Spectral behavior of contractive noise

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We study the behavior of the spectra corresponding to quantum systems subjected to a contractive noise, i.e., the environment reduces the accessible phase space of the system, but the total probability is conserved. We find that the number of long-lived resonances grows as a power law in $\hbar$, but surprisingly there is no relationship between the exponent of this power law and the fractal dimension of the corresponding classical attractor. This is in disagreement with the predictions of the fractal Weyl law which has been established for open systems, where the probability is lost under the effect of a projective opening.

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I. INTRODUCTION

In recent years there has been a great interest in the study of open quantum systems. The motivation for this upsurge can be found in the recent development of quantum computation and information [1,2] (even though some related aspects were under investigation long ago, as in scattering theory, for example). The noise coming from the environment has always been the major drawback for any realistic attempt to implement a quantum computer and a serious obstacle for quantum information purposes. At the same time we have realized that we know much less about quantum open systems than about their closed counterparts. Therefore, their study has become an extremely active topic in fundamental physics [3].

It is in this context and for the case of scattering systems that the fractal Weyl law for the number of long-lived resonances was conjectured and tested by means of several examples [4]. The classical invariant distribution in these cases, i.e., the fractal hyperbolic set of all the trajectories nonescaping in the past and future (the repeller), plays a fundamental role with respect to the quantum spectrum of quasibound states (the resonances). Indeed, the aforementioned law says that [5] the number of resonances that decay at the slowest pace (the long-lived ones) grows as $\hbar^{-(1+d_H)}$, where d_H is the partial Hausdorff dimension of the repeller. We can trace back these studies to the proposal and proof of a fractal Weyl upper bound for a Hamiltonian flow showing a fractal trapped set [6]. Later, results that seem to strongly support the validity of the conjectured law have been obtained for different kinds of systems ranging from smooth to hard-wall potentials [7]. But it is in open quantum maps that the fractal Weyl law was more easily tested [8]. In these systems, paradigmatic in the quantum chaos literature, the resonances have been found to grow as $\hbar^{-d}$, where d is the partial fractal dimension of the repeller. However, explorations of the spectral behavior of quantum systems having different kinds of invariant classical distributions associated with them are very scarce. In this sense it is very interesting to ask what happens in the case of a nonprojective kind of opening, i.e., a contractive environment, for example. Here we will focus on dissipative quantum operations, whose action can be interpreted as a phase space contraction leading to dissipative dynamics [9]. There were very few attempts to study the spectral properties of this kind of system in previous works. The only antecedent we were able to find in the literature is the paper by Ermann and Shepelyansky [10]. But in this case only the spectrum of a discretized Perron-Frobenius operator was considered. For this approximation to the classical problem (i.e., not a quantization of it), the authors found that the behavior of the corresponding long-lived resonances follows the fractal Weyl law.

In this work we analyze the dissipative baker map. To obtain the quantum counterpart we have implemented a standard procedure in which the noise superoperator is written in terms of appropriately defined Kraus operators [11]. For this model, all classical initial conditions fall asymptotically on a strange attractor. At the quantum level, this is represented by a resonance with eigenvalue 1 and the rest of the spectrum lying inside the unit circle. In sharp contrast to what has been found for the discretized classical dynamics, the number of long-lived resonances of the superoperator grows as a power law in $\hbar$, but the exponent is rather insensitive to the dimension of the fractal invariant set.

This paper is organized as follows: In Sec. II we present the definition and details of the model we have used to study this kind of spectrum. In Sec. III the numerical results are analyzed, and we present an interpretation of them supported by further studies. Finally, we draw the conclusions in Sec. IV.

II. MODEL SYSTEM

One of the simplest models one can think of for studying complex systems are chaotic maps. Regardless of their simplicity they capture all the essential features of chaotic behavior and their quantization allows these advantages to be extended to quantum mechanics. All this turned them into paradigmatic models for quantum chaos and the theory of dissipative systems [12–15]. We have investigated the spectral behavior of the dissipative baker map. The classical map is defined on the two-torus $\mathcal{T}^2 = [0,1) \times [0,1)$ by

$$\mathcal{B}(q,p) = \begin{cases} (2q, \epsilon\, p/2) & \text{if} \quad 0 \leqslant q < 1/2, \\ (2q-1, (\epsilon\, p+1)/2) & \text{if} \quad 1/2 \leqslant q < 1. \end{cases} \quad (1)$$

This transformation is an area-contracting, piecewise-linear map. The map contracts the torus in the p direction by a factor ϵ, stretches the unit square by a factor of 2 in the q direction, squeezes it by the same factor in the p direction, and then stacks the right half onto the left one. Any initial distribution falls asymptotically into a fractal set, the strange attractor.

When quantizing this system any state $|\psi\rangle$ must satisfy periodic boundary conditions on the torus, for both the position and momentum representations. This amounts to taking $\langle q+1|\psi\rangle = e^{i2\pi\chi_q}\langle q|\psi\rangle$ and $\langle p+1|\psi\rangle = e^{i2\pi\chi_p}\langle p|\psi\rangle$, with $\chi_q,\chi_p \in [0,1)$. This implies a Hilbert space of finite dimension $N = (2\pi\hbar)^{-1}$. The discrete set of position and momentum eigenstates is given by $|q_j\rangle = |(j+\chi_q)/N\rangle$ $(j = 0,1,\ldots,N-1)$, and $|p_k\rangle = |(k+\chi_p)/N\rangle$ $(k = 0,1,\ldots,N-1)$, labeled by the corresponding eigenvalues q_j,p_k. They are related by a discrete Fourier transform, i.e.,

$$(G_N)_{kj} \equiv \langle p_k|q_j\rangle = \frac{1}{\sqrt{N}}\exp\left(\frac{-i2\pi}{N}(j+\chi_q)(k+\chi_p)\right).$$

Throughout the paper we assume a phase space with antisymmetric boundary conditions ($\chi_q = \chi_p = 1/2$). For an even-N-dimensional Hilbert space, the quantum baker map is defined in the momentum representation in terms of the discrete Fourier transform as [16,17]

$$B_N = \begin{pmatrix} G_{N/2} & 0 \\ 0 & G_{N/2} \end{pmatrix} G_N^{-1}, \qquad (2)$$

with B_N a unitary matrix that represents the quantum dynamics of the closed baker map.

As discussed in Ref. [9], quantum dissipative processes can be described by nonunital quantum operations. In this work the dissipative noise is implemented by an $N^2 \times N^2$ Kraus superoperator of the form

$$M = \sum_{\mu=0}^{N-1} A^{\mu} \otimes A^{\mu\dagger}, \qquad (3)$$

where

$$A^{\mu} = \sum_{i=\mu}^{N-1} \sqrt{\begin{pmatrix} i \\ i-\mu \end{pmatrix} \epsilon^{i-\mu}(1-\epsilon)^{\mu}}\, |p_{i-\mu}\rangle\langle p_i| \qquad (4)$$

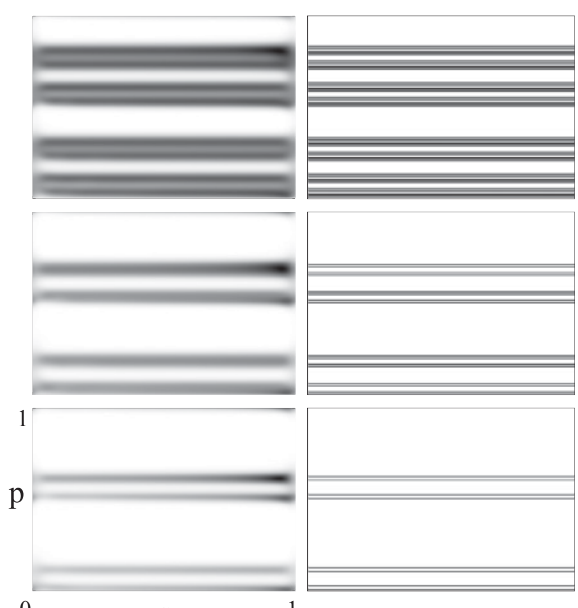

FIG. 1. Husimi representation of the invariant state of the quantum dissipative baker map for (from top to bottom) $\epsilon = 0.8,0.6,0.4$ and $N = 180$ (left column), and the corresponding classical attractors (right column).

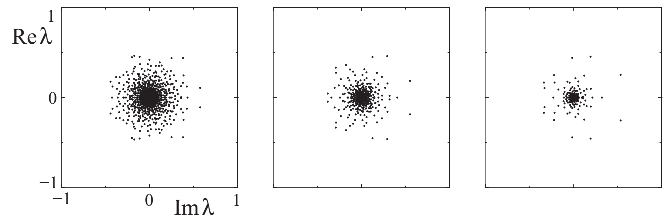

FIG. 2. Eigenvalues of the quantum dissipative baker map \$ in the complex plane for three different values of the dissipation. From left to right: $\epsilon = 0.8,0.6,0.4$. In all cases $N = 180$.

are operators accounting for transitions toward the momentum state $|p_{i=0}\rangle$. The coupling constant ϵ coincides with the dissipation parameter of the corresponding classical map. M is a trace-preserving ($\sum_\mu A_\mu^\dagger A_\mu = 1$) and nonunital ($\sum_\mu A_\mu A_\mu^\dagger \neq 1$) superoperator, which describes a process in which phase space volume is not preserved. Superoperators of this type are obtained from the integration of master equations derived from modeling a microscopic interaction of a system (an oscillator, a large spin, etc.) with a thermal bath at zero temperature representing the environment [18–20].

To describe the noisy evolution of the density matrix of the system we compose M, modeled by its Kraus superoperator form, with the unitary map (2), according to

$$\$ = (B_N \otimes B_N^\dagger) \circ M. \qquad (5)$$

An evolution of the density matrix specified in this way is known in the literature as a quantum operation [1,2].

III. RESULTS

The spectrum obtained by the diagonalization of the superoperator of Eq. (5) consists of a leading eigenvalue $\lambda = 1$ and $N^2 - 1$ complex eigenvalues λ [with $|\lambda| = \exp(\frac{-\gamma}{2})$] inside the unit circle [1,2]. Due to its non-normality, \$ has distinct left ψ_λ^L and right ψ_λ^R eigenoperators corresponding to each eigenvalue. As expected, the Husimi function of the invariant state ψ_0^R, having eigenvalue $\lambda = 1$, closely follows the fractal structure of the attractor corresponding to the classical dissipative map. This is shown in Fig. 1, for several values of ϵ. In the classical case the dissipation parameter ϵ determines the dimension of the attractor, $d = 1 + \ln(2)/[\ln(2) - \ln(\epsilon)]$ [21]. In the quantum case ϵ is related to the quantum phase space

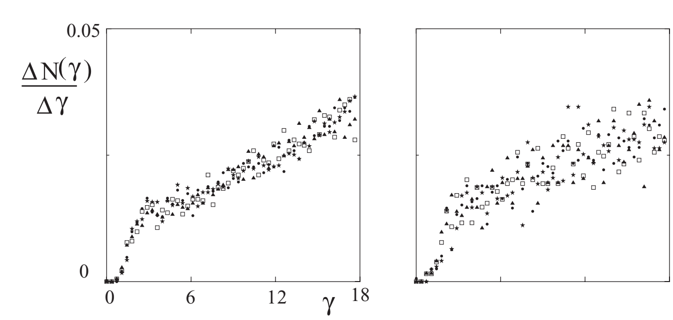

FIG. 3. Density of states $\Delta N(\gamma)/\Delta\gamma$ as a function of γ for different values of N [90 (▲),100 (□),150 (●),180 (⋆)] and two values of the dissipation $\epsilon = 0.8$ (left) and $\epsilon = 0.6$ (right). The densities are normalized according to $\int_0^{18} [\Delta N(\gamma)/\Delta\gamma] = 1$.

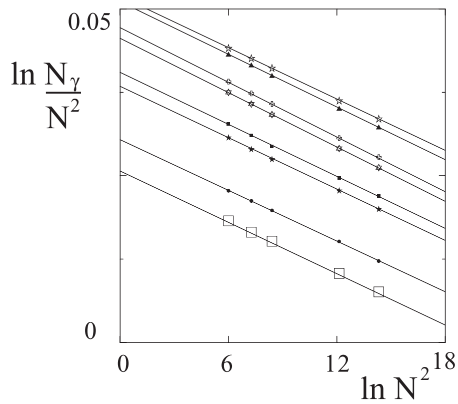

FIG. 4. Log-log plot for the fraction of long-lived resonances (with $\gamma < \gamma_{\rm cut}$) as a function of N^2 for $\epsilon = 0.8$ and eight values of the cutoff valuein the range $4 \leqslant \gamma_{\rm cut} \leqslant 20$. The fitted slopes are in the interval $0.76 \leqslant \beta \leqslant 0.82$.

contraction rate [9]. We have verified that, as N increases, the phase space representation of ψ_0^R reveals finer details of the attractor, reflecting the quantum-to-classical correspondence.

As a first approach to analyzing the behavior of the resonances, in Fig. 2 we show different spectra of $ for three values of the dissipation parameter $\epsilon = 0.8,0.6,0.4$ and a Hilbert space dimension $N = 180$. We notice that the longest-lived eigenvalues, although they change their positions in the complex plane, more or less keep their moduli, while the radius r_λ of the dense circle where most of the eigenvalues concentrate strongly shrinks for increasing values of the dissipation. However, this radius cannot be directly related to the parameter ϵ, in contrast to what is observed in Ref. [10] for the spectra of the discretized Perron-Frobenius operator. This gives to the quantum version a seemingly more contractive character in spectral terms.

To better show this point the differential radial density distributions defined as $\frac{\Delta N(\gamma)}{\Delta\gamma}$ are plotted in Fig. 3, for different values of N. These densities, normalized according to $\int_0^{18} \frac{\Delta N(\gamma)}{\Delta\gamma} = 1$, are practically N independent for the dimensions we are considering. They present a relative maximum at a decay rate $\gamma \ll -2\ln\epsilon$, and then they smoothly increase at larger values of γ. These profiles are very different from the ones obtained in Ref. [10] for the spectrum of the Perron-Frobenius operator, which peak around $\gamma = -2\ln\epsilon$, that is, the global relaxation rate to the strange attractor, and decay rapidly for larger γ.

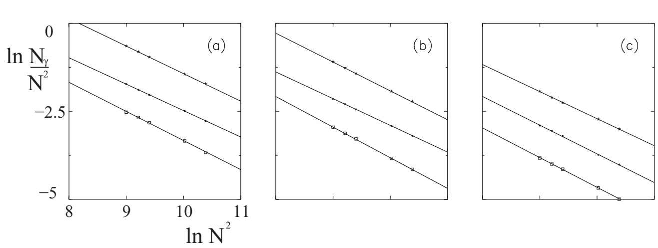

FIG. 5. Log-log plot for the fraction of long-lived resonances (with $\gamma < \gamma_{\rm cut}$) as a function of N^2 for $\epsilon = 0.8$ ($\star$), 0.6 ($\bullet$), and 0.4 ($\square$), with (a) $\gamma_{\rm cut} = 20$, (b) $\gamma_{\rm cut} = 15.2$, and (c) $\gamma_{\rm cut} = 9.2$. The fitted slopes are $\beta = 0.83,0.75,0.78$ for (a), $\beta = 0.87,0.76,0.82$ for (b), and $\beta = 0.84,0.82,0.77$ for (c).

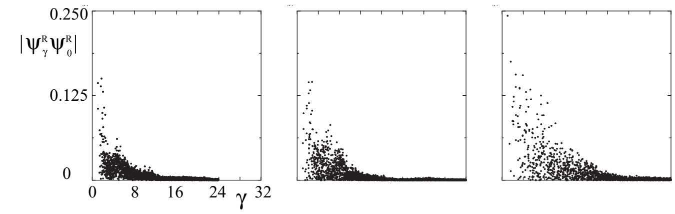

FIG. 6. Overlap $|(\psi_\lambda^R \psi_0^R)|$ as a function of γ for different values of the dissipation parameter $\epsilon = 0.8,0.6,0.4$ (from left to right) and $N = 180$.

We now come to the central point of this paper. We want to see how the differences between a full quantization and a discretization procedure of phase space which are suggested by the comparison of the corresponding spectra affect the adherence to the fractal Weyl law. Figure 4 displays the fraction of long-lived resonances $\frac{N_{\gamma<\gamma_{\rm cut}}}{N^2}$ as a function of N^2 for $\epsilon = 0.8$ and values of $\gamma_{\rm cut}$ ranging from 4 to 20. It can be clearly seen that this fraction scales as $\frac{N_{\gamma<\gamma_{\rm cut}}}{N^2} \propto (N^2)^{-\beta}$ with an exponent $\beta \sim 0.8$ fairly independent of the cutoff value $\gamma_{\rm cut}$ used to distinguish between the long-lived states and the short-lived ones.

The existence of a power law dependence for $\frac{N_{\gamma<\gamma_{\rm cut}}}{N^2}$ and the insensitivity of the exponent to the choice of the cutoff value $\gamma_{\rm cut}$ are fundamental properties of the fractal Weyl law which was originally conjectured for projective openings. However, in the case of our contractive noise the exponent which fits the numerical results cannot be related in a direct way to the dimension of the classical attractor. This dimension is equal to $d = 1.756$ for $\epsilon = 0.8$ and thus the relation $\beta = 2 - d$ does not hold.

Moreover, the exponent β turns out to be almost insensitive to the value of the dissipation parameter ϵ. This remarkable fact can be appreciated in Fig. 5 where $\frac{N_{\gamma<\gamma_{\rm cut}}}{N^2}(N^2)$ is shown for different values of the coupling constant $\epsilon = 0.8,0.6,0.4$ (and three choices of $\gamma_{\rm cut}$). In all cases the data follow a power law, with $\beta \sim 0.8$, independently of the fractal dimension of the corresponding classical attractor ($d = 1.756$ for $\epsilon = 0.8$, $d = 1.576$ for $\epsilon = 0.6$, and $d = 1.431$ for $\epsilon = 0.4$).

For comparison we have applied the superoperator formalism to the well-studied case of an open Baker map subjected

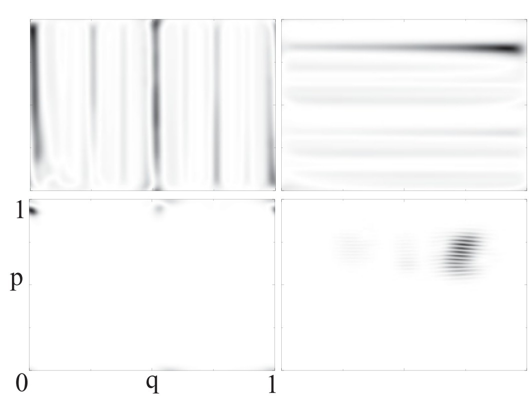

FIG. 7. Husimi representations of the (a) left and (b) right eigenstates for the excited state closest to the unit circle, $|\lambda| = 0.5845$ (upper panel), and for a state with a large decay rate, $|\lambda| = 9.278 \times 10^{-5}$ (lower panel); $\epsilon = 0.8, N = 180$.

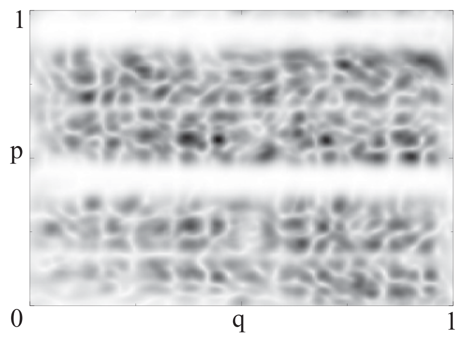

FIG. 8. Husimi representation for the projector of Eq. (6), with $\lambda_{\rm cut} = 0.1348$ (or $\gamma_{\rm cut} = 4$), $\epsilon = 0.8$, and $N = 180$.

to a projective opening [22,23] modeled by a nonunitary operator $U_{\rm open}$, by defining $\$_{\rm open} = U_{\rm open} \otimes U_{\rm open}^{\dagger}$. Since the eigenvalues of $\$_{\rm open}$ are just products of two eigenvalues of $U_{\rm open}$, the fraction of long-lived resonances of $\$_{\rm open}$ will be $(\frac{N_{ll}}{N})^2$ (where $\frac{N_{ll}}{N}$ is the fraction of long-lived resonances of $U_{\rm open}$) and will scale with N^2 as does the fraction of resonances of $U_{\rm open}$ with N. This was verified (for values of N in the same range as for the contractive case), confirming the validity of the fractal Weyl law in the open system.

In order to shed more light on these results we turn to analyzing the eigenvectors. To characterize the eigenstates of \$, we consider the overlap of the right eigenstates with the invariant state, by defining the measure $|(\psi_\lambda^R \psi_0^R)| = |{\rm Tr}(\psi_\lambda^{R\dagger} \psi_0^R)|$. Figure 6 displays the dependence of this measure with respect to γ, for different values of ϵ. It is clear that the overlap is on average larger for slow-decaying states, and decreases as the decay rate increases. But it is always small, thus giving further support to the strong contraction of the spectrum we have observed.

The distinction between slow- and fast-decaying states is also noticeable if we look at their distributions in phase space. For this, we compute the Husimi representation of the right and left eigenstates of \$, $\langle z|\psi_\lambda^{R,L}|z\rangle = {\rm Tr}(\psi_\lambda^{R,L\dagger},|z\rangle\langle z|)$, where $|z\rangle$ are coherent states centered at $z = (q,p)$. As shown in Fig. 7 for a typical slow-decaying state(upper panel), the Husimi density of the right eigenstate reasonably follows the structure of the attractor (although being much more localized than the invariant state), while the left one has a delocalized pattern. For a state with a large decay rate (lower panel), the probability pattern becomes difficult to associate with the attractor.

Although the long-lived eigenfunctions are morphologically different from the invariant state, we have verified that they have support on the phase space region corresponding to the classical attractor by making use of a recently developed representation especially suited for open systems [24]. We have built the sum

$$\sum_{\lambda=\lambda_{\rm cut}}^{1} \frac{\langle z|\psi_\lambda^R|z\rangle\langle z|\psi_\lambda^L|z\rangle}{\langle\psi_\lambda^L|\psi_\lambda^R\rangle} \qquad (6)$$

and obtained the distribution shown in Fig. 8, indicating a clear localization of the long-lived states on the attractor region in the semiclassical limit.

In view of these results we have found that, although the classical support of the long-lived resonances is the classical attractor, their number scales with N at a different pace than that predicted by the fractal Weyl law.

IV. CONCLUSIONS

We have studied the spectra of a paradigmatic model of the theories of quantum chaos and dissipative systems, the baker map with dissipation, by following a standard quantization procedure based on the Kraus representation of superoperators. In contrast to what happens for the discretized Perron-Frobenius operator in Ref. [10], we found that the standard fractal Weyl law does not hold in this case (we underline that in Ref. [10] there was no intention of quantizing the classical system). Even if the fraction of long-lived resonances does scale with the dimension following a power law which is roughly insensitive to the cutoff value of the decay rate considered for the statistics, the exponent (which is approximately constant) is not directly related to the fractal dimension of the classical attractor. We checked that the same behavior is present when the baker is replaced with the cat map [25].

In order to give an interpretation for this intriguing result, we investigated the morphology of the eigenstates of the system. In particular we built the Husimi representation of the projector constituted by the eigenfunctions with slow escape rate [24]. We found that its density concentrates on the attractor region, strongly suggesting that one should expect, as for the projective case, a connection between the statistics of these long-lived resonances and the fractal dimension of the strange attractor.

A possible cause for the lack of such natural connection might be in the nonorthogonality of the eigenfunctions localized on the strange attractor. In fact, the fractal Weyl law supposes the quasiorthogonality of the eigenfunctions supported by the fractal set, since it is based on a Planck-cell partitioning of this set. This works in the case of projective openings, but might not be the case for contractive ones. This is an open problem and we are currently studying [25] the degree of nonorthogonality induced by a contractive dynamics as an explanation for the apparent nonvalidity of the standard Weyl law revealed by the present numerical investigation.

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