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Cyclic homology, tight crossed products, and small stabilizations

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Abstract. In [1] (arXiv:1212.5901) we associated an algebra $\Gamma^\infty(\mathfrak{A})$ to every bornological algebra $\mathfrak{A}$ and an ideal $I_{S(\mathfrak{A})} \lhd \Gamma^\infty(\mathfrak{A})$ to every symmetric ideal $S \lhd \ell^\infty$. We showed that $I_{S(\mathfrak{A})}$ has K-theoretical properties which are similar to those of the usual stabilization with respect to the ideal $J_S \lhd \mathcal{B}$ of the algebra $\mathcal{B}$ of bounded operators in Hilbert space which corresponds to S under Calkin's correspondence. In the current article we compute the relative cyclic homology $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})})$. Using these calculations, and the results of *loc. cit.*, we prove that if $\mathfrak{A}$ is a C^*-algebra and c_0 the symmetric ideal of sequences vanishing at infinity, then $K_*(I_{c_0(\mathfrak{A})})$ is homotopy invariant, and that if $* \geq 0$, it contains $K_*^{\mathrm{top}}(\mathfrak{A})$ as a direct summand. This is a weak analogue of the Suslin–Wodzicki theorem ([20]) that says that for the ideal $\mathcal{K} = J_{c_0}$ of compact operators and the C^*-algebra tensor product $\mathfrak{A} \mathbin{\widetilde{\otimes}} \mathcal{K}$, we have $K_*(\mathfrak{A} \mathbin{\widetilde{\otimes}} \mathcal{K}) = K_*^{\mathrm{top}}(\mathfrak{A})$. Similarly, we prove that if $\mathfrak{A}$ is a unital Banach algebra and $\ell^{\infty-} = \bigcup_{q<\infty} \ell^q$, then $K_*(I_{\ell^{\infty-}(\mathfrak{A})})$ is invariant under Hölder continuous homotopies, and that for $* \geq 0$ it contains $K_*^{\mathrm{top}}(\mathfrak{A})$ as a direct summand. These K-theoretic results are obtained from cyclic homology computations. We also compute the relative cyclic homology groups $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})})$ in terms of $HC_*(\ell^\infty(\mathfrak{A}) : S(\mathfrak{A}))$ for general $\mathfrak{A}$ and S. For $\mathfrak{A} = \mathbb{C}$ and general S, we further compute the latter groups in terms of algebraic differential forms. We prove that the map $HC_n(\Gamma^\infty(\mathbb{C}) : I_{S(\mathbb{C})}) \to HC_n(\mathcal{B} : J_S)$ is an isomorphism in many cases.

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1. Introduction

Let $\ell^2 = \ell^2(\mathbb{N})$ be the Hilbert space of square-summable sequences of complex numbers and $\mathcal{B} = \mathcal{B}(\ell^2)$ the algebra of bounded operators. Calkin's theorem in [3, Theorem 1.6], as restated by Garling in [15, Theorem 1], establishes an isomorphism

$$S \mapsto J_S$$

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between the lattice of proper symmetric ideals of the algebra ℓ^∞ of bounded sequences and that of proper two-sided ideals of the algebra $\mathcal{B} = \mathcal{B}(\ell^2)$ of bounded operators. In [1] we introduced a subalgebra $\Gamma^\infty \subset \mathcal{B}$ and showed that the above lattices are also isomorphic to the lattice of proper two-sided ideals of Γ^∞, via the correspondence

$$S \mapsto I_S = J_S \cap \Gamma^\infty.$$

More generally, we associated to each bornological algebra $\mathfrak{A}$, an algebra $\Gamma^\infty(\mathfrak{A})$ which contains an ideal $I_{S(\mathfrak{A})}$ for each symmetric ideal $S \lhd \ell^\infty$. We showed that the algebra $I_{S(\mathfrak{A})}$ has K-theoretical properties which are analogous to those of the usual stabilization with respect to J_S, at least when S is one of the following:

$$S \in \{c_0, \ell^{p-}, \ell^q, \ell^{q+} \quad (p \le \infty, q < \infty)\}. \tag{1.1}$$

Here c_0 is the ideal of sequences vanishing at infinity, ℓ^q consists of the q-summable sequences, and

$$\ell^{p-} = \bigcup_{r<p} \ell^r, \quad \ell^{q+} = \bigcap_{s>q} \ell^s.$$

We proved that for S as in (1.1), there is a long exact sequence:

$$\begin{array}{ccc} KH_{n+1}(I_{S(\mathfrak{A})}) & \longrightarrow & HC_{n-1}(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) \\ & & \downarrow \\ KH_n(I_{S(\mathfrak{A})}) & \longleftarrow & K_n(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) \end{array} \tag{1.2}$$

If furthermore, $S \neq c_0$, then $KH_*(I_{S(\mathfrak{A})}) = KH_*(I_{\ell^1(\mathfrak{A})})$. We proved that the functor $KH_*(I_{c_0(\mathfrak{A})})$ is invariant under arbitrary continuous homotopies of bornological algebras, and that $KH_*(I_{\ell^1(\mathfrak{A})})$ is invariant under Hölder continuous homotopies. We also showed that if $* \ge 0$ and either $\mathfrak{A}$ is a C^*-algebra and $S = c_0$ or $\mathfrak{A}$ is a local Banach algebra and $S = \ell^1$, then $KH_*(I_{S(\mathfrak{A})})$ contains $K_*^{\mathrm{top}}(\mathfrak{A})$ as a direct summand. In the current article we study the groups $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})})$ for general S and $\mathfrak{A}$. We show for example that if $\mathfrak{A}$ is a C^*-algebra then $I_{c_0(\mathfrak{A})}$ is H-unital and

$$HC_*(\Gamma^\infty(\mathfrak{A}) : I_{c_0(\mathfrak{A})}) = 0.$$

It follows from this, excision, and the exact sequence (1.2), that the comparison map

$$K_*(I_{c_0(\mathfrak{A})}) \to KH_*(I_{c_0(\mathfrak{A})}) \tag{1.3}$$

is an isomorphism. In particular, if $\mathfrak{A}$ is a C^*-algebra, then $K_*(I_{c_0(\mathfrak{A})})$ is homotopy invariant, and if $* \ge 0$, it contains $K_*^{\mathrm{top}}(\mathfrak{A})$ as a direct summand. This again shows that $I_{c_0(-)}$ has properties analogous to those of $J_{c_0} = \mathcal{K}$, the ideal of compact operators. Indeed, the result above is a weak analogue of the Suslin–Wodzicki theorem (Karoubi's conjecture) which says that if $\mathfrak{A}$ is a C^*-algebra then $K_*(\mathfrak{A}\,\widetilde{\otimes}\,\mathcal{K}) = K_*^{\mathrm{top}}(\mathfrak{A})$. We also show that if $\mathfrak{A}$ is a unital Banach algebra then $I_{\ell^\infty-(\mathfrak{A})}$ is H-unital and

$$HC_*(\Gamma^\infty(\mathfrak{A}) : I_{\ell^\infty-(\mathfrak{A})}) = 0.$$

Thus the comparison map

$$K_*(I_{\ell^\infty-(\mathfrak{A})}) \to KH_*(I_{\ell^\infty-(\mathfrak{A})}) \tag{1.4}$$

is an isomorphism. Again this is analogous to a similar property of stabilization with respect to $J_{\ell^\infty-} = \bigcup_p \mathcal{L}^p$, the union of all Schatten ideals (see [24, pp. 490], [9, Theorem 8.2.5]). In [24], M. Wodzicki studied the relative cyclic homology groups $HC_n(\mathcal{B} : J_S)$. For S as in (1.1), the following integer was computed by Wodzicki in [24, Corollary to Theorem 8]

$$m = m_S = \min\{n : HC_n(\mathcal{B} : J_S) \neq 0\}.$$

We prove in Proposition 7.3 that

$$m = \min\{n : HC_n(\Gamma^\infty : I_S) \neq 0\}, \tag{1.5}$$

and that the natural map is an isomorphism for $n = m$:

$$HC_m(\Gamma^\infty : I_S) \xrightarrow{\cong} HC_m(\mathcal{B} : J_S). \tag{1.6}$$

The techniques used in this article to establish the results above about $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})})$ are similar to those used in [24] to study the relative cyclic homology of stabilizations by J_S. We also obtain more results about the groups $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})})$ using a different technique, which involves a description of Γ^∞ and I_S as crossed products, established in [1, Proposition 6.12]. The inverse monoid Emb of all partially defined injections

$$\mathbb{N} \supset \operatorname{dom} f \xrightarrow{f} \mathbb{N}.$$

acts on $\ell^\infty(\mathfrak{A})$ by

$$f_*(\alpha)_n = \begin{cases} \alpha_m & \text{if } f(m) = n \\ 0 & \text{else.} \end{cases} \tag{1.7}$$

By definition, an ideal $S \lhd \ell^\infty$ is symmetric if the action above maps S to itself. Observe that if $A, B \subset \mathbb{N}$ are disjoint then the inclusions $p_A : A \to \mathbb{N}$ and $p_B : B \to \mathbb{N}$ satisfy

$$(p_{A\cup B})_* = (p_A)_* + (p_B)_*$$

In other words, the action above is *tight* in the sense of Exel [14]. Thus $\ell^\infty(\mathfrak{A})$ is a module over the ring

$$\Gamma = \mathbb{Z}[\mathrm{Emb}]/\langle p_A + p_B - p_{A\cup B} : A \cap B = \emptyset\rangle$$

Let $\mathcal{P} \subset \Gamma$ be the subring generated by all the p_A with $A \subset \mathbb{N}$. Note that $\mathcal{P}$ is isomorphic to the subring of $\ell^\infty(\mathfrak{A})$ consisting of those sequences $\alpha : \mathbb{N} \to \mathbb{Z}$ which take finitely many distinct values. In particular (1.7) makes $\mathcal{P}$ into a Γ-module. Moreover $\ell^\infty(\mathfrak{A})$ is a $\mathcal{P}$-algebra, and the map

$$HC(\ell^\infty(\mathfrak{A}) : S(\mathfrak{A})) \to HC((\ell^\infty(\mathfrak{A})/\mathcal{P} : S(\mathfrak{A}))/\mathcal{P}) \tag{1.8}$$

is a quasi-isomorphism (see Example 6.6 and (6.6.3)). Furthermore the action of Emb on $\ell^\infty(\mathfrak{A})$ extends to a tight action on $HC(\ell^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})})$, and we show that

$$HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) = \mathbb{H}_*(\Gamma/\mathcal{P} : HC((\ell^\infty(\mathfrak{A}) : S(\mathfrak{A}))/\mathcal{P})). \tag{1.9}$$

Here the hyperhomology groups $\mathbb{H}_*(\Gamma/\mathcal{P}, -)$ are the hyperderived functors of the functor

$$\Gamma - \mathrm{Mod} \to \mathfrak{Ab},\ \ M \mapsto H_0(\Gamma^\infty/\mathcal{P}, M) := M \otimes_\Gamma \mathcal{P}.$$

We show in Proposition 6.3 that

$$\begin{aligned} H_0(\Gamma/\mathcal{P}, M) &= M_{\mathcal{E}} \\ &= M/\mathrm{span}\{m - f_*(m) : m \in M, f \in \mathrm{Emb} \text{ such that } \mathrm{dom} f = \mathbb{N}\}. \end{aligned} \tag{1.10}$$

It follows from (1.8) and (1.9) that there is a first quadrant spectral sequence

$$E^2_{p,q} = H_p(\Gamma/\mathcal{P}, HC_q(\ell^\infty(\mathfrak{A}) : S(\mathfrak{A}))) \Rightarrow HC_{p+q}(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}).$$

In particular

$$HC_0((\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) = H_0(\Gamma/\mathcal{P} : \ell^\infty(\mathfrak{A})/[\ell^\infty(\mathfrak{A}) : S(\mathfrak{A})]).$$

Specializing to $\mathfrak{A} = \mathbb{C}$ and using (1.10) and [13, Theorem 5.12] we obtain

$$HC_0(\Gamma^\infty : I_S) = S_{\mathcal{E}} = HC_0(\mathcal{B} : J_S) \tag{1.11}$$

for every symmetric ideal $S \lhd \ell^\infty$. Another application of (1.9) is that for $\mathfrak{A}$ commutative the groups $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})})$ carry a natural Hodge decomposition. Indeed, the usual Hodge decomposition of the cyclic chain complex [17] gives an Emb-equivariant direct sum decomposition

$$HC((\ell^\infty(\mathfrak{A}) : S(\mathfrak{A}))/\mathcal{P}) = \bigoplus_{p\geq 0} HC^{(p)}((\ell^\infty(\mathfrak{A}) : S(\mathfrak{A}))/\mathcal{P}).$$

Thus for

$$HC^{(p)}(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) = \mathbb{H}(\Gamma/\mathcal{P}, HC^{(p)}((\ell^\infty(\mathfrak{A}) : S(\mathfrak{A}))/\mathcal{P}))$$

we have

$$HC_n(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) = \bigoplus_{p=0}^{n} HC_n^{(p)}(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}). \tag{1.12}$$

In Theorem 7.7 we obtain a description of $HC_n^{(p)}(\Gamma^\infty : I_S)$ in terms of differential forms which we shall presently explain. Let Ω_{ℓ^∞} be the de Rham complex of absolute –i.e. $\mathbb{Z}$-linear– algebraic differential forms. For $p \geq 0$ consider the subcomplex

$$(\mathcal{F}_p(S))^q = \begin{cases} S^{p-q+1}\Omega^q_{\ell^\infty} & p \geq q \\ \Omega^q_{\ell^\infty} & q > p. \end{cases}$$

We show in Theorem 7.7 that

$$HC_*^{(p)}(\Gamma^\infty : I_S) = \mathbb{H}_{*+p}(\Gamma/\mathcal{P}, \mathcal{F}_{(p)}(S)). \tag{1.13}$$

It follows that there is a spectral sequence (Corollary 7.8)

$${}_pE^1_{m,n} = H_n(\Gamma/\mathcal{P}, S^{m+1}\Omega^{p-m}_{\ell^\infty}) \Rightarrow HC^{(p)}_{m+n+p}(\Gamma^\infty : I_S).$$

Using this spectral sequence, we obtain (Corollary 7.9)

$$HC_n^{(n)}(\Gamma^\infty : I_S) = \left(S\Omega^n_{\ell^\infty}/d(S^2\Omega^{n-1}_{\ell^\infty})\right)_{\mathcal{E}}$$

for every symmetric ideal $S \lhd \ell^\infty$. In the particular cases (1.1) we can say more (see Proposition 7.12). We show, for example, that if $p \in \mathbb{Z}$, then

$$HC_n^{(q)}(\Gamma^\infty : I_{\ell^p}) = \begin{cases} 0 & n < q + p - 1 \\ (\ell^1\Omega^{q-p}_{\ell^\infty}/d(\ell^{p/p+1}\Omega^{q-p}_{\ell^\infty}))_{\mathcal{E}} & n = q + p - 1. \end{cases} \tag{1.14}$$

In particular, by (1.5) and (1.6) we have

$$HC_{2p-2}(\mathcal{B} : \mathcal{L}^p) = HC_{2p-2}(\Gamma^\infty : I_{\ell^p}) = HC^{(p-1)}_{2p-2}(\Gamma^\infty : I_{\ell^p}) = \ell^1_{\mathcal{E}}.$$

The rest of this paper is organized as follows. In Section 2 we recall some material from [1], including, in particular, the crossed product decomposition $I_{S(\mathfrak{A})} = S(\mathfrak{A})\#_{\mathcal{P}}\Gamma$ (Proposition 2.3). This crossed product is just the tensor product $S(\mathfrak{A}) \otimes_{\mathcal{P}} \Gamma$ with multiplication twisted by the action of Emb on $S(\mathfrak{A})$

$$(a\#f)(b\#g) = af_*(b)\#fg.$$

In particular

$$\Gamma^\infty(\mathfrak{A}) = I_{\ell^\infty(\mathfrak{A})} = \ell^\infty(\mathfrak{A})\#_{\mathcal{P}}\Gamma).$$

In Section 3 we show that every two-sided ideal of Γ^∞ is flat (Proposition 3.3). Furthermore, if S is closed under taking square roots of positive elements (e.g. if $S = c_0, \ell^{\infty-}$) then $I_{S(\mathfrak{A})}$ is a flat ideal of $\Gamma^\infty(\mathfrak{A})$ for every unital Banach algebra $\mathfrak{A}$ (Proposition 3.5). Section 4 concerns the algebra $\mathcal{P}$. We show that $\mathcal{P}$ is a filtering colimit of separable $\mathbb{Z}$-algebras (Proposition 4.1) and that if k is a field then $\mathcal{P}(k) = \mathcal{P} \otimes k$ is von Neumann regular (Corollary 4). Hence if k is a field then every $\mathcal{P}(k)$-module is flat. Further, we show that for any unital ring R, $\Gamma(R) = \Gamma \otimes R$ is flat as a module over $\mathcal{P}(R)$ (Proposition 4.2). The next section concerns excision. We call a ring A K-excisive if it satisfies excision in algebraic K-theory. It was proved by Suslin and Wodzicki [20] that a ring having a certain triple factorization property (TFP) is K-excisive. We prove in Proposition 5.1 that if $\mathfrak{A}$ is a bornological algebra and $S \lhd \ell^\infty$ is a symmetric ideal such that $S(\mathfrak{A})$ has the TFP, then $I_{S(\mathfrak{A})}$ is K-excisive. This applies, for example, when $\mathfrak{A}$ is a C^*-algebra and $S = c_0$ (Example 5.2), and also when $\mathfrak{A}$ is a unital Banach algebra and $S = \ell^{\infty-}$ (Example 5.3). Section 6 is concerned with the homology of crossed products of the form $R\#_{\mathcal{P}}\Gamma$ where R is unital. The identity (1.10) is proved in Proposition 6.3. The quasi-isomorphism (1.8) follows from the case $k = \mathbb{Q}$ of Example 6.6, which says that if k is a field, A is a unital $\mathcal{P}(k)$-algebra, and N is an $A \otimes_{\mathcal{P}(k)} A^{op}$-module, then the map of Hochschild complexes

$$HH(A/k, N) \to HH(A/\mathcal{P}(k), N)$$

is a quasi-isomorphism. In Proposition 6.7 we compute the Hochschild homology of a crossed product $R\#_{\mathcal{P}}\Gamma$ with coefficients in a bimodule of the form $M\#_{\mathcal{P}}\Gamma$. We show that there is a quasi-isomorphism

$$\mathbb{H}(\Gamma/\mathcal{P}, HH(R/\mathcal{P}(k), M)) \xrightarrow{\sim} HH(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k), M\#_{\mathcal{P}}\Gamma).$$

As an application, we obtain the isomorphism (1.11) in Corollary 6.5. Using this, the calculations of [24] compute $HC_0(\Gamma^\infty : I_S)$ for $S \in \{\ell^p, \ell^{\pm p}\}$ (Lemma 6.9). Theorem 6.11 shows that if k is a field and R is unital then there is a quasi-isomorphism

$$\mathbb{H}(\Gamma/\mathcal{P}, HC(R/\mathcal{P}(k))) \xrightarrow{\sim} HC(R\#_{\mathcal{P}}\Gamma/k).$$

The identity (1.9) follows from this (Corollary 6.6). In the particular case when R is a commutative $\mathbb{Q}$-algebra, we obtain (in Subsection 6.7) a Hodge decomposition

$$HC_n(R\#_{\mathcal{P}}\Gamma) = \bigoplus_{p=0}^{n} HC_n^{(p)}(R\#_{\mathcal{P}}\Gamma) = \bigoplus_{p=0}^{n} \mathbb{H}_n(\Gamma/\mathcal{P} : HC^{(p)}(R/\mathcal{P})).$$

The decomposition (1.12) follows from this. In Section 7 we study the groups $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})})$. The identities (1.5) and (1.6) are proved in Proposition 7.3. Theorem 7.5 proves that the comparison map (1.3) is an isomorphism when $\mathfrak{A}$ is a C^*-algebra and that (1.4) is an isomorphism when $\mathfrak{A}$ is a unital Banach algebra. The identity (1.13) is proved in Theorem 7.7. The latter is deduced from a computation of $HC_*^{(p)}(\ell^\infty/S)$ (Theorem 7.6) which, we think, is of independent interest. The identity (1.14) is included in Proposition 7.12, which considers also the case when $p \notin \mathbb{Z}$ and computes some of the groups $HC_n^{(q)}(\Gamma^\infty : I_{\ell^{\pm p}})$.

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2. Preliminaries

2.1. Symmetric sequence ideals and the algebra $\Gamma^\infty(\mathfrak{A})$. Throughout this paper we work in the setting of bornological spaces and bornological algebras; a quick introduction to the subject is given in [12, Chapter 2]. Recall that a (complete, convex) bornological vector space over the field $\mathbb{C}$ of complex numbers is a filtering union $\mathbb{V} = \cup_D \mathbb{V}_D$ of Banach spaces, indexed by the disks of $\mathbb{V}$, such that the inclusions $\mathbb{V}_D \subset \mathbb{V}_{D'}$ are bounded. A subset of $\mathbb{V}$ is *bounded* if it is a bounded subset of some $\mathbb{V}_D$. Let X be a nonempty set. A map $X \to V$ is *bounded* if its image is contained in a bounded subset. We write $\ell^\infty(X, \mathbb{V})$ for the bornological vector space of bounded maps $X \to \mathbb{V}$ where $B \subset \ell^\infty(X, \mathbb{V})$ is bounded if $\bigcup_{b\in B} b(X)$ is. The inverse monoid $\operatorname{Emb}(X)$ of partially defined embeddings $X \to X$ acts on $\ell^\infty(X, \mathbb{V})$ by means of the following action

$$(f_*(\alpha))_x = \begin{cases} \alpha_{f^\dagger(x)} & \text{if } x \in \operatorname{ran}(f) \\ 0 & \text{otherwise.} \end{cases}$$

When $X = \mathbb{N}$ or $\mathbb{V} = \mathbb{C}$, we omit it from our notation; thus $\operatorname{Emb} = \operatorname{Emb}(\mathbb{N})$, $\ell^\infty(\mathbb{V}) = \ell^\infty(\mathbb{N}, \mathbb{V})$, $\ell^\infty(X) = \ell^\infty(X, \mathbb{C})$ and $\ell^\infty = \ell^\infty(\mathbb{N}, \mathbb{C})$. A subspace $S \lhd \ell^\infty$ is called *symmetric* if it is stable under the action of Emb. If $S \subset \ell^\infty$ is a symmetric subspace and $\mathbb{V}$ is a bornological vector space, then

$$S(\mathbb{V}) := \{\alpha \in \ell^\infty(\mathbb{V}) : (\exists D)\, \alpha(\mathbb{N}) \subset \mathbb{V}_D \text{ and } ||\alpha||_D \in S\}$$

is a symmetric subspace of $\ell^\infty(\mathbb{V})$.

We will often work with sequences indexed by infinite countable sets other than $\mathbb{N}$. A bijection $u : \mathbb{N} \to X$ gives rise to a bounded isomorphism $\alpha \mapsto \alpha u$ between $\ell^\infty(X, \mathbb{V})$ and $\ell^\infty(\mathbb{V})$. If $S \subset \ell^\infty$ is a symmetric subspace, we define $S(X,\mathbb{V}) = \{su^{-1} : s \in S(\mathbb{V})\}$. Because S is symmetric by assumption, this definition does not depend on the choice of u.

Recall a bornological algebra is a bornological vector space $\mathfrak{A}$ with an associative bounded multiplication. If $\mathfrak{A}$ is a bornological algebra, then pointwise multiplication makes $\ell^\infty(\mathfrak{A})$ into a bornological algebra, and if $S \lhd \ell^\infty$ is a symmetric ideal, then $S(\mathfrak{A}) \lhd \ell^\infty(\mathfrak{A})$ is a symmetric two-sided ideal.

Let R be a ring and $A : \mathbb{N} \times \mathbb{N} \to R$ a countably infinite square matrix with entries in R. For $i, j \in \mathbb{N}$, consider the following elements of $\mathbb{Z} \cup \{\infty\}$:

$$r_i(A) = \#\{j : A_{ij} \neq 0\}, c_j(A) = \#\{i : A_{ij} \neq 0\},$$
$$N(A) := \sup\{r_i(A), c_i(A) : i \in \mathbb{N}\}.$$

Let $\mathfrak{A}$ be a bornological algebra, and $S \lhd \ell^\infty(\mathfrak{A})$ an ideal. Following [1, Definition 3.5], we set

$$I_{S(\mathfrak{A})} = \{A = (A_{ij})_{i,j\in\mathbb{N}} : \{A_{ij}\} \in S(\mathbb{N}\times\mathbb{N}) \text{ and } N(A) < \infty\} \tag{2.1.1}$$
$$\text{and } \Gamma^\infty(\mathfrak{A}) = I_{\ell^\infty(\mathfrak{A})}.$$

2.2. Crossed products with Γ. Let R be a ring. *Karoubi's cone* of the ring R is the ring

$$\Gamma(R) = \{A \in M_{\mathbb{N}}(R) : N(A) < \infty \text{ and } \#\{A_{i,j} : (i,j) \in \mathbb{N}\times\mathbb{N}\} < \infty\}.$$

We also consider the ring of all locally constant sequences

$$\mathcal{P}(R) = \{\alpha \in R^{\mathbb{N}} : \#\{\alpha_n : n \in \mathbb{N}\} < \infty\}.$$

Observe that $\alpha \in \mathcal{P}(R)$ if and only if the diagonal matrix $\operatorname{diag}(\alpha) \in \Gamma(R)$. We shall identify $\mathcal{P}(R)$ with $\operatorname{diag}(\mathcal{P}(R)) \subset \Gamma(R)$. When $R = \mathbb{Z}$ we omit it from our notation; we set

$$\Gamma = \Gamma(\mathbb{Z}),\ \ \mathcal{P} = \mathcal{P}(\mathbb{Z}).$$

By [8, Lemma 4.7.1] the map

$$\phi : \Gamma \otimes R \to \Gamma(R),\ \ \phi(A \otimes x)_{i,j} = A_{i,j}x \tag{2.2.1}$$

is an isomorphism. It follows from this that Γ and $\mathcal{P}$ are flat $\mathbb{Z}$-modules. By [1, Remark 6.8] the restriction of ϕ induces an isomorphism

$$\mathcal{P} \otimes R \xrightarrow{\cong} \mathcal{P}(R). \tag{2.2.2}$$

There is a monoid homomorphism

$$U : \mathrm{Emb} \to \Gamma,\ \ (U_f)_{i,j} = \begin{cases} 1 & \text{if } j \in \operatorname{dom}(f) \text{ and } f(j) = i \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.3}$$

Observe that the idempotent submonoid of Emb is isomorphic to the monoid $2^{\mathbb{N}}$ of subsets of $\mathbb{N}$ with intersection of subsets as multiplication. If $p^2 = p$ and $A = \operatorname{Im} p$, then $U_p = \operatorname{diag}(\chi_A)$ is a diagonal matrix. We will often identify p, U_p and χ_A. We also consider the monoid rings $\mathbb{Z}[2^{\mathbb{N}}]$ and $\mathbb{Z}[\mathrm{Emb}]$, and the two-sided ideals

$$I = \langle\{\chi_{A\sqcup B} - \chi_A - \chi_B : A, B \subset \mathbb{N},\ \ A \cap B = \emptyset\}\rangle \lhd \mathbb{Z}[2^{\mathbb{N}}], \tag{2.2.4}$$

$$J = \langle\{\chi_{A\sqcup B} - \chi_A - \chi_B : A, B \subset \mathbb{N},\ \ A \cap B = \emptyset\}\rangle \lhd \mathbb{Z}[\mathrm{Emb}]. \tag{2.2.5}$$

The following lemma follows from [1, Lemma 5.4 and Remark 6.8].

Lemma 2.1. *Let R be a ring. The maps* (2.2.3), (2.2.1) *and* (2.2.2) *induce the following isomorphisms:*
i) $\mathcal{P}(R) = R[2^{\mathbb{N}}]/R \otimes I$.
ii)$\Gamma(R) = R[\mathrm{Emb}]/R \otimes J$.

Remark 2.2. Given a monoid M and a unital ring R, a representation of M in R-modules is the same thing as a module over the monoid algebra $R[M]$. In view of Lemma 2.1, the modules over $\mathcal{P}(R)$ and $\Gamma(R)$ correspond to those representations of the inverse monoids $2^{\mathbb{N}}$ and Emb which are tight in the sense of Exel (see [14, Def. 13.1 and Prop. 11.9]).

Because Emb is a monoid, if $\mathcal{A}$ is a ring on which Emb acts by algebra endomorphisms we can form the *crossed product* $\mathcal{A}\#\mathrm{Emb}$. As an abelian group, $\mathcal{A}\#\mathrm{Emb} = \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathrm{Emb}]$ with multiplication given by

$$(a\#f)(b\#g) = af_*(b)\#fg. \tag{2.2.6}$$

Here $\# = \otimes$ and $f_*(b)$ denotes the action of f on Emb. Now assume that the Emb-ring $\mathcal{A}$ is also a $\mathcal{P}$-algebra, that is, it is a ring and a $\mathcal{P}$-bimodule, and these operations are compatible in the sense that

$$(ap)b = a(pb) \ \ (a, b \in \mathcal{A},\ \ p \in \mathcal{P}).$$

Further assume that $\mathcal{A}$ is central as a $\mathcal{P}$-bimodule, i.e. $pa = ap$ ($a \in \mathcal{A}$, $p \in \mathcal{P}$), and that

$$pa = p_*(a) \qquad (p \in 2^{\mathbb{N}}).$$

Under all these conditions, we say that $\mathcal{A}$ is an Emb-*bundle* (cf. [2, Def. 2.1]). For $J \lhd \mathbb{Z}[\mathrm{Emb}]$ as in (2.2.5), we have

$$\mathcal{A}\#\mathrm{Emb} \rhd \mathcal{A}\#J = \operatorname{span}\{r\#j : r \in \mathcal{A}, j \in J\} \text{ and}$$
$$\mathcal{A}\#\mathrm{Emb} \rhd L = \operatorname{span}\{rp\#h - r\#ph : r \in \mathcal{A}, p \in \mathcal{P}, h \in \mathrm{Emb}\}.$$

Set

$$\mathcal{A}\#_{\mathcal{P}}\Gamma = \mathcal{A}\#\mathrm{Emb}/(L + \mathcal{A}\#J). \tag{2.2.7}$$

Thus, $\mathcal{A}\#_{\mathcal{P}}\Gamma = \mathcal{A} \otimes_{\mathcal{P}} \Gamma$ as left $\mathcal{P}$-modules, and the product is that induced by (2.2.6); we have

$$(a\#U_f)(b\#U_g) = af_*(b)\#U_{fg} \in \mathcal{A}\#_{\mathcal{P}}\Gamma. \tag{2.2.8}$$

Proposition 2.3. ([1, Proposition 6.11]) *Let $\mathfrak{A}$ be a bornological algebra. The map*

$$\ell^{\infty}(\mathfrak{A})\#_{\mathcal{P}}\Gamma \to \Gamma^{\infty}(\mathfrak{A}), \quad \alpha\#U_f \mapsto \operatorname{diag}(\alpha)U_f \tag{2.2.9}$$

is an isomorphism of $\mathcal{P}$-algebras. If $S \lhd \ell^{\infty}$ is a symmetric ideal, then (2.2.9) *sends $S(\mathfrak{A})\#_{\mathcal{P}}\Gamma$ isomorphically onto $I_{S(\mathfrak{A})} \lhd \Gamma^{\infty}(\mathfrak{A})$.*

3. Flat ideals of Γ^{∞} and ℓ^{∞}

Proposition 3.1. *Every finitely generated ideal of ℓ^{∞} is principal and projective.*

Proof. The fact that the finitely generated ideals of ℓ^{∞} are projective follows from [18, Corollary 2.4]. We will prove that they are principal. Given $\alpha \in \ell^{\infty}$, set

$$\nu_{\alpha}(n) = \begin{cases} 0, & \text{if } \alpha(n) = 0 \\ \frac{\alpha(n)}{|\alpha(n)|}, & \text{otherwise.} \end{cases} \tag{3.1}$$

Notice that ν_{α} is the partial isometry in the polar decomposition of α. In fact, we have

$$\alpha = \nu_{\alpha}|\alpha|, \quad |\alpha| = \overline{\nu}_{\alpha}\alpha.$$

It follows that, for any ideal I in ℓ^{∞}, $\alpha \in I$ if and only if $|\alpha| \in I$. Now let I be an ideal of ℓ^{∞} generated by $\{\alpha_0, \alpha_1\}$, and set

$$\mu(n) = \max\{|\alpha_0(n)|, |\alpha_1(n)|\}.$$

For $i = 0, 1$, let

$$\gamma_i(n) = \begin{cases} 1/2 & \text{if } |\alpha_0(n)| = |\alpha_1(n)| \\ 1 & \text{if } |\alpha_i(n)| > |\alpha_{1-i}(n)| \\ 0 & \text{otherwise.} \end{cases}$$

We have $\mu = \gamma_0|\alpha_0| + \gamma_1|\alpha_1|$; thus $\mu \in I$. Now set

$$\tau_i(n) = \begin{cases} 0 & \text{if } \mu(n) = 0 \\ \frac{\alpha_i(n)}{\mu(n)} & \text{otherwise.} \end{cases}$$

Then $\alpha_i = \tau_i\mu$, $(i = 0, 1)$. Notice that $\tau_i \in \ell^{\infty}$, since $|\tau_i(n)| \leq 1$ for all $n \in \mathbb{N}$, $i = 0, 1$. Therefore, μ generates I. The general case can now be proven by induction on the number of generators. □

Corollary. *Every ideal of ℓ^{∞} is flat.*

Proposition 3.2. *Let $\mathfrak{A}$ be a unital Banach algebra and $S \lhd \ell^{\infty}$ a symmetric ideal. Assume that*

$$\alpha \in S \Rightarrow \sqrt{|\alpha|} \in S.$$

Then $S(\mathfrak{A}) \lhd \ell^{\infty}(\mathfrak{A})$ is flat both as a right and as a left $\ell^{\infty}(\mathfrak{A})$-module.

Proof. Consider the following homomorphism of $\ell^\infty(\mathfrak{A})$-modules

$$\mu : \ell^\infty(\mathfrak{A}) \otimes_{\ell^\infty} S \to S(\mathfrak{A}), \;\; \mu(\alpha \otimes \beta)_n = \alpha_n \beta_n.$$

We claim that μ is an isomorphism. To prove it is surjective, for $\alpha \in S(\mathfrak{A})$ let ν_α be as in (3.1). Then $\nu_\alpha \in \ell^\infty(\mathfrak{A})$ and

$$\alpha = \mu(\nu_\alpha \otimes ||\alpha||).$$

Thus μ is surjective. To prove it is also injective, let

$$\eta = \sum_{i=1}^{n} \alpha^i \otimes \beta^i \in \ker \mu.$$

By Proposition 3.1, the ideal $\langle \beta^1, \dots, \beta^n \rangle \lhd \ell^\infty$ is principal. Let β be a generator; we may and do choose it so that $\beta = |\beta|$. By bilinearity, we may rewrite η as a single elementary tensor and we have

$$\eta = \alpha \otimes \beta, \;\; \alpha\beta = 0.$$

But $\alpha\beta = 0$ implies $\alpha\sqrt{\beta} = 0$, whence

$$\eta = \alpha\sqrt{\beta} \otimes \sqrt{\beta} = 0.$$

Thus the claim is proved. It follows that $S(\mathfrak{A})$ is flat as a left $\ell^\infty(\mathfrak{A})$-module, since it is the scalar extension of S, which is a flat ℓ^∞-module by Corollary 3. The proof that $S(\mathfrak{A})$ is flat on the right is similar. □

Examples 3.2. The hypothesis of Proposition 3.2 are satisfied, for example, when S is either of $\ell^{\infty-}$, c_0.

Proposition 3.3. *Every two-sided ideal of Γ^∞ is flat both as a left and as a right Γ^∞-module.*

Proof. Let $I \lhd \Gamma^\infty$. By [1, Theorem 4.5] there is a symmetric ideal S such that $I = I_S$. Observe that

$$I_S = S \otimes_{\mathcal{P}} \Gamma = S \otimes_{\ell^\infty} \ell^\infty \otimes_{\mathcal{P}} \Gamma = S \otimes_{\ell^\infty} \Gamma^\infty.$$

Thus $I_S \otimes_{\Gamma^\infty} = S \otimes_{\ell^\infty}$ is exact by Corollary 3. Hence I is flat as a right module and therefore also as a left module, since Γ^∞ is a $*$-algebra. □

Remark 3.4. By [1, Proposition 4.6], if k is a field, then $M_\infty k$ is the only proper two-sided ideal of $\Gamma(k)$. Observe that $M_\infty k$ is projective both as a left and as a right module, since it is isomorphic to an infinite sum of copies of the principal ideal generated by the idempotent $E_{1,1}$.

Proposition 3.5. *Let $\mathfrak{A}$ be a unital Banach algebra and $S \lhd \ell^\infty$ a symmetric ideal as in Proposition 3.2. Then $I_{S(\mathfrak{A})}$ is flat both as a left and as a right $\Gamma^\infty(\mathfrak{A})$-module.*

Proof. By Proposition 2.3 and the proof of Proposition 3.2 we have the following canonical isomorphisms of right $\Gamma^\infty(\mathfrak{A})$-modules

$$I_{S(\mathfrak{A})} = S(\mathfrak{A}) \otimes_{\mathcal{P}} \Gamma = S \otimes_{\ell^\infty} \ell^\infty(\mathfrak{A}) \otimes_{\mathcal{P}} \Gamma = S \otimes_{\ell^\infty} \Gamma^\infty(\mathfrak{A}).$$

This, together with Corollary 3, proves that $I_{S(\mathfrak{A})}$ is flat as a right $\Gamma^\infty(\mathfrak{A})$-module. The proof that it is also flat on the left is similar. □

4. Flatness properties of $\mathcal{P}$

Let k be a commutative ring. Recall that a k-algebra A which is projective as an $A \otimes_k A^{op}$-module is called *separable*.

Proposition 4.1. *The k-algebra $\mathcal{P}(k)$ is a filtering union of separable algebras.*

Proof. We shall show that $\mathcal{P}$ is a filtering union of finite products of copies of $\mathbb{Z}$, indexed by the finite partitions of $\mathbb{N}$. Here a finite partition of $\mathbb{N}$ is a finite set $\pi = \{A_1, \dots, A_n\}$ of subsets of $\mathbb{N}$ such that $\mathbb{N} = A_1 \sqcup \cdots \sqcup A_n$. We say that a partition $\rho = \{B_1, \dots, B_m\}$ is *finer* than π if the following condition is satisfied:

$$(\forall 1 \le i \le m)(\exists j) \quad B_i \subset A_j.$$

Note that if π and π' are any two finite partitions, then

$$\pi \wedge \pi' = \{B \subset \mathbb{N} : (\exists A \in \pi, A' \in \pi')B = A \cap A'\}.$$

is a finite partition and is finer than each of them. Thus the set

$$\mathrm{Part}(\mathbb{N}) = \{\pi \text{ finite partition of } \mathbb{N}\}.$$

is a filtered partially ordered set. If $\pi \in \mathrm{Part}(\mathbb{N})$ has n elements, put

$$\mathcal{P} \supset R_\pi = \bigoplus_{i=1}^{n} \mathbb{Z} P_{A_i}.$$

Observe that $R_\pi \cong \mathbb{Z}^n$ and that $\mathcal{P} = \bigcup_\pi R_\pi$. This proves the proposition in the case $k = \mathbb{Z}$. The general case follows from this using the isomorphism $\mathcal{P} \otimes k \xrightarrow{\cong} \mathcal{P}(k)$. □

Corollary. *If k is a field, then $\mathcal{P}(k)$ is a von Neumann regular ring. In other words, every $\mathcal{P}(k)$-module is flat.*

Proposition 4.2. *Let R be a unital ring. Then $\Gamma(R)$ is flat, both as a left and as a right $\mathcal{P}(R)$-module.*

Proof. We prove that $\Gamma(R)$ is flat as a right $\mathcal{P}(R)$-module; the proof that it is also flat on the left is similar. If M is a $\mathcal{P}(R)$-module, then

$$\Gamma(R) \otimes_{\mathcal{P}(R)} M = \Gamma \otimes R \otimes_{\mathcal{P}\otimes R} M = \Gamma \otimes_{\mathcal{P}} M.$$

Hence it suffices to consider the case $R = \mathbb{Z}$. In view of Proposition 4.1 and its proof, we have

$$\Gamma \otimes_{\mathcal{P}} M = \operatorname*{colim}_{\pi \in \mathrm{Part}(\mathbb{N})} \Gamma \otimes_{R_\pi} M.$$

Hence it suffices to show that Γ is flat as a module over R_π, for each $\pi \in \mathrm{Part}(\mathbb{N})$. We have

$$R_\pi = \bigoplus_{A\in\pi} \mathbb{Z} P_A.$$

Hence

$$\Gamma \otimes_{R_\pi} M = \bigoplus_{A\in\pi} \Gamma p_A \otimes p_A M.$$

Thus it suffices to show that Γp_A is flat as an abelian group. Since Γp_A is a direct summand of Γ, we are reduced to showing that Γ is $\mathbb{Z}$-flat. As said above, the map (2.2.1) is an isomorphism for every ring; in particular this applies to show that if M is any abelian group—regarded as a ring with trivial multiplication—then $\Gamma \otimes M = \Gamma(M)$. Since $M \to \Gamma(M)$ is clearly exact, this conlcudes the proof. □

5. Excision

A ring A is called *K-excisive* if for every ideal embedding $A \lhd B$ the map $K_*(A) \to K_*(B : A)$ is an isomorphism. It was proved by Suslin and Wodzicki [20, Theorem C] that if a ring A satisfies the following property then it is K-excisive.

$$\forall n, \forall a \in A^{\oplus n}, \exists b \in A^{\oplus n},\ \ c, d \in A,\ \text{such that } a = cdb \text{ and such that}$$
$$(0 :_A d)_r := \{v \in A : dv = 0\} = (0 :_A cd)_r.$$

The right ideal $(0 :_A d)_r$ is called the *right annihilator* of d in A. The property above is the so-called left *triple factorization property* (TFP). A ring is K-excisive if and only if its opposite ring A^{op} is ([20, Remark (1) pp 53]), so rings satisfying the right TFP are excisive also. Further results of Wodzicki ([23, Theorems 1.1 and 3.1]) and of Suslin–Wodzicki ([20, Theorem B]) establish that a $\mathbb{Q}$-algebra A is

K-excisive if and only if it is excisive for cyclic homology, and that this happens if and only if the *bar complex* $(C_*^{bar}(A), b')$ is exact. Here

$$b' : C_{n+1}^{bar}(A) = A^{\otimes n+2} \to A^{\otimes n+1} = C_n^{bar}(A) \qquad (n \geq 0)$$
$$b'(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

The tensor products above are taken over $\mathbb{Z}$ or, equivalently, over $\mathbb{Q}$, since A is assumed to be a $\mathbb{Q}$-algebra. The $\mathbb{Q}$-algebras whose bar homology vanishes—that is, the K-excisive ones—are also called *H-unital*.

Proposition 5.1. *Let $\mathfrak{A}$ be a bornological algebra and $S \lhd \ell^\infty$ a symmetric ideal. Assume that $S(\mathfrak{A})$ has the (left or right) triple factorization property. Then $I_{S(\mathfrak{A})}$ is K-excisive.*

Proof. Assume that $S(\mathfrak{A})$ has the left TFP. We have to prove that $I_{S(\mathfrak{A})}$ is H-unital. Let $n \geq 0$ and let $z \in C_n^{bar}(I_{S(\mathfrak{A})})$ be a cycle. We may write

$$z = \sum_{i=1}^{m} \mathrm{diag}(\alpha^{0,i}) U_{f_{0,i}} \otimes \cdots \otimes \mathrm{diag}(\alpha^{n,i}) U_{f_{n,i}},$$

where $\mathrm{supp}(\alpha^{j,i}) = \mathrm{ran}(f_{j,i})$ for all i, j. By TFP, there are elements γ, δ and $\beta^1, \ldots, \beta^m$ in $S(\mathfrak{A})$ such that $\alpha^{0,i} = \gamma\delta\beta^i$ $(1 \leq i \leq m)$, and such that

$$(0 :_{S(\mathfrak{A})} \gamma\delta)_r = (0 :_{S(\mathfrak{A})} \delta)_r. \tag{5.1}$$

Now observe that if $\theta \in S(\mathfrak{A})$ then, by our definition of $I_{S(\mathfrak{A})}$ (2.1.1), we have

$$(0 :_{I_{S(\mathfrak{A})}} \mathrm{diag}(\theta))_r = \{T \in I_{S(\mathfrak{A})} : (\forall j)\ T_{*,j} \in (0 :_{S(\mathfrak{A})} \theta)_r\}.$$

Hence, (5.1) implies that

$$(0 :_{I_{S(\mathfrak{A})}} \mathrm{diag}(\gamma\delta))_r = (0 :_{I_{S(\mathfrak{A})}} \mathrm{diag}(\delta))_r. \tag{5.2}$$

Put

$$y = \sum_i \mathrm{diag}(\beta^i) U_{f_{0,i}} \otimes \mathrm{diag}(\alpha^{1,i}) U_{f_{1,i}} \otimes \cdots \otimes \mathrm{diag}(\alpha^{n,i}) U_{f_{n,i}}.$$

Consider the following element of $C_{n+1}^{bar}(I_{S(\mathfrak{A})})$

$$w = \mathrm{diag}(\gamma) \otimes \mathrm{diag}(\delta) y.$$

We have

$$b'(w) = z - \mathrm{diag}(\gamma) \otimes \mathrm{diag}(\delta) b'(y).$$

If $n = 0$ then $b'(y) = 0$, so this proves that z is a boundary. We have to show that $\mathrm{diag}(\delta)b'(y) = 0$ if $n \geq 1$. Choose a basis $\{v_l\}$ of the $\mathbb{Q}$-vector space $C^{bar}_{n-1}(I_{S(\mathfrak{A})})$. Then $y = \sum_l T_l \otimes v_l$ for unique $T_l \in I_{S(\mathfrak{A})}$, and

$$0 = b'(z) = \mathrm{diag}(\gamma\delta)b'(y) = \sum_l \mathrm{diag}(\gamma\delta)T_l \otimes v_l.$$

Hence we must have $\mathrm{diag}(\gamma\delta)T_l = 0$ for all l, and therefore $\mathrm{diag}(\delta)b'(y) = 0$ by (5.2). □

Example 5.2. Any Banach algebra with a bounded left approximate unit satisfies the Cohen-Hewitt factorization property; thus it has the left TFP ([6, Lemma 6.5.1]). In particular, this applies to C^*-algebras. If $\mathfrak{A}$ is a C^*-algebra then $c_0(\mathfrak{A})$ is again a C^*-algebra; hence $I_{c_0(\mathfrak{A})}$ is K-excisive, by Proposition 5.1.

Example 5.3. If $\mathfrak{A}$ is a unital Banach algebra then $\ell^{\infty-}(\mathfrak{A})$ has the TFP. To see this, let $\alpha^1, \dots, \alpha^m \in \ell^{\infty-}$. Choose p such that $\alpha^i \in \ell^p(\mathfrak{A})$ for all i. For each n put

$$\gamma_n = \max_{1\leq i\leq m} ||\alpha^i_n||, \quad \beta^i_n = \begin{cases} \alpha^i_n/\gamma_n^{1/2} & \text{if } \gamma_n \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $||\beta^i_n|| \leq ||\alpha^i_n||^{1/2}$ and therefore $\beta^i \in \ell^{2p}(\mathfrak{A})$. Similarly $\gamma^{1/4} \in \ell^{4p}(\mathfrak{A})$. One checks that the factorization $\alpha^i = \gamma^{1/4}\gamma^{1/4}\beta^i$ satisfies the requirements of the TFP.

6. Homology of crossed products with Γ

6.1. Homology of augmented algebras. In this subsection A and B will be unital rings; furthermore, B will be an A-algebra, that is, B will be a ring together with a unital ring homomorphism $\iota : A \to B$. Further assume that A is equipped with a left B-module structure and a surjective B-module homomorphism $\pi : B \to A$ such that $\pi\iota = id_A$. Observe that the triple (B, A, π) is an augmented ring in the sense of Cartan-Eilenberg [4, Chapter VIII,§1]. Since in addition, B is an A-algebra, we call the triple (B, A, π) an *augmented algebra*. Let M be a right B-module. Consider the simplicial A-module $\perp (B/A, M)$ given in dimension n by

$$\perp_n (B/A, M) = M \otimes_A B^{\otimes_A n},$$

with face and degeneracy maps defined as follows ($n \geq 0$)

$$\partial_i : \perp_{n+1} (B/A, M) \to \perp_n (B/A, M),$$

$$\partial_i(x_0 \otimes \dots \otimes x_{n+1}) = \begin{cases} x_0 \otimes \dots \otimes x_i x_{i+1} \otimes \dots \otimes x_{n+1} & i \leq n \\ x_0 \otimes \dots \otimes x_n \pi(x_{n+1}) & i = n+1 \end{cases}$$

$$\delta_i : \perp_n (B/A, M) \to \perp_{n+1} (B/A, M), \quad (0 \leq i \leq n)$$

$$\delta_i(x_0 \otimes \dots \otimes x_n) = x_0 \otimes \dots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \dots \otimes x_n.$$

The homology of $(B/A, M)$ relative to (A, B, π), denoted $H_*(B/A, M)$, is the homotopy of the simplicial module $\perp (B/A, M)$;

$$H_*(B/A, M) = \pi_*(\perp (B/A, M)) = H_*(\perp (B/A, M), \partial).$$

Here

$$\partial = \sum_{i=0}^{n+1} (-1)^i \partial_i : \perp_{n+1} (B/A, M) \to \perp_n (B/A, M)$$

is the alternating sum of the face maps. We have

$$H_0(B/A, M) = M \otimes_B A.$$

Let $P(B/A) = \perp (B/A, B)$; $\pi : P(B/A) \to A$ is a resolution which is projective relative to B/A, and $\perp (B/A, N) = N \otimes_B P(B/A)$. Hence if B is flat both as a left and as a right A-module, then

$$H_*(B/A, M) = \operatorname{Tor}_*^B(M, A).$$

Without flatness assumptions, we may regard the groups $H_*(B/A, M)$ as relative Tor groups.

Lemma 6.1. *Let N be a right B-module. Consider $N^2 = N^{1\times 2}$ as a right module over $M_2 B$ via the matrix product. View $M_2 B$ as an $A \oplus A$-algebra through the diagonal embedding $(a_1, a_2) \mapsto E_{11} a_1 + E_{22} a_2$. Then the map*

$$\iota : \perp (B/A, N) \to \perp (M_2(B)/A \oplus A, N \oplus N)$$
$$\iota(x_0 \otimes \cdots \otimes x_n) = E_{11} x_0 \otimes \cdots \otimes E_{11} x_n$$

is a quasi-isomorphism.

Proof. Consider the maps

$$\iota' : P(B/A)^{2\times 1} \to P(M_2 B/A^2),$$
$$\iota'(E_{i1}(x_0 \otimes \cdots \otimes x_n)) = E_{i1} x_0 \otimes E_{11} x_1 \otimes \cdots \otimes E_{11} x_n,$$
$$\text{and } p' : P(M_2 B/A^2) \to P(B/A)^{2\times 1},$$
$$p'(E_{i_0,i_1} x_0 \otimes \cdots \otimes E_{i_n,i_{n+1}} x_n) = E_{i_0 1}(x_0 \otimes \cdots \otimes x_n).$$

One checks that both ι' and p' are $M_2 B$-linear chain homomorphisms, and that $p'\iota' = 1$. In particular $\pi^{2\times 1} : P(B/A)^{2\times 1} \to A^{2\times 1}$ is a projective resolution relative to $M_2 A/A^2$, whence

$$\iota = N^{1\times 2} \otimes_{M_2 B} \iota'$$

is a quasi-isomorphism, as claimed. □

6.2. The augmented algebra $(\Gamma, \mathcal{P}, \epsilon_l)$**.** Regarding the elements of $2^{\mathbb{N}}$ as sequences of zeros and ones, there is an obvious action $\mathrm{Emb}\times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$, $(f, p) \mapsto f_*(p)$. It agrees with the inner action; we have

$$f_*(p) = fpf^{\dagger}.$$

Thus $\mathbb{Z}[2^{\mathbb{N}}]$ is a $\mathbb{Z}[\mathrm{Emb}]$-module. Note that, if $A, B \subset \mathbb{N}$ are disjoint, then for $I \subset \mathbb{Z}[2^{\mathbb{N}}]$ as in (2.2.4) and $q \in 2^{\mathbb{N}}$, we have

$$\begin{aligned} f_*((p_{A\sqcup B} - p_A - p_B)q) \\ &= \left(p_{f((A\sqcup B)\cap \mathrm{dom}(f))} - p_{f(A\cap \mathrm{dom}(f))} - p_{f(B\cap \mathrm{dom}(f))}\right) f_*(q) \in I, \end{aligned}$$

$$\begin{aligned} (f(p_{A\sqcup B} - p_A - p_B)g)_*(q) &= f_*((p_{A\sqcup B} - p_A - p_B)_*(g_*(q))) \\ &= f_*((p_{A\sqcup B} - p_A - p_B)\cdot g_*(q)) \in I. \end{aligned}$$

Thus $\mathcal{P}$ is a Γ-module. Let $f \in \mathrm{Emb}$; put

$$\epsilon_l(f) = p_{\mathrm{ran}(f)} \in 2^{\mathbb{N}} \ni \epsilon_r(f) = \epsilon_l(f^{\dagger}) = p_{\mathrm{dom}(f)}.$$

Note that

$$\epsilon_l(fg)(n) = p_{\mathrm{ran}(fg)}(n) = \left\{\begin{matrix} 1 & \text{if } n \in f(\mathrm{dom}(f) \cap \mathrm{ran}(g)) \\ 0 & \text{otherwise} \end{matrix}\right\} = f_*(\epsilon_l(g))(n).$$

Thus the induced linear map $\epsilon_l : \mathbb{Z}[\mathrm{Emb}] \to \mathbb{Z}[2^{\mathbb{N}}]$ is a homomorphism of left $\mathbb{Z}[\mathrm{Emb}]$-modules. In particular, if $A, B \subset \mathbb{N}$ are disjoint, we have

$$\epsilon_l(f(p_{A\sqcup B} - p_A - p_B)g) = f_*(p_{A\sqcup B} - p_A - p_B)\epsilon_l(g) \in I.$$

Hence ϵ_l induces a homomorphism of left Γ-modules

$$\epsilon_l : \Gamma \to \mathcal{P}.$$

Observe that the canonical inclusion $\mathcal{P} \subset \Gamma$, which is an algebra homomorphism, but not a Γ-module homomorphism, is a section of ϵ_l. Thus we are in the augmented algebra setting described above. Moreover Γ is flat over $\mathcal{P}$, by Proposition 4.2. Hence

$$H_*(\Gamma/\mathcal{P}, M) = \mathrm{Tor}_*^{\Gamma}(M, \mathcal{P}). \tag{6.2.1}$$

Observe also that if k is any commutative ring and M is a $\Gamma(k)$-module, then

$$C(\Gamma/\mathcal{P}, M) = C(\Gamma(k)/\mathcal{P}(k), M).$$

In particular,

$$H_*(\Gamma/\mathcal{P}, M) = H_*(\Gamma(k)/\mathcal{P}(k), M).$$

In the next lemma and below we consider the following submonoids of Emb

$$\mathrm{Emb} \supset \mathcal{E} = \{f : \mathrm{dom} f = \mathbb{N}\} \supset \mathcal{E}^* = \{f \in \mathcal{E} : \mathrm{ran}(f) = \mathbb{N}\}.$$

If M is a Γ-module and $\mathfrak{S} \in \{\mathcal{E}, \mathcal{E}^*\}$ we write

$$M_{\mathfrak{S}} = M/\mathrm{span}\{m - f_*(m) : f \in \mathfrak{S}\}.$$

Here the span is $\mathbb{Z}$-linear.

Lemma 6.2. *The kernel of $\epsilon_l : \Gamma \to \mathcal{P}$ is generated, as a left $\mathcal{P}$-module, by the elements $U_f - 1$, $f \in \mathcal{E}^*$.*

Proof. Let $K = \ker(\epsilon_l)$. It is clear that K is generated, as an abelian group, by the elements $U_f - p_{\mathrm{ran} f}$, $f \in \mathrm{Emb}$. Assume that $f \in \mathrm{Emb}$ but $f \notin \mathcal{E}^*$. We claim that we may choose a subset $A \subset \mathrm{dom}(f)$ such that $B = \mathbb{N}\backslash A$ is bijectable to $\mathbb{N}\backslash f(A)$, and such that $\mathbb{N}\backslash(\mathrm{dom} f \cap B)$ is bijectable to $\mathbb{N}\backslash f(\mathrm{dom} f \cap B)$. Indeed if $\mathbb{N}\backslash\mathrm{dom} f$ is already bijectable to $\mathbb{N}\backslash\mathrm{ran} f$, we may take $A = \mathrm{dom} f$. Otherwise $\mathrm{dom} f$ is infinite, so we may split it into two disjoint infinite pieces, and take A to be one of them. Thus the claim is proved. For such A, there exist $g, h \in \mathcal{E}^*$ such that $g_{|A} = f_{|A}$ and $h_{|\mathrm{dom}(f)\cap B} = f_{|\mathrm{dom}(f)\cap B}$. We have

$$p_{\mathrm{ran} f} = p_{f(A)} + p_{f(\mathrm{dom} f \cap B)} \text{ and}$$

$$U_f = p_{f(A)} U_{f_{|A}} + p_{f(\mathrm{dom}(f)\cap B)} U_{f_{\mathrm{dom}(f)\cap B}} = p_{f(A)} U_g + p_{f(\mathrm{dom}(f)\cap B)} U_h.$$

Thus

$$U_f - p_{\mathrm{ran} f} = p_{f(A)}(U_g - 1) + p_{f(\mathrm{dom} f \cap B)}(U_h - 1).$$

□

Proposition 6.3. *Let M be a Γ-module. Then*

$$H_0(\Gamma/\mathcal{P}, M) = M_{\mathcal{E}} = M_{\mathcal{E}^*}.$$

Proof. Immediate from Lemma 6.2. □

6.3. Hochschild homology. We recall the basic definitions for Hochschild homology of algebras over a noncommutative base ring ([17, §1.2.11]). If N is a $B \otimes B^{op}$-module, we write

$$\begin{aligned}
[b, x] &= bx - xb \qquad (b \in B, x \in N),\\
[B, N] &= \{\sum_{i=1}^{n} [b_i, x_i] : b_i \in B, x_i \in N, n \geq 1\},\\
N_B &= N/[B, N].
\end{aligned}$$

Next let $A \to B$ be a unital ring homomorphism. Recall from [17, §1.2.11] that the *Hochschild* homology of B relative to A with coefficients in N, $HH_*(B/A, N) = \pi_* C(B/A, M)$, is the homotopy of the simplicial $\mathbb{Z}$-module which is given in dimension n by

$$C_n(B/A, N) = (N \otimes_A B^{\otimes_A n})_A,$$

with the following face and degeneracy maps

$$\mu_i : C_{n+1}(B/A, N) \to C_n(B/A, N),$$

$$\mu_i(x_0 \otimes \cdots \otimes x_{n+1}) = \begin{cases} x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1} & i \leq n \\ x_{n+1}x_0 \otimes \cdots \otimes x_n & i = n+1 \end{cases}$$

$$s_i : C_n(B/A, N) \to C_{n+1}(B/A, N), \quad (0 \leq i \leq n)$$

$$s_i(x_0 \otimes \cdots \otimes x_n) = x_0 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_n.$$

We write b for the alternating sum of the face maps, and $HH(B/A, N)$ for the resulting chain complex. Thus

$$HH_*(B/A, N) = H_*(HH(B/A, N))$$

is the Hochschild homology of B/A with coefficients N. If A is commutative and B is central as an A-bimodule, then $B \otimes_A B^{op}$ is a ring. If furthermore, B happens to be flat as a left A-module, then

$$HH_*(B/A, N) = \operatorname{Tor}_*^{B \otimes_A B^{op}}(B, N).$$

Note this is the case, for example, if A is a field. We shall write $HH_*(B, N)$ for $HH_*(B/\mathbb{Z}, N)$.

Remark 6.4. If A and B are commutative and M is a central bimodule, then $C(B/A, M) = M \otimes_B C(B/A, B)$.

Lemma 6.5. *(cf.* [17, Theorem 1.12.13]*) Let k be a field, $A \to B$ a homomorphism of unital k-algebras, and N a $B \otimes_k B^{op}$-module. Assume that A is a filtering colimit of separable k-algebras. Then*

$$HH_*(B/k, N) = HH_*(B/A, N).$$

Proof. It suffices to show that $B \otimes_A B^{op}$ is flat as a $B \otimes_k B^{op}$-module. By hypothesis $A = \operatorname{colim}_i A_i$ is a filtering colimit of separable algebras. Hence $B \otimes_A B^{op} = \operatorname{colim}_i B \otimes_{A_i} B^{op}$, so it suffices to prove that if $k \subset A$ is separable then $B \otimes_A B$ is flat over $B \otimes_k B^{op}$, and this is well known. □

Example 6.6. If k is a field, A is a unital $\mathcal{P}(k)$-algebra, and N is an $A \otimes_k A^{op}$-module, then $HH_*(A/k, N) = HH_*(A/\mathcal{P}(k), N)$, by Proposition 4.1 and Lemma 6.5. If $A \supset \mathbb{Q}$, then $HH_*(A, N) = HH_*(A/\mathbb{Q}, N)$ and $HH_*(A/\mathcal{P}, N) = HH_*(A/\mathcal{P}(\mathbb{Q}), N)$, whence we also have $HH_*(A, N) = HH_*(A/\mathcal{P}, N)$.

6.4. Hochschild homology of crossed products with Γ. In this subsection k is a field and, as in (2.2.7), R is an Emb-*bundle over* k; that is, R is a k-algebra with a k-linear action of Emb so that R is an Emb-bundle. We also fix an R-bimodule M, central as a $\mathcal{P}$-bimodule, together with a left action of Emb

$$\mathrm{Emb} \times M \to M, \ \ (f, m) \mapsto f_*(m).$$

We require that this action induce a Γ-module structure on M which is *covariant* in the sense that

$$f_*(rms) = f_*(r) f_*(m) f_*(s) \quad (r, s \in R, m \in M). \tag{6.4.1}$$

In this situation, we can form the crossed product $M\#_{\mathcal{P}}\Gamma$; this is the $R\#_{\mathcal{P}}\Gamma$-bimodule consisting of $M \otimes_{\mathcal{P}} \Gamma$ equipped with the following left and right actions of $R\#_{\mathcal{P}}\Gamma$

$$(a\#U_f)(m\#U_g) = af_*(m)\#U_{fg}, \qquad (m\#U_g)(a\#U_f) = mg_*(a)\#U_{gf}.$$

Observe that, as R is assumed to be a k-algebra, $M\#_{\mathcal{P}}\Gamma = M\#_{\mathcal{P}(k)}\Gamma(k)$. We are interested in the Hochschild homology of $R\#_{\mathcal{P}}\Gamma$ with coefficients in $M\#_{\mathcal{P}}\Gamma$, which by Example 6.6 is computed by the simplicial $\mathcal{P}(k)$-module $C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k), M\#_{\mathcal{P}}\Gamma)$. On the other hand it is not hard to check, using (6.4.1) and the definition of Emb-bundle, that the diagonal action of Emb on $C(R/k)$ descends to an action of Γ on $C(R/\mathcal{P}(k))$. Hence we may also consider the bisimplicial module $\perp(\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M))$ which results from applying the functor $\perp(\Gamma/\mathcal{P}, -)$ dimension-wise to the simplicial module $C(R/\mathcal{P}(k), M)$. The diagonal of this bisimplicial module is

$$\begin{aligned}&\mathrm{diag}(\perp(\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M)))_n\\ &\qquad =\perp^n(\Gamma/\mathcal{P}, C_n(R/\mathcal{P}(k), M)) = \left(M \otimes_{\mathcal{P}} R^{\otimes_{\mathcal{P}(k)} n}\right)_{\mathcal{P}} \otimes_{\mathcal{P}} \Gamma^{\otimes_{\mathcal{P}} n},\end{aligned}$$

with faces $\mu_i \partial_i$ and degeneracies $s_i \delta_i$. The simplicial module

$$\mathrm{diag}(\perp(\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M)))$$

is a model for the hyperhomology of $\Gamma/\mathcal{P}$ with $C(R/\mathcal{P}(k), M)$ coefficients. Hence, if $\mathbb{H}(\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M))$ is any other such model, we have a quasi-isomorphism

$$\mathbb{H}(\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M)) \xrightarrow{\sim} \mathrm{diag}(\perp(\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M)).$$

Observe that any element of $\mathrm{diag}(\perp(\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M)))_n$ can be written as a sum of congruence classes of elementary tensors of the form

$$x = a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes f_1 \otimes \cdots \otimes f_n, \tag{6.4.2}$$

where $a_0 \in M$, $a_i \in R$, and $f_i \in \mathrm{Emb}$ ($i \geq 1$) are such that

$$\begin{aligned}\epsilon_r(f_i) &= \epsilon_l(f_{i+1}) \quad (1 \leq i \leq n-1),\\ a_j \epsilon_l(f_1) &= a_j \quad (0 \leq j \leq n).\end{aligned}$$

Next we define a map

$$\phi : \mathrm{diag}(\perp(\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M)) \to C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k), M\#_{\mathcal{P}}\Gamma).$$

For x as in (6.4.2), we put

$$\phi([x]) = [a_0\#f_1 \otimes f_1^{\dagger}(a_1)\#f_2 \otimes \cdots \otimes (f_1 \cdots f_n)^{\dagger}(a_n)\#(f_1 \cdots f_n)^{\dagger}]. \tag{6.4.3}$$

Here [] denotes congruence class.

Proposition 6.7. *The assignment* (6.4.3) *gives a simplicial isomorphism*

$$\phi : \mathrm{diag}(\perp(\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M))) \xrightarrow{\cong} C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k), M\#_{\mathcal{P}}\Gamma).$$

In particular, we have a quasi-isomorphism

$$\mathbb{H}(\Gamma/\mathcal{P}, HH(R/\mathcal{P}(k), M)) \xrightarrow{\sim} HH(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k), M\#_{\mathcal{P}}\Gamma).$$

Proof. First of all, we must check that (6.4.3) gives a well-defined simplicial homomorphism. To do this, one checks first that formula (6.4.3) defines a simplicial homomorphism

$$\hat{\phi} : \mathrm{diag}(\perp(\mathbb{Z}[\mathrm{Emb}], C(R, M))) \to C(R\#\mathrm{Emb}, M\#\mathrm{Emb}).$$

Then one observes that it passes down to the quotient, inducing a map $\phi : \mathrm{diag}(\perp(\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M))) \to C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k), M\#_{\mathcal{P}}\Gamma)$. Next note that the image of $\hat{\phi}$ is contained in the simplicial subgroup

$$S \subset C(R\#\mathrm{Emb}, M\#\mathrm{Emb})$$

given in dimension n by

$$S_n = \mathrm{span}\{[a_0\#f_0 \otimes \cdots \otimes a_n\#f_n] : f_i \in \mathrm{Emb},\ \ a_i \in R,\ \ f_0 \cdots f_n \in 2^{\mathbb{N}}\}.$$

To prove that ϕ is surjective, we must show that

$$S \to C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k), M\#_{\mathcal{P}}\Gamma)$$

is surjective. Any element of $C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k), M\#_{\mathcal{P}}\Gamma)$ can be written as a linear combination of classes of elementary tensors of the form

$$y = a_0\#f_0 \otimes \cdots \otimes a_n\#f_n, \tag{6.4.4}$$

such that the following conditions are satisfied for $0 \le i \le n-1$ and $0 \le j \le n$:

$$\epsilon_r(f_i) = \epsilon_l(f_{i+1}), \quad \epsilon_r(f_n) = \epsilon_l(f_0) \quad a_j = a_j\epsilon_l(f_j). \tag{6.4.5}$$

Let $f = f_0 \cdots f_n$; then $\mathrm{dom}(f) = \mathrm{ran}(f) = \mathrm{ran}(f_0) = \mathrm{dom}(f_n)$. Let

$$\mathbb{N} \supset A = \{x \in \mathrm{dom}(f) : f(x) = x\}.$$

If $A = \mathrm{dom}(f)$ then $f \in 2^{\mathbb{N}}$, and thus the element (6.4.4) belongs to S. Otherwise, by Zorn's Lemma, there exists $\emptyset \neq B \subset \mathrm{dom}(f)$ maximal with the property that $f(B) \cap B = \emptyset$. Clearly $A \cap B = \emptyset$; let $C = \mathrm{dom}(f)\backslash(A \sqcup B)$. Then $f(B) \subset C$, $f(C) \subset B$, and $p_{\mathrm{dom}(f)} = p_A + p_B + p_C$. Hence we have

$$[y] = [p_{\mathrm{dom}(f)} y p_{\mathrm{dom}(f)}] = [p_A y p_A] = [a_0 \# g_0 \otimes \cdots \otimes a_n \# g_n],$$

for $g_n = (f_n)_{|A}$ and $g_i = (f_i)_{|f_{i+1}\cdots f_n(A)}$ $(0 \leq i \leq n-1)$. In particular $g_0 \cdots g_n = p_A$. Thus ϕ is surjective. To prove it is injective, define a map

$$\psi : C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k), M\#_{\mathcal{P}}\Gamma) \to \mathrm{diag}(\perp (\Gamma/\mathcal{P}, C(R/\mathcal{P}(k), M)))$$

as follows. For y as in (6.4.4) satisfying the conditions (6.4.5) and such that $f_0 \cdots f_n \in 2^{\mathbb{N}}$, put

$$\psi([y]) = [a_0 \otimes f_0(a_1) \otimes \cdots \otimes (f_0 \cdots f_{n-1})(a_n) \otimes f_0 \otimes \cdots \otimes f_{n-1}].$$

One checks that ψ is well-defined and that $\psi\phi = id$. □

Corollary. *Assume that R is commutative and that M is a central R-bimodule. Then*

$$HH_0(R\#_{\mathcal{P}}\Gamma, M\#_{\mathcal{P}}\Gamma) = M_{\mathcal{E}}.$$

Proof. By Proposition 6.7,

$$HH_0(R\#_{\mathcal{P}}\Gamma, M\#_{\mathcal{P}}\Gamma) = H_0(\Gamma/\mathcal{P}, HH_0(R, M)).$$

By our assumptions on R and M, $HH_0(R, M) = M$. Finally we have $H_0(\Gamma/\mathcal{P}, M) = M_{\mathcal{E}}$, by Proposition 6.3. □

6.5. Comparing the 0^{th}-homology of (Γ^∞, I_S) and that of $(\mathcal{B} : J_S)$.

Proposition 6.8. *Let $S \lhd \ell^\infty$ be a symmetric ideal and let $J_S \lhd \mathcal{B} = \mathcal{B}(\ell^2)$ be the corresponding ideal of bounded operators in ℓ^2. Then the inclusion $\Gamma^\infty \subset \mathcal{B}$ induces an isomorphism*

$$HH_0(\Gamma^\infty, I_S) \xrightarrow{\cong} HH_0(\mathcal{B}, J_S).$$

Proof. By Proposition 2.3 Corollary 6.4, the inclusion diag : $S \to I_S$ descends to a bijection

$$S_{\mathcal{E}} \xrightarrow{\cong} HH_0(\Gamma^\infty, I_S). \tag{6.5.1}$$

By [13, Theorem 5.12] the composite of (6.5.1) with the map induced by the inclusion $I_S \subset J_S$ is an isomorphism. □

Corollary. *The map $HC_0(\Gamma^\infty : I_S) \to HC_0(\mathcal{B} : J_S)$ is an isomorphism.*

Proof. It follows from Proposition 6.8 and the fact that, if R is a unital ring and $I \lhd R$ is an ideal then

$$HH_0(R : I) = HC_0(R : I) = I/[R, I].$$

□

Lemma 6.9. *Let $p > 0$. Then:*

$$HC_0(\Gamma^\infty : I_{\ell^p+}) = \begin{cases} \mathbb{C} & p < 1 \\ 0 & p \geq 1 \end{cases}$$

$$HC_0(\Gamma^\infty : I_{\ell^p-}) = \begin{cases} \mathbb{C} & p \leq 1 \\ 0 & p > 1 \end{cases}$$

$$HC_0(\Gamma^\infty : I_{\ell^p}) = \begin{cases} \mathbb{C} & p < 1 \\ \mathbb{C} \oplus \mathbb{V} & p = 1 \\ 0 & p > 1. \end{cases}$$

Here $\mathbb{V}$ is a $\mathbb{C}$-vector space of uncountable dimension.

Proof. It follows from Corollary 6.5 and [24, pp. 492–493]. □

6.6. Cyclic homology of $R\#_{\mathcal{P}}\Gamma$. Now we go back to the general situation of Subsection 6.4. So k is a field and R is an Emb-bundle over k. Let M be a right Γ-module. Consider the simplicial module $\perp(\Gamma/\mathcal{P}, M)$. Every element of $\perp_n(\Gamma/\mathcal{P}, M)$ can be written as a sum of elementary tensors

$$x = m \otimes f_1 \otimes \cdots \otimes f_n$$

with $m \in M$, $f_i \in \mathrm{Emb}$, and $\mathrm{dom}(f_i) = \mathrm{ran}(f_{i+1})$ $(i < n)$. For x as above, put

$$\tau_n(x) = (-1)^n m(f_1 \cdots f_n) \otimes (f_1 \cdots f_n)^\dagger \otimes f_1 \otimes \cdots \otimes f_{n-1}. \tag{6.6.1}$$

One checks that the assignment (6.6.1) gives a well-defined endomorphism of $\perp_n(\Gamma/\mathcal{P}, M)$, and that the cyclic identities [17, 2.5.1.1] hold. Thus the simplicial (k-)module $\perp(\Gamma/\mathcal{P}, M)$, equipped with the cyclic operators τ_n $(n \geq 0)$, is a *cyclic module*. In general if $\mathcal{C}$ is any cyclic module, then we can equip $\mathcal{C}$ with a map $B : \mathcal{C} \to \mathcal{C}[+1]$ called the Connes' operator, which, together with the usual boundary $b : \mathcal{C} \to \mathcal{C}[-1]$ given by the alternating sum of the face maps, satisfy $b^2 = B^2 = [b, B] = 0$. When $\mathcal{C} = \perp(\Gamma/\mathcal{P}, M)$, we write ∂ and $\mathcal{B}$ for the operators b and B. The *Hochschild complex* of a cyclic module $\mathcal{C}$ is $HH(\mathcal{C}) = (\mathcal{C}, b)$. The *cyclic* and *negative cyclic* complexes are the complexes given in dimension n by $HC(\mathcal{C})_n = \bigoplus_{m\geq 0} \mathcal{C}_{n-2m}$ and $HN(\mathcal{C})_n = \prod_{m\geq 0} \mathcal{C}_{n+2m}$;

they are equipped with the boundary $b + B$. Observe that $HC(\mathcal{C})$ is also equipped with a chain map $S : HC(\mathcal{C}) \to HC(\mathcal{C})[-2]$ defined by the obvious projections $HC(\mathcal{C})_n \to HC(\mathcal{C})_{n-2}$. If C is another chain complex equipped with a chain map $S : C \to C[-2]$, then by a *map of S-complexes* $C \to HC(\mathcal{C})$ we understand a chain map which commutes with S.

Proposition 6.10. *There is a natural quasi-isomorphism of S-complexes* $(HC(\perp (\Gamma/\mathcal{P}, M)), \partial) \to (HC(\perp (\Gamma/\mathcal{P}, M)), \partial + \mathcal{B})$.

Proof. View $\mathcal{C} = \perp (\Gamma/\mathcal{P}, M)$ as a cyclic module. Consider the projection

$$\pi : HN(\mathcal{C})_n = \prod_{m \geq 0} \mathcal{C}_{n+2m} \to \mathcal{C}_n = HH(\mathcal{C})_n.$$

Observe that $\pi(b + \mathcal{B}) = b\pi$. Proceed as in [11, §3.1] to define a chain map $\Upsilon : HH(\mathcal{C}) \to HN(\mathcal{C})$ such that $\pi\Upsilon = 1$. We have a chain map $\theta^n : HN(\mathcal{C}) \to HC(\mathcal{C})[2n]$ $(n \geq 0)$ given by the composite

$$\theta^n : HN(\mathcal{C})_p = \prod_{m \geq 0} \mathcal{C}_{p+2m} \twoheadrightarrow \bigoplus_{m=0}^{n} \mathcal{C}_{p+2m} \subset \bigoplus_{q \geq 0} \mathcal{C}_{p+2(n-q)} = HC(\mathcal{C})_{p+2n}.$$

The map of the proposition is

$$\sum_{n=0}^{\infty} \theta^n \Upsilon : (HC(\mathcal{C}), \partial) = \bigoplus_{n \geq 0} HH(\mathcal{C})[-2n] \to (HC(\mathcal{C}), b + \mathcal{B}).$$

□

Theorem 6.11. *Let k be a field and R an* Emb*-bundle over k. There is a natural zig-zag of quasi-isomorphisms*

$$\mathbb{H}(\Gamma/\mathcal{P}, HC(R/\mathcal{P}(k))) \xrightarrow{\sim} HC(R\#_{\mathcal{P}}\Gamma/k).$$

Proof. Consider the bicyclic module

$$\mathcal{C}_{*,*} : ([m], [n]) \mapsto \perp_m (\Gamma/\mathcal{P}, C_n(R/\mathcal{P}(k))). \tag{6.6.2}$$

It follows from Proposition 6.10 that the total cyclic complex

$$T = (HC(\mathcal{C}_{*,*}), b + \partial + B + \mathcal{B})$$

is quasi-isomorphic to

$$(HC(\mathcal{C}_{*,*}), b + \partial + B),$$

which in turn is a model for $\mathbb{H}(\Gamma/\mathcal{P}, HC(R/\mathcal{P}(k)))$. By the cylindrical version of the Eilenberg-Zilber theorem ([16, Theorem 3.1]), the complex T is S-equivalent to the HC-complex of the diagonal Δ of (6.6.2). By Proposition (6.7), the map (6.4.3) is an isomorphism of simplicial modules $\Delta \xrightarrow{\cong} C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k))$; one checks that it is actually an isomorphism of cyclic modules. Finally, by Example 6.6, the projection $C(R\#_{\mathcal{P}}\Gamma/k) \to C(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k))$ induces a quasi-isomorphism

$$HC(R\#_{\mathcal{P}}\Gamma/k) \to HC(R\#_{\mathcal{P}}\Gamma/\mathcal{P}(k)). \tag{6.6.3}$$

□

Corollary. *Let $\mathfrak{A}$ be a bornological algebra and $S \lhd \ell^\infty$ a symmetric ideal. Then*

$$HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) = \mathbb{H}_*(\Gamma/\mathcal{P} : HC((\ell^\infty(\mathfrak{A}) : S(\mathfrak{A}))/\mathcal{P})).$$

Proof. By Proposition 2.3, we have $\Gamma^\infty(\mathfrak{A}) = \ell^\infty(\mathfrak{A})\#_{\mathcal{P}}\Gamma$ and $I_{S(\mathfrak{A})} = S(\mathfrak{A})\#_{\mathcal{P}}\Gamma$. Now apply Theorem 6.11 and take fibers. □

6.7. Hodge decomposition. If R is a commutative $\mathbb{Q}$-algebra, then there are defined Adams operations on $C(R)$, and we have an eigenspace decomposition [17, Theorems 4.5.10 and 4.6.7]

$$C(R) = \bigoplus_{p\geq 0} C^{(p)}(R), \tag{6.7.1}$$

called the *Hodge decomposition*. We have $C_n^{(p)} = 0$ for $n < p$ and each $C^{(p)}$ is a graded R-submodule, closed under the Hochschild boundary map b. Thus, if M is a central R-bimodule, for $HH^{(p)}(R, M) = M \otimes_R (C^{(p)}(R), b)$ we have

$$HH_n(R, M) = \bigoplus_{p\geq 0}^{n} HH_n^{(p)}(R, M).$$

The Connes operator B sends $C^{(p)}$ to $C^{(p+1)}$. Thus, we have a direct sum decomposition of the cyclic complex

$$HC(R) = \bigoplus_{p=0}^{\infty} HC^{(p)}(R)$$

where

$$HC^{(p)}(R)_n = \bigoplus_{p\geq 0}^{n} C_{n-2p}^{(n-p)}(R).$$

Hence for $HC_*^{(p)}(R) = H_*(HC^{(p)}(R))$,

$$HC_n(R) = \bigoplus_{p=0}^{n} HC_n^{(p)}(R).$$

Let (Ω_R^*, d) be the DGA of (absolute) Kähler differential forms. There is a natural map of mixed complexes

$$\mu : (C(R), b, B) \to (\Omega_R, 0, d)$$
$$\mu(x_0 \otimes \cdots \otimes x_n) = (1/n!)x_0 dx_1 \wedge \cdots \wedge dx_n. \tag{6.7.2}$$

Let M be a central R-bimodule; the map μ induces isomorphisms

$$HH_n^{(n)}(R, M) = M \otimes_R \Omega_R^n \tag{6.7.3}$$
$$\text{and } HC_n^{(n)}(R) = \Omega_R^n / d(\Omega_R^{n-1}). \tag{6.7.4}$$

We say that R is *homologically smooth* if (6.7.2) is a quasi-isomorphism.

Remark 6.12. If R happens to also be an algebra over $\mathcal{P}$, then the Hodge decomposition above induces a similar decomposition on $HH(R/\mathcal{P}, M)$ and $HC(R/\mathcal{P})$, so that $HH^{(p)}(R, M) \to HH^{(p)}(R/\mathcal{P}, M)$ and $HH^{(p)}(R, M) \to HH^{(p)}(R/\mathcal{P})$ are quasi-isomorphisms. Moreover $\Omega_R \to \Omega_{R/\mathcal{P}}$ is an isomorphism.

Example 6.13. Let R be a unital commutative complex C^*-algebra over $\mathbb{C}$. It was proved in [10, Thm. 8.2.6] that R, regarded as a $\mathbb{Q}$-algebra, is homologically smooth. In particular this applies when $R = \ell^\infty$. Moreover, by [10, proof of Prop. 5.2.2], ℓ^∞ is a filtering colimit of smooth $\mathbb{C}$-algebras. It follows that $\Omega_{\ell^\infty}^n$ is a flat ℓ^∞-module for every n. Hence

$$HH_n(\ell^\infty, M) = M \otimes_{\ell^\infty} \Omega_{\ell^\infty}^n$$

for every central bimodule M.

Now assume that the commutative $\mathbb{Q}$-algebra R is an Emb-bundle. Then by Proposition 6.7, Theorem 6.11, and naturality of the Hodge decomposition, we have quasi-isomorphisms

$$HH(R\#_{\mathcal{P}}\Gamma, M\#_{\mathcal{P}}\Gamma) \xrightarrow{\sim} \bigoplus_{p\geq 0} \mathbb{H}(\Gamma/\mathcal{P}, HH^{(p)}(R/\mathcal{P}, M)) \tag{6.7.5}$$
$$\text{and } HC(R\#_{\mathcal{P}}\Gamma) \xrightarrow{\sim} \bigoplus_{p\geq 0} \mathbb{H}(\Gamma/\mathcal{P}, HC^{(p)}(R/\mathcal{P})). \tag{6.7.6}$$

Put

$$HH_n^{(p)}(R\#_{\mathcal{P}}\Gamma, M\#_{\mathcal{P}}\Gamma) = \mathbb{H}_n(\Gamma/\mathcal{P}, HH^{(p)}(R/\mathcal{P}, M)), \tag{6.7.7}$$
$$HC_n^{(p)}(R\#_{\mathcal{P}}\Gamma) = \mathbb{H}_n(\Gamma/\mathcal{P}, HC^{(p)}(R/\mathcal{P})).$$

We have decompositions

$$HH_n(R\#_{\mathcal{P}}\Gamma, M\#_{\mathcal{P}}\Gamma) = \bigoplus_{p=0}^{n} HH_n^{(p)}(R\#_{\mathcal{P}}\Gamma, M\#_{\mathcal{P}}\Gamma),$$
$$HC_n(R\#_{\mathcal{P}}\Gamma) = \bigoplus_{p=0}^{n} HC_n^{(p)}(R\#_{\mathcal{P}}\Gamma).$$

If follows from (6.7.3), (6.7.4), and Proposition 6.3 that

$$HH_n^{(n)}(R\#_{\mathcal{P}}\Gamma, M\#_{\mathcal{P}}\Gamma) = (M \otimes_R \Omega_R^n)_{\mathcal{E}}, \qquad (6.7.8)$$
$$HC_n^{(n)}(R\#_{\mathcal{P}}\Gamma) = (\Omega_R^n / d\Omega_R^{n-1})_{\mathcal{E}}.$$

7. The relative cyclic homology $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})})$

7.1. The Quillen spectral sequence. Let R be a unital $\mathbb{Q}$-algebra and $I \lhd R$ a two-sided ideal, flat both as a right and as a left ideal. Then

$$I^{\otimes_R^n} \cong I^n.$$

Using the isomorphism above and flatness again we see that if $P \xrightarrow{\sim} I$ is a projective bimodule resolution, then $Q = P^{\otimes_R^n} \xrightarrow{\sim} I^n$ is again a resolution. Hence modding out Q by the commutator subspace $[Q, R]$ we obtain a complex which computes $HH_*(R, I^n)$ and which has a natural action of $\mathbb{Z}/n\mathbb{Z}$ via permutation of factors. Following Quillen [19, pp. 210] we shall write $HH_*(R, I^n)_\sigma$ for the coinvariants of this action. Quillen introduced a first quadrant spectral sequence (see [19, Proposition 2.16 and Theorem 4.3]),

$$E^1_{p,q} = \begin{cases} HC_q(R) & p = 0 \\ HH_{q-p+1}(R, I^p)_\sigma & p \geq 1, \end{cases} \qquad (7.1.1)$$

which converges to $HC_{p+q}(R/I)$. For example, every ideal $J \lhd \mathcal{B} = \mathcal{B}(\ell^2)$ of the algebra of bounded operators is flat; M. Wodzicki has used this spectral sequence, together with the results of [13], to study the relative cyclic homology groups $HC_*(\mathcal{B} : J)$. By Proposition 3.3, every ideal of Γ^∞ is flat; by Proposition 3.5 and Examples 3.2, the same is true of $I_{c_0(\mathfrak{A})}$ and $I_{\ell^{\infty-}(\mathfrak{A})}$ for every unital Banach algebra $\mathfrak{A}$. In this subsection we shall use Quillen's spectral sequence to study the cyclic homology groups $HC_*(\Gamma^\infty : I_S)$. Proposition 7.1 below will play a role akin to that played by [24, Theorem 8] in the context of operator ideals. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and let $\hat{\otimes}$ be the projective tensor product. We have maps

$$\Gamma \otimes \Gamma \to \Gamma(\mathbb{N} \times \mathbb{N}), \;\; U_f \otimes U_g \mapsto U_{f \times g}, \qquad (7.1.2)$$
$$\boxtimes : \ell^\infty(\mathfrak{A}) \otimes \ell^\infty(\mathfrak{B}) \to \ell^\infty(\mathbb{N} \times \mathbb{N}, \mathfrak{A}\hat{\otimes}\mathfrak{B}), \;\; (\alpha \boxtimes \beta)_{m,n} = \alpha_n \hat{\otimes} \beta_m. \qquad (7.1.3)$$

These two maps together induce

$$\Gamma^\infty(\mathfrak{A})\otimes\Gamma^\infty(\mathfrak{B})\to$$
$$\Gamma^\infty(\mathbb{N}\times\mathbb{N},\mathfrak{A}\hat{\otimes}\mathfrak{B}):=\ell^\infty(\mathbb{N}\times\mathbb{N},\mathfrak{A}\hat{\otimes}\mathfrak{B})\#_{\mathcal{P}(\mathbb{N}\times\mathbb{N})}\Gamma(\mathbb{N}\times\mathbb{N}).$$

We write $\Gamma^\infty(\mathbb{N}\times\mathbb{N})=\Gamma^\infty(\mathbb{N}\times\mathbb{N},\mathbb{C})$. In particular we have a map

$$\Gamma^\infty\otimes\Gamma^\infty\to\Gamma^\infty(\mathbb{N}\times\mathbb{N}). \tag{7.1.4}$$

Proposition 7.1. (cf. [24, Theorem 8]) *Let $S,T\lhd\ell^\infty$ be symmetric ideals, and let $\mathfrak{B}$ be a unital Banach algebra. Assume that*

(i) *The map* (7.1.3) *sends $S\otimes T\to T(\mathbb{N}\times\mathbb{N})$.*

(ii) $S_\mathcal{E}=0$.

Then

$$HH_*(\Gamma^\infty(\mathfrak{B}),I_{T(\mathfrak{B})})=0.$$

Proof. Proceeding as in the proof of [1, Proposition 7.3.4], we obtain a commutative diagram

$$\begin{array}{ccc}\Gamma^\infty\otimes\Gamma^\infty(\mathfrak{B}) & \longrightarrow & M_2\Gamma^\infty(\mathfrak{B})\\ {\scriptstyle E_{1,1}\otimes -}\uparrow & \nearrow & \\ \Gamma^\infty(\mathfrak{B}) & & \end{array}$$

By hypothesis (i) this restricts to a commutative diagram

$$\begin{array}{ccc}I_S\otimes I_{T(\mathfrak{B})} & \longrightarrow & M_2 I_{T(\mathfrak{B})}\\ {\scriptstyle E_{1,1}\otimes -}\uparrow & \nearrow & \\ I_{T(\mathfrak{B})} & & \end{array}$$

Now use hypothesis (ii), Morita invariance and the Künneth formula for Hochschild homology ([?Theorem~1.2.4]lod and [22, Proposition 9.4.1]), and induction, to conclude that $HH_*(\Gamma^\infty(\mathfrak{A}),I_{T(\mathfrak{A})})=0$. □

We shall need the following result of Dykema, Figiel, Weiss and Wodzicki, which follows by combining [13, Theorem 5.11(ii) and Theorem 5.12].

Proposition 7.2. ([13]) *Let $S\lhd\ell^\infty$ be a symmetric ideal and let $\omega=(1/n)_{n\ge 1}$ be the harmonic sequence. Then*

$$S_\mathcal{E}=0\iff\omega\boxtimes S\subset S(\mathbb{N}\times\mathbb{N}).$$

Proposition 7.3.

(i) $HC_*(\Gamma^\infty : I_{c_0}) = HC_*(\mathcal{B} : J_{c_0}) = 0$.

(ii) $HC_*(\Gamma^\infty : I_{\ell^{\infty -}}) = HC_*(\mathcal{B} : J_{\ell^{\infty -}}) = 0$.

(iii) *Let* $0 < p < \infty$*,* $S \in \{\ell^p, \ell^{p-}, \ell^{p+}\}$*,*

$$\begin{aligned} m &= \min\{n : HC_n(\Gamma^\infty : I_S) \neq 0\}, \\ \text{and} \quad m' &= \min\{n : HC_n(\mathcal{B} : J_S) \neq 0\}. \end{aligned}$$

Then $m = m'$ *and the map* $HC_m(\Gamma^\infty : I_S) \to HC_m(\mathcal{B} : J_S)$ *is an isomorphism.*

Proof. Consider the spectral sequence (7.1.1) in the cases $R = \Gamma^\infty, \mathcal{B}$ and $I = I_S, J_S$ for each of the symmetric ideals S of the proposition. We have $E^1_{0,*} = 0$ since both Γ^∞ and $\mathcal{B}$ are rings with infinite sums [1, §5]. In both (i) and (ii), we have $S^2 = S$ and $\omega \boxtimes S \subset S(\mathbb{N} \times \mathbb{N})$ whence $E^1_{*,*} = 0$, by Propositions 7.2 and 7.1 and [24, Theorem 8]. This gives (i) and (ii). In each of the cases considered in part (iii), we have $S \boxtimes S \subset S(\mathbb{N} \times \mathbb{N})$. Since $\omega \in \ell^p$ if and only if $p > 1$ and since $(\ell^p)^n = \ell^{p/n}$, we have $HH_*(\Gamma^\infty, I_{(\ell^p)^n}) = HH_*(\mathcal{B}, (\mathcal{L}^p)^n) = 0$ for $p/n > 1$, again by Propositions 7.2 and 7.1 and [24, Theorem 8]. The case $S = \ell^p$ follows from this and from Corollary 6.8. The remaining cases follow similarly. □

Remark 7.4. Proposition 7.12 below provides a more detailed computation of $HC_n(\Gamma^\infty : I_S)$ for S as in case iii) of Proposition 7.3 above.

Theorem 7.5. *The comparison map* $K_*(I_{S(\mathfrak{A})}) \to KH_*(I_{S(\mathfrak{A})})$ *is an isomorphism in the following cases:*

(i) $S = c_0$ *and* $\mathfrak{A}$ *is a* C^**-algebra.*

(ii) $S = \ell^{\infty -}$ *and* $\mathfrak{A}$ *is a unital Banach algebra.*

Proof. By Proposition 5.1 and Examples 5.2 and 5.3, $I_{S(\mathfrak{A})}$ is H-unital in both cases. Hence by (1.2) it suffices to show that $HC_*(\Gamma^\infty(\mathfrak{A}) : I_{S(\mathfrak{A})}) = 0$. As explained in the proof of Proposition 7.3, Proposition 7.2 implies that $S_\mathcal{E} = 0$. Hence if $\mathfrak{A}$ is unital we are done by Propositions 3.5 and 7.1; in particular, part (ii) is proved. The nonunital case of (i) follows from the unital case using excision. □

7.2. Computing $HC^{(p)}(\Gamma^\infty : I_S)$ in terms of differential forms.

Let $S \lhd \ell^\infty$ be an ideal. Consider the subcomplex

$$\mathcal{F}_p(S) \subset \Omega_{\ell^\infty} \tag{7.2.1}$$

$$(\mathcal{F}_p(S))^q = \begin{cases} S^{p-q+1}\Omega^q_{\ell^\infty} & p \geq q \\ \Omega^q_{\ell^\infty} & q > p. \end{cases}$$

Write

$$D^{(p)}(S)_q = (\Omega^{-q}_{\ell^\infty}/(\mathcal{F}^{-q}_p(S)) \tag{7.2.2}$$

$$L^{(p)}(S)_q = \mathcal{F}^{-q}_{p-1}(S)/\mathcal{F}^{-q}_p(S). \tag{7.2.3}$$

Note $L^{(p)}(S)$ and $D^{(p)}(S)$ are nonpositive chain complexes.

Theorem 7.6. *Let $S \lhd \ell^\infty$ be a symmetric ideal. Then there are* Emb*-equivariant quasi-isomorphisms*

$$HH^{(p)}(\ell^\infty/S) \xrightarrow{\sim} L^{(p)}(S)[p]$$
$$HC^{(p)}(\ell^\infty/S) \xrightarrow{\sim} D^{(p)}(S)[p].$$

Proof. Consider the skew-commutative graded algebra $\Lambda = \ell^\infty \oplus S$ with grading $\Lambda_0 = \ell^\infty$, $\Lambda_1 = S$. The inclusion $S \subset \ell^\infty$ defines a homogeneous ℓ^∞-linear derivation $\partial : \Lambda \to \Lambda[-1]$. Thus Λ is a chain DGA, and the projection $\ell^\infty \to \ell^\infty/S$ defines a quasi-isomorphism of cyclic modules $C(\Lambda, \partial) \xrightarrow{\sim} C(\ell^\infty/S)$. By [7, Thms. 2.6 and 3.3] and Proposition 3.1, there are quasi-isomorphisms $C(\Lambda, \partial) \xrightarrow{\sim} \bigoplus_p L^{(p)}(S)[p]$ and $\mathfrak{B}(\Lambda, \partial) \xrightarrow{\sim} \bigoplus_p D^{(p)}(S)[p]$; by [21] they are compatible with the Hodge decomposition. Finally, all these quasi-isomorphisms are natural, and thus Emb-equivariant. □

Theorem 7.7.

$$HC^{(p)}_*(\Gamma^\infty : I_S) = \mathbb{H}_{*+p}(\Gamma/\mathcal{P}, \mathcal{F}_{(p)}(S))$$
$$HH^{(p)}_*(\Gamma^\infty : I_S) = \mathbb{H}_{*+p+1}(\Gamma/\mathcal{P}, L_{(p)}(S)).$$

Proof. It follows from (6.7.7) using Theorem 7.6 and the fact that Γ^∞ is an infinite sum ring ([1, §5]). □

Corollary 7.8. *There is a first quadrant homological spectral sequence*

$${}_pE^1_{m,n} = H_n(\Gamma/\mathcal{P}, S^{m+1}\Omega^{p-m}_{\ell^\infty}) \Rightarrow HC^{(p)}_{m+n+p}(\Gamma^\infty : I_S).$$

Proof. This is the spectral sequence associated to $\mathbb{H}(\Gamma/\mathcal{P}, \mathcal{F}_{(p)}(S))$. It is located in the first quadrant because as Γ^∞ is an infinite sum ring,

$$HH^{(q)}_*(\Gamma^\infty) = H_{*+q}(\Gamma/\mathcal{P}, \Omega^q_{\ell^\infty}) = 0.$$

□

Corollary 7.9.

$$HC^{(n)}_n(\Gamma^\infty : I_S) = (S\Omega^n_{\ell^\infty}/d(S^2\Omega^{n-1}_{\ell^\infty}))_{\mathcal{E}}.$$

Proof. It follows from inspection of the second term of the spectral sequence of Corollary 7.8, by using the fact that $H_0(\Gamma/\mathcal{P}, -) = (\ )_{\mathcal{E}}$ is right exact. □

7.3. The cases $S = \ell^p, \ell^{p\pm}$.

Lemma 7.10. *Let $S \lhd \ell^\infty$ be a symmetric ideal. Then the map*

$$C(\Gamma/\mathcal{P}, S\Omega^p_{\ell^\infty}) \to C(\Gamma(\mathbb{N} \sqcup \mathbb{N})/\mathcal{P}(\mathbb{N} \sqcup \mathbb{N}), S(\mathbb{N} \sqcup \mathbb{N})\Omega^p_{\ell^\infty(\mathbb{N}\sqcup\mathbb{N})})$$

induced by the inclusion $\mathbb{N} \subset \mathbb{N} \sqcup \mathbb{N}$ into the first copy, is a quasi-isomor- phism.

Proof. Recall from Corollary 3 that every ideal of ℓ^∞ is flat, and from Example 6.13 that $\Omega^p_{\ell^\infty}$ is a flat ℓ^∞-module. It follows that the map $S \otimes_{\ell^\infty} \Omega^p_{\ell^\infty} \to S\Omega^p_{\ell^\infty}$ is an isomorphism for every ideal S. Now the proof is immediate from [1, Lemma 7.3.1] and Lemma 6.1. □

Lemma 7.11. *Let $0 \neq S_1, S_2 \subset \ell^\infty$ be symmetric ideals. Assume that $(S_1)_\mathcal{E} = 0$ and that the map $\ell^\infty \otimes \ell^\infty \to \ell^\infty(\mathbb{N} \times \mathbb{N})$ sends $S_1 \otimes S_2 \to S_2(\mathbb{N} \times \mathbb{N})$. Then $H_*(\Gamma/\mathcal{P}, S_2\Omega^p_{\ell^\infty}) = 0$ $(p \geq 0)$.*

Proof. The proof follows using Lemma 7.10 and the argument of the proof of Proposition 7.1. □

Let $p \in \mathbb{R}$; the following notation is used in the proposition below.

$$[p] = \max\{n \in \mathbb{Z} : n \leq p\}, \quad \lfloor p \rfloor = \begin{cases} p-1 & p \in \mathbb{Z} \\ [p] & p \notin \mathbb{Z}. \end{cases}$$

Proposition 7.12.

(i) *Let $p > 0$ and let S_p be either ℓ^p or ℓ^{p-}. Then*

$$HC_n^{(q)}(\Gamma^\infty : I_{S_p}) = \begin{cases} 0 & n < q + \lfloor p \rfloor \\ (S_{(p/(\lfloor p \rfloor+1))}\Omega^{q-\lfloor p \rfloor}_{\ell^\infty}/d(S_{(p/(\lfloor p \rfloor+2))}\Omega^{q-\lfloor p \rfloor-1}_{\ell^\infty}))_\mathcal{E} & n = q + \lfloor p \rfloor. \end{cases}$$

In particular, the first nonzero group is

$$HC_{2\lfloor p \rfloor}(\Gamma^\infty : I_{S_p}) = HC^{\lfloor p \rfloor}_{2\lfloor p \rfloor}(\Gamma^\infty : I_{S_p}) = HC_0(\Gamma^\infty : I_{S_{p/(\lfloor p \rfloor+1)}})$$

which was computed in 6.9.

(ii)

$$HC_n^{(q)}(\Gamma^\infty : I_{\ell^{p+}}) = \begin{cases} 0 & n < q + [p] \\ ((\ell^{(p/([p]+1))+}\Omega^{q-[p]}_{\ell^\infty}/d(\ell^{(p/([p]+2))+}\Omega^{q-[p]-1}_{\ell^\infty}))_\mathcal{E} & n = q + [p]. \end{cases}$$

In particular, the first nonzero group is

$$HC_{2[p]}(\Gamma^\infty : I_{\ell^{p+}}) = HC^{([p])}_{2[p]}(\Gamma^\infty : I_{\ell^{p+}}) = HC_0(\Gamma^\infty : I_{\ell^{(p/([p]+1))+}}) = \mathbb{C}$$

Proof. This is a straightforward application of the spectral sequence of Corollary 7.8 together with Lemma 7.11 and Proposition 7.2. □

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