VALUATIONS OF SKEW QUANTUM POLYNOMIALS

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Abstract
In this paper we extend some results obtained by Artamonov and Sabitov for quantum polynomials to skew quantum polynomials and quasi–commutative bijective skew PBW extensions. Moreover, we find a counterexample to the conjecture proposed in [6].

Keywords: Skew PBW extensions, skew quantum polynomials, Ore domains, valuations, completions.

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1 Introduction

This section is divided into three subsections, we recall the definition of $\Gamma$-valuation, valuation and quantum polynomials. We review some fundamental properties of valuations and valuations of quantum polynomials (see [4] and [6]).

1.1 Valuations

Let $D$ be a division ring, $D^*$ the multiplicative group and $\Gamma$ is a totally ordered group (with additive notation and not necessarily commutative).

Definition 1.1. A function $\nu : D^* \to \Gamma$ is a $\Gamma$-valuation of $D^*$ if:

i) $\nu$ is surjective,

ii) $\nu(ab) = \nu(a) + \nu(b)$,

iii) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$.

Proposition 1.2. [14, 9] If $\nu$ is a $\Gamma$-valuation of $D^*$, then:

1) If $\nu(a) \neq \nu(b)$, then $\nu(a + b) = \min\{\nu(a), \nu(b)\}$.

2) $\Lambda_{\nu} := \{a \in D; a = 0 \text{ or } \nu(a) \geq 0\}$ is a subring of $D$.

3) The group of units $U_{\nu} := \{a \in D^*; \nu(a) = 0\}$ is a subgroup of $D^*$.

4) $W_{\nu} := \{a \in D, a = 0 \text{ or } \nu(a) > 0\}$ is a completely prime ideal of $\Lambda_{\nu}$ and $W_{\nu} = \Lambda_{\nu} - U_{\nu}$.

5) $\Lambda_{\nu}$ is a local ring with unique maximal ideal $W_{\nu}$.

1.2 Valuations with values on $\Gamma \cup \{\infty\}$

Proposition 1.3. Let $\Gamma$ be a totally ordered group with additive notation ordere. Then $\Gamma \cup \{\infty\}$ is an ordered additive monoid such that

$$x + \infty := \infty =: \infty + x, \text{ for all } \Gamma \cup \{\infty\},$$

and $\infty > x$ for all $x \in \Gamma$.

Definition 1.4 ([8]). Let $R$ be a ring. By a valuation on $R$ with values in a totally ordered group $\Gamma$, the value group, we shall understand a function $\nu$ on $R$ with values in $\Gamma \cup \{\infty\}$ subject to the conditions:

i) $\nu(a) \in \Gamma \cup \{\infty\}$ and $\nu$ assumes at least two values,

ii) $\nu(ab) = \nu(a) + \nu(b)$,
Proposition 1.5. [8, 9] If $\nu$ is a valuation of $R$, then:

1) $\ker \nu := \{a \in R; \nu(a) = \infty\}$ is an ideal of $R$.

2) If $\nu(a + b) > \min\{\nu(a), \nu(b)\}$, then $\nu(a) = \nu(b)$.

3) $\Lambda_\nu := \{a \in R; \nu(a) \geq 0\}$ is a subring of $R$.

4) The group of units $U_\nu := \{a \in R^*; \nu(a) = 0\}$ is a subgroup of $R^*$.

5) $\mathcal{W}_\nu := \{a \in R, \nu(a) > 0\}$ is an ideal of $\Lambda_\nu$.

6) $\ker \nu$ is a completely prime ideal of $R$ and $R/\ker \nu$ is an integral domain.

Proposition 1.6 ([8]). If $\nu$ is a $\Gamma$—valuation of $D$. Then $\Gamma$ is abelian, if and only if $\nu(a) = 0$ for all $a \in [D^*, D^*]$.

1.3 Quantum polynomials

Let $D$ be a division ring with a fixed set $\alpha_1, \alpha_2, \ldots, \alpha_n, n \geq 2$, of automorphisms. Also, we have $q_{ij}$ in $D^*$ for $i, j = 1, 2, \ldots, n$ fix elements, satisfying the relations:

\[ q_{ii} = q_{ij}q_{ji} = q_{ijr}q_{rij} = 1 \]
\[ \alpha_i(\alpha_j(d)) = q_{ij}\alpha_j(\alpha_i(d)) q_{ji}, \]

where $q_{ijr} = q_{ij}\alpha_j(q_{ir})$ and $d \in D$. We set $q = (q_{ij}) \in \mathcal{M}(n, D)$ and $

\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Definition 1.7. The elements $q_{ij}$ of the matrix $q$ are called system of multiparameters.

Definition 1.8 (Quantum polynomial ring). Denote by

\[ \mathcal{O}_q := D_q,\alpha \left[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n\right] \]

the associative ring generated by $D$ and by elements $x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n$ subject to the defining relations

\[ x_ix_i^{-1} = x_i^{-1}x_i = 1, 1 \leq i \leq r, \]
\[ x_id = \alpha_i(d)x_i, d \in D, i = 1, 2, \ldots, n, \]
\[ x_ix_j = q_{ij}x_jx_i, i, j = 1, 2, \ldots, n. \]
Definition 1.9. Let $N$ be the subgroup in the multiplicative group $D^*$ of the ring $D$ generated by the derived subgroup $[D^*, D^*]$ and by the set of all elements of the form $z^{-1}\sigma_i(z)$ where $z \in R^*$ and $i = 1, \ldots, n$. $\Lambda := D_{q,a}[x_1, x_2, \ldots, x_n]$ is a general (generic) quantum polynomials ring if the images of all multiparameters $q_{ij}$, $1 \leq i < j \leq n$, are independent in the multiplicative Abelian group $D^*/N$.

The ring $O_q$ is a left and right Noetherian domain, it satisfies Ore Condition and it has a division ring of fractions $F := D_q(x_1, \ldots, x_n)$. We consider $\nu : F^* \rightarrow \Gamma$ a $\Gamma$-valuation with $\nu(D^*) = 0$.

Theorem 1.10 ([6]). A valuation of a quantum division ring $D$, is Abelian in the sense that the group $\Gamma$ is Abelian.

Definition 1.11 ([4], [6]). Let $\nu_1 : D^* \rightarrow \Gamma_1$ and $\nu_2 : D^* \rightarrow \Gamma_2$ be two valuations. Set $\nu_1 \geq \nu_2$ if there exists an epimorphism of ordered groups $\tau : \Gamma_1 \rightarrow \Gamma_2$ such that $\tau \nu_1 = \nu_2$. It means that the diagram

$$
\begin{array}{ccc}
D^* & \xrightarrow{\nu_1} & \Gamma_1 \\
\downarrow{\nu_2} & & \downarrow{\tau} \\
\Gamma_2 & & 
\end{array}
$$

is commutative.

Definition 1.12 ([4], [6]). A valuation $\nu$ has a maximal rank if $\tau$ is an isomorphism in the previous definition.

Theorem 1.13 ([4]). A valuation $\nu : F^* \rightarrow \Gamma$ of a general quantum division ring $O_q$ is has maximal rank if only if $\Gamma \cong \mathbb{Z}^n$.

2 Completions of quantum polynomials

In this section $\nu : F^* \rightarrow \mathbb{Z}^n$ is a maximal $\mathbb{Z}^n-$valuation.

Definition 2.1 ([6]). Let $F$ be the set of all maps $f : \mathbb{Z}^n \rightarrow k$ and the zero element such that $\text{supp } f := \{m \in \mathbb{Z}^n; f(m) \neq 0\}$ is Artinian with respect to the lexicographic order on $\mathbb{Z}^n$.

Theorem 2.2. $F$ is a division ring containing $F$.


Expand the valuation $\nu$ to $f \in F$ in the following way. If $f \in F$ then $\nu(f)$ the least element from $\text{supp } f$.

Definition 2.3 ([6]). The division ring $F$ is called a completion of $F$ with respect to $\nu$. 4
Remark 2.10. If $O := \{ f \in F ; \nu(f) \geq 0 \}$ and $m := \{ f \in F ; \nu(f) > 0 \}$, then $O$ is a subring in $F$ and $m$ is an ideal in $O$. Moreover, $O/m \cong k$.

Let $\mathbb{R}^n$ be a vector space of all rows $(r_1, \ldots, r_n)$, $r_i \in \mathbb{R}$, of a length $n$ and $\mathbb{R}^n$ is equipped with the lexicographic order.

Theorem 2.5 ([10]). Let $\leq_{\mathbb{Z}^n}$ be a totally order in the additive group $\mathbb{Z}^n$. Then there exists order preserving group embedding $\mathbb{Z}^n \to \mathbb{R}^n$.

Definition 2.6. [6] A totally order $\leq_{\mathbb{Z}^n}$ is essentially lexicographic if it belongs to the orbit of the standard embedding of $\mathbb{Z}^n$ in to $\mathbb{R}^n$ under the action of the group $GL(n, \mathbb{Z})$. i.e., if $a, b \in \mathbb{Z}^n$, $a \leq_{\mathbb{Z}^n} b$ if and only if $aA \leq bA$ for some fixed $A$ in $GL(n, \mathbb{Z})$ and $\leq$ is the lexicographic order.

Conjecture 2.7 ([6]). A valuation $\nu$ is associated to an essentially lexicographic order on $\mathbb{Z}^n$ if and only if $\cap_{i \geq 1} m^i = 0$.

In the study of this conjecture we obtain the following results partial:

Proposition 2.8. If $\nu : R \to \Gamma \cup \{ \infty \}$ is a valuation of a ring $R$ and $\Gamma$ is an Archimedean group with $\mathcal{W}_\nu := \{ a \in R ; \nu(a) > 0 \}$, $\inf \{ \nu(W_\nu) \} \neq 0$ and $\cap_{i \geq 1} W_\nu^i := I$, then $\nu(I) = \infty$.

Proof. Let $A_i := \nu(W_\nu^i)$ and $\lambda_i := \inf \{ A_i \}$ be, then $\lambda_1 < \lambda_2 < \ldots < \lambda_i$ and $i\lambda_1 \leq \lambda_i$, indeed: (by induction over $i$) as $\inf \{ \nu(W_\nu) \} \neq 0$ then $0 < \lambda_1 \leq \nu(a)$ for all $a \in W_\nu$, hence $\lambda_1 < 2\lambda_1 \leq \nu(ab)$ for all $a, b \in W_\nu$, therefore $2\lambda_1 \leq \lambda_2$, suppose that $\lambda_{i-1} < \lambda_i$ and $i\lambda_1 \leq \lambda_i$, then $i\lambda_1 < (i+1)\lambda_1 \leq \lambda_i + \lambda_1 \leq \nu(a) + \nu(b) = \nu(ab)$ for all $a \in W_\nu^i$ and $b \in W_\nu$, therefore $\lambda_i < \lambda_{i+1}$ and $(i+1)\lambda_1 \leq \lambda_{i+1}$.\hfill $\Box$

Now, suppose there exists $b \in I$ such that $\nu(b) = \lambda < \infty$, so $\lambda_1 < \lambda$ and as $\Gamma$ is Archimedean there exists an integer $m$ such that $m\lambda_1 > \lambda$, therefore $\lambda \notin A_m$, hence $b \notin W_\nu^m$, contradicting that $b \in I$.\hfill $\Box$

Corollary 2.9. If $\nu : D \to \Gamma \cup \{ \infty \}$ is a valuation of a division ring $D$ and $\Gamma$ is an Archimedean group with $\inf \{ \nu(W_\nu) \} \neq 0$, then $\cap_{i \geq 1} W_\nu^i = 0$.

Proof. $0 = \nu(1) = \nu(aa^{-1}) = \nu(a) + \nu(a^{-1})$ for all $a \in D^*$, therefore $\nu(a) < \infty$ for all $a \in D^*$, therefore $\nu(a) = \infty$ if only if $a = 0$.\hfill $\Box$

Remark 2.10. In the Proposition 2.8 the condition $\inf \{ \nu(W_\nu) \} \neq 0$ can be replaced by $\inf \{ \nu(W_\nu) \} \neq 0$ for any $i > 0$ in $\mathbb{N}$.

Example 2.11. If we take lexicographic order on $\mathbb{Z}^2$ the order does not have intersection property: consider $A := \{(x, y) \in \mathbb{Z}^2 ; (0, 0) < (x, y) \}$ and $nA := \sum_{i=1}^n A$ with $n > 0$, then $nA = \{(x, y) \in \mathbb{Z}^2 ; (0, n) \leq (x, y) \}$. By induction over $n$: If $n = 2$, then $2A = A \setminus \{(0, 1)\}$, indeed: as $\min \{ A \} = (0, 1)$ then $(0, 2) \leq (x, y)$ with $(x, y) \in 2A$, thus $2A \subseteq A \setminus \{(0, 1)\}$. Now, if $(x, y)$
in $2A$, then $(x, y - 1) \in A$, because $x > 0$ or $x = 0$ and $y \geq 2$.

Suppose that $nA = (n - 1)A \setminus \{(0, n - 1)\}$, as $\min\{nA\} = (0, n)$ then $(0, n + 1) \leq (x, y)$ with $(x, y) \in (n + 1)A$, thus $(n + 1)A \subseteq nA \setminus \{(0, n)\}$.

Now, if $(x, y)$ in $(n + 1)A$, then $(x, y - 1) \in nA$, because $x > 0$ or $x = 0$ and $y \geq n + 1$. Consequently $(n + 1)A = \{(x, y) \in \mathbb{Z}^2; (0, n + 1) \leq (x, y)\}$.

Hence, as $(1, 0) \in nA$ for every $n \geq 1$ since $(0, n) < (1, 0)$, then $(1, 0) \in \bigcap_{n > 0} nA$.

It follows a counterexample to the conjecture, since a lexicographic order is essentially lexicographic.

3 Skew PBW extensions

In this section we recall the definition and some basic properties of skew PBW (Poincaré-Birkhoff-Witt) extensions, introduced in [11]. Some ring-theoretic and homological properties of these class of noncommutative rings have been studied in [12].

**Definition 3.1.** Let $R$ and $A$ be rings. We say that $A$ is a skew PBW extension of $R$ (also called a $\sigma$ - PBW extension of $R$) if the following conditions hold:

(i) $R \subseteq A$.

(ii) There exists finitely many elements $x_1, \ldots, x_n \in A$ such $A$ is a left $R$-free module with basis

$$\text{Mon}(A) := \{x^u = x_1^{u_1} \cdots x_n^{u_n} \mid u = (u_1, \ldots, u_n) \in \mathbb{N}^n\}.$$  

In this case it also says that $A$ is a left polynomial ring over $R$ with respect to $\{x_1, \ldots, x_n\}$ and $\text{Mon}(A)$ is the set of standard monomials of $A$. Moreover, $x_1^0 \cdots x_n^0 := 1 \in \text{Mon}(A)$.

(iii) For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r - c_{i,r} x_i \in R.$$  \hspace{1cm} (3.1)

(iv) For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n.$$  \hspace{1cm} (3.2)

Under these conditions we will write $A := \sigma(R) \langle x_1, \ldots, x_n \rangle$. 

6
Proposition 3.2. Let $A$ be a skew PBW extension of $R$. Then, for every $1 \leq i \leq n$, there exists an injective ring endomorphism $\sigma_i : R \to R$ and a $\sigma_i$-derivation $\delta_i : R \to R$ such that

$$x_i r = \sigma_i(r)x_i + \delta_i(r),$$

for each $r \in R$.

Proof. See [11], Proposition 3. $\square$

The previous proposition gives the notation and the alternative name given for the skew PBW extensions.

Definition 3.3. Let $A$ be a skew PBW extension.

(a) $A$ is quasi-commutative if the conditions (iii) and (iv) in Definition 3.1 are replaced by

(iii') For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that

$$x_i r = c_{i,r} x_i.$$ (3.3)

(iv') For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that

$$x_j x_i = c_{i,j} x_i x_j.$$ (3.4)

(b) $A$ is bijective if $\sigma_i$ is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

Definition 3.4. Let $A$ be a skew PBW extension of $R$ with endomorphisms $\sigma_i, 1 \leq i \leq n$, as in Proposition 3.2.

(i) For $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$, $\sigma^u := \sigma_1^{u_1} \cdots \sigma_n^{u_n}$, $|u| := u_1 + \cdots + u_n$. If $v = (v_1, \ldots, v_n) \in \mathbb{N}^n$, then $u + v := (u_1 + v_1, \ldots, u_n + v_n)$.

(ii) For $X = x^u \in \text{Mon}(A)$, $\exp(X) := u$ and $\deg(X) := |u|$.

(iii) If $f = c_1 X_1 + \cdots + c_t X_t$, with $X_i \in \text{Mon}(A)$ and $c_i \in R - \{0\}$, then $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$.

Theorem 3.5. Let $A$ be a left polynomial ring over $R$ w.r.t. $\{x_1, \ldots, x_n\}$. $A$ is a skew PBW extension of $R$ if and only if the following conditions hold:

(a) For every $x^u \in \text{Mon}(A)$ and every $0 \neq r \in R$ there exist unique elements $r_u := \sigma^u(r) \in R - \{0\}$ and $p_{u,r} \in A$ such that

$$x^u r = r_u x^u + p_{u,r},$$ (3.5)

where $p_{u,r} = 0$ or $\deg(p_{u,r}) < |u|$ if $p_{u,r} \neq 0$. Moreover, if $r$ is left invertible, then $r_u$ is left invertible.
(b) For every $x^u, x^v \in \text{Mon}(A)$ there exist unique elements $c_{u,v} \in R$ and $p_{u,v} \in A$ such that
\[
x^u x^v = c_{u,v} x^{u+v} + p_{u,v},
\]
where $c_{u,v}$ is left invertible, $p_{u,v} = 0$ or $\deg(p_{u,v}) < |u + v|$ if $p_{u,v} \neq 0$.

Proof. See [11], Theorem 7.

Proposition 3.6. Let $A$ be a skew PBW extension of a ring $R$. If $R$ is a domain, then $A$ is a domain.

Proof. See [12].

The next theorem characterizes the quasi-commutative skew PBW extensions.

Theorem 3.7. Let $A$ be a quasi-commutative skew PBW extension of a ring $R$. Then,

(i) $A$ is isomorphic to an iterated skew polynomial ring of endomorphism type, i.e.,

\[ A \cong R[z_1; \theta_1] \cdots [z_n; \theta_n]. \]

(ii) If $A$ is bijective, then each endomorphism $\theta_i$ is bijective, $1 \leq i \leq n$.

Proof. See [12].

Corollary 3.8. Let $A$ be a bijective and quasi-commutative skew PBW extension of a ring $R$. If $R$ is a left Ore domain, then $A$ is a left Ore domain.

Proof. By Theorem 3.7, $A$ is isomorphic to an iterated skew polynomial ring of automorphism type over a left Ore domain $R$.

Theorem 3.9. Let $A$ be an arbitrary skew PBW extension of $R$. Then, $A$ is a filtered ring with filtration given by
\[
F_m := \begin{cases} R & \text{if } m = 0 \\ \{ f \in A \mid \deg(f) \leq m \} & \text{if } m \geq 1 \end{cases}
\]

and the corresponding graded ring $\text{Gr}(A)$ is a quasi-commutative skew PBW extension of $R$. Moreover, if $A$ is bijective, then $\text{Gr}(A)$ is a quasi-commutative bijective skew PBW extension of $R$.

Proof. See [12].

Theorem 3.10 (Hilbert Basis Theorem). Let $A$ be a bijective skew PBW extension of $R$. If $R$ is a left (right) Noetherian ring then $A$ is also a left (right) Noetherian ring.

Proof. See [12].
3.1 Skew quantum polynomials

In this subsection we recall the definition and some basic properties of skew quantum polynomials ring over $R$, introduced in [12]. We mention some results generalized for valuations of skew quantum polynomials and bijective and quasi-commutative skew PBW extension.

**Definition 3.11.** Let $R$ be a ring with matrix of parameters $q := [q_{ij}] \in M_n(R)$, $n \geq 2$, such that $q_{ii} = 1 = q_{ij}q_{ji}$ for each $1 \leq i, j \leq n$ and suppose also that is given a system $\sigma_1, \ldots, \sigma_n$ of automorphisms of $R$. The skew quantum polynomials ring over $R$, denoted by $R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$, is defined with the following conditions:

1. $R \subseteq R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$,
2. $R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$ is a free left $R$-module with basis $\{x^u; x^u = x_1^{u_1} \cdots x_n^{u_n}, u_i \in \mathbb{Z}, 1 \leq i \leq r$ and $u_i \in \mathbb{N}$ for $r+1 \leq i \leq n\}$,
3. The $x_1, \ldots, x_n$ elements satisfy the defining relations

$$x_i x_i^{-1} = 1 = x_i^{-1} x_i, \quad 1 \leq i \leq r,$$

$$x_i x_j = q_{ij} x_j x_i, \quad 1 \leq i, j \leq n,$$

$$x_i r = \sigma_i(r) x_i, \quad r \in R \ y 1 \leq i \leq n.$$  

When all automorphisms are trivial, we write $R_q[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$ and this ring is called the ring of quantum polynomials over $R$. If $R = K$ is a field, then $K_q[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n]$ is the algebra of skew quantum polynomials. For trivial automorphisms we get the algebra of quantum polynomials simply.

If $r = n$, $R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is called the $n$-multiparametric skew quantum torus over $R$, when all automorphisms are trivial, is called the $n$-multiparametric quantum torus over $R$. If $r = 0$, $R_{q,\sigma}[x_1, \ldots, x_n]$ is called the $n$-multiparametric skew quantum space over $R$, when all automorphisms are trivial is called $n$-multiparametric quantum space over $R$.

The algebra of quantum polynomials can be defined as a quasi-commutative bijective skew PBW extension of the $r$-multiparameter quantum torus, or also, as a localization of a quasi-commutative bijective skew PBW extension.
**Theorem 3.12.** \( R_{q,\sigma}[x_1, \ldots, x_n] \cong R[z_1; \theta_1] \cdots [z_n; \theta_n] \), where

i) \( \theta_1 = \sigma_1 \),

ii) \( \theta_i : R[z_1; \theta_1] \cdots [z_{i-1}; \theta_{i-1}] \to R[z_1; \theta_1] \cdots [z_{i-1}; \theta_{i-1}] \),

iii) \( \theta_i(z_i) = q_{ij} z_i, 1 \leq i < j \leq n, \theta_i(r) = \sigma_i(r) \) for \( r \in R \).

In particular, \( R_{q}[x_1, \ldots, x_n] \cong R[z_1] \cdots [z_n] \).

**Proof.** See [12]. \( \square \)

**Theorem 3.13.** \( R_{q,\sigma}[x_1^1, \ldots, x_r^1, x_r+1, \ldots, x_n] \) is a ring of fractions of \( B := R_{q,\sigma}[x_1, \ldots, x_n] \) with respect to the multiplicative subset

\[
S = \{ rx^n; r \in R^*, x^n \in Mon\{x_1, \ldots, x_r\} \},
\]

i.e.,

\[ R_{q,\sigma}[x_1^1, \ldots, x_r^1, x_{r+1}, \ldots, x_n] \cong S^{-1}B. \]

**Proof.** See [12]. \( \square \)

**Remark 3.14.** Let \( Q_{q,\sigma}^{r,n}(R) := R_{q,\sigma}[x_1^1, \ldots, x_r^1, x_{r+1}, \ldots, x_n] \) and \( R \) be a left (right) Noetherian ring, then \( Q_{q,\sigma}^{r,n}(R) \) is left (right) Noetherian by Theorem 3.10. Moreover, if \( R \) is a domain, then \( Q_{q,\sigma}^{r,n}(R) \) is also a domain by Theorem 3.6. Thus, if \( R \) is a left (right) Noetherian domain, then \( Q_{q,\sigma}^{r,n}(R) \) is a left (right) Ore domain.

Thus, \( Q_{q,\sigma}^{r,n}(R) \) has a total division ring of fractions

\[ Q(Q_{q,\sigma}^{r,n}(R)) \cong Q(A) := \sigma(R)(x_1, \ldots, x_n), \]

where \( \sigma(R)(x_1, \ldots, x_n) \) denotes the rational fractions of \( A := \sigma(R)(x_1, \ldots, x_n) \).

### 3.2 Some properties

**Definition 3.15.** Let \( N \) be the subgroup in the multiplicative group \( R^* \) of the ring \( R \) generated by the derived subgroup \([R^*, R^*] \) and by the set of all elements of the form \( z_i^{-1} \sigma_i(z) \) where \( z \in R^* \) and \( i = 1, \ldots, n \).

**Remark 3.16.** \( N \) is a normal subgroup in \( R^* \).

**Definition 3.17.** If the images of \( q_{ij} \) with \( 1 \leq i < j \leq n \) are independent in the multiplicative Abelian group \( \tilde{R} = R^*/N \) then, \( R_{q,\sigma}[x_1^1, \ldots, x_r^1, x_{r+1}, \ldots, x_n] \) is a generic skew quantum polynomials ring.

**Remark 3.18.** If \( n = 2 \) in \( R_{q,\sigma}[x_1^1, \ldots, x_r^1, x_{r+1}, \ldots, x_n] \), of the previous definition \( q = q_{12} \) is not a root of unity.

**Proposition 3.19.** For each \( a \in R^* \) and \( \sigma \) endomorphism over \( R \), \( \sigma^k(a) = an \) with \( k \in \mathbb{N} \) and \( n \in \mathbb{N} \).
Proof.

\[
\sigma^k(a) = a(a^{-1}\sigma(a))((\sigma(a))^{-1}\sigma^2(a)) \cdots ((\sigma^{k-1}(a))^{-1}\sigma^k(a)) = an, \text{ with } n \in N. \tag{3.12}
\]

Proposition 3.20. If \( u, v \in \mathbb{Z}^r \times \mathbb{N}^{n-r} \) and \( \lambda, \mu \in R^* \), then

(1) \( x_i x^u = \left( \prod_{j=1}^n q_{ji}^u \right) n_u \cdot x^u x_i \), for some \( n_u \in N \) and for any \( 1 \leq i \leq n \).

(2) \( (x^u)(x^v) = \left( \prod_{i<j} q_{ji}^{u,v} \right) n_{u+v} \cdot x^{u+v} \), with \( n_{u+v} \in N \).

(3) \( (\lambda x^u)(\mu x^v) = \lambda \mu \left( \prod_{i<j} q_{ji}^{u,v} \right) n' \cdot x^{u+v} \), with \( n' \in N \).

Proof. Applying the Proposition 3.19 and note that \( x_i x^{-1} = q_{ji}^{-1} x_i^{-1} x_i \) with \( 1 \leq j \leq r \). \( \square \)

Proposition 3.21. Let \( f := \sum_{u \in \mathbb{Z}} \lambda_u x^u \) be in \( R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}, \ldots, x_n] \) and \( x_i \) with \( 1 \leq i \leq r \).

(1) If \( \lambda_u \in R \), then \( x_i f x_i^{-1} = \sum_{u \in \mathbb{Z}^n} \sigma_i(\lambda_u) \lambda'_u x^u \),

where \( \lambda'_u := \left( \prod_{j=1}^n q_{ji}^u \right) n_u \in R^* \).

(2) If \( \lambda_u \in R^* \), then \( x_i f x_i^{-1} = \sum_{u \in \mathbb{Z}^n} \lambda'_u x^u \),

where \( \lambda'_u \in R^* \).

Proof. (1) Note that \( N \subseteq R^* \) and

\[
x_i f x_i^{-1} = \sum_{u \in \mathbb{Z}^n} \sigma_i(\lambda_u) x_i x^u x_i^{-1} = \sum_{u \in \mathbb{Z}^n} \sigma_i(\lambda_u) \left( \prod_{j=1}^n q_{ji}^u \right) n_u x^u,
\]

where \( n_u \in N \).

(2) By item (1), \( \sigma_i(\lambda_u) \lambda'_u \in R^* \). \( \square \)
Remark 3.22. If \(Q(Q_{q,\sigma}^n(R))\) exists and \(G\) denotes the multiplicative subgroup in \(Q(Q_{q,\sigma}^n(R))^*\) generated by \(R^*\) and \(x_1, ..., x_n\). Then \(R^* \triangleleft G\) and \(G/R^*\) is a free abelian group with the base \(x_1R^*, ..., x_nR^*\).

**Proposition 3.23.** Let \(R\) be a left Ore domain and \(\sigma\) automorphisms over \(R\), then \(\sigma\) can be extended to \(Q(R)\) and is also an automorphism.

**Proof.** By universal property we have the following commutative diagram:

\[
\begin{array}{ccc}
R & \xrightarrow{\psi} & Q(R) \\
\downarrow{\sigma} & & \downarrow{	ilde{\sigma}} \\
R & \xrightarrow{\psi} & Q(R)
\end{array}
\]

where \(\psi, \sigma\) are injective and \(\tilde{\sigma}\left(\frac{a}{b}\right) = \frac{\sigma(a)}{\sigma(b)}\) for \(a, b \neq 0 \in R\). Therefore, \(\psi \circ \sigma\) is injective and so is \(\tilde{\sigma}\).

If \(\frac{a}{b} \in Q(R)\), then \(\frac{a}{b} = \psi(b)^{-1}\psi(a) = \psi(\sigma(b_0))^{-1}\psi(\sigma(a_0))\) for \(a_0, b_0 \neq 0 \in R\), consequently,

\[
\frac{a}{b} = \psi(\sigma(b_0))^{-1}\psi(\sigma(a_0)) \\
= \tilde{\sigma}(\psi(b_0))^{-1}\tilde{\sigma}(\psi(a_0)) \\
= \tilde{\sigma}(\psi(b_0)^{-1}\psi(a_0)) \\
= \tilde{\sigma}\left(\frac{a_0}{b_0}\right).
\]

\[\square\]

**Theorem 3.24.** Let \(R\) be a left Ore domain and \(S = R - \{0\}\), then

\[
S^{-1}(R_{q,\sigma}[x_1, ..., x_n]) \cong Q(R)_{\tilde{q},\tilde{\sigma}}[x_1, ..., x_n],
\]

where \(\tilde{q} = (\frac{q_1}{q_2}) \in M(n, Q(R))\).

**Proof.** By Theorem 3.12 \(R_{q,\sigma}[x_1, ..., x_n] \cong R[z_1; \theta_1] \cdots [z_n; \theta_n]\), with each \(\theta_i\) bijective. Thus, if \(S = R - \{0\}\) then

\[
S^{-1}(R_{q,\sigma}[x_1, ..., x_n]) \cong S^{-1}(R[z_1; \theta_1] \cdots [z_n; \theta_n]) \\
\cong S^{-1}(R[z_1; \tilde{\theta}_1] \cdots [z_n; \tilde{\theta}_n]) \\
= Q(R)[z_1; \tilde{\theta}_1] \cdots [z_n; \tilde{\theta}_n]
\]
where

\[ \tilde{\theta}_i : Q(R) \rightarrow Q(R) \]
\[ \frac{a}{b} \mapsto \tilde{\theta}_i \left( \frac{a}{b} \right) = \frac{\theta_i(a)}{\theta_i(b)} = \frac{\sigma_i(a)}{\sigma_i(b)} = \tilde{\sigma}_i \left( \frac{a}{b} \right), \]

and

\[ \tilde{\theta}_i : Q(R) [z_1; \tilde{\theta}_1] \cdots [z_{i-1}; \tilde{\theta}_{i-1}] \rightarrow Q(R) [z_1; \tilde{\theta}_1] \cdots [z_{i-1}; \tilde{\theta}_{i-1}] \]

with
\[ \tilde{\theta}_i \left( \frac{a}{b} \right) = \tilde{\sigma}_i \left( \frac{a}{b} \right) \quad y \tilde{\theta}_i (z_i) = \frac{q_{ij}}{1} z_i. \]

Therefore,

\[ S^{-1}(R_{q,\sigma}[x_1, \ldots, x_n]) \cong Q(R)_{\tilde{q}, \tilde{\sigma}}[x_1, \ldots, x_n], \]

where \( \tilde{q} = (\frac{a}{b}) \in \mathcal{M}(n, Q(R)). \)

\[ \Box \]

**Proposition 3.25.** Let \( R \) be a left Ore domain, there exists

\[ \phi : R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \rightarrow Q(R)_{\tilde{q}, \tilde{\sigma}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \]

an injective ring homomorphism.

**Proof.** Let \( B_R := R_{q,\sigma}[x_1, \ldots, x_n] \) and \( B_{Q(R)} := Q(R)_{\tilde{q}, \tilde{\sigma}}[x_1, \ldots, x_n] \) be, by Theorem 3.13 \( R_{q,\sigma}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \cong S^{-1}_1 B_R \) with \( S_1 = \{ rx^u : r \in R^*, x^u \in Mon\{x_1, \ldots, x_n\} \} \), and \( Q(R)_{\tilde{q}, \tilde{\sigma}}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \cong S^{-1}_1 B_{Q(R)} \) with \( S_1' = \{ rx^u : r \in Q(R)^*, x^u \in Mon\{x_1, \ldots, x_n\} \} \).

Now, consider the following diagram of ring homomorphisms:

\[
\begin{array}{ccc}
R & \xrightarrow{\psi} & R_{q,\sigma}[x_1, \ldots, x_n] \\
\downarrow \psi' & & \downarrow \psi
\end{array}
\]

\[
\begin{array}{ccc}
Q(R) & \xrightarrow{\psi'} & Q(R)_{\tilde{q}, \tilde{\sigma}}[x_1, \ldots, x_n] \\
\downarrow \psi' & & \downarrow \psi
\end{array}
\]

\[
\begin{array}{ccc}
& \xrightarrow{\psi'} & S^{-1}_1 B_R \\
& \downarrow \varphi & \downarrow \varphi
\end{array}
\]

\[
\begin{array}{ccc}
& \xrightarrow{\psi'} & S^{-1}_1 B_{Q(R)} \\
& \downarrow \varphi & \downarrow \varphi
\end{array}
\]

where \( \psi \) is the injection for the localisation of \( R \) to the total ring fractions \( Q(R) \), \( \psi' \) the injection determined by the isomorphism of Theorem 3.24 where \( \psi'(ax^u) = \frac{a}{1} x^u \), and \( \psi_1, \psi_1' \) injections determined by the localizations for \( B_R \) and \( B_{Q(R)} \) respectively.

As \( \psi'(S_1) \subseteq S_1' \), then \( \psi_1'(\psi'(S_1)) \subseteq \psi_1'(S_1') \subseteq (S^{-1}_1 B_{Q(R)})^* \), therefore, by universal property there exists \( \varphi \). If \( f = \sum a_n x^u \in R_{q,\sigma}[x_1, \ldots, x_n] \) and \( rx^u \in S_1 \) then,
\[
\varphi \left( \frac{f}{rx^v} \right) = \varphi \left( \frac{\sum a_u x^u}{rx^v} \right) \\
= \psi_1 \left( \psi'(rx^v) \right)^{-1} \psi_1' \left( \psi' \left( \sum a_u x^u \right) \right) \\
= \psi_1' \left( \frac{r}{1} \right)^{-1} \psi_1' \left( \sum \frac{a_u}{1} x^u \right) \\
= \frac{1}{r} \sum \frac{a_u}{1} x^u \\
= \sum \frac{a_u}{1} x^u \\
= \frac{r}{1} x^v \\
= \psi'(f) \\
= \psi'(rx^v).
\]

Also, \( \varphi \) is injective by \( \psi' \) and \( \psi_1' \) are injective.

\[\square\]

Need the following result for the subsequent theorem:

**Proposition 3.26.** Let \( R \) be a ring and \( S \subset R \) a multiplicative subset. If \( Q := S^{-1}R \) exists, then any finite set \( \{q_1, \ldots, q_n\} \) of elements of \( Q \) posses a common denominator, i.e., there exists \( r_1, \ldots, r_n \in R \) and \( s \in S \) such that \( q_i = \frac{r_i}{s}, 1 \leq i \leq n \).

**Proof.** See [13], Lemma 2.1.8. \( \square \)

**Theorem 3.27.** Let \( R \) be a left Ore domain, then \( Q(Q(Q_{n,n}(R))) \cong Q(Q_{n,n}(Q(R))). \)

**Proof.** With the notation of the proof in the Proposition 3.25 consider the following diagram of ring homomorphisms

\[
\begin{array}{ccc}
S_1^{-1}B_R & \xrightarrow{\psi_2} & Q(S_1^{-1}B_R) \\
\varphi \downarrow & & \varphi' \downarrow \\
S_1^{-1}B_{Q(R)} & \xrightarrow{\psi_2'} & Q(S_1^{-1}B_{Q(R)})
\end{array}
\]

where \( \psi_2, \psi_2' \) are injections determined by the localizations of \( S_1^{-1}B_R \) and \( S_1^{-1}B_{Q(R)} \) respectively and \( \varphi \) the injection of the Proposition 3.25.

By Remark 3.14, \( S_1^{-1}B_R \) and \( S_1^{-1}B_{Q(R)} \) are domain, now, if \( \frac{p_1}{q_1}, \frac{p_2}{q_2} \in S_1^{-1}B_R \) with \( \frac{p_1}{q_1} \neq 0 \), then \( p_1 \neq 0 \) and there exist \( f_1 \neq 0 \) and \( f_2 \in B_R \) such that \( f_1p_1 = f_2p_2 \). Then, \( \frac{f_1}{p_1} \frac{p_1}{q_1} = \frac{f_2}{q_2} \frac{p_2}{f_2} = \frac{f_2}{q_2} \frac{p_2}{f_2} \neq 0 \), therefore \( S_1^{-1}B_R \) is a Ore domain, similarly it has to \( S_1^{-1}B_{Q(R)} \). Thus, if \( S_2 = S_1^{-1}B_R - \{0\} \) and \( S_2' = S_1^{-1}B_{Q(R)} - \{0\} \) as \( \varphi(S_2) \subseteq S_2' \), then
ψ_2(ϕ(S_2)) ⊆ ψ_2(S_2) ⊆ (Q(S_t^{-1}B_Q(R)))^*, hence, by universal property there exists ϕ' injective ring homomorphism.

Note that if f, g ∈ B_R and ax^u, bx^v ∈ S_1, then

\[
\frac{f}{g} = \frac{g}{ax^u}f = bx^v \frac{f}{g}
\]

and

\[
\frac{f'}{g'} = \frac{1}{g'} \frac{f'}{1} = \left(\frac{g'}{1}\right)^{-1} \frac{f'}{1} = \frac{f'}{q'},
\]

where f', g' ∈ B_R by Remark 3.14 with r = 0. Similarly is obtained for Q(S_t^{-1}B_Q(R)).

Therefore,

\[
\phi' \left(\frac{f}{g}\right) = \psi_2^* \left(\phi \left(\frac{g}{1}\right)^{-1}\right) \psi_2^* \left(\phi \left(\frac{f}{1}\right)\right)
\]

\[
= \psi_2^* \left(\phi'(g)^{-1}\right) \psi_2^* \left(\phi'(f)^{-1}\right)
\]

\[
= \frac{1}{\phi'(g)} \phi'(f)
\]

Now, if f, 0 ≠ g ∈ S_t^{-1}B_Q(R), applying Theorem 3.26 must be

\[
\frac{f}{g} = \frac{\sum a_u x^u}{\sum \phi a_u x^v} = \frac{1}{s} \sum a_u x^u = \left(\sum c_{\nu} x^v\right)^{-1} \left(\frac{1}{s}\right)^{-1} \frac{1}{s} \sum a_u x^u
\]

\[
= \left(\sum c_{\nu} x^v\right)^{-1} \left(\frac{1}{s'}\right) \sum a_u x^u = \left(\sum c_{\nu} x^v\right)^{-1} \left(\frac{r'}{r}\right) \sum a_u x^u
\]

\[
= \left(\sum r c_{\nu} x^v\right)^{-1} \left(\sum r' a_u x^u\right)
\]

\[
= \frac{\sum r' a_u x^u}{\sum r c_{\nu} x^v} = \psi'(f') \psi'(g')
\]

\[
= \phi' \left(\frac{f'}{g'}\right).
\]

where f' = \sum (r' a_u) x^u and g' = \sum (r c_{\nu}) x^v, then φ is surjective. Hence Q(Q_{q,s}(R)) ∼= Q(Q_{q,s}^u(Q(R))).
3.3 Valuations of skew quantum polynomials.

Theorem 3.28. Let $R$ be a left Ore domain and $\nu : Q(Q_{q,\sigma}^{n,n}(R))^* \rightarrow \Gamma$ is a valuation with $\nu(Q(R)^*) = 0$, then $\Gamma$ is Abelian.

Proof. $Q(R)$ is a division ring and $Q(Q_{q,\sigma}^{n,n}(R)) \cong Q(Q_{q,\sigma}^{n,n}(Q(R)))$, by Theorem 1.10. $\Gamma$ is Abelian.

Corollary 3.29. Let $R$ be a left Ore domain, $\nu : Q(Q_{q,\sigma}^{n,n}(R))^* \rightarrow \Gamma$ a valuation with $\nu(Q(R)^*) = 0$ and $Q_{q,\sigma}^{n,n}(Q(R))$ generic, then $\Gamma$ is Abelian.

Theorem 3.30. Let $R$ be a left Ore domain, a valuation $\nu : Q(Q_{q,\sigma}^{n,n}(R))^* \rightarrow \Gamma$ with $\nu(Q(R)^*) = 0$ and $Q_{q,\sigma}^{n,n}(Q(R))$ generic. The valuation $\nu$ has maximal rank if only if $\Gamma \cong \mathbb{Z}^n$.

Proof. By Theorem 3.27. $Q(Q_{q,\sigma}^{n,n}(R)) \cong Q(Q_{q,\sigma}^{n,n}(Q(R)))$ with $Q(R)$ a division ring, by Theorem 1.13 the valuation $\nu$ has maximal rank if only if $\Gamma \cong \mathbb{Z}^n$.

3.4 Valuations of skew PBW extension.

Theorem 3.31. Let $A = \sigma(R)(x_1, \ldots, x_n)$ be a bijective and quasi-commutative skew PBW extension of a ring $R$. If $R$ is a left Ore domain and $\nu : Q(A)^* \rightarrow \Gamma$ a valuation with $\nu(Q(R)^*) = 0$, then $\Gamma$ is Abelian.

Proof. By Theorem 3.8 $A$ is an Ore domain then, $Q(A)$ exists and is a division ring, by Remark 3.14. $Q(A) \cong Q(Q_{q,\sigma}^{n,n}(R))$ (in particular $r = 0$) and by Theorem 3.28 $\Gamma$ is abelian.

Corollary 3.32. Let $A$ be a bijective skew PBW extension of a ring $R$. If $R$ is a left Ore domain and $\nu : Q(Gr(A))^* \rightarrow \Gamma$ a valuation with $\nu(Q(R)^*) = 0$, then $\Gamma$ is Abelian.

Proof. By Theorem 3.9 $Gr(A)$ is bijective and quasi-commutative.

References


