_l) = 2(i'_{1,l})_*[M'_{1,l}] \in H_1(\mathcal{K}(X))$. Thus $\mathcal{K}(j_l)_*(\beta_l) \otimes 1_{\mathbb{Q}}$ is twice an element of $\mathcal{B}_1$, the chosen basis for $H_1(\mathcal{K}(X); \mathbb{Q})$.

Since $(n+1)!N \geq 1$, for each $1 \leq l \leq d_1$ the l-th copy of $\mathcal{K}(\{x, y, z, w\})$ is mapped by $\mathcal{K}(f)$ into L' or into $\mathcal{K}(\mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}})$ for some l'.

If $\mathcal{K}(f)(\mathcal{K}(\{x, y, z, w\})) \subseteq L'$, then it is contained in a contractible subcomplex since any closed edge-path of four edges in a barycentric subdivision is contained in the star of a vertex, and then $\mathcal{K}(f)_*\mathcal{K}(j_l)_*(\beta_l) = 0 \in H_1(\mathcal{K}(X))$. If $\mathcal{K}(f)(\mathcal{K}(\{x, y, z, w\}))$ is contained in $\mathcal{K}(\mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}})$ for some $l' \neq l$, then $\mathcal{K}(fj_l) : \mathcal{K}(\{x, y, z, w\}) \to \mathcal{K}(\mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}})$. The codomain of this map deformation retracts to $M'_{1,l'}$ by a retraction r (which is not simplicial). Then $i'_{1,l'} r \mathcal{K}(fj_l) : \mathcal{K}(\{x, y, z, w\}) \to \mathcal{K}(X)$ is homotopic to $\mathcal{K}(f)\mathcal{K}(j_l)$ and therefore $\mathcal{K}(f)_*\mathcal{K}(j_l)_*(\beta_l)$ is an integer multiple of $(i'_{1,l'})_*[M'_{1,l'}]$. In any of the cases considered so far, the matrix of $\mathcal{K}(f)_*$ in the basis $\mathcal{B}_1$ has a zero in the entry (l, l). Suppose then that $\mathcal{K}(f)(\mathcal{K}\{x, y, z, w\})$ is contained in $\mathcal{K}(\mathcal{X}(C^{bc}_{1,l}) \cup B_{h_l})$. Since $C^{bc}_{1,l}$ deformation retracts to $M_{1,l}$ by a simplicial retraction, $\mathcal{K}(\mathcal{X}(C^{bc}_{1,l}) \cup B_{h_l})$ deformation retracts to $\mathcal{K}(B_{h_l})$ by a simplicial retraction. Moreover, $\mathcal{K}(B_{h_l})$ deformation retracts to $\mathcal{K}(\mathfrak{K})$ by a simplicial retraction which maps $M'_{1,l}$ into $\mathcal{K}(\{x', y', z', w'\})$. Then $\mathcal{K}(\mathcal{X}(C^{bc}_{1,l}) \cup B_{h_l})$ deformation retracts to $\mathcal{K}(\mathfrak{K})$ by a simplicial retraction $R : \mathcal{K}(\mathcal{X}(C^{bc}_{1,l}) \cup B_{h_l}) \to \mathcal{K}(\mathfrak{K})$ which maps $(C^{bc}_{1,l})'$ into $\mathcal{K}(\{x', y', z', w'\})$. Thus, $\mathcal{K}(f)_*\mathcal{K}(j_l)_*(\beta_l) = i_* R_* \mathcal{K}(fj_l)_*(\beta_l)$ where $i : \mathcal{K}(\mathfrak{K}) \to \mathcal{K}(X)$ denotes the inclusion. Then the homology class $\mathcal{K}(j_l)_*(\beta_l) = 2(i'_{1,l})_*[M'_{1,l}]$ is mapped by $\mathcal{K}(f)$ to an integer multiple of the generator $(i'_{1,l})_*[M'_{1,l}]$ of the image of i_*. Therefore, $\mathcal{K}(j_l)_*(\beta_l)$ is mapped to an integer multiple of itself. Since f is a self map of a finite set, the powers of f induce only finite morphisms in homology, so $\mathcal{K}(j_l)_*(\beta_l)$ is mapped to 0, $\mathcal{K}(j_l)_*(\beta_l)$ or $-\mathcal{K}(j_l)_*(\beta_l)$. By the Lefschetz fixed point theorem we can assume that for some $1 \leq l \leq d_1$, $\mathcal{K}(j_l)_*(\beta_l)$ is mapped to itself. Then $R_*\mathcal{K}(fj_l)_*(zx+xw+wy+yz) \in Z_1(\mathcal{K}(\mathfrak{K}))$ is a 1-cycle of norm at most 4 which represents the double of the generator of $H_1(\mathcal{K}(\mathfrak{K}))$. By the assertion in Lemma 10 there is a unique cycle satisfying these conditions, which is $zx+xw+wy+yz$. Therefore $R\mathcal{K}(fj_l)$ maps $\{x,y,z,w\}$ into itself and then $f(\{x,y,z,w\}) = \{x,y,z,w\}$. Thus, the set of points smaller than one of those four points, $\{x,y,z,w,p_1,p_2,\dots,p_6\}$, is mapped also to itself and then f maps $\mathfrak{K}$ into $\mathfrak{K}$. By Lemma 10, f has a fixed point. □

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