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Global study of the simple pendulum by the homotopy analysis method

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Abstract
Techniques are developed to find all periodic solutions in the simple pendulum by means of the homotopy analysis method (HAM). This involves the solution of the equations of motion in two different coordinate representations. Expressions are obtained for the cycles and periods of oscillations with a high degree of accuracy in the whole range of amplitudes. Moreover, the convergence of the method is easily checked. The aim of this work is to show how the dynamics of a simple example of oscillatory systems may be studied globally with the HAM and to incentivize the interest of advanced undergraduate students in this type of techniques.

(Some figures may appear in colour only in the online journal)

1. Introduction

The simple pendulum consists of a mass point, constrained to move in a vertical plane at a fixed distance from a pivot, and subject to gravity. In the absence of friction it is a conservative system.

Despite the simplicity of this system it has dynamic characteristics that make it very interesting. On the one hand, it is a nonlinear system that can be explicitly integrated, but only after resorting to the theory of Jacobi elliptic functions [1, 2]. On the other hand, its description as a dynamical system shows a phase space which has the structure of a manifold with non-trivial topology; it is a cylinder. These features make it an example in almost every book on mechanics, geometric mechanics or dynamical systems.

The simple pendulum is usually studied near its trivial equilibrium by linearization. This approximation turns out to be a harmonic oscillator. It can be easily solved but it gives solutions that are nearly exact only for oscillations of amplitude close to zero. In this work, we propose to apply the homotopy analysis method (HAM) [3–5]. With this method we can calculate analytical approximations of all oscillatory solutions (even for large amplitudes) and also for rotating solutions, usually not studied. We believe that the efficacy of this technique will be of great interest for advanced undergraduate students of science and engineering who are interested in dynamics and solving differential equations.

The HAM is a non-perturbative method that solves a wide range of nonlinear problems. The system considered here is especially interesting from the point of view of resolution with this method. There are mainly two reasons for this. The first is that the structure of the phase space allows us to illustrate how a change of coordinates can be used to calculate the rotations in the domain of a single chart. The second is that the presence of the function $\sin\theta$ in the equation gives rise to an algebraic problem referring to the starting conditions of the method. We solved this problem by resorting to the series expansion of Jacobi–Anger [2].

In section 2, we show the characteristics of a well-known oscillatory system, the simple pendulum, in which the proposed methodology will be applied later. In section 3, we implement the HAM to find periodic solutions in this system. The method could be systematized in the following steps. Write the equation in a suitable coordinate system. Find a change of coordinates to normalize the solutions so that they have unit frequency and amplitude. Choose the harmonic oscillator of unit frequency as a linear operator. Determine the starting conditions of the HAM by solving certain algebraic equations. Obtain the terms of the series for the cycle, the frequency and amplitude up to an arbitrary order by solving linear differential equations. Then, check the convergence by plotting certain curves obtained from the solutions (h-curves). The last step permits us to give a value to the h parameter of the method. In sections 4 and 5 the method is applied to obtain approximations of oscillatory and rotary solutions and their respective frequencies. All these calculations can easily be performed with Mathematica.

The potential of the method for finding periodic orbits, without using complex mathematical results, makes the HAM highly attractive for those students interested in the study of dynamics and differential equations.

2. The simple pendulum

Consider the equation of a simple pendulum with distance L from the pivot,

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0, \tag{1}$$

where g is the acceleration of gravity. After introducing the parameter $\overline{\omega} = \sqrt{g/L}$ and changing the time variable to $\overline{\omega}t$, we obtain the dimensionless equation

$$\ddot{\theta} + \sin\theta = 0. \tag{2}$$

The simple pendulum shows two equilibria. The first is the trivial equilibrium which is a centre according to the dynamical systems theory [6] (this is because it is a centre of the linearization and the system is conservative). The orbits around this equilibrium are cycles homotopic to the identity of the fundamental group of the cylinder. These movements are called vibrations. The second is a saddle equilibrium that corresponds to the inverted pendulum. There are two homoclinic orbits connecting this unstable equilibrium to itself. Separated by the latter are the so-called rotations, orbits that are not homotopic to identity. Figure 1 shows several oscillating and rotating solutions together with homoclinic orbits. Two different representations of the phase space are shown: to the left it is represented on the plane and to the right on the cylinder. In the following sections, we show why the last representation is the most appropriate to study periodic solutions of the pendulum.

Because the energy

$$e = \frac{\dot{\theta}^2}{2} + 1 - \cos\theta \tag{3}$$

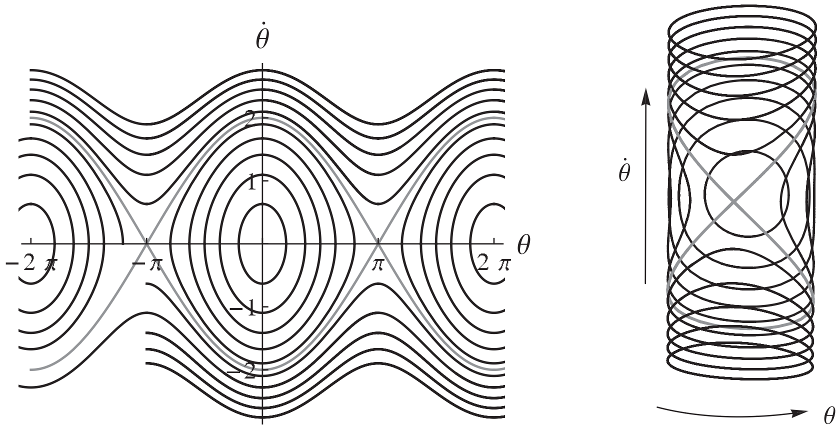

Figure 1. Solutions in phase space $(\theta, \dot{\theta})$ (left) and in the cylinder (right).

is constant, the solutions of the simple pendulum can be calculated by direct integration (see for example [1], [7] or [8] for a quantum version). As we mentioned earlier, there are three types of solutions depending on the value of the velocity v_0 for $\theta = 0$.

(1) If $|v_0| < 2$ the motion is oscillatory. The angle has the following expression:

$$\theta(t) = 2\arcsin(\lambda\ \mathrm{sn}(t + \phi; \lambda)), \tag{4}$$

where $\lambda = v_0/2$ and $\phi = K(\lambda)$. K is the complete elliptic integral of the first kind,

$$K(\lambda) = \int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\sqrt{1 - \lambda^2 \sin^2 \varphi}}, \qquad 0 < \lambda < 1, \tag{5}$$

and $\mathrm{sn}(u; \lambda)$ is a Jacobi elliptic function. The period of this movement is

$$T = 4K\left(\frac{v_0}{2}\right). \tag{6}$$

(2) If $|v_0| > 2$ the movement is a rotation. The expression for the angle is

$$\theta(t) = 2\arcsin(\mathrm{sn}(\lambda t; 1/\lambda)). \tag{7}$$

The sign of v_0 determines the direction of rotation; if $v_0 > 0$, it is counterclockwise, otherwise it is clockwise. The period in this case is

$$T = \frac{4}{v_0} K\left(\frac{2}{v_0}\right). \tag{8}$$

(3) If $|v_0| = 2$ there are two homoclinic orbits connecting the unstable equilibrium with itself. In this case, the motion is not periodic (it may be considered as the limit of a cycle when the period goes to infinity), and the expression is as follows:

$$\theta(t) = 4\arctan(\tanh(t/2)). \tag{9}$$

The three types of solutions mentioned above are depicted in figure 1.

3. Periodic solutions with the homotopy analysis method

Now we show how to apply the HAM to calculate periodic solutions of nonlinear ordinary differential equations.

Consider, for example, the second-order differential equation given by

$$f(y''(\tau),\ y'(\tau),\ y(\tau)) = 0, \tag{10}$$

where f is a nonlinear function. Suppose that there is a periodic solution of this equation with frequency ω and amplitude a (the maximum displacement of the periodic solution). After making the replacements $t = \omega\tau$ and $y = ax$, the normalized equation becomes

$$f(\omega^2 a\ddot{x}(t),\ \omega a\dot{x}(t),\ ax(t)) = 0. \tag{11}$$

In the new variables the periodic solution has unit frequency and amplitude; the last equation can be written as $N[x, \omega, a] = 0$, N being an appropriate nonlinear differential operator, algebraically dependent on ω and a. We can perform a similar change of variables in other differential equations and even in systems of equations. Different unknowns, such as frequency and amplitude, are transformed into constants to be determined.

In the general case, we consider a nonlinear differential operator

$$N[x, g_1, g_2, \ldots, g_m] = 0, \tag{12}$$

where $x(t) \in \mathbb{R}^n$ and $g_i \in \mathbb{R}$, $i = 1, \ldots, m$, are unknown constants. The system has given initial conditions, which we call IC.

To find a periodic solution $x_P(t)$ of (12) we consider the family of operators $\mathcal{H}_q$ which depend on a deformation parameter $q \in [0, 1]$:

$$\mathcal{H}_q[\phi] = (1-q)\,\mathcal{L}[\phi - x_0] - q\,h\,\mathcal{N}_q[\phi], \tag{13}$$

where $\phi(t, q)$ is an homotopy that we construct with the method, $h \neq 0$ is a real parameter, $x_0(t)$ is an initial approximation which verifies the ICs and $\mathcal{L}$ is a linear operator associated with the system (12). In each equation studied, $x_0(t)$, h and $\mathcal{L}$ must be properly chosen. Finally, $\mathcal{N}_q$ is the nonlinear operator

$$\mathcal{N}_q[\phi] = N[\phi(t, q), \gamma_1(q), \gamma_2(q), \ldots, \gamma_m(q)]. \tag{14}$$

The procedure is based on a search for functions $\phi(t, q), \gamma_1(q), \ldots, \gamma_m(q)$, analytical in q such that

(i) $\mathcal{H}_q[\phi] = 0$ for $q \in [0, 1]$,
(ii) $\phi(t, q)$ verify the ICs for $q \in [0, 1]$.

If these functions exist, then taking $q = 0$ we have

$$\mathcal{H}_0[\phi] = \mathcal{L}[\phi(t, 0) - x_0(t)] = 0. \tag{15}$$

Then, as $\phi(t, q)$ and $x_0(t)$ verify the same ICs, we obtain $\phi(t, 0) = x_0(t)$. In addition, if $q = 1$,

$$\mathcal{H}_1[\phi] = -h\mathcal{N}_1[\phi(t, 1)] = 0, \tag{16}$$

thereafter $x_P(t) = \phi(t, 1)$, $g_1 = \gamma_1(1), \ldots, g_m = \gamma_m(1)$ will be the solution of the system (12). Thus, when the parameter q varies from 0 to 1, the function $\phi(t, q)$ varies from the initial approximation $x_0(t)$ to the desired solution $x_P(t)$.

To find the analytic functions $\phi(t, q), \gamma_1(q), \ldots, \gamma_m(q)$ we consider their series expansions

$$\phi(t, q) = \sum_{k=0}^{+\infty} x_k(t) q^k, \quad \gamma_1(q) = \sum_{k=0}^{+\infty} g_{1k} q^k, \ldots, \gamma_m(q) = \sum_{k=0}^{+\infty} g_{mk} q^k. \tag{17}$$

Replacing these series in $\mathcal{H}_q[\phi] = 0$ and taking the kth derivative with respect to q at $q = 0$, we obtain

$$\mathcal{L}[x_k(t) - (1 - \delta_{1k})x_{k-1}(t)] = \frac{h}{(k-1)!} \left.\frac{\partial^{k-1}\mathcal{N}_q[\phi]}{\partial q^{k-1}}\right|_{q=0}. \tag{18}$$

Similarly, taking the series expansion of ϕ and considering that $\phi(t, 0) = x_0(t)$ verifies the ICs, we obtain $x_k(0) = x_k'(0) = 0$ for $k \geqslant 1$.

The terms $x_k(t)$ in the series expansion of $\phi(t, 1)$ are calculated by solving the above equation with the stated conditions. As the solution we want to find is periodic, each term should be. Then, depending on the linear operator we take, the right-hand side of (18) must verify certain conditions to ensure that the kth term does not contain non-periodic functions (of type $t \cos t$ or $t \sin t$). These conditions allow us to calculate the terms g_{ik}, $i = 1, \dots, m$. For $k = 1$, it yields a nonlinear system of equations with unknowns $g_{10}, \dots, g_{m0}$, while for $k \geqslant 2$ the system is linear. For each k the corresponding system is solved and $x_k(t)$ is calculated; this procedure is repeated up to the desired order.

It remains to determine an appropriate value for h. The approximations of the constants g_i, $i = 1, \dots, m$, are polynomials in h, and so will be $x_P(t)$ and its derivatives for fixed t. As shown in Liao's book [3] observation of the behaviour of these polynomials is necessary to select an appropriate value for h. For values of this parameter for which the series is convergent, such polynomials converge to a value that is independent of h, as the order goes to infinity. Thus, the polynomial plots give us a rough idea of the place where these intervals are found and therefore we select an appropriate value for h.

4. Vibrations

Consider equation (2); suppose that there is a periodic solution with frequency and amplitude ω and a, respectively. Rescaling the variables t and θ (but keeping their names to simplify the notation), we obtain

$$\omega^2 a\ddot{\theta}(t) + \sin(a\theta(t)) = 0. \tag{19}$$

The new periodic solution $\theta_P(t)$ will be of unit frequency and amplitude. In particular, we can impose the following initial conditions:

$$\theta_P(0) = 1 \quad \text{and} \quad \dot{\theta}_P(0) = 0. \tag{20}$$

From equation (19) we define

$$\begin{aligned}\mathcal{N}_q[\phi] &= N[\phi(t, q), \Omega(q), A(q)] \\ &= \Omega(q)^2 A(q)\frac{\partial^2\phi(t, q)}{\partial t^2} + \sin(A(q)\phi(t, q))\end{aligned} \tag{21}$$

and take the linear operator

$$\mathcal{L}[\phi] = \frac{\partial^2\phi}{\partial t^2} + \phi. \tag{22}$$

Using the HAM described in section 3, $\theta_0(t)$ being the initial approximation, and replacing in $\mathcal{H}_0[\phi] = 0$ we have

$$\phi(t, 0) - \theta_0(t) = c_1 \cos t + c_2 \sin t, \qquad c_1,\ c_2 \in \mathbb{R}, \tag{23}$$

but as $\phi(t, q)$ and $\theta_0(t)$ must verify conditions (20), then $\phi(t, 0) = \theta_0(t)$.

Moreover, the periodic solution of (19) is

$$\theta_P(t) = \phi(t, 1) = \sum_{k=0}^{+\infty} \theta_k(t), \tag{24}$$

and we have

$$\omega = \Omega(1) = \sum_{k=0}^{+\infty} \omega_k, \qquad a = A(1) = \sum_{k=0}^{+\infty} a_k. \tag{25}$$

To start with the method we need to define an initial function $\theta_0(t)$. Considering the initial conditions (20) we choose $\theta_0(t) = \cos t$. Then, according to operators (21) and (22), equation (18) for $k = 1$ is

$$\ddot{\theta}_1(t) + \theta_1(t) = h\big(- \omega_0^2 a_0 \cos t + \sin(a_0 \cos t)\big), \tag{26}$$

with ICs $\theta_1(0) = \dot{\theta}_1(0) = 0$. This equation can be solved by the method of variation of parameters; we obtain

$$\theta_1(t) = \frac{h}{a_0} \cos t (\cos a_0 - \cos(a_0 \cos t)) + h \sin t \left(-\frac{1}{2}\omega_0^2 a_0 t + \int_0^t \cos s \sin(a_0 \cos s)\, \mathrm{d}s \right). \tag{27}$$

It is straightforward to see that the term $\theta_1(t)$ thus defined is not necessarily periodic. We can make non-periodic terms disappear by setting certain values of ω_0 and a_0. More precisely, if we cancel the coefficients of $\cos t$ and $\sin t$ on the right-hand side of (26), we obtain a periodic function $\theta_1(t)$. Using the Jacobi–Anger expansion in (26), we obtain

$$\ddot{\theta}_1(t) + \theta_1(t) = h \left(-\omega_0^2 a_0 \cos t + 2 \sum_{n=0}^{+\infty} (-1)^n J_{2n+1}(a_0) \cos((2n+1)t) \right), \tag{28}$$

where J_n is the Bessel function of the first kind of order n. Then, the term $\theta_1(t)$ will be periodic if the coefficient of $\cos t$ vanishes, that is, if $-\omega_0^2 a_0 + 2J_1(a_0) = 0$, which implies

$$\omega_0 = \sqrt{\frac{2J_1(a_0)}{a_0}}. \tag{29}$$

Setting a value of a_0 the above equation allows us to obtain a value of ω_0 and to calculate the term $\theta_1(t)$. We obtain

$$\theta_1(t) = 2h \sum_{n=1}^{+\infty} (-1)^n \frac{J_{2n+1}(a_0)}{1 - (2n+1)^2} (\cos((2n+1)t) - \cos t). \tag{30}$$

Now, we are able to find the equation for the term $\theta_2(t)$. Considering $k = 2$ in (18), we obtain

$$\begin{aligned} \ddot{\theta}_2(t) + \theta_2(t) = \ddot{\theta}_1(t) + \theta_1(t) + h\big(&a_0 \omega_0^2 \ddot{\theta}_1(t) - (\omega_0^2 a_1 + 2\omega_0\omega_1 a_0) \cos t \\ &+ (a_0 \theta_1(t) + a_1 \cos t) \cos(a_0 \cos t)\big). \end{aligned} \tag{31}$$

After replacing ω_0 defined in (29), $\theta_1(t)$ already calculated and using the Jacobi–Anger expansion of $\cos(a_0 \cos t)$, we can calculate $\theta_2(t)$ by solving the equation with ICs $\theta_2(0) = \dot{\theta}_2(0) = 0$. As in the previous case, the term $\theta_2(t)$ will be periodic if the coefficient of $\cos t$ vanishes. This gives a linear condition involving ω_1 and a_1. Setting a value of a_1, we can obtain a value of ω_1. At this point, the hand calculations become very laborious; however, they can be easily performed with symbolic computation programs.

The calculations for values of $k \geqslant 2$ are performed with Mathematica considering the following Jacobi–Anger expansions:

$$\begin{aligned} \sin(a_0 \cos t) &= 2(J_1(a_0) \cos(t) - J_3(a_0) \cos(3t)) \\ \cos(a_0 \cos t) &= J_0(a_0) + 2(-J_2(a_0) \cos(2t) + J_4(a_0) \cos(4t)). \end{aligned} \tag{32}$$

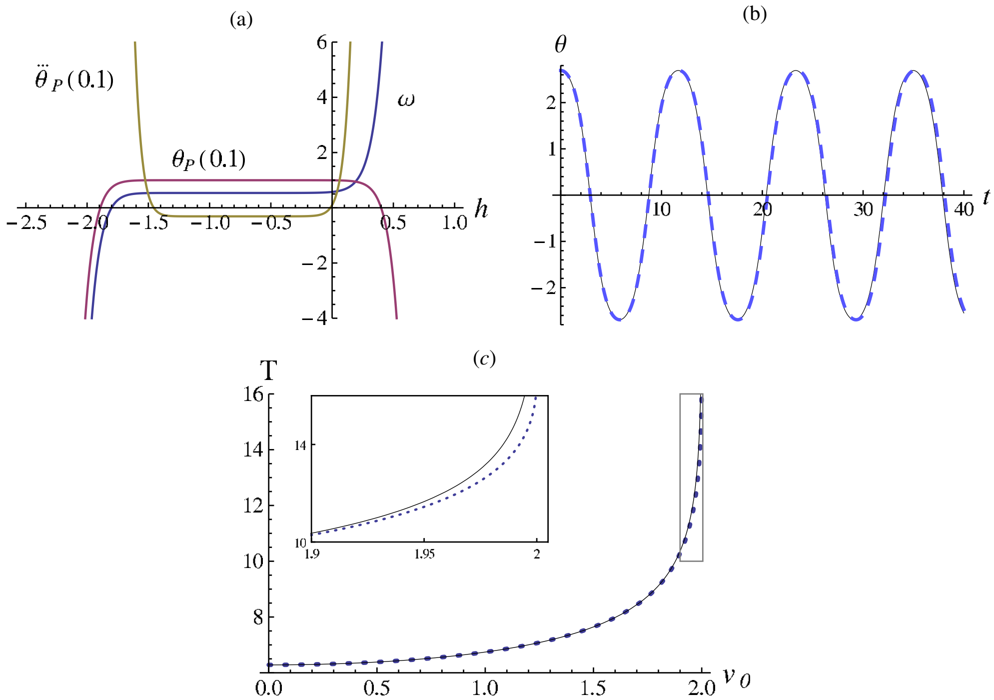

Figure 2. (a) Polynomial function of h for $v_0 = 1.95$, (b) (——) exact and (- - - -) approximate solution with the HAM for $h = -0.8$ and (c) comparison periods: (——) exactly as in (6) and $(\cdots\cdots)$ $2\pi/\omega^{\text{vib}}$.

The mean square error (in a period) for this approximations is less than 0.01 for all $a_0 < 2.7$.

We see that for each k the condition of periodicity gives a single equation (corresponding to cancel the coefficient of $\cos t$ in the right-hand side of (18)), which provides the relationship that must verify the ω_{k-1} and a_{k-1} values. Therefore, as in the case $k = 1$, by setting the value of a_{k-1} we can obtain ω_{k-1} and $\theta_k(t)$, and continue the process until the desired order.

But, how do we set the values of a_k for $k \geqslant 0$? The simple pendulum equation has a centre at the trivial equilibrium. Moreover, for each IC $\theta(0) \in (0, \pi)$, $\dot{\theta}(0) = 0$, there is a periodic solution of (2). Then, if we take $a_0 \in (0, \pi)$ and $a_k = 0$ for $k \geqslant 1$, the HAM allows us to calculate the solution $\theta_P(t)$ of (19) and the frequency ω corresponding to the amplitude $a = a_0$.

4.1. Numerical results

For an oscillation of amplitude a, the energy equation (3) gives us the corresponding velocity v_0 at $\theta = 0$. Then, setting $a = \arccos\left(1 - v_0^2/2\right)$ and using the above process, we obtain expressions of the solution and the frequency depending on the initial velocity $0 < v_0 < 2$.

As an example, we obtain the corresponding approximation of the solution up to order 15 for $v_0 = 1.95$. Polynomials in h for ω, $\theta_P(0.1)$ and $\dddot{\theta}_P\ (0.1)$ are shown in figure 2(a).

The expressions of these polynomials are

$$\omega = 0.270\,673h^{15} + 3.123\,59h^{14} + 16.9264h^{13} + 57.1904h^{12} + 134.906h^{11} + 235.701h^{10} + 315.718h^{9} + 331.011h^{8} + 274.825h^{7} + 181.545h^{6} + 95.2529h^{5} + 39.3305h^{4} + 12.5197h^{3} + 2.944\,37h^{2} + 0.465\,156h + 0.573\,929$$

$$\theta_P(0.1) = -0.091\,4114h^{16} - 1.0801h^{15} - 6.001\,75h^{14} - 20.828h^{13} - 50.5537h^{12} - 91.0661h^{11} - 126.052h^{10} - 136.917h^{9} - 118.12h^{8} - 81.3688h^{7} - 44.7303h^{6} - 19.4903h^{5} - 6.635\,02h^{4} - 1.719\,43h^{3} - 0.323\,528h^{2} - 0.040\,1146h + 0.995\,004 \tag{33}$$

$$\ddot{\theta}_P\,(0.1) = 40.5402h^{16} + 469.514h^{15} + 2553.42h^{14} + 8657.97h^{13} + 20492.h^{12} + 35911.1h^{11} + 48221.1h^{10} + 50637.2h^{9} + 42054.h^{8} + 27737.4h^{7} + 14497.3h^{6} + 5950.09h^{5} + 1883.72h^{4} + 445.934h^{3} + 74.7369h^{2} + 7.96799h + 0.0998334.$$

As mentioned in the previous section, we can easily choose an appropriate value for h, for example we can choose $h = -0.8$; the solution for this value gives

$$\theta_P(t) = 2.88495\cos(0.536173t) - 0.20478\cos(1.60852t) + 0.0154932\cos(2.68086t) - 0.00246577\cos(3.75321t) + 0.000277749\cos(4.82555t) - 0.0000354327\cos(5.8979t) + 5.09679\times10^{-6}\cos(6.97024t) - 7.10179\times10^{-7}\cos(8.04259t) + 1.01719\times10^{-7}\cos(9.11494t) - 1.43853\times10^{-8}\cos(10.1873t) + 2.00188\times10^{-9}\cos(11.2596t) - 2.71269\times10^{-10}\cos(12.332t) + 3.57771\times10^{-11}\cos(13.4043t) - 3.68269\times10^{-12}\cos(14.4767t) + 1.66554\times10^{-13}\cos(15.549t) + 1.96159\times10^{-14}\cos(16.6214t) - 9.50801\times10^{-15}\cos(17.6937t). \tag{34}$$

Figure 2(b) shows the calculated solution along with the exact one given in (4).

On the other hand, using the method described here it is possible to calculate an approximation to the frequency ω as a function of the amplitude and therefore of the initial velocity v_0. So we can obtain the following approximation of order 2:

$$\omega^{\mathrm{vib}} = \sqrt{\frac{2J_1(a)}{a}} - \frac{1}{10}\frac{(-aJ_0(a) + 2J_1(a) + aJ_4(a))J_3(a)}{\sqrt{2aJ_1(a)}}, \tag{35}$$

where $a = \arccos(1 - v_0^2/2)$.

Different approximations for the exact period (6) are known. They can be calculated by the series expansion of the elliptic integral, harmonic balance method, etc (see [9], [10] or [11] and references therein). Figure 2(c) compares the period (6) with our result $2\pi/\omega^{\mathrm{vib}}$. Note the almost exact match for the speed $v_0 < 1.9$; the relative error is less than 0.5%.

5. Rotations

The method as developed in section 4 is based on setting the amplitudes of the vibrations. The definition of the amplitudes we have used does not even make sense in rotations. Also the function $\theta(t)$ that represents a rotation is not strictly periodic because the movement does not occur within a single coordinate chart. In order to apply the HAM for finding rotating solutions of the pendulum in a straightforward manner we map the cylinder into the punctured plane, i.e. the plane without the origin. The aim of this change is to solve the equations for a complete rotation in the domain of a single chart. The relationship with the angular coordinates is as follows:

$$\begin{cases} u = \mathrm{e}^{\dot\theta}\cos\theta \\ v = \mathrm{e}^{\dot\theta}\sin\theta \end{cases}. \tag{36}$$

The pendulum equations in the new coordinates are

$$\begin{cases} \dot{u} = -uv(u^2+v^2)^{-1/2} - \frac{1}{2}v\ln(u^2+v^2) \\ \dot{v} = -v^2(u^2+v^2)^{-1/2} + \frac{1}{2}u\ln(u^2+v^2) \end{cases}. \tag{37}$$

Suppose that there is a periodic solution with frequency ω such that $(u(0), v(0)) = (e^{\xi}, 0)$. We normalize the system (37) so that the new periodic solution $(u_P(t), v_P(t))$ has frequency 1 and verifies $(u_P(0), v_P(0)) = (1, 0)$ (to simplify the notation we keep the names of variables after the normalization); hence, we obtain

$$\begin{cases} \omega\dot{u} = -uv\left(u^2+v^2\right)^{-1/2} - v\xi - \frac{1}{2}v\ln(u^2+v^2) \\ \omega\dot{v} = -v^2(u^2+v^2)^{-1/2} + u\xi + \frac{1}{2}u\ln(u^2+v^2) \end{cases}. \tag{38}$$

To find periodic solutions of the previous system we apply the HAM with the operator $\mathcal{L}$ defined by

$$\mathcal{L}[\phi_1, \phi_2] = \begin{pmatrix} \frac{\partial}{\partial t} & 1 \\ -1 & \frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \phi_1}{\partial t} + \phi_2 \\ -\phi_1 + \frac{\partial \phi_2}{\partial t} \end{pmatrix}, \tag{39}$$

and $\mathcal{N}_q$ given by

$$\begin{aligned} \mathcal{N}_q[\phi_1, \phi_2] &= N[(\phi_1(t,q), \phi_2(t,q)), \Omega(q), \Xi(q)] = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \\ &= \begin{pmatrix} \Omega\,\partial\phi_1/\partial t + \phi_1\phi_2\left(\phi_1^2+\phi_2^2\right)^{-1/2} + \phi_2\Xi + \frac{1}{2}\phi_2\ln\left(\phi_1^2+\phi_2^2\right) \\ \Omega\,\partial\phi_2/\partial t + \phi_2^2\left(\phi_1^2+\phi_2^2\right)^{-1/2} - \phi_1\Xi - \frac{1}{2}\phi_1\ln\left(\phi_1^2+\phi_2^2\right) \end{pmatrix}. \end{aligned} \tag{40}$$

Thus, replacing in $\mathcal{H}_0[\phi_1, \phi_2] = 0$ we have

$$\begin{pmatrix} \phi_1(t,0) \\ \phi_2(t,0) \end{pmatrix} - \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix} = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}, \qquad c_1, c_2 \in \mathbb{R}, \tag{41}$$

but as $(\phi_1(t,q), \phi_2(t,q))^T$ and $(u_0(t), v_0(t))^T$ must verify the same conditions, $(\phi_1(t,0), \phi_2(t,0))^T = (u_0(t), v_0(t))^T$. Further, according to the HAM the periodic solution of (38) is

$$\begin{pmatrix} u_P(t) \\ v_P(t) \end{pmatrix} = \begin{pmatrix} \phi_1(t,1) \\ \phi_2(t,1) \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{+\infty} u_k(t) \\ \sum_{k=0}^{+\infty} v_k(t) \end{pmatrix}, \tag{42}$$

and we obtain

$$\omega = \Omega(1) = \sum_{k=0}^{+\infty} \omega_k, \qquad \xi = \Xi(1) = \sum_{k=0}^{+\infty} \xi_k. \tag{43}$$

According to the conditions of the periodic solutions at $t = 0$, we take $(u_0(t), v_0(t))^T = (\cos t, \sin t)^T$. Equation (18) for $k = 1$ is

$$\begin{cases} \dot{u}_1(t) + v_1(t) = h(-\omega_0\sin t + \cos t\sin t + \xi_0\sin t) \\ -u_1(t) + \dot{v}_1(t) = h(\omega_0\cos t + \sin^2 t - \xi_0\cos t) \end{cases}, \tag{44}$$

with initial conditions $u_1(0) = v_1(0) = 0$. Considering the inverse operator of $\mathcal{L}$, the term $(u_1(t), v_1(t))^T$ will be periodic if the coefficients of $\cos t$ and $\sin t$ vanish in the following expression:

$$\left.\left(\frac{\partial}{\partial t}N_1 - N_2\right)\right|_{q=0} = 2(-\omega_0 + \xi_0)\cos(t) + \frac{1}{2}(-1 + 3\cos(2t)); \tag{45}$$

then, the term is periodic if $\omega_0 = \xi_0$. Now, we can solve equation (44) to obtain

$$\begin{cases} u_1(t) = h(\cos(t) + \frac{1}{2}(-1 - \cos(2t))) \\ v_1(t) = h\big(\sin(t) - \frac{1}{2}\sin(2t)\big) \end{cases}. \tag{46}$$

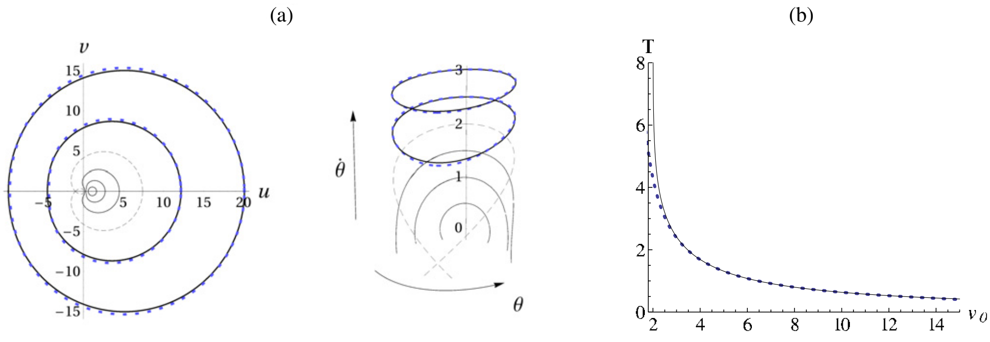

Figure 3. (a) (——) exact rotating solutions and (- - - -) approximate solution with HAM in both coordinate systems, (b) comparison periods: (——) exactly as in (8), and $(\cdots\cdots)$ $2\pi/\omega^{\rm rot}$.

For each $k \geqslant 2$, taking the inverse operator of $\mathcal{L}$, the term $(u_k(t), v_k(t))^T$ will be periodic if the coefficients of $\cos t$ and $\sin t$ must vanish in the following expression:

$$\left(\frac{\partial}{\partial t}\left(\frac{\partial^{k-1}N_1}{\partial q^{k-1}}\right) - \frac{\partial^{k-1}N_2}{\partial q^{k-1}}\right)\bigg|_{q=0}. \tag{47}$$

As in the previous section, one of these coefficients vanishes for all k and the remaining give a relation between ω_{k-1} and ξ_{k-1}. The first of these conditions means that $\omega_0 = \xi_0$. It is appropriate to fix the value $\xi_0 = \dot\theta(0) > 2$, and $\xi_k = 0$ if $k \geqslant 1$. If $\dot\theta(0) < -2$ the direction of the solutions changes and in this case the appropriate initial function is $(u_0(t), v_0(t))^T = (\cos t, -\sin t)^T$. Given the symmetry of the rotational solutions we only consider the case $\xi_0 > 2$.

The terms ω_{k-1} and $(u_k(t), v_k(t))^T$ can be calculated for a fixed ξ_{k-1}. Then, the solution $(u_P(t), v_P(t))^T$ and frequency ω can be determined for the initial condition $(\mathrm{e}^{\xi}, 0)$ to the desired order. In the original coordinates the solution found corresponds to the rotational solution with velocity $v_0 = \xi > 2$ for $\theta = 0$.

5.1. Numerical results

Given a value $v_0 > 2$, we can calculate the corresponding rotation. As in the previous case, we choose a suitable value for h after performing the calculations until the desired order. Figure 3(a) shows the exact solution (7) together with the approximated one calculated by the HAM to order 10 for $v_0 = 2.5$ and 3. The solutions are shown in space (u, v) and in the cylinder. We choose $h = -0.3$ for the first of these values of v_0 and the expression of the solution is

$$\begin{aligned}
u_P(t) &= 1.00598 + 7.61297\cos(1.99018t) + 2.83201\cos(3.98036t)\\
&+ 0.613502\cos(5.97053t) + 0.10334\cos(7.96071t) + 0.0133044\cos(9.95089t)\\
&+ 0.00129487\cos(11.9411t) + 0.0000891894\cos(13.9312t)\\
&+ 4.33837\times 10^{-6}\cos(15.9214t) + 1.163544\times 10^{-7}\cos(17.9116t)\\
&+ 4.043190\times 10^{-9}\cos(19.9018t) - 1.267310\times 10^{-10}\cos(21.892t)\\
v_P(t) &= 7.61297\sin(1.99018t) + 2.79792\sin(3.98036t) + 0.610272\sin(5.97053t)\\
&+ 0.103039\sin(7.96071t) + 0.0132836\sin(9.95089t) + 0.00129386\sin(11.9411t)\\
&+ 0.0000891479\sin(13.9312t) + 4.338662\times 10^{-6}\sin(15.9214t)\\
&+ 1.162647\times 10^{-7}\sin(17.9116t) + 4.055239\times 10^{-9}\sin(19.9018t)\\
&- 1.267310\times 10^{-10}\sin(21.892t).
\end{aligned} \tag{48}$$

In addition for $v_0 = 3$ we choose $h = -0.2$. The expression of the periodic solution in this case is

$$u_P(t) = 1.70325 + 14.0746\cos(2.61725t) + 3.68573\cos(5.2345t) + 0.551584\cos(7.85176t) + 0.0644068\cos(10.469t) + 0.00556285\cos(13.0863t) + 0.000358114\cos(15.7035t) + 0.0000152767\cos(18.3208t) + 5.073228\times 10^{-7}\cos(20.938t) + 4.111218\times 10^{-9}\cos(23.5553t) + 4.416404\times 10^{-10}\cos(26.1725t) \tag{49}$$

$$v_P(t) = 14.0746\sin(2.61725t) + 3.65976\sin(5.2345t) + 0.549794\sin(7.85176t) + 0.0643009\sin(10.469t) + 0.00555781\sin(13.0863t) + 0.000357997\sin(15.7035t) + 0.0000152709\sin(18.3208t) + 5.074780\times 10^{-7}\sin(20.938t) + 4.101773\times 10^{-9}\sin(23.5553t) + 4.418700\times 10^{-10}\sin(26.1725t).$$

Moreover, in this case it is also possible to calculate the frequency approximations ω, for the initial velocity v_0. The approximation to order 8 is

$$\omega^{\mathrm{rot}} = -1.40935 + 1.74363v_0 - 0.214217v_0^2 + 0.0381421v_0^3 - 0.00436724v_0^4 + 0.000317095v_0^5 - 0.0000134445v_0^6 + 2.56289\times 10^{-7}v_0^7. \tag{50}$$

Figure 3(b) compares the period (8) with that calculated from the previous approximation. For values of v_0 between 3.2 and 15, the relative error of the approximation does not exceed 1%. For values $v_0 < 3.2$ the error increases considerably, but can be improved by calculating higher order approximations.

6. Conclusions

In this paper, we have applied the HAM to find analytical expressions for oscillating and rotating periodic solutions of the simple pendulum. We mapped the cylinder to the punctured plane to calculate the rotating solutions and apply the HAM to study the resulting system of differential equations. The approximations of both types of solutions are very good compared with the exact solutions, which are calculated using Jacobi elliptic functions. In addition, the method allows us to obtain expressions for the frequency of the periodic solutions. It is noted that the corresponding period is in very good agreement with the exact one, for a wide range of initial velocities. If necessary, this approach can be improved by increasing the number of calculated terms. Therefore we show how a comprehensive study, in a pedagogical example, can be made using the HAM, without the use of complex mathematics. We show that it is a suitable and effective method for studying the dynamics and solutions of differential equations.

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