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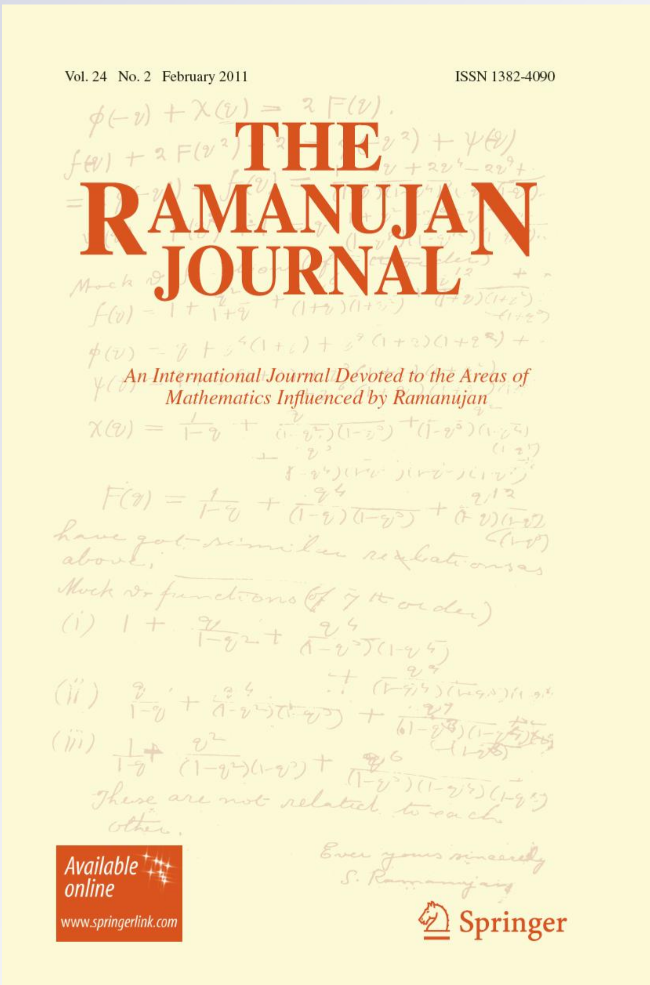

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Small quotients in Euclidean algorithms

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Abstract Numbers whose continued fraction expansion contains only small digits have been extensively studied. In the real case, the Hausdorff dimension σ_M of the reals with digits in their continued fraction expansion bounded by M was considered, and estimates of σ_M for $M \to \infty$ were provided by Hensley (J. Number Theory 40:336–358, 1992). In the rational case, first studies by Cusick (Mathematatika 24:166–172, 1997), Hensley (In: Proc. Int. Conference on Number Theory, Quebec, pp. 371–385, 1987) and Vallée (J. Number Theory 72:183–235, 1998) considered the case of a fixed bound M when the denominator N tends to ∞. Later, Hensley (Pac. J. Math. 151(2):237–255, 1991) dealt with the case of a bound M which may depend on the denominator N, and obtained a precise estimate on the cardinality of rational numbers of denominator less than N whose digits (in the continued fraction expansion) are less than $M(N)$, provided the bound $M(N)$ is large enough with respect to N. This paper improves this last result of Hensley towards four directions. First, it considers various continued fraction expansions; second, it deals with various probability settings (and not only the uniform probability); third, it studies the case of all possible sequences $M(N)$, with the only restriction that $M(N)$ is at least equal to a given constant M_0; fourth, it refines the estimates due to Hensley, in the cases that are studied by Hensley. This paper also generalises previous estimates due to Hensley (J. Number Theory 40:336–358, 1992) about the Hausdorff dimension

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σ_M to the case of other continued fraction expansions. The method used in the paper combines techniques from analytic combinatorics and dynamical systems and it is an instance of the Dynamical Analysis paradigm introduced by Vallée (J. Théor. Nr. Bordx. 12:531–570, 2000), and refined by Baladi and Vallée (J. Number Theory 110:331–386, 2005).

Keywords Continued fractions · Hausdorff dimension · Maps of the interval · Euclidean algorithms

Mathematics Subject Classification (2000) 11K50 · 11K55 · 37C30

1 Introduction

This paper aims to estimate the probability that the continued fraction expansion of a rational number only contains "small" quotients. Every $x \in]0, 1]$ admits a finite or infinite (CF)-continued fraction expansion of the form

$$x = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{\ddots + \cfrac{1}{m_n + \cdots}}}}. \tag{1.1}$$

Ordinary continued fraction expansions can be viewed as trajectories of a one-dimensional dynamical system, the Gauss map $T : [0, 1] \to [0, 1]$,

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \quad \text{for } x \neq 0, \ T(0) = 0. \tag{1.2}$$

Here, $\lfloor x \rfloor$ is the integer part of x. For an irrational x, the trajectory $\mathcal{T}(x) = (x, T(x), T^2(x), \ldots, T^n(x), \ldots)$ never meets 0 and is encoded by the infinite sequence of *digits* $(m_1(x), m_2(x), m_3(x), \ldots, m_n(x), \ldots)$, defined by

$$m_i(x) := m\big(T^{i-1}(x)\big) \quad \text{with } m(x) := \left\lfloor \frac{1}{x} \right\rfloor.$$

For a rational number $x = u/v$, the trajectory $\mathcal{T}(x)$ reaches 0 in a finite number of steps $p(x)$, and describes the execution of the Euclid's algorithm on the pair (u, v), the number $p(x)$ being equal to the number of iterations of the algorithm. The digits coincide with the quotients obtained during the execution of the Euclid's algorithm on the input pair (u, v). For an irrational number x, we let $p(x) := +\infty$.

Here, we wish to study the distribution of the function $D : [0, 1] \to [0, +\infty]$ defined as

$$D(x) := \sup\big\{m_i(x) : 1 \le i \le p(x)\big\},$$

and for an "integer" $M \le \infty$, we wish to study the probability of the event $[D < M]$. We focus on the case when x is rational and aim to relate the distribution of D with the denominator of the rational x.

Previous results

There are, in fact, three possible studies, depending on whether we are interested in the real case (with a fixed bound M possible infinite) or in the rational case. The last case gives rise to two possibilities: the bound M may be fixed or it may depend on the denominator of the rational.

Real case It is well-known that reals with small digits in the standard continued fraction expansion are badly approximable. This fact has promoted the study of the variable D. The Hausdorff dimension of the set $\mathcal{R}_M := \{x \in [0,1];\ D(x) < M\}$ of the real numbers whose digits are less than M is denoted by σ_M. The asymptotics of σ_M when M goes to infinity has been studied by Hensley in [12], and he proved the following:

Theorem A (Hensley) *The Hausdorff dimension of the set* $\mathcal{R}_M := \{x \in [0,1];\quad D(x) < M\}$ *of the real numbers whose digits in their continued fraction expansion are less than* M *satisfies*

$$\sigma_M < 1, \quad 2(\sigma_M - 1) = -\frac{2}{\zeta(2)}\frac{1}{M} - \frac{4}{\zeta(2)^2}\frac{\log M}{M^2} + O\left(\frac{1}{M^2}\right) \quad (M \to \infty).$$

Rational case—fixed M The set $\mathcal{O}^{[M]}$ of rationals $u/v \in [0,1]$ whose all digits are less than some fixed M has been studied by Cusick [4], Hensley [9] and Vallée [18]. These authors consider the set of rationals

$$\Omega_N := \left\{x = \frac{u}{v} \in]0,1];\ 0 < v \le N,\ \gcd(u,v) = 1\right\} \tag{1.3}$$

endowed with the uniform probability $\mathbb{P}_N$, together with the subset

$$\mathcal{O}_N^{[M]} := \Omega_N \cap \mathcal{O}^{[M]} := \{x \in \Omega_N : D(x) < M\},$$

and they proved the following:

Theorem B (Cusick, Hensley, Vallée) *For each fixed* $M \ge 2$, *the probability of the set* $\mathcal{O}_N^{[M]}$ *satisfies*

$$\mathbb{P}_N(\mathcal{O}_N^{[M]}) = C_M N^{2(\sigma_M - 1)}[1 + \epsilon_M(N)], \quad \epsilon_M(N) \to_{N\to\infty} 0.$$

Here, σ_M *is the Hausdorff dimension of* $\mathcal{R}_M$, *and* C_M *is a positive sequence which satisfies* $C(M) = 1 + O(\log M/M)$.

Remark The sequence $\epsilon_M(N)$ depends on M on a way which is not elucidated until now. This entails that the result cannot be extended in a direct way to the case where M depends on N. The present paper aims to deal with this case.

Rational case—bound $M(N)$ depends on the denominator of rationals Hensley [11] was the first to consider the case when $M(N)$ may depend on the denominator of the rational N. He studied the set $\mathcal{O}_N^{[M(N)]}$ of Ω_N formed with rationals x for which $D(x)$ is less than $M(N)$, i.e.

$$\mathcal{O}_N^{[M(N)]} := \{x \in \Omega_N : D(x) < M(N)\},$$

and obtained the following result:

Theorem C (Hensley) *For any integer N, denote by $n := \log N$. Consider a sequence $M(N) > 2$ which satisfies*

$$M(N) = a(n)n \quad \text{with } a(n) \geq 4/\log n. \tag{1.4}$$

Then, as $N \to \infty$,

$$\mathbb{P}_N\left(\mathcal{O}_N^{[a \log N]}\right) = \exp\left(-\frac{12}{a(n)\pi^2}\right)\left[1 + \exp\left(\frac{24}{a(n)\pi^2}\right)\left(1 + \frac{1}{a^2(n)}\right)O\left(\frac{\log n}{n}\right)\right].$$

Our results

The paper provides a generalisation and an improvement of these last three results (Theorems A, B, C). We consider a class of Continued Fraction Expansions, a class of probabilistic models, we deal with the all possible sequences $M(N)$ with $M(N) \geq M_0$ [and not only the sequences which satisfy (1.4)], and we improve the remainder term obtained by Hensley for any sequence satisfying (1.4).

A class of continued fractions The standard continued fraction has several variants adapted to different applications, for example, the computation of the Jacobi symbol. In this paper, we consider, together with the standard continued fraction, two variants: the centred and the odd continued fractions. The Euclidean algorithms corresponding to these variants and to the standard one are fast in the sense explained in [20]. This is due to the fact that these three continued fractions systems share the same framework: There exist an interval $\mathcal{I}$ which contains 0, and a map $T : \mathcal{I} \to \mathcal{I}$ of the form

$$T(x) := \left|\frac{1}{x} - A\left(\frac{1}{x}\right)\right|, \quad x \neq 0, \ T(0) = 0.$$

We have already seen that the standard continued fraction is defined by the interval $\mathcal{I} := [0, 1]$ and the map $A(x) := \lfloor x \rfloor$ which is the integer part of x. The centred continued fraction is defined by the interval $\mathcal{I} := [0, 1/2]$ and the map $A(x) := \lfloor x \rceil$ which is the nearest integer to x. The odd continued fraction is defined by the interval $\mathcal{I} := [0, 1]$ and the map $A(x) := \lfloor x \rceil_O$ which is the nearest odd integer to x. Each system produces, for the real $x \in \mathcal{I}$, a sequence of digits $m_i(x)$ defined by the relation

$$m_i(x) := m\left(T^{i-1}(x)\right) \quad \text{with } m(x) := A\left(\frac{1}{x}\right). \tag{1.5}$$

When restricted to rational numbers $x = u/v \in \mathcal{I}$, the trajectory $\mathcal{T}(x)$ reaches 0 after a finite number of steps $p(x)$, and describes the execution of the corresponding variant of the Euclid algorithm on the pair (u, v), the number $p(x)$ being equal to the number of iterations of the algorithm. Each of them is related to a particular type of integer divisions. The centred division, of the form $v = mu + \epsilon r$ produces a quotient $m \geq 2$ and a remainder r such that $0 \leq r \leq u/2$. The odd division, also of the form $v = mu + \epsilon r$, produces an odd quotient m and a remainder r with $0 \leq r \leq u$. In the three cases (standard, centred, odd), the divisions are defined by pairs $q = (m, \epsilon)$, which are called the partial quotients.

It is natural to study the function $D : \mathcal{I} \to [1, +\infty]$ related to each continued fraction variant and defined as

$$D(x) := \sup\{m_i(x) : 1 \leq i \leq p(x)\},$$

and, for an "integer" $M \leq \infty$, to describe the probability of the event $[D < M]$. We first obtain a generalisation of Theorem A, and prove that the same asymptotic expansion holds for each variant.

A class of probabilistic models We deal with the subset

$$\Omega_N := \{x = u/v \in \mathcal{I};\ \gcd(u, v) = 1,\ v \leq N\},$$

endowed with the probability $\mathbb{P}_{N,f}$ associated to some strictly positive density f of class $\mathcal{C}^1$ on the interval $\mathcal{I}$, as

$$\mathbb{P}_{N,f}(x_0) := \frac{f(x_0)}{\sum_{x \in \Omega_N} f(x)},$$

and we recover, when $f \equiv 1$, the case of the uniform probability.

The uniform model $f \equiv 1$ is not always the most natural. It may be interesting and useful to study what happens in the "middle" of an execution of the Euclidean Algorithm. Since the density evolves with the execution of the algorithm, this leads to considering non-uniform densities, even if one starts with a uniform density. Such a situation occurs when one studies the Divide-and-Conquer version of the Euclidean algorithm (the Knuth–Schönage algorithm).

The condition on the sequence $M(N)$: the integer M_0 and the exponent α We consider a large class of possible sequences $N \mapsto M(N)$, whereas Hensley only deals with sequences which satisfy Condition (1.4). In fact, our result is valid as soon as M is at most equal to some integer M_0. What is this integer M_0?

Our result strongly depends on the width 2γ of a vertical strip of the form $\mathcal{S}_\gamma := \{s; |\Re s - 1| \leq \gamma\}$ where some crucial property—the US Property—holds. The US Property (US is a shorthand name for "Uniformity on Strips") means that there exists a vertical strip $\mathcal{S}_\gamma$ where a certain Dirichlet series has a unique pole and is of polynomial growth for $\Im s \to \infty$. The existence of such a vertical strip is precisely stated in Theorem D, Sect. 2. And the integer M_0 (which depends on the width γ) is related to the behaviour of two sequences with respect to the vertical strip $\mathcal{S}_\gamma$: first, the sequence σ_M, already mentioned, which is the central object of our study; second, another sequence of functions $M \mapsto r_M(s)$, defined in (3.6). More precisely, in our Theorem 1, the integer $M_0 = M_0(\gamma)$ satisfies[1] the following: For any $M \geq M_0$, the two conditions are fulfilled:

(i) the real σ_M belongs to the real interval $]1-\gamma, 1+\gamma[$,
(ii) the sequence $r_M(s)$ is strictly less than 1 on $[1-\gamma, 1+\gamma]$.

Then, the exponent α of the remainder term is just the minimal distance $\alpha := \sigma_{M_0} - (1-\gamma)$.

Conjecture *Previous results of Dolgopyat* [5] *and Baladi–Vallée* [2] *have shown the existence of such a US-strip, but the maximal possible width is not known, it is even thought to be at most* $1/2$. *Suppose that the maximal width can be chosen to be equal to any* $\gamma < 1/2$. *The Hausdorff dimension* σ_3, *which is the smallest possible* σ_M, *satisfies* $\sigma_3 \approx 0.53128$. *On the other hand, the results due to Mayer show that* $r_\infty(s)$ *is strictly less than* 1 *on* $]1/2, 1]$. *Then, if the sequence* $M \mapsto r_M$ *is increasing, it is perhaps possible to choose* $M_0 = 3$. *If it is the case, our result would take into account all possible sequences* $N \mapsto M(N)$.

Our main result is as follows:

Theorem 1 *For each of the three continued fraction expansions* (*standard, centred, odd*), *there are an integer* $M_0 = M_0 \geq 3$ *and a real* α *with* $0 < \alpha < 1/2$ *so that, for any* $N \geq 1$, $M \geq M_0$, *the probability that a rational with a denominator at most* N *has all its digits less than* M *satisfies, for any density* f *in* $\mathcal{C}^1(\mathcal{I})$,

$$\mathbb{P}_{N,f}\big(\mathcal{O}_N^{[M]}\big) = C_M(f) N^{2(\sigma_M - 1)}\big[1 + O\big(N^{-\alpha}\big)\big],$$
$$\textit{with } C_M(f) = 1 + O\left(\frac{\log M}{M}\right).$$

Here, σ_M *and* $C_M(1) = C_M$ *are the constants of Theorem* B, *the constants in the* O-*terms only depend on the density* f, *and the following asymptotic expansion holds for* σ_M,

$$2(\sigma_M - 1) = -\frac{2}{\zeta(2)}\frac{1}{M} - \frac{4}{\zeta(2)^2}\frac{\log M}{M^2} + O\left(\frac{1}{M^2}\right) \quad (M \to \infty).$$

This result exhibits a threshold phenomenon (already obtained by Hensley) depending on the relative order of $\sigma_M - 1$ (of order $O(1/M)$) with respect to $n := \log N$:

(a) If $M/n \to +\infty$, then, almost everywhere, any rational of Ω_N has all its *CFE*-digits less than M.
(b) If $M/n \to 0$, then, almost everywhere, any rational of Ω_N has at least one of its *CFE*-digits greater than M.

[1] We prove that such an integer M_0 exists.

More precisely, there are several cases of interest, according to the behaviour of the sequence $M(N)$. As previously, we let $n = \log N$.

(i) If $M(N) = an$ for some constant a, then

$$\mathbb{P}_{N,f}\big(\mathcal{O}_N^{[a \log N]}\big) = \exp\left(-\frac{12}{a\pi^2}\right)\left[1 + O\left(\frac{\log n}{n}\right)\right].$$

In this case, we obtain the same estimates as Hensley, but in a more general framework.

(ii) If $M(N) = a(n)n$ for a sequence $a(n) \to \infty$, then the probability of the subset tends to 1 and, more precisely,

$$\mathbb{P}_{N,f}\big(\mathcal{O}_N^{[a(n) \log N]}\big) = \exp\left(-\frac{12}{a(n)\pi^2}\right)\left[1 + \frac{1}{a^2(n)n} O\big(\log n + \log a(n)\big)\right].$$

A natural instance is provided by the case $M(N) = n^b$ with $b > 1$, where the two remainders may be compared:

$$\left[1 + O\left(\frac{\log n}{n^b}\right)\right] \quad \text{(this paper)}, \qquad \left[1 + O\left(\frac{\log n}{n}\right)\right] \quad \text{(Hensley)}.$$

(iii) Finally, if $M(N) = a(n)n$, with $a(n) \to 0$ and $M(N) \to \infty$, then the probability of the subset tends to 0. For instance, if $M(N) = n^b$ with $1/2 < b < 1$, one has

$$\mathbb{P}_{N,f}\big(\mathcal{O}_N^{[a(n) \log N]}\big) = \exp\left(-\frac{12}{\pi^2} n^{1-b}\right)\left[1 + O\left(\frac{\log n}{n^{2b-1}}\right)\right].$$

We remark that Hensley cannot deal with this case.

(iv) Our framework also applies to the case of a constant sequence M provided that M is large enough, $M \geq M_0$.

Motivations and methods

We use methods which are more direct than those used by Hensley. Hensley uses generating functions (even if he does not use explicitly the name), in particular generating functions for rationals whose all digits are less than M. He studies their continuants, and uses a quasi-multiplicativity property for continuants which allows him to relate the generating functions of interest to powers of the Riemann zeta function. He then uses the Perron Formula for extracting coefficients from these Dirichlet series. The quasi-multiplicativity of continuants (which is not an exact multiplicativity property) creates an additional error term in his estimates.

Like Hensley, we deal with generating functions of Dirichlet type from which we extract coefficients via the Perron Formula. However, we *directly* use an *exact* alternative expression for our generating functions, by means of the transfer operator of the underlying dynamical system, and we apply the dynamical analysis paradigm to this problem.

The dynamical analysis methodology was introduced by Vallée around 1995 with the aim of studying the average-complexity of a whole class of Euclidean Algorithms.

First used in the average-case analysis (e.g. [19]), it was later extended by Baladi and Vallée [2] to the distributional analysis. The Dynamical Analysis method proceeds in three main steps: First, each discrete algorithm is extended into a continuous process which can be defined in terms of the corresponding dynamical system. Then the transfer operator associated to the dynamical system $\mathbf{H}_s$ explains how the distribution evolves, but only in the continuous world. The executions of the Euclidean algorithms are now described by particular trajectories (i.e. trajectories of "rational" points), and a transfer "from continuous to discrete" must be finally performed, by means of Dirichlet Series.

To estimate the probability of the subset $\mathcal{O}_N^{[M]}$, we first use a generating Dirichlet series, which is proven to be exactly related with the (restricted) transfer operator $\mathbf{H}_{M,s}$ of the dynamical system "constrained" by M (see, for instance, Proposition 1). This relation is not new and has been already applied in previous works (see, for instance, [18]), where it constitutes a crucial step for the analysis. Then, for instance, in [18], the extraction of coefficients was made with plain Tauberian Theorems (which do not provide explicit remainder terms) and only needed few (easy) properties of the quasi-inverse $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ of the (plain) operator near $s = \sigma_M$, which entails the results cited in Theorem B.

Here, we wish to obtain remainder terms (uniform with respect to M), and we enter the framework of Distributional Dynamical Analysis. Such an analysis is classically based on a precise knowledge of the quasi-inverse $(\mathrm{Id} - \mathbf{H}_s)^{-1}$ when the parameter s belongs to a vertical strip on the left of $\Re s = 1$. A crucial point is the US Property for the quasi-inverse $(\mathrm{Id} - \mathbf{H}_s)^{-1}$ of the (plain) transfer operator [there exists a vertical strip where the quasi-inverse has a unique pole and is of polynomial growth for $\Im s \to \infty$]. Thus, we need to extend this type of the results to the quasi-inverse $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ of the restricted operator, with estimates uniform with respect to M. We mainly use perturbation theory (since the operator $\mathbf{H}_{M,s}$ is a small perturbation of $\mathbf{H}_s$, when $M \to \infty$), and estimate the speed of convergence of the spectral objects of $\mathbf{H}_{M,s}$ to those of $\mathbf{H}_s$ by extending to our present framework methods due to Cesaratto and Vallée [3] and Hensley [12].

Outline of the paper

Section 2 describes the main objects—dynamical systems, (restricted) transfer operators, Dirichlet series—and the central relation between these objects. The US Property for the operator $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ is stated. Then Sect. 3 is devoted to the proof of this property when s is "near the real axis", whereas Sect. 4 considers the case when s is "far from" the real axis. Finally, our main Theorem is proved in Sect. 5.

Remarks about notations In this paper, the notation $A_M(x) \ll B_M(x)$ means that A is less than B up to absolute multiplicative constants. This means that there exists some absolute constant k such that for every x of interest, $A_M(x) \le k B_M(x)$. It is synonymous with $A(x) = O(B(x))$ with an absolute O-term.

2 Dynamical methods

This section describes the three Euclidean dynamical systems and their main geometric properties. Then, it introduces the two main tools of the paper; first, the transfer operators with their constrained and unconstrained versions; second, the generating functions (of Dirichlet type). Finally, it exhibits the fundamental relation between these two objects, which is the base of the whole analysis.

2.1 Geometric properties of the three Euclidean dynamical systems

Each of the three dynamical systems, whose graphs are presented in Table 1, possesses the same three main properties:

(i) They are related to piecewise complete maps of the interval.
(ii) They belong to the so-called Good Class which gathers expanding maps with bounded distortion. The notion of a Good Class will be made more precise in Property 2.
(iii) They satisfy the UNI Property: Their branches are not "too often too close".

Property 1 (Piecewise complete maps of the interval) *For each of the three systems, the map $T:\mathcal{I}\to\mathcal{I}$ is piecewise complete, i.e. there exist a (finite or countable) set $\mathcal{Q}$ and a partition $\{\mathcal{I}_q\}_{q\in\mathcal{Q}}$ (modulo a countable set) of the interval $\mathcal{I}$ into open subintervals $\mathcal{I}_q$ such that the restriction of T to $\mathcal{I}_q$ extends to a bijective mapping of the class $\mathcal{C}^2$ from the closure of $\mathcal{I}_q$ to $\mathcal{I}$.*

For each Euclidean dynamical system, the set $\mathcal{Q}$ is the set of all possible quotients $q=(m,\epsilon)$ and it is described in Table 1. The set $\mathcal{H}=\{h_{[q]}\}$ of branches of the inverse function T^{-1} is then naturally indexed by the set $\mathcal{Q}$. Each inverse branch relative to $q=(m,\epsilon)$ is a linear fractional transformation (LFT) of the form $h_{[m,\varepsilon]}=1/(m+\varepsilon x)$. The set of the inverse branches of the iterate T^k is $\mathcal{H}^k$; its elements are of the form $h_{[q_1]}\circ h_{[q_2]}\circ\cdots\circ h_{[q_k]}$ where k is called the *depth* of the branch. Setting $\mathcal{H}^0=\{\text{Id}\}$, the set $\mathcal{H}^\star:=\bigcup_{k\geq 0}\mathcal{H}^k$ is the semi-group generated by $\mathcal{H}$. Each interval $h(\mathcal{I})$ for h of depth k is called a *fundamental interval of depth k*. It gathers all the reals x which have the same continued fraction expansion (CFE) of depth k.

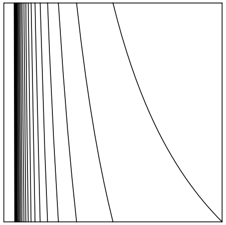
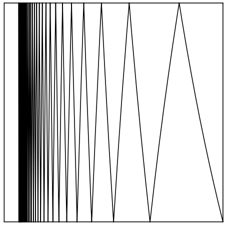
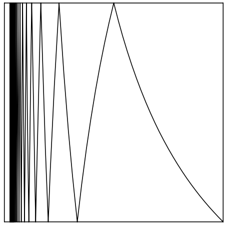
Any Euclidean algorithm, whose execution on the input (u,v) involves the partial quotients (m_1,ε_1), (m_2,ε_2), …, (m_p,ε_p), builds a CFE of the rational u/v as

$$u/v=h_{[m_1,\varepsilon_1]}\circ h_{[m_2,\varepsilon_2]}\circ\cdots\circ h_{[m_p,\varepsilon_p]}(0).$$

The last step uses a particular set of digits, the final set described in Table 1. Let $\mathcal{F}$ be the set of inverse branches related to the final set. The previous decomposition is unique and defines a bijection between the set $\mathbb{Q}\cap\mathcal{I}$ and the set $\mathcal{H}^\star\times\mathcal{F}$.

The three Euclidean systems $(\mathcal{I},T)$ corresponding to the Standard, Centred and Odd algorithms are instances of fractional systems in Schweiger's sense that are extensively studied in [17]. The three systems also belong to a subclass of piecewise complete mappings, the Good Class. Systems that belong to this Class enjoy "nice"

Table 1 The three Euclidean systems ($\phi = (1 + \sqrt{5})/2$)

CFE	Standard	Centred	Odd
Intervals	$\mathcal{I} = [0, 1]$	$\mathcal{I} = [0, 1/2]$	$\mathcal{I} = [0, 1]$
Set $\mathcal{Q}$ of pairs (m, ϵ)	$m \geq 1$, $\epsilon = +1$	$m \geq 2$, $\epsilon = \pm 1$ if $m = 2$ then $\epsilon = +1$	$m \geq 1$ odd, $\epsilon = \pm 1$ if $m = 1$ then $\epsilon = +1$
Final set	$m \geq 2$	$\epsilon = +1$	$\epsilon = +1$
Graph of $T(x) = \left\| \frac{1}{x} - A\left(\frac{1}{x}\right) \right\|$			
Function $A(y)$	Integer part of y	Nearest integer to y, i.e. m s.t. $y - m \in [-1/2, +1/2[$	Nearest odd integer to y, i.e. m odd s.t. $y - m \in [-1, +1[$
Contraction ratio	$\rho = 1/\phi^2$	$\rho = 1/(\sqrt{2} + 1)^2$	$\rho = 1/\phi^2$
Invariant density f_1	$\frac{1}{\log 2} \frac{1}{1+x}$	$\frac{1}{\log \phi}\left[\frac{1}{\phi+x} + \frac{1}{\phi^2-x}\right]$	$\frac{1}{3\log \phi}\left[\frac{1}{\phi-1+x} + \frac{1}{\phi^2-x}\right]$
Entropy	$\frac{\pi^2}{6\log 2}$	$\frac{\pi^2}{6\log \phi}$	$\frac{\pi^2}{9\log \phi}$

ergodic properties such as chaotic behaviour of trajectories. For a more precise discussion about the behaviour of Euclidean algorithms and dynamical properties of the corresponding map T, see [21].

Property 2 (Good Class) *Each Euclidean dynamical system belongs to the Good Class which gathers the dynamical systems satisfying the following*:

(i) *T is piecewise uniformly expanding, i.e. there are C and $\widehat{\rho} < 1$ so that $|h'(x)| \le C\widehat{\rho}^n$ for every inverse branch h of T^n, all n and all $x \in \mathcal{I}$. The infimum of such ρ is called the contraction ratio, and satisfies*

$$\rho = \limsup_{n\to\infty} \big(\max\big\{\big|h'(x)\big|; h \in \mathcal{H}^n, x \in \mathcal{I}\big\}\big)^{1/n}. \tag{2.1}$$

(ii) *There is $\widehat{K} > 0$, called the distortion constant, so that every inverse branch h of T satisfies $|h''(x)| \le \widehat{K}|h'(x)|$ for all $x \in \mathcal{I}$.*

(iii) *There is $\sigma_0 < 1$ such that $\sum_{h\in\mathcal{H}} \sup|h'|^{\sigma} < \infty$ for all real $\sigma > \sigma_0$.*

For each of the three Euclidean systems, the estimate $|h'_{[m,\epsilon]}| = \Theta(m^{-2})$ entails the equality $\sigma_0 = 1/2$. We will see that the abscissa σ_0 is a lower bound for the sequence $(\sigma_M)_{M\ge 3}$.

Finally, each system satisfies the UNI Condition, as Baladi and Vallée [2] already proved it. If $\Delta(h,k)$ denotes the "distance" between two inverse branches h and k of same depth, defined as

$$\Delta(h,k) = \inf_{x\in\mathcal{I}} \big|\Psi'_{h,k}(x)\big| \quad \text{with } \Psi_{h,k}(x) = \log\left|\frac{h'(x)}{k'(x)}\right|, \tag{2.2}$$

the UNI Condition expresses that the inverse branches are not "too often too close". It uses the "ball" of centre h and radius $\eta > 0$, formed with the fundamental intervals related to inverse branches whose distance to h is less than η,

$$\text{for } h \text{ in } \mathcal{H}^n, \quad J(h,\eta) := \bigcup_{k\in\mathcal{H}^n, \Delta(h,k)\le\eta} k(\mathcal{I}) \tag{2.3}$$

and is stated as follows:

Property 3 (Condition *UNI*) *Each Euclidean dynamical system, with contraction ratio ρ, fulfils the UNI Condition: Each inverse branch of T extends to a $\mathcal{C}^3$ function and*

(a) *For any a $(0 < a < 1)$ we have $|J(h,\rho^{an})| \ll \rho^{an}, \forall n, \forall h \in \mathcal{H}^n$.*

(b) *$Q := \sup\{|\Psi''_{h,k}(x)|; n \ge 1, h,k \in \mathcal{H}^n, x \in \mathcal{I}\} < \infty$.*

2.2 Transfer operators

If $\mathcal{I}$ is endowed with an initial probability density g_0 with respect to Lebesgue measure, T acts on it and transforms it into a new density g_1. The operator $\mathbf{H}$ such that

$g_1 = \mathbf{H}[g_0]$ is called the density transformer, or the Perron–Frobenius operator (acting now on L^1 functions, soon we shall restrict its domain). An application of the change of variable formula gives

$$\mathbf{H}[f](x) := \sum_{h \in \mathcal{H}} \left|h'(x)\right| f \circ h(x).$$

To generate Dirichlet Series it is useful to deal with a more general operator, the transfer operators $\mathbf{H}_s, \mathbf{F}_s$ which depend on a complex parameter s:

$$\mathbf{H}_s[f](x) := \sum_{h \in \mathcal{H}} \left|h'(x)\right|^s \cdot f \circ h(x), \qquad \mathbf{F}_s[f](x) := \sum_{h \in \mathcal{F}} \left|h'(x)\right|^s \cdot f \circ h(x)$$

where $\mathcal{H}$ and $\mathcal{F}$ are the sets of inverse branches defined in Property 1. (Note that $\mathbf{H}_1 = \mathbf{H}$.) If $\sigma := \Re s > 1/2$, then $\mathbf{H}_s, \mathbf{F}_s$, act boundedly on the Banach space $\mathcal{C}^1(\mathcal{I})$ of $\mathcal{C}^1$ functions on $\mathcal{I}$ endowed with the norm

$$\|f\|_{1,1} = \|f\|_0 + \left\|f'\right\|_0, \quad \text{with } \|f\|_0 := \sup |f|.$$

2.3 Constrained continued fractions

Here, we are not interested in constraints on the ε_i, and we only study constraints on the digits $m_i(x)$ defined in (1.5). The set $\mathcal{R}_M$ of the reals of the interval $\mathcal{I}$ for which all the digits m_i are less than M is associated to the "constrained" dynamical system $T_M : \mathcal{R}_M \to \mathcal{R}_M$, where T_M is the restriction of T to $\mathcal{R}_M$. The set $\mathcal{R}_M$ is a classical instance of a "fractal set" and its Hausdorff dimension denoted by σ_M has been largely studied (see [10, 12, 18]).

For any $M \geq 3$, the sets $\mathcal{H}_M := \{h_{m,\varepsilon} : \ m < M\}$, $\mathcal{F}_M := \mathcal{H}_M \cap \mathcal{F}$ gather the inverse branches associated to digits less than M. The restricted transfer operators, defined as

$$\mathbf{H}_{M,s}[f](x) := \sum_{h \in \mathcal{H}_M} \left|h'(x)\right|^s \cdot f \circ h(x), \qquad \mathbf{F}_{M,s}[f](x) := \sum_{h \in \mathcal{F}_M} \left|h'(x)\right|^s \cdot f \circ h(x), \tag{2.4}$$

are well-adapted to deal with continued fractions whose digits are less than M. We recover the case of the unconstrained operator when $M = \infty$, and we often let $\mathbf{H}_{\infty,s} := \mathbf{H}_s$, $\mathbf{F}_{\infty,s} := \mathbf{F}_s$.

For any complex s with $\Re s > 1/2$, the operators $\mathbf{H}_{M,s}, \mathbf{F}_{M,s}$ act boundedly on the Banach space of $\mathcal{C}^1(\mathcal{I})$ endowed with the norm $\|.\|_{1,1}$.

2.4 Dirichlet series and transfer to the discrete setting

We wish to analyse the distribution of $D := \max\{m_i(x) : \ 1 \leq i \leq p(x)\}$. For any fixed integer M, we consider the subsets

$$\Omega := \mathcal{I} \cap \mathbb{Q}, \qquad \mathcal{O}^{[M]} := \left\{x \in \Omega : D(x) < M\right\},$$

and, for any pair N, M of integers, the subsets

$$\Omega_N := \left\{ x = \frac{u}{v} \in \mathcal{I} \cap \mathbb{Q} : \gcd(u, v) = 1, v \le N \right\},$$

$$\mathcal{O}_N^{[M]} := \left\{ x \in \Omega_N : D(x) < M \right\}.$$

We remark the equality $\mathcal{O}_N^{[\infty]} = \Omega_N$. The probability of $\mathcal{O}_N^{[M]}$ can be expressed as

$$\mathbb{P}_{N,f}\left(\mathcal{O}_N^{[M]}\right) = \frac{\phi_M(N)}{\phi_\infty(N)} \quad \text{with } \phi_M(N) := \sum_{x \in \mathcal{O}_N^{[M]}} f(x). \tag{2.5}$$

In order to study ϕ_M for $M \le \infty$, we introduce the probability Dirichlet series:

$$F_M(s) := \sum_{\substack{(u,v) \\ u/v \in \mathcal{O}^{[M]}}} \frac{1}{v^s} f\left(\frac{u}{v}\right) = \sum_{v \ge 1} \frac{1}{v^s} c_M(v) \quad \text{with } c_M(v) := \sum_{\substack{u \\ u/v \in \mathcal{O}^{[M]}}} f\left(\frac{u}{v}\right).$$

The following (easy) proposition relates the Dirichlet series $F_M(s)$ to the operator $\mathbf{H}_{M,s}$. This will be a central tool of the paper.

Proposition 1 *For any $M \le \infty$, there is an alternative expression of the Dirichlet series $F_M(s)$ as a function of the quasi-inverse of the transfer operator $\mathbf{H}_{M,s}$,*

$$F_M(2s) = \mathbf{F}_{M,s} \circ (\mathrm{Id} - \mathbf{H}_{M,s})^{-1}[f](0). \tag{2.6}$$

Moreover, for any density of class $\mathcal{C}^1$ on $\mathcal{I}$, and s near 1, one has

$$F_\infty(2s) \sim_{s \to 1} \frac{\zeta(2s-1)}{\zeta(2s)}.$$

Proof Any rational decomposes in a unique way as a continued fraction of the form

$$x = h(0), \quad \text{with } h = h_{[m_1,\varepsilon_1]} \circ h_{[m_2,\varepsilon_2]} \circ \cdots \circ h_{[m_p,\varepsilon_p]}.$$

Here, the digit m_i is less than M if and only $h_{[m_i,\varepsilon_i]}$ belongs to $\mathcal{H}_M$, and the last $h_{[m_p,\varepsilon_p]}$ belongs to $\mathcal{F}_M$. Since all elements of $\mathcal{H}$ are linear fractional transformations, with determinant equal to ± 1, the relations

$$\frac{1}{v^2} = \left|h'(0)\right| \quad \text{and} \quad f(x) = f\big(h(0)\big)$$

provide the desired expressions for the Dirichlet series in terms of transfer operators. Finally, using Euler–Maclaurin formula (which compares finite sums and integrals) leads to the equality

$$F_\infty(2s) = \sum_{(u,v)\in\Omega^2} \frac{1}{v^{2s}} f\left(\frac{u}{v}\right) = \frac{1}{\zeta(2s)} \sum_{v\geq 1} \frac{1}{v^{2s-1}} \sum_{\substack{u \\ u/v\in\mathcal{I}}} \frac{1}{v} f\left(\frac{u}{v}\right)$$
$$= \frac{\zeta(2s-1)}{\zeta(2s)} \int_{\mathcal{I}} f(t)\, dt + R(s),$$

where $R(s)$ is analytic for $\Re s > 1/2$, whereas the first term has a pole at $s = 1$. This proves that, near $s = 1$, $F_\infty(2s)$ behaves as a quotient of the Riemann ζ functions. □

2.5 Dynamical analysis

We wish to study the asymptotics of the sum $\phi_M(N)$ of the first N coefficients of the Dirichlet series $F_M(s)$ [see (2.5)] when N tends to ∞. In a general setting, the asymptotics of coefficients is related to the position and the nature of the dominant singularity of the function $F_M(s)$. This explains the importance of Proposition 1 which shows that the singularities of $F_M(s)$ are related to values s for which the transfer operator $\mathbf{H}_{M,s}$ has a spectral value equal to 1. This is why we first study the operators $\mathbf{H}_{M,s}$ and their spectral properties. This will provide precise information on the singularities of $F_M(s)$. Then, we need to transfer this knowledge on coefficients of this Dirichlet series. To achieve this, we rely on convenient "extractors" which express the coefficients of the series as a function of the series itself. The Perron Formula of order two (see, e.g. [7]) is valid for a Dirichlet series $F(s) = \sum_{n\geq 1} a_n n^{-s}$ and a vertical line $\Re s = L$ inside the convergence domain of F,

$$\psi(T) := \sum_{n\leq T} a_n(T-n) = \frac{1}{2\pi i} \int_{L-i\infty}^{L+i\infty} F(s) \frac{T^{2s+1}}{s(2s+1)}\, ds. \tag{2.7}$$

It is next natural to modify the integration contour $\Re s = L$ into a contour which contains the singularities of $F(s)$.

Works of Dolgopyat [5], made precise by Baladi–Vallée [2], related the behaviour of the quasi-inverse $(\mathrm{Id} - \mathbf{H}_s)^{-1}$ of the unrestricted operator on vertical strips [summarised by the US Property] to the geometry of the Dynamical System [summarised by the UNI (Uniform non-Integrability) Condition]. We now state the US Property.

Theorem D (US property for the (unrestricted) transfer operator $\mathbf{H}_s$ relative to each Euclidean dynamical system) *For any* $0 < \xi < 1/10$, *there exist a* $\gamma > 0$, $t_0 > 0$, $C > 0$ *for which the following holds, for any* $f \in \mathcal{C}^1(\mathcal{I})$,

- (i) *In the vertical strip* $\mathcal{S} := \{s = \sigma + it, |\sigma - 1| \leq \gamma\}$, *the quasi-inverse* $(\mathrm{Id} - \mathbf{H}_s)^{-1}$ *is a meromorphic function with a unique pole. This pole is simple and located at* $s = 1$.
- (ii) *In the domain* $\mathcal{S}' := \{s = \sigma + it, |\sigma - 1| \leq \gamma, |t| > t_0\}$, *the quasi-inverse satisfies*

$$\sup_{x\in\mathcal{I}} \left|(\mathrm{Id} - \mathbf{H}_s)^{-1}[f](x)\right| \leq C|t|^{\xi} \cdot \|f\|_{1,1}.$$

What is known about the width γ? From the works of Mayer [15], Efrat [6], the quasi-inverse $(\mathrm{Id} - \mathbf{H}_s)^{-1}$ of the plain operator $\mathbf{H}_s$ relative to the standard Euclid algorithm (when acting on a nice functional space $\mathcal{F}$ of analytic functions) has a unique pole located at $s = 1$ in the half-plane $\Re s > 1/2$. The other singularities of the quasi-inverse are located on the line $\Re s = 1/2$ or at values s for which the Riemann zeta function satisfies $\zeta(2s) = 0$. Then, for any $\gamma < 1/2$, the vertical strip $\mathcal{S}_\gamma := \{s, |\Re s - 1| \le \gamma\}$ contains only one pole of the quasi-inverse $(\mathrm{Id} - \mathbf{H}_s)^{-1}$, located at $s = 1$; this is closely related to Property (i) of Theorem D. But this does not mean that the US-strip can be chosen as $\mathcal{S}_\gamma$, for two main reasons: first, we do not know if the quasi-inverse (even if it acts on $\mathcal{F}$) has polynomial growth on $\mathcal{S}_\gamma$ when $|\Im s|$ tends to ∞. Moreover, the quasi-inverse $(\mathrm{Id} - \mathbf{H}_s)^{-1}$ (when it acts on $\mathcal{C}^1$) may possess many other singularities than when it acts on $\mathcal{F}$.

To extract the coefficients of $F_M(s)$, we need a US property for the quasi-inverse of the constrained transfer operator. Furthermore, since later M will depend on N, we need this property to be uniform with respect to M. We will obtain the central result which shows that Theorem D extends to all the restricted operators,

Theorem 2 (Property US for the restricted operator) *Let $(\mathcal{I}, T)$ be one of the three Euclidean systems of interest. Denote by $\mathbf{H}_{M,s}$ the associated constrained transfer operator. For any $0 < \xi < 1/10$, there exists $\underline{\gamma} > 0$, $\underline{C} > 0$, $\underline{t_0} > 0$ and an integer M_0 for which, for any $f \in \mathcal{C}^1(\mathcal{I})$, the following holds*:

- (i) *For any $M \ge M_0$, the quasi-inverse $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}[f](0)$ is a meromorphic function in $\mathcal{S} := \{s; |\Re s - 1| \le \underline{\gamma}\}$ and has a unique pole on $\mathcal{S}$. This pole is simple and located at $s = \sigma_M$.*
- (ii) *In the domain $\mathcal{S}' := \{s = \sigma + it, |\sigma - 1| \le \underline{\gamma}, |t| > \underline{t_0}\}$ and for any M, the quasi-inverse satisfies*

$$\sup_{x \in \mathcal{I}} \left| (\mathrm{Id} - \mathbf{H}_{M,s})^{-1}[f](x) \right| \le \underline{C}, |t|^{\xi} \cdot \|f\|_{1,1}.$$

There are three main regions in a vertical strip to deal with. First, in the next section, we consider the behaviour near the real axis, and we prove Theorems 3 and 4. Then, in Sect. 4, we focus on the behaviour far from the real axis, and we prove Theorem 5. There remains an intermediary region which will be considered in Lemma 5. With these three results at hand, we obtain the proof of Property US for the quasi-inverse of the restricted operator (Theorem 2). We then return to our main Theorem 1 in Sect. 5.

3 Near the real axis

The spectral properties of the operators $\mathbf{H}_{M,s}$ in a neighbourhood of the real axis are well-known and summarised in the next proposition. But they are not sufficient for our purpose, where we need spectral properties to be "uniform" on M: We have to prove the existence of a neighbourhood of $s = 1$, the same neighbourhood for all integers M, where all the quasi-inverses $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ are meromorphic with only one possible pole for each of them. This is obtained in Theorem 3. We also obtain in Theorem 4 the extension of Theorem A to the other two Euclidean dynamical systems.

3.1 Classical spectral properties

First, we recall the definition of quasi-compact operators for bounded operators. Let $\mathbf{L}$ be a bounded operator on a Banach space. Denote by $\mathrm{Sp}\,\mathbf{L}$ the spectrum of $\mathbf{L}$, by $\mathrm{R}(\mathbf{L})$ its spectral radius, and by $\mathrm{R}^{(e)}(\mathbf{L})$ its *essential spectral radius*, i.e. the smallest $r \geq 0$ such that any $\lambda \in \mathrm{Sp}(\mathbf{L})$ with modulus $|\lambda| > r$ is an isolated eigenvalue of finite multiplicity. An operator $\mathbf{L}$ is *quasi-compact* if $\mathrm{R}^{(e)}(\mathbf{L}) < \mathrm{R}(\mathbf{L})$ holds.

Proposition 2 (Spectral properties for the operator $\mathbf{H}_{M,s}$) *For a fixed M, let $\mathbf{H}_{M,s}$ be the* (*constrained*) *transfer operators associated to a dynamical system with contraction radius ρ and abscissa of convergence σ_0* (*quantities defined in Property* 2). *Let Σ_0 be the interval $]\sigma_0, \infty[$.*

(i) (Quasi-compactness) *Let $\rho < \widehat{\rho} < 1$. If $\sigma := \Re s \in \Sigma_0$, then $\mathbf{H}_{M,s}$ acts boundedly on $\mathcal{C}^1(\mathcal{I})$. The spectral radius $\mathrm{R}_M(s)$ of $\mathbf{H}_{M,s}$ and its essential spectral $\mathrm{R}_M^{(e)}(s)$ satisfy*

$$\mathrm{R}_M(s) \leq \mathrm{R}_M(\sigma), \qquad \mathrm{R}_M^{(e)}(s) \leq \widehat{\rho} \cdot \mathrm{R}_M(\sigma);$$

in particular, $\mathbf{H}_{M,s}$ is quasi-compact for real s.

(ii) (Unique dominant eigenvalue) *For real $\sigma \in \Sigma_0$, $\mathbf{H}_{M,\sigma}$ has a unique eigenvalue $\lambda_M(\sigma)$ of maximal modulus, which is real and simple, the dominant eigenvalue. The associated eigenfunction $f_{M,\sigma}$ is strictly positive, and the associated eigenvector $\nu_{M,\sigma}$ of the adjoint operator $\mathbf{H}^*_{M,\sigma}$ is a Radon measure. With the normalisation conditions, $\nu_{M,\sigma}[1] = 1$, $\nu_{M,\sigma}[f_{M,\sigma}] = 1$, the measure $\nu_{M,\sigma}$ is defined in a unique way. In particular, $\nu_{\infty,1}$ is Lebesgue measure, with $\lambda_\infty(1) = 1$.*

(iii) (Spectral gap) *For real parameters $\sigma \in \Sigma_0$, there is a spectral gap, i.e. the subdominant spectral radius $r_M(\sigma)$ defined by*

$$r_M(\sigma) := \sup\bigl\{|\lambda|; \lambda \in \mathrm{Sp}(\mathbf{H}_{M,\sigma}), \lambda \neq \lambda_M(\sigma)\bigr\}$$

satisfies $r_M(\sigma) < \lambda_M(\sigma)$.

(iv) (Analyticity in compact sets) *The operator $\mathbf{H}_{M,s}$ depends analytically on s for $\Re s \in \Sigma_0$. Thus, $\lambda_M(\sigma)^{\pm 1}$, $f_{M,\sigma}^{\pm 1}$, $f'_{M,\sigma}$, depend analytically on $\sigma \in \Sigma_0$.*

Sketches. Proofs of these properties can be found in [1] for the usual transfer operator. In [18], the author considers the constrained operator in the standard continued fraction context acting on the space of analytic functions. The spectrum of constrained operators in the continued fraction context acting on different Banach spaces is also studied in [8]. All these proofs are easily extended to the constrained transfer operator associated to dynamical systems of the Good Class.

We give here sketches of proofs for Assertions (i) and (iv), since their arguments will be central in other proofs of this paper. In particular, many operators of interest in this paper can be written as a sum of operators R_h of the form $R_h : f \mapsto r_h \cdot f \circ h$, the sum being taken over a subset of $\mathcal{H}^\star$. In most cases, the functions r_h equal $|h'|^s$ or $|h'(x)|^s \log|h'(x)|$. Note that

$$\begin{aligned} \|r_h \cdot f \circ h\|_0 &\le \|r_h\|_0 \cdot \|f\|_0, \\ \left\|(r_h \cdot f \circ h)'\right\|_0 &\le \left\|r_h'\right\|_0 \|f\|_0 + \|r_h\|_0 \left\|h'\right\|_0 \left\|f'\right\|_0, \end{aligned} \tag{3.1}$$

so that

$$\|R_h\|_{1,1} \le \|r_h\|_0 \cdot \left[1 + \left\|h'\right\|_0\right] + \left\|r_h'\right\|_0. \tag{3.2}$$

Assertion (i). It is based on the existence of a Lasota–Yorke bound, which is uniform with respect to M. With Hennion's Theorem, this kind of bound entails the relation $\mathrm{R}^{(e)}(\mathbf{H}_{M,s}) \le \widehat{\rho} \cdot \mathrm{R}(\mathbf{H}_{M,s})$.

(Lasota–Yorke bounds) *For every compact subset $\mathcal{L}$ of Σ_0, there exists $C > 0$ so that for all s with $\Re s \in \mathcal{L}$, and all $f \in \mathcal{C}^1(\mathcal{I})$, for all $M \le \infty$, for all $n \ge 1$*

$$\left\|\mathbf{H}^n_{M,s}[f]\right\|_1 \le C \left\|\mathbf{H}^n_{M,\sigma}\right\|_1 \left(|s|\|f\|_0 + \widehat{\rho}^n \|f\|_1\right), \tag{3.3}$$

and $\|\mathbf{H}^n_{M,\sigma}\|_1$ is uniformly bounded for $M \ge 3$.

We now prove this bound: The quantity $\mathbf{H}^n_{M,s}[f]$ can be written as a sum over $h \in \mathcal{H}^n_M$ of functions $r_h \cdot f \circ h$ with $r_h := |h'|^s$. The bounded distortion property entails

$$\left|r_h'\right| \le |s|\left|h''\right|\left|\left|h'\right|^{s-1}\right| \le |s|\widehat{K}\left|\left|h'\right|^s\right| = |s|K\left|h'\right|^\sigma \quad (\sigma := \Re s),$$

and, with the definition of contraction ratio, one has $|h'| \le \widehat{C}\rho^n$. Finally, with (3.1), we obtain the bound (3.3).

Assertion (iv). Consider the operator $\mathbf{G}_{M,s,z} := \mathbf{H}_{M,s} - \mathbf{H}_{M,z} - (s-z)\mathbf{H}'_{M,s}$, with

$$\mathbf{H}'_{M,s}[f] := \sum_{h \in \mathcal{H}_M} \left|h'\right|^s \log\left|h'\right| \cdot f \circ h. \tag{3.4}$$

The operator $\mathbf{H}'_{M,s}$ can be written as a sum over $\mathcal{H}_M$ of terms $r_h \cdot f \circ h$, with $r_h = |h'|^s \log|h'|$. With the distortion property, the estimate $|h'_{[m,\varepsilon]}(x)| = \Theta(m^{-2})$, together with (3.1), this entails

$$\left\|\mathbf{H}'_{M,s}\right\|_{1,1} \le C\zeta'(2\sigma) \quad (\sigma := \Re s). \tag{3.5}$$

In the same vein, the operator $\mathbf{G}_{M,s,z}$ can be written as a sum over $\mathcal{H}_M$ of terms $r_h \cdot f \circ h$, where the functions r_h defined as

$$r_h = \left|h'\right|^s - \left|h'\right|^z - (s-z)\left|h'\right|^s \log|h'|$$

satisfy, with $\sigma := \min(\Re s, \Re z)$,

$$|r_h| \le |s-z|^2 |\log\left|h'\right|^2 \left|h'\right|^\sigma, \qquad \left|r_h'\right| \le \frac{|h''|}{|h'|}|z||r_h|,$$

and, with the distortion property, the estimate $|h'_{[m,\varepsilon]}(x)| = \Theta(m^{-2})$, together with (3.2), this entails

$$\left\|\mathbf{G}_{M,s,z}[f]\right\|_0 \le C|s-z|^2\,\zeta''(2\sigma)\|f\|_0,$$
$$\left\|\left(\mathbf{G}_{M,s,z}[f]\right)'\right\|_0 \le C|s-z|^2\left(K|z|\zeta''(2\sigma)\|f\|_0 + \zeta''(2\sigma+1)\|f\|_1\right)$$

where the constant C may depend on the system. These two relations prove that $s \mapsto \mathbf{H}_{M,s}$ is analytic, with a derivative equal to $\mathbf{H}'_{M,s}$ defined in (3.4), whose norm $\|\mathbf{H}'_{M,s}\|_{1,1}$ admits via (3.5) an upper bound independent of M. □

This proposition together with analytic perturbation theory [13] entails the following spectral decomposition for each $\mathbf{H}_{M,s}$, on a neighbourhood of the real axis, that a priori depends on M: For each $M \le \infty$, there exists $\mathcal{U}_M$ so that for all $s \in \mathcal{U}_M$, one has

$$\mathbf{H}_{M,s} = \lambda_M(s)\mathbf{P}_{M,s} + \mathbf{N}_{M,s} \tag{3.6}$$

where $\mathbf{P}_{M,s}$ is the projector for the dominant eigenvalue $\lambda_M(s)$, the two operators satisfy $\mathbf{N}_{M,s} \circ \mathbf{P}_{M,s} = \mathbf{P}_{M,s} \circ \mathbf{N}_{M,s} = 0$, and the dominant spectral radius $r_M(s)$ satisfies $r_M(s) \le \theta_M|\lambda_M(s)|$ for some $\theta_M < 1$. Therefore,

$$\mathbf{H}^n_{M,s}[f](x) = \lambda^n_M(s)\mathbf{P}_{M,s}[f](x) + \mathbf{N}^n_{M,s}[f](x) \quad \forall n \ge 1. \tag{3.7}$$

Moreover, all the cited objects—except the subdominant spectral radius $r_M(s)$—are analytic functions of s. The subdominant spectral radius $r_M(s)$ is a continuous function of s.

As we already said, we need this spectral decomposition to hold on a common neighbourhood of the real axis, in order to obtain in the sequel bounds "uniform" with respect to M. This is achieved in the next section. We first prove a perturbation result, from which we deduce the uniform spectral decomposition, the convergence of the dominant spectral objects of $\mathbf{H}_{M,s}$ to those of the $\mathbf{H}_s$ and a variety of uniform bounds.

3.2 Uniform spectral decomposition of $\mathbf{H}_{M,s}$ around $s = 1$

The next result of Continuous Perturbation Theory (a simplified version of Theorem 3.16 from Chap. IV of the book of Kato [13]) is well suited to obtain an uniform spectral decomposition.

Theorem E (Kato) *Let X be a Banach space and $\mathbf{T}$ a bounded operator on X. Suppose that $\mathbf{T}$ has a simple eigenvalue doubly separated from the rest of the spectrum by two curves Γ_- and Γ_+. This means that $\mathbf{T}$ has a simple eigenvalue outside Γ_+, no element of the spectrum between Γ_- and Γ_+, and the rest of the spectrum inside Γ_-. Then, there exists a $\delta > 0$ (which depends on $\mathbf{T}$ and $\Gamma_\pm$) with the following property: any bounded operator S which satisfies $\|\mathbf{S} - \mathbf{T}\| \le \delta$ has a simple eigenvalue doubly separated from the rest of the spectrum by Γ_+ and Γ_-.*

The following result proves that the hypotheses needed to apply Kato's Theorem are fulfilled.

Lemma 1 *Let* $(\mathcal{I}, T)$ *be any of the three Euclidean dynamical systems of the Good Class. There exists a* $C > 0$ *such that the following holds*:

(a) *For any* s *with* $\sigma := \Re s > 1/2$, *one has*

$$\|\mathbf{H}_{M,s} - \mathbf{H}_s\|_{1,1} \le C\frac{1}{M^{2\sigma-1}}, \qquad \left\|\mathbf{H}'_{M,s} - \mathbf{H}'_s\right\|_{1,1} \le C\frac{\log M}{M^{2\sigma-1}}. \tag{3.8}$$

(b) *For any* (s, z) *with* $\sigma := \min(\Re s, \Re z) > 1/2$, *for any* M, *one has*

$$\|\mathbf{H}_{M,s} - \mathbf{H}_{M,z}\|_{1,1} \le C\zeta'(2\sigma)|s - z|. \tag{3.9}$$

Proof The first operator $\mathbf{H}_{M,s} - \mathbf{H}_s$ is written as a sum of operators R_h of the form $R_h : f \mapsto r_h \cdot f \circ h$ with $r_h = |h'^s|$, the sum being taken over $\mathcal{H} \setminus \mathcal{H}_M$. Together with (3.1), and the bounded distortion property, the estimates $|h'_{[m,\varepsilon]}(x)| = \Theta(m^{-2})$ entail the bounds

$$\sum_{h\in\mathcal{H}\setminus\mathcal{H}_M} \|r_h\|_0 \le \frac{C}{M^{2\sigma-1}}, \qquad \sum_{h\in\mathcal{H}\setminus\mathcal{H}_M} \left\|r_h \cdot \left|h'\right|\right\|_0 \le \frac{C}{M^{2\sigma+1}},$$

$$\sum_{h\in\mathcal{H}\setminus\mathcal{H}_M} \left\|r'_h\right\|_0 \le KC|s|\frac{1}{M^{2\sigma-1}},$$

which prove the first inequality.

The second operator $\mathbf{H}'_{M,s} - \mathbf{H}'_s$ is written as a sum of operators R_h of the form $R_h : f \mapsto r_h \cdot f \circ h$ with $r_h = |h'^s| \log|h'|$, the sum being taken over $\mathcal{H} \setminus \mathcal{H}_M$. Together with (3.1), the estimates $|h'_{[m,\varepsilon]}(x)| = \Theta(m^{-2})$ entail the bounds

$$\sum_{h\in\mathcal{H}\setminus\mathcal{H}_M} \|r_h\|_0 \le C\frac{\log M}{M^{2\sigma-1}}, \qquad \sum_{h\in\mathcal{H}\setminus\mathcal{H}_M} \left\|r_h \cdot \left|h'\right|\right\|_0 \le \frac{C}{M^{2\sigma+1}},$$

$$\sum_{h\in\mathcal{H}\setminus\mathcal{H}_M} \left\|r'_h\right\|_0 \le KC|s|\frac{\log M}{M^{2\sigma-1}},$$

which prove the second inequality.

The third inequality is just a consequence of the analyticity of $s \mapsto \mathbf{H}_{M,s}$ [Assertion (iv) of Proposition 2] together with the bound (3.5). □

3.3 The first result: near the real axis

With this lemma, together with Kato's Theorem, we now prove the first important result of this paper which constitutes the first step for proving part (ii) of the US Property for the constrained transfer operator (Theorem 2).

Theorem 3 *Denote by* $\mathbf{H}_{M,s}$ *the constrained operator relative to one of the three Euclidean dynamical systems. The following holds*:

(i) *For any* $\gamma < 1/2$, *there exists a* $t_1 > 0$ *such that, in the rectangle* $\mathcal{V}_\gamma^{(1)} := [1-\gamma, 1+\gamma] \times [-t_1, +t_1]$ *and for all* $M \geq 3$, *the quasi-inverses* $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ *admit the spectral decomposition*

$$(\mathrm{Id} - \mathbf{H}_{M,s})^{-1} = \frac{\lambda_M(s)}{1-\lambda_M(s)} \mathbf{P}_{M,s} + (\mathrm{Id} - \mathbf{N}_{M,s})^{-1}.$$

(ii) *For any* $\gamma < 1/2$, *there exists a real* $t_2 > 0$ (*with* $t_2 < t_1$) *and an integer* $M_2 \geq 3$ *such that, for any integer* $M \geq M_2$, *the functions* $\lambda_M(s) - 1$ *possess a unique zero in the rectangle* $\mathcal{V}_\gamma^{(2)} := [1-\gamma, 1+\gamma] \times [-t_2, +t_2]$, *real and simple, located at* $s = \sigma_M$. *The integer* M_2 *is chosen as*

$$M_2 = M_2(\gamma) := \min\{M; \sigma_M > 1-\gamma\},$$

so that $\alpha := \min\{\sigma_M - (1-\gamma);\ M \geq M_2\}$ *is strictly positive.*

(iii) *Define the real* γ_0 *as the supremum of the reals* $\gamma < 1/2$ *for which the subdominant spectral radius* $r(s)$ *of the plain operator* $\mathbf{H}_s$ *is strictly less than* 1 *on the interval* $[1-\gamma, 1+\gamma]$. *Then, for any* $\gamma < \gamma_0$, *there exists a real* $t_3 > 0$ (*with* $t_3 < t_1$) *and an integer* $M_3 \geq 3$ *such that, for any integer* $M \geq M_3$, *for any* s *in the rectangle* $\mathcal{V}_\gamma^{(3)} = [1-\gamma, 1+\gamma] \times [-t_3, +t_3]$, *the quasi-inverses* $(\mathrm{Id} - \mathbf{N}_{M,s})^{-1}$ *are analytic on* $\mathcal{V}_\gamma^{(3)}$.

(iv) *For any* $\gamma < \gamma_0$, *there exists a real* $t_4 > 0$ *and an integer* $M_0 \geq 3$ *such that, for any integer* $M \geq M_0$, *the quasi-inverses* $\mathbf{F}_{M,s}(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ *are meromorphic in the rectangle* $\mathcal{V}_\gamma^{(4)} = [1-\gamma, 1+\gamma] \times [-t_4, +t_4]$, *with a unique pole at* $s = \sigma_M$. *The residue of the function* $s \mapsto F_M(s) := (\mathrm{Id} - \mathbf{H}_{M,s})^{-1}[f](0)$ *at* $s = \sigma_M$ *is equal to*

$$\mathrm{Res}_{s=\sigma_M} F_M(s) = \frac{1}{2\lambda'_M(\sigma_M)} \mathbf{F}_{M,\sigma_M} \circ \mathbf{P}_{M,\sigma_M}[f](0).$$

In particular, for any density of class $\mathcal{C}^1$ *on* $\mathcal{I}$,

$$\mathrm{Res}_{s=1} F_\infty(s) = \frac{1}{2\lambda'_\infty(1)} \mathbf{F}_{\infty,1}[f_{\infty,1}](0) = \frac{1}{2\zeta(2)}. \tag{3.10}$$

Furthermore, there exist a real $t_0 \leq t_4$ *and a constant* C *such that, for any* $M \geq M_0$, *on the left line of the rectangle* $\mathcal{V}_\gamma := [1-\gamma, 1+\gamma] \times [-t_0, +t_0]$ (*i.e. for any* s *of the form* $s = 1-\gamma + it$ *with* $|t| \leq t_0$), *one has* $|F_M(s)| \leq C$.

Proof Assertion (i). Consider $\gamma < 1/2$ and two real constants θ_-, θ_+ with

$$\sup\left\{\frac{r_\infty(s)}{\lambda_\infty(s)}, s \in [1-\gamma, 1+\gamma]\right\} < \theta_- < \theta_+ < 1.$$

Consider any s of the real interval $[1-\gamma, 1+\gamma]$, and apply Kato's Theorem with the operator $\mathbf{H}_s$ and the two circles $\Gamma_s^\pm$ having centre 0 and radius $\theta_\pm \lambda_\infty(s)$. This entails the existence of some δ_s. Then, Lemma 1 proves the existence of two strictly positive reals $a_s, t_{(s)}$, together with an integer $M_{(s)}$, for which one has, for $M \geq M_{(s)}$,

$$\|\mathbf{H}_{M,z} - \mathbf{H}_s\|_{1,1} \leq \delta_s \quad \text{for } z \in [s - a_s, s + a_s] \times [-t_{(s)}, +t_{(s)}].$$

By Theorem E and Proposition 2, the following spectral decomposition is valid for all $z \in [s - a_s, s + a_s] \times [-t_{(s)}, +t_{(s)}]$ and $M \geq M_{(s)}$,

$$\mathbf{H}_{M,z}[f](x) = \lambda_M(z)\mathbf{P}_{M,z}[f](x) + \mathbf{N}_{M,z}[f](x). \tag{3.11}$$

Here, $\mathbf{P}_{M,z}$ is the projector associated to the dominant eigenvalue $\lambda_M(z)$, the operators $\mathbf{N}_{M,z}$, $\mathbf{P}_{M,z}$ satisfy $\mathbf{N}_{M,z} \circ \mathbf{P}_{M,z} = \mathbf{P}_{M,z} \circ \mathbf{N}_{M,z} = 0$, and the subdominant spectral radius $r_M(z)$ satisfies

$$r_M(z) < \theta_- \lambda_\infty(s) < \theta_+ \lambda_\infty(s) < \left|\lambda_M(z)\right|, \qquad \frac{r_M(z)}{|\lambda_M(z)|} < \frac{\theta_-}{\theta_+}.$$

The intervals $]s - a_s, s + a_s[$ form an open covering of the compact $[1 - \gamma, 1 + \gamma]$. Then, there exists a finite sub-covering of $[1 - \gamma, 1 + \gamma]$ associated to a finite family of points s_i. With $t_1' := \min\{t_{(s_i)}\}$, $M_1 := \max\{M_{(s_i)}\}$, this proves that the spectral decomposition (3.11) holds for each operator $\mathbf{H}_{M,s}$ on the rectangle $\mathcal{V}_\gamma := [1 - \gamma, 1 + \gamma] \times [-t_1', +t_1']$, for any $M \geq M_1$. Moreover, the dominant eigenvalue and the subdominant spectral radius satisfy

$$\frac{r_M(z)}{|\lambda_M(z)|} < \frac{\theta_-}{\theta_+} < 1.$$

Then, on the rectangle $[1 - \gamma, 1 + \gamma] \times [-t_1', +t_1']$, and for any $M \geq M_1$, the quasi-inverse of each operator decomposes as

$$(\mathrm{Id} - \mathbf{H}_{M,s})^{-1} = \frac{\lambda_M(s)}{1 - \lambda_M(s)}\mathbf{P}_{M,s} + (\mathrm{Id} - \mathbf{N}_{M,s})^{-1}. \tag{3.12}$$

Consider now any integer M with $M < M_1$. There exists, for each such M, a rectangle of the form $[1-\gamma, 1+\gamma] \times [-t_{[M]}, +t_{[M]}]$ on which the quasi-inverse $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ decomposes. It is then sufficient to choose $t_1 := \min(t_1', \min\{t_{[M]}, M < M_1\})$ to obtain the conclusion.

Assertion (ii). The solutions of the equation $\lambda_M(s) = 1$ give rise to poles for the quasi-inverse $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$. This last equation has been deeply studied because its solution $s = \sigma_M$ is the Hausdorff dimension of the set $\mathcal{R}_M$ (see [10, 12, 18]). Here we summarise the most important properties of the equation $\lambda_M(s) = 1$ and refer to the cited papers for the full proofs and deeper results.

For any $M \leq \infty$, the function $\sigma \mapsto \lambda_M(\sigma)$ of the real variable σ is strictly decreasing. The two inequalities $\lambda_M(1/2) > 1$ and $\lambda_M(1) \leq 1$ (see [18]) entail that the function $\lambda_M(s) - 1$ has a unique zero $s = \sigma_M$ in the interval $]1/2, 1]$. Moreover, the sequence $M \mapsto \lambda_M(\sigma)$ is strictly decreasing, which implies that the sequence $M \mapsto \sigma_M$ of solutions of the equation $\lambda_M(s) = 1$ is strictly decreasing, too. Denote by M_2 the smallest integer M for which σ_M is larger than $1 - \gamma$.

Furthermore, due to the inequalities of Theorem 4, the sequence of analytic functions $M \mapsto \lambda_M(s)$ converges to $\lambda_\infty(s)$ uniformly on $\mathcal{V}_\gamma$, and this is the same for the sequence of derivatives $M \mapsto \lambda'_M(s)$ which converges to $\lambda'(s)$. Since $\lambda_\infty(s) - 1$ has a simple zero at $s = 1$, there exists a neighbourhood of $s = 1$ of the form $\mathcal{W} :=]1-\delta, 1+\delta[\times]-t'_2, +t'_2[$ such that all the functions $\lambda_M(s) - 1$ have a unique zero on $\mathcal{W}$ for $M \geq M_2$. Due to decreasing properties of the functions $\sigma \mapsto \lambda_M(\sigma)$, there exists a $t_2 > 0$ (with $t_2 < t'_2$) such that, for any $M \geq M_2$, the functions $\lambda_M(s) - 1$ have a unique zero on $]1-\gamma, 1+\gamma[\times]-t_2, +t_2[$ for $M \geq M_0$.

Assertion (iii). Consider, for $\gamma < \gamma_0$, a constant θ such that

$$\sup\{r(s); s \in [1-\gamma, 1]\} < \theta < 1.$$

Consider any s of the real interval $[1-\gamma, 1]$, and apply Kato's Theorem with the operator $\mathbf{H}_s$ and the circle Γ having centre 0 and radius θ. The inequality $\lambda_\infty(s) \geq 1$ for $s \in [1-\gamma, 1]$ entails that the circle Γ is convenient for applying Kato's Theorem. This proves the existence of some δ'_s. Then, Lemma 1 proves the existence of two strictly positive reals $a'_s, t'_{(s)}$, together with an integer $M'_{(s)}$, for which one has, for any $M \geq M'_{(s)}$,

$$\|\mathbf{H}_{M,z} - \mathbf{H}_s\|_{1,1} \leq \delta'_s \quad \text{for } z \in \left[s - a'_s, s + a'_s\right] \times \left[-t'_{(s)}, +t'_{(s)}\right].$$

By Theorem E and Proposition 2, this entails that $r_M(z) < \theta$ for $z \in [s - a'_s, s + a'_s] \times [-t'_{(s)}, +t'_{(s)}]$ and $M \geq M'_{(s)}$. In the same vein as previously, there is a finite subcovering formed with points s'_i, which defines $t_3 := \min\{t'_{(s'_i)}\}$, $M_3 := \max\{M'_{(s'_i)}\}$. Finally, one has $r_M(z) < \theta$ when z belongs to the rectangle $[1-\gamma, 1+\gamma] \times [-t_3, +t_3]$ and $M \geq M_3$.

Assertion (iv). Choosing $t_4 := \min(t_2, t_3)$, $M_0 := \max(M_2, M_3)$ entails that the quasi-inverse $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ fulfils the three previous assertions. The residue at the only pole σ_M of the function $F_M(s)$ is easily computed with the alternative expression of $F_M(s)$ provided in Proposition 1. This is also true when M is infinite, and, in this case, the two different expressions are also provided by Proposition 1.

On the left line of the rectangle $\mathcal{V}_\gamma^{(4)}$, the subdominant spectral radius $r_M(z)$ satisfies $r_M(z) < \theta$, whereas the projector $\mathbf{P}_{M,z}$ is bounded from above (uniformly with respect to M). Moreover, the functions $|\lambda_M(s) - 1|$ (for $M_0 \leq M \leq +\infty$) admit a lower bound on the left line of the rectangle $\mathcal{V}_\gamma^{(4)}$. This is due to the fact that the functions $|\lambda_M(s) - 1|$ are continuous, strictly positive and the sequence $|\lambda_M(s) - 1|$ converges uniformly to $|\lambda(s) - 1|$. □

3.4 Speed of convergence of σ_M to 1 for $M \to \infty$

As we already said, the speed of convergence of the dominant spectral objects of $\mathbf{H}_{M,\sigma}$ to those of $\mathbf{H}_1$ when $M \to \infty$ and $\sigma \to 1$ is crucial. We provide an extension of the result of Hensley to the other Euclidean Dynamical Systems (centred and odd), with methods slightly different from Hensley's, since we do not deal with the same functional space. Inside this subsection, we deal only with the real values of the parameter s, and we use σ instead of s.

Theorem 4 *The following holds*:

(i) $\big(\lambda(\sigma)-\lambda_M(\sigma)\big)\int_{\mathcal{I}} f_{M,\sigma}(x)\,d\nu_\sigma(x)=\int_{\mathcal{I}}(\mathbf{H}_\sigma-\mathbf{H}_{M,\sigma})[f_{M,\sigma}](x)\,d\nu_\sigma(x),$
$\lambda'_M(\sigma)=\int_{\mathcal{I}}\mathbf{H}'_{M,\sigma}[f_{M,\sigma}](x)\,d\nu_{M,\sigma}(x).$

(ii) $\lambda(\sigma)-\lambda_M(\sigma)=O\big(M^{1-2\sigma}\big), \qquad |\sigma_M-1|=O\big(M^{-1}\big).$

(iii) $\|f_{M,\sigma}-f_\sigma\|_{1,1}$ *and* $\|\nu_{M,\sigma}-\nu_\sigma\|_{1,1}$ *are both* $O\big(M^{1-2\sigma}\big)$.
$\|f_\sigma-f_1\|_{1,1}$ *and* $\|\nu_\sigma-\nu_1\|_{1,1}$ *are both* $O\big(|\sigma-1|\big)$.

(iv) $\lambda(\sigma)-\lambda_M(\sigma)=\beta_M(\sigma)\big(1+|\sigma-1|+O\big(M^{1-2\sigma}\big)\big)$
with $\beta_M(\sigma):=\int_{\mathcal{I}}(\mathbf{H}_\sigma-\mathbf{H}_{M,\sigma})[f_1](x)\,dx.$

(v) $\sigma_M-1=-\dfrac{1}{\zeta(2)}\dfrac{1}{M}-\dfrac{2}{\zeta(2)^2}\dfrac{\log M}{M^2}+O\left(\dfrac{1}{M^2}\right) \quad (M\to\infty).$

(vi) $\|f_{M,\sigma_M}-f_1\|_{1,1}=O\left(\dfrac{1}{M}\right), \qquad \|\nu_{M,\sigma_M}-\nu_1\|_{1,1}=O\left(\dfrac{1}{M}\right),$
$\big|\lambda'_M(\sigma_M)-\lambda'(1)\big|=O\left(\dfrac{\log M}{M}\right).$

Proof *Assertion* (i). With the two relations

$$\mathbf{H}_{M,\sigma}[f_{M,\sigma}]=\lambda_M(\sigma)\,f_{M,\sigma}, \qquad \mathbf{H}_\sigma[f_\sigma]=\lambda(\sigma)\,f_\sigma,$$

the following equality holds

$$\begin{aligned}&\big(\lambda(\sigma)-\lambda_M(\sigma)\big)f_{M,\sigma}+\lambda(\sigma)(f_\sigma-f_{M,\sigma})\\&\quad=(\mathbf{H}_\sigma-\mathbf{H}_{M,\sigma})[f_{M,\sigma}]+\mathbf{H}_\sigma[f_\sigma-f_{M,\sigma}].\end{aligned}$$

We consider the integral with respect to measure ν_σ. Since ν_σ is an eigenvector of the dual operator $\mathbf{H}^\star_\sigma$, it satisfies

$$\int_{\mathcal{I}}\mathbf{H}_\sigma[f_\sigma-f_{M,\sigma}](t)\,d\nu_\sigma(x)=\lambda(\sigma)\int_{\mathcal{I}}\big[f_\sigma(x)-f_{M,\sigma}(x)\big]\,d\nu_\sigma(x),$$

which provides the equality

$$\big(\lambda(\sigma)-\lambda_M(\sigma)\big)\int_{\mathcal{I}}f_{M,\sigma}(x)\,d\nu_\sigma(x)=\int_{\mathcal{I}}(\mathbf{H}_\sigma-\mathbf{H}_{M,\sigma})[f_{M,\sigma}](x)\,d\nu_\sigma(x).$$

In the same vein, taking the derivative (with respect to σ) of the relation $\mathbf{H}_{M,\sigma}[f_{M,\sigma}]=\lambda_M(\sigma)\,f_{M,\sigma}$ leads to the second equality of Assertion (i).

Assertion (ii). The functions $f_{M,\sigma}$ are positive and uniformly bounded from above and below for M sufficiently large. This is the same for the integrals $\int_{\mathcal{I}} f_{M,\sigma}(x)\,d\nu_\sigma(x)$, and finally,

$$\lambda(\sigma)-\lambda_M(\sigma)=O\big(\|\mathbf{H}_\sigma-\mathbf{H}_{M,\sigma}\|_0\big)=O\big(M^{1-2\sigma}\big).$$

Assertion (i) at $\sigma = 1$ entails the estimate $1 - \lambda_M(1) = O(M^{-1})$, and, with the Mean Value Theorem, the equality

$$\lambda_M(1) - 1 = \lambda_M(1) - \lambda_M(\sigma_M) = (1 - \sigma_M)\lambda'_M(\tau_M)$$

with $\tau_M \in]\sigma_M, 1[$. Finally, when $M \to \infty$, $|\lambda'_M(\tau_M)|$ tends to the entropy h. One obtains

$$|\sigma_M - 1| = O\left(\frac{1}{M}\right). \tag{3.13}$$

Assertion (iii). Denote by $\widehat{\mathbf{H}}_{M,\sigma}$ the operator

$$\widehat{\mathbf{H}}_{M,\sigma} := \frac{1}{\lambda_M(\sigma)} \cdot \mathbf{H}_{M,\sigma}.$$

The dominant eigenvalue $\widehat{\lambda}_M(\sigma)$ of $\widehat{\mathbf{H}}_{M,\sigma}$ is constant and equal to 1, and $f_{M,\sigma}$ is the dominant eigenfunction of $\widehat{\mathbf{H}}_{M,\sigma}$ relative to the dominant eigenvalue 1. Since $\nu_{M,\sigma}$ is an eigenvector of the dual operator $\mathbf{H}^\star_{M,\sigma}$, it satisfies

$$\int_{\mathcal{I}} \widehat{\mathbf{H}}_{M,\sigma}[f_{M,\sigma} - f_\sigma](x)\, d\nu_{M,\sigma}(x) = \int_{\mathcal{I}} \big[f_{M,\sigma}(x) - f_\sigma(x)\big]\, d\nu_{M,\sigma}(x).$$

This entails the equality $\int_{\mathcal{I}} g_{M,\sigma}(x)\, d\nu_{M,\sigma}(x) = 0$ with

$$\begin{aligned} g_{M,\sigma} &:= \big(\widehat{\mathbf{H}}_{M,\sigma} - \mathrm{Id}\big)[f_{M,\sigma} - f_\sigma] \\ &= \frac{1}{\lambda_M(\sigma)}\big((\mathbf{H}_\sigma - \mathbf{H}_{M,\sigma})[f_\sigma] + \big(\lambda(\sigma) - \lambda_M(\sigma)\big)[f_\sigma]\big). \end{aligned}$$

The projection of $g_{M,\sigma}$ on the dominant eigensubspace of $\widehat{\mathbf{H}}_{M,\sigma}$ equals 0. Denote by $\widehat{\mathbf{N}}_{M,\sigma}$ the operator $\widehat{\mathbf{N}}_{M,\sigma} := (1/\lambda_M(\sigma)) \cdot \mathbf{N}_{M,\sigma}$ with $\mathbf{N}_{M,\sigma}$ defined in (3.6). Then, for all $n \geq 1$, one has $\widehat{\mathbf{H}}^n_{M,\sigma}[g_{M,\sigma}] = \widehat{\mathbf{N}}^n_{M,\sigma}[g_{M,\sigma}]$. Now, the quasi-compacticity of $\widehat{\mathbf{H}}_{M,\sigma}$ proves that the series of general term $\widehat{\mathbf{H}}^n_{M,\sigma}[g_{M,\sigma}]$ is convergent, with a sum equal to $f_\sigma - f_{M,\sigma}$. Finally,

$$f_\sigma - f_{M,\sigma} = \big(\mathrm{Id} - \widehat{\mathbf{N}}_{M,\sigma}\big)^{-1}[g_{M,\sigma}].$$

Now, the norm $\|\widehat{\mathbf{N}}_{M,\sigma}\|_{1,1}$ is at most $\beta < 1$ for σ sufficiently close to 1, the function f_σ is uniformly bounded as well as $\lambda_M(\sigma)$. Then, with (ii) and (3.8), the norm $\|f_\sigma - f_{M,\sigma}\|_{1,1}$ satisfies

$$\|f_\sigma - f_{M,\sigma}\|_{1,1} = O\big(\|\mathbf{H}_{M,\sigma} - \mathbf{H}_\sigma\|_{1,1} + \big|\lambda_M(\sigma) - \lambda(\sigma)\big|\big) = O\big(M^{1-2\sigma}\big).$$

The proof is exactly the same for the dominant eigenmeasure $\nu_{M,\sigma}$ of the dual operator $\mathbf{H}^*_{M,\sigma}$.

Assertions (iv) and (v). With (i) and (iii), we have

$$\lambda(\sigma) - \lambda_M(\sigma) = \beta_M(\sigma)\big[1 + O\big(|\sigma - 1| + M^{1-2\sigma}\big)\big] \tag{3.14}$$

with

$$\beta_M(\sigma) := \int_{\mathcal{I}} (\mathbf{H}_\sigma - \mathbf{H}_{M,\sigma})[f_1](x)\,dx = \int_{\mathcal{I}} \sum_{h\in\mathcal{H}\setminus\mathcal{H}_M} \left|h'(x)\right|^\sigma f_1 \circ h(x)\,dx.$$

A change a variables provides

$$\beta_M(\sigma) = \int_{I_M} u^{2(\sigma-1)} f_1(u)\,du, \quad \text{with } I_M := \bigcup_{h\in\mathcal{H}\setminus\mathcal{H}_M} h(\mathcal{I}).$$

Since the interval I_M is of the form $[0, a_M]$, with $Ma_M \to 1$ for $M \to \infty$ and writing $f_1(u)$ as $f_1(u) = f_1(0) + ug(u)$ with $g(u) = \Theta(1)$, the integral $\beta_M(\sigma)$ decomposes as

$$\beta_M(\sigma) = \frac{1}{2\sigma - 1} f_1(0) M^{1-2\sigma} + \Theta\big(M^{-2\sigma}\big).$$

Consider now the special value $\sigma = \sigma_M$, for which one has $|\sigma_M - 1| = O(M^{-1})$ [Assertion (ii)]. Then,

$$\beta_M(\sigma_M) = f_1(0)\frac{1}{M}\big[1 - 2(\sigma_M - 1)\log M\big] + O\big(M^{-2}\big). \tag{3.15}$$

Now, the Taylor expansion $\lambda(\sigma) = 1 + \lambda'(1)(\sigma - 1) + O((\sigma - 1)^2)$, together with the estimate $|\sigma_M - 1| = O(M^{-1})$, relations (3.14), (3.15) entail

$$0 = \lambda'(1)(\sigma_M - 1) - f_1(0)\frac{1}{M} + O\left(\frac{\log M}{M^2}\right).$$

Now, with the relation (3.10) which links $f_1(0)$, the entropy $-\lambda'(1)$ and $\zeta(2)$ (see Table 1), one obtains a first estimate of $\sigma_M - 1$, namely

$$\sigma_M - 1 = \frac{-1}{\zeta(2)}\frac{1}{M} + \Theta\left(\frac{\log M}{M^2}\right). \tag{3.16}$$

Putting the estimate obtained in Relation (3.16) into Relation (3.15) provides a refinement of the estimate about $\beta(\sigma_M)$ which permits obtaining the final estimate about $\sigma_M - 1$.

Assertion (vi). The first two estimates are just consequences of Assertion (iii) and (v). Indeed, one has

$$M^{1-2\sigma_M} = \frac{1}{M}\exp\big[2(1 - \sigma_M)\log M\big] = O\left(\frac{1}{M}\right).$$

For the last estimate, one uses the expression of the derivative obtained in Assertion (i), together with the two first estimates of Assertion (vi), and finally the decomposition

$$\big\|\mathbf{H}'_{M,\sigma_M} - \mathbf{H}'_1\big\|_{1,1} \le \big\|\mathbf{H}'_{M,\sigma_M} - \mathbf{H}'_{M,1}\big\|_{1,1} + \big\|\mathbf{H}'_{M,1} - \mathbf{H}'_1\big\|_{1,1},$$

which, with Lemma 1, proves the last estimate. □

4 Far from the real axis

In this section, we aim to prove the US property for the quasi-inverse $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ of the constrained transfer operator with uniform bounds with respect to M. We have already obtained in Theorems 3 and 4 precise information about the behaviour of the quasi-inverse $(\mathrm{Id} - \mathbf{H}_{M,s})$ near the real axis. We now wish to obtain a bound for the norm of the quasi-inverse when parameter s is on a vertical line on the left of $s = 1$, sufficiently far from the real axis (Theorem 2).

Estimates of this type have been previously obtained by Dolgopyat [5] for the (unrestricted) transfer operators related to dynamical systems satisfying the UNI Condition. Baladi and Vallée extended this results for maps with an infinite number of branches [2].

The aim of this section is to obtain bounds on the norm of the quasi-inverse $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ for large values of $\Re s$, and uniformly with respect to M. For dealing with large values of the imaginary part $t := \Im s$, Dolgopyat introduced a family of equivalents norms in $\mathcal{C}^1(\mathcal{I})$, and, for $t > 0$, he defined

$$\|f\|_{1,t} := \|f\|_0 + \frac{1}{t}\|f'\|_0.$$

This section is devoted to obtaining Dolgopyat-type estimates for the $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ that are uniform with respect to M:

Theorem 5 (Dolgopyat-type estimates for the constrained transfer operators) *Let $\mathbf{H}_{M,s}$ be the constrained transfer operator acting on $\mathcal{C}^1(\mathcal{I})$. For any ξ, with $0 < \xi < 1/10$, there are a real interval $\Sigma = [1-\gamma_1, 1+\gamma_1]$ of 1, $t_5 > 0$, and $C > 0$ such that for all $s = \sigma + it$ with $\sigma \in \Sigma$ and $|t| \geq t_5$, and any $M \geq 3$,*

$$\left\|(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}\right\|_{1,t} \leq C \cdot |t|^{\xi}. \tag{4.1}$$

In the proof, we shall take profit from the three following facts:

($s1$) The operator $\mathbf{H}_{M,s}$ is a small perturbation of $\mathbf{H}_1$ for M large and s near 1.
($s2$) The operators $\mathbf{H}_{M,\sigma}$ are "smaller" than $\mathbf{H}_\sigma$ for real σ, that is,

$$\mathbf{H}_{M,\sigma}[f](x) \leq \mathbf{H}_\sigma[f](x), \quad \text{for any } f \in \mathcal{C}^1(\mathcal{I}),\ f \geq 0, M \geq 3. \tag{4.2}$$

($s3$) The constrained dynamical system is a restriction of the unconstrained dynamical system, for which the UNI Condition holds. Even if we do not make any explicit use of the UNI condition, most of the partial results of [2] which we use strongly rely on this condition.

Due to ($s2$), the (truncated) vertical strip $\{|\Re s - 1| \leq \gamma, |t| \geq t_5\}$ obtained for the quasi-inverse of the restricted operator $\mathbf{H}_{M,s}$ contains the (truncated) vertical strip obtained in the proof of Baladi–Vallée [2] for the quasi-inverse of the plain operator $\mathbf{H}_s$, as we now prove it, in the sequel of this section.

4.1 Preparatory material

Here we prove three lemmas, whose proofs are slight modifications of the proofs of Lemma 1, 2, 3 from [2]. We deal with the normalised transfer operators,

$$\widetilde{\mathbf{H}}_s[f] := \frac{1}{\lambda(\sigma) f_\sigma}\mathbf{H}_s[f_\sigma . f], \qquad \widetilde{\mathbf{H}}_{M,s}[f] := \frac{1}{\lambda(\sigma) f_{M,\sigma}}\mathbf{H}_{M,s}[f_{M,\sigma} . f]$$

whose nth iterates satisfy

$$\widetilde{\mathbf{H}}^n_s[f] := \frac{1}{\lambda^n(\sigma) f_\sigma}\mathbf{H}^n_s[f_\sigma . f], \qquad \widetilde{\mathbf{H}}^n_{M,s}[f] := \frac{1}{\lambda^n(\sigma) f_{M,\sigma}}\mathbf{H}^n_{M,s}[f_{M,\sigma} . f].$$

They have a spectral radius of at most 1, and $\widetilde{\mathbf{H}}_\sigma$ fixes the constant function 1. The following two inequalities

$$\left\|\widetilde{\mathbf{H}}_{M,s}[f]\right\|_0 \le \left\|\widetilde{\mathbf{H}}_{M,\sigma}[1]\right\|_0 \|f\|_0 \quad \text{and} \quad \left\|\widetilde{\mathbf{H}}_s[f]\right\|_0 \le \left\|\widetilde{\mathbf{H}}_\sigma[1]\right\|_0 \|f\|_0$$

imply the useful bound

$$\|\widetilde{\mathbf{H}}_{M,s}\|_0 \le 1 \quad \text{for } M \le \infty, \Re s > 1/2. \tag{4.3}$$

The following lemma compares the behaviour of $\widetilde{\mathbf{H}}_{M,\sigma}$ and $\widetilde{\mathbf{H}}_1$ when $\sigma \to 1$. It generalises the same result already obtained in [2] for $M = \infty$.

Lemma 2 *Let $\mathcal{L}$ be a compact subset of $\Sigma_0 :=]1/2, +\infty[$. For any $\sigma \in \mathcal{L}$, for any $f \in \mathcal{C}^1(\mathcal{I})$, for any $n \ge 1$, and for any $M \ge 3$, one has*

$$\left\|\widetilde{\mathbf{H}}_{M,\sigma}\left[|f|\right]\right\|_0^2 \ll A_\sigma^{2n} \left\|\widetilde{\mathbf{H}}_1\left[f^2\right]\right\|_0 \quad \textit{with } A_\sigma := \frac{\lambda(2\sigma-1)^{1/2}}{\lambda(\sigma)}. \tag{4.4}$$

The constants involved depend only on $\mathcal{L}$, and the function $\sigma \mapsto A_\sigma$ is continuous with $A_1 = 1$.

Proof Use the result of Baladi–Vallée that deals with the case $M = \infty$ and extend it to the case $M < \infty$ with Inequality (4.2). □

First use of the $(1,t)$-norm In the bound (3.3), two terms appear: the first one contains a factor $|s|$, while the other one is exponentially decreasing in n. In order to suppress the effect of the factor $|s|$, Dolgopyat uses the family of norms

$$\|f\|_{1,t} := \|f\|_0 + \frac{1}{|t|}\left\|f'\right\|_0 = \sup|f| + \frac{1}{|t|}\sup\left|f'\right|, \quad t \ne 0,$$

which appear in the statement of Proposition 5. With this norm and (3.3), together with (4.3), we obtain the first (easy) result:

Lemma 3 *For any $t_1 > 0$, for every compact subset $\mathcal{L}$ of Σ_0, there is a $C_0 > 0$ such that for all $n \ge 1$, all s for which $\Re s \in \mathcal{L}$ and $|\Im s| \ge t_1$ we have $\|\widetilde{\mathbf{H}}^n_{M,s}\|_{1,\Im s} \le C_0$.*

4.2 Estimates of the L^2-norm

In [2], Lemmas 4 and 5 compare the L^2 norm of $\widetilde{\mathbf{H}}_s^n[f]$ with the $(1,t)$-norm of f. Writing $s = \sigma + it$, the term $|\widetilde{\mathbf{H}}_s^n[f](x)|^2$ can be expressed as a sum taken over $\mathcal{H}^n \times \mathcal{H}^n$ of the form,

$$\left|\widetilde{\mathbf{H}}_s^n[f](x)\right|^2 \ll \frac{1}{\lambda(\sigma)^{2n}} \sum_{(h,k)\in\mathcal{H}^n\times\mathcal{H}^n} \exp\left[it\Psi_{h,k}(x)\right] \cdot R_{h,k}(x)$$

$$\text{with } \Psi_{h,k}(x) := \log\frac{|h'(x)|}{|k'(x)|} \quad \text{and} \tag{4.5}$$

$$R_{h,k}(x) = \left|h'(x)\right|^\sigma \left|k'(x)\right|^\sigma \frac{1}{f_\sigma^2(x)} (f \cdot f_\sigma) \circ h(x) \cdot \left(\overline{f} \cdot f_\sigma\right) \circ k(x). \tag{4.6}$$

Dolgopyat, then Baladi and Vallée [2] estimate the oscillatory integrals

$$\widehat{I}(h,k) := \int_{\mathcal{I}} \exp\left[it\Psi_{h,k}(x)\right] R_{h,k}(x)\, dx,$$

and their Lemmas 4 and 5 are summarised as follows:

Lemma 4 *Consider a dynamical system that satisfies the UNI condition. Letting* $\lceil x \rceil$ *denote the smallest integer greater than* x, *set*

$$n_0 := \left\lceil \frac{1}{|\log\rho|} \log|t| \right\rceil. \tag{4.7}$$

Then, for any interval $[1-\gamma, 1+\gamma]$, *and for any* s *with* $\sigma = \Re s \in \mathcal{L}$ *and* $|t| \geq 1/\rho^2$, *for any* a, *with* $0 < a < 1/2$, *one has*

$$\sum_{h,k\in\mathcal{H}^{n_0}\times\mathcal{H}^{n_0}} \left|\widehat{I}(h,k)\right| \ll \left(\max\left\{\rho^{(1-2a)}, A_\sigma \rho^{a/2}\right\}\right)^{n_0} \|f\|_{1,t}^2$$

where A_σ *is defined in Lemma* 2.

Lemma 4 can be extended for the case of a finite M, as we now explain. First, notice that (4.5) can be extended to the case when M is finite,

$$\int_{\mathcal{I}} \left|\widetilde{\mathbf{H}}_{M,s}^n[f](x)\right|^2 dx \ll \frac{1}{\lambda(\sigma)^{2n}} \sum_{h,k\in\mathcal{H}_M^n\times\mathcal{H}_M^n} \left|\widehat{I}(h,k)\right|.$$

Now, the inclusion $\mathcal{H}_M^n \subset \mathcal{H}^n$, together with Lemma 4, entails the following inequality

$$\int_{\mathcal{I}} \left|\widetilde{\mathbf{H}}_{M,s}^{n_0}[f](x)\right|^2 dx \ll \left(\max\left\{\rho^{(1-2a)}, A_\sigma \rho^{a/2}\right\}\right)^{n_0} \|f\|_{1,t}^2$$

for n_0 and t as in Lemma 4. Note that the "hidden" constants do not depend on M.

For $a \in]2/5, 1/2[$, the inequality $(a/2) > 1-2a > 0$ holds, and there is a real neighbourhood Σ_1 of $\sigma = 1$, defined as

$$\Sigma_1 := \left\{\sigma;\ A_\sigma \rho^{a/2} \le \rho^{1-2a}\right\} \tag{4.8}$$

which does not depend on M. Finally, the inequality

$$\int_{\mathcal{I}} \left|\widetilde{\mathbf{H}}^{n_0}_{M,s}[f](x)\right|^2 dx \ll \rho^{(1-2a)n_0} \|f\|^2_{1,t}, \quad M \ge 2, \tag{4.9}$$

holds for $\Re s \in \Sigma_1$ with constants which do not depend on M.

4.3 End of the proof of Dolgopyat-type estimates

The end of the proof of the Dolgopyat-type estimates for the quasi-inverse of the operator $\mathbf{H}_{M,s}$ follows the same lines as in [2]. It is necessary to operate transfers between various norms. The uniform bounds are a consequence of the uniform bounds obtained in Lemma 2, (3.3), Lemma 3, and (4.9).

From the L^2-norm to the sup-norm Since the normalised density transformer $\widetilde{\mathbf{H}}_1$ is quasi-compact with respect to the $(1,1)$-norm, and fixes the constant function 1, it satisfies

$$\left\|\widetilde{\mathbf{H}}_1^k\left[|g|^2\right]\right\|_0 = \left(\int_{\mathcal{I}} |g|^2(x)\, dx\right) + O\left(r_1^k\right)\left\|g^2\right\|_{1,1}, \tag{4.10}$$

where r_1 is the subdominant spectral radius of $\mathbf{H}_1$.

Consider an iterate $\widetilde{\mathbf{H}}^n_{M,s}$ with $n \ge n_0$. Then

$$\left\|\widetilde{\mathbf{H}}^n_{M,s}[f]\right\|_0^2 \ll \left\|\widetilde{\mathbf{H}}^{n-n_0}_{M,\sigma}[g_M]\right\|_0^2 \quad \text{with } g_M = \left|\widetilde{\mathbf{H}}^{n_0}_{M,s}[f]\right|.$$

Now, using (4.4) from Lemma 2 and (4.10) with $k := n - n_0$, together with the bound (4.9) for the L^2-norm, and finally Lasota–Yorke bounds (3.3) to evaluate $\|g_M^2\|_{1,1}$, one obtains

$$\left\|\widetilde{\mathbf{H}}^n_{M,s}[f]\right\|_0^2 \ll A_\sigma^{2(n-n_0)}\left[\rho^{(1-2a)n_0} + r_1^{n-n_0}|t|\right]\|f\|^2_{1,t}$$

and the hidden constant does not depend on M. We now choose $n = n_1$ as a function of t so that the two terms $\rho^{(1-2a)n_0}$ and $r_1^{n-n_0}|t|$ are almost equal (with $n_0(t)$ defined in (4.7)):

$$n_1 = (1+\eta)n_0 \quad \text{with } \eta := 2(1-a)\frac{\log \rho}{\log r_1} > 0. \tag{4.11}$$

Choose now d such that $0 < \eta(5a-2) < d < 1-2a < 1/5$ (which is possible if a is of the form $a = 2/5 + \epsilon$, with a small $\epsilon > 0$). We then obtain, when $\sigma := \Re s$ is in Σ_1 defined in (4.8), for $n_1(t)$ and η defined in (4.11)

$$\left\|\widetilde{\mathbf{H}}^{n_1}_{M,s}[f]\right\|_0 \ll \rho^{n_1 b}\|f\|_{1,t} \quad \text{with } b := \frac{1-2a-d}{1+\eta}. \tag{4.12}$$

From the sup-norm to the $\|.\|_{1,t}$*-norm* Applying Lasota–Yorke bounds (3.3) twice and using (4.12) yield the inequality

$$\begin{aligned}\left\|\widetilde{\mathbf{H}}_{M,s}^{2n_1}[f]\right\|_1 &\ll |s|\left\|\widetilde{\mathbf{H}}_{M,s}^{n_1}[f]\right\|_0+\rho^{n_1}\left\|\widetilde{\mathbf{H}}_{M,s}^{n_1}[f]\right\|_1\\ &\ll |s|\rho^{n_1 b}\|f\|_{1,t}+\rho^{n_1}|t|\left(\frac{|s|}{|t|}\|f\|_0+\rho^{n_1}\frac{\|f\|_1}{|t|}\right)\\ &\ll |t|\rho^{n_1 b}\|f\|_{1,t}, \end{aligned}\tag{4.13}$$

which finally entails that there is a constant C_1 such that, for any $t \geq 1/\rho^2$, and $n_2 = 2n_1$ (with $n_1(t)$ as above),

$$\left\|\widetilde{\mathbf{H}}_{M,s}^{n_2}\right\|_{1,t} \leq C_1\rho^{n_2 b/2} \quad (\Re s \in \Sigma_1). \tag{4.14}$$

Now choose t sufficiently large, namely $|t| \geq t_5 := C_1^{1/(2(1-2a-d))}$, to ensure the inequality $C_1 < \rho^{-n_2 b/4}$ for any $n_2(t)$ with $|t| \geq t_5$. Finally, one has

$$\left\|\widetilde{\mathbf{H}}_{M,s}^{n_2}\right\|_{1,t} \leq \rho^{n_2 b/4} \quad \left(\Re s \in \Sigma_1, |t| \geq t_5\right). \tag{4.15}$$

The last step in Theorem 5 For fixed t with $|t| > t_5$, any integer n can be written $n = kn_2 + \ell$ with $\ell < n_2(t)$. Then (4.15) and Lemma 3 entail

$$\left\|\widetilde{\mathbf{H}}_{M,s}^{n}\right\|_{1,t} \leq C_2\left\|\widetilde{\mathbf{H}}_{M,s}^{n_2}\right\|_{1,t}^{k} \leq C_2\rho^{bkn_2/4} \leq C_2\rho^{bn/4}\rho^{-bn_2/4}.$$

Since $bn_2/4 = bn_1/2 = (1-2a-d)n_0/2$, with n_0 defined in (4.7), we finally obtain

$$\left\|\widetilde{\mathbf{H}}_{M,s}^{n}\right\|_{1,t} \leq C_3|t|^{\xi}\gamma^n, \qquad \text{with } \xi := \frac{1-2a-d}{2},\ b := \frac{2\xi}{1+\eta},\ \gamma := \rho^{b/4}.$$

Then ξ is any value between 0 and $1/10$. Therefore, returning to the operator $\mathbf{H}_{M,s}$, we have shown

$$\left\|\mathbf{H}_{M,s}^{n}\right\|_{1,t} \leq C_3 \cdot \gamma^n \cdot |t|^{\xi} \cdot \lambda(\sigma)^n, \quad \forall n, \forall M, \forall |t| \geq t_5. \tag{4.16}$$

Finally, with

$$a \in]2/5, 1/2[, \qquad \eta := 2(1-a)\frac{\log\rho}{\log r_1}, \qquad \eta(5a-2) < d < 1-2a, \qquad \xi := \frac{1-2a-d}{2},$$

we take a refinement of the neighbourhood Σ_1 defined in (4.8) and define the neighbourhood Σ of $\sigma = 1$ as

$$\Sigma := \left\{\sigma;\, A_\sigma < \rho^{-(2-5a)/2}, \lambda(\sigma) < \rho^{-(1-2a-d)/8(1+\eta)}\right\} \supset]1-\gamma_1, 1+\gamma_1].$$

Then, for $\Re s \in \Sigma$, one has $\gamma\lambda(\sigma) \leq \rho^{(1-2a-d)/8(1+\eta)} = \widehat{\gamma} < 1$. This finally proves Theorem 5 with $C := C_3/(1-\widehat{\gamma})$, and γ_1 defined via the neighbourhood Σ.

4.4 Intermediate compact region

In this section, we deal with the intermediate region, and we wish to prove the following result:

Lemma 5 *Consider the constrained operator* $\mathbf{H}_{M,s}$ *relative to one of the three Euclidean systems. For any pair of fixed real numbers* t_0, t_5, *with* $t_5 > t_0 > 0$, *there exist* $\gamma_2 > 0, \theta < 1$ *so that the spectral radius satisfies, for any* M,

$$\mathrm{R}_M(s) \le \theta \quad \text{for all } s \in \mathcal{A} := \left\{ s = \sigma + it : t_0 \le |t| \le t_5 \text{ and } |\Re s - 1| \le \gamma_2 \right\}$$

and, for any $\xi > 0$, *there exists a* $C_2 > 0$ *such that the quasi-inverse* $(\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ *satisfies*

$$\left\| (\mathrm{Id} - \mathbf{H}_{M,s})^{-1} \right\|_{1,t} \le C_2 \cdot |t|^{\xi} \quad \text{for all } s \in \mathcal{A}.$$

Proof The same result is valid in the case of the plain operator $\mathbf{H}_s$ (see [2, Lemma 8]): there are $\gamma_3 > 0, \theta_3 < 1$ so that the spectral radius $\mathrm{R}(s)$ satisfies

$$\mathrm{R}(s) \le \theta_3 \quad \text{for all } s \in \mathcal{A}_3 := \left\{ s = \sigma + it : t_0 \le |t| \le t_5 \text{ and } |\Re s - 1| \le \gamma_3 \right\}.$$

Now, we extend this property to the case of finite M, by using the upper-semi-continuity of the spectrum under small (continuous) perturbations, as it is described in [13, Chap. IV, §3, Remark 3.3].

(Upper semi-continuity of the Spectrum) *For any* $\mathbf{T}$ *bounded and* $\varepsilon > 0$, *there is* $\delta > 0$ *such that* $\sup_{\lambda \in \mathrm{Sp}(\mathbf{S})} |\lambda - \mathrm{Sp}(\mathbf{T})| < \varepsilon$ *if* $\|\mathbf{S} - \mathbf{T}\| < \delta$.

We then apply the previous Property to $\mathbf{T} := \mathbf{H}_s$ with the choice $\varepsilon := (1 - \theta_3)/2$. This entails the existence of some δ. Then, Lemma 1 proves the existence of an integer M_4 for which $\|\mathbf{H}_s - \mathbf{H}_{M,s}\| < \delta$ for all $M \ge M_4$ and $s \in \mathcal{A}_3$. Finally, for $M \ge M_4$, the spectral radius $\mathrm{R}_M(s)$ is at most $\theta_2 := (1 + \theta_3)/2$ for all $s \in \mathcal{A}_3 := \{s = \sigma + it : t_1 \le |t| \le t_5 \text{ and } |\Re s - 1| \le \gamma_3\}$.

Now, for each fixed $M < M_4$, the spectral radius $\mathrm{R}_M(s)$ is strictly less than $\lambda_M(\Re s)$ (see [18]). This implies that, for each M, there are $\gamma_{[M]} > 0$, $\theta_{[M]} < 1$ so that the spectral radius satisfies

$$\mathrm{R}_M(s) \le \theta_{[M]} \quad \text{for all } s \in \mathcal{A}_{[M]} := \left\{ s = \sigma + it : t_0 \le |t| \le t_5 \text{ and } |\Re s - 1| \le \gamma_{[M]} \right\}.$$

Then, choosing

$$\theta := \max\left\{\theta_2, \max\{\theta_{[M]}, M \le M_4\}\right\}, \qquad \gamma_2 := \min\left\{\gamma_3, \min\{\gamma_{[M]}, M \le M_4\}\right\}$$

leads to the proof. □

4.5 End of the proof of Theorem 2

We now gather the conclusions of Theorem 3, Theorem 5, and Lemma 5. This will provide the proof of Theorem 2.

First, consider any γ less than $\min(\gamma_0, \gamma_1)$ where γ_0 is defined in Theorem 3 and γ_1 is defined in Theorem 5. Then, Theorem 3 defines a real t_0, and Theorem 5 defines a real t_5 together with a constant C. Then, Lemma 5 associates to this pair (t_0, t_5) a real γ_2, and a constant C_2. Finally, we let $\underline{\gamma} := \min(\gamma_0, \gamma_1, \gamma_2)$. Then, for any $\gamma < \underline{\gamma}$, Theorem 3 defines an integer $M_0 = M_0(\overline{\gamma})$. Then, it follows that, for $M \geq M_0$, the map $s \mapsto (\mathrm{Id} - \mathbf{H}_{M,s})^{-1}$ is meromorphic on $|\Re s - 1| \leq \underline{\gamma}$ with an unique pole at $s = \sigma_M$, and has a polynomial growth on the vertical strip $|\overline{\Re} s - 1| < \underline{\gamma}$, $|t| \geq \underline{t}_0$, with $\underline{t}_0 = t_0$ and a constant $\underline{C} := \max(C, C_2)$. This polynomial growth is thus uniform with respect to $M \geq M_0$. This ends the proof of Theorem 2.

5 Proof of Theorem 1

In this section, we complete the proof of Theorem 1. Recall that we have introduced in Sect. 2.4 the probability Dirichlet generating functions and we have obtained a fundamental relation between this Dirichlet series and the quasi-inverse of transfer operators in Proposition 1.

5.1 The sums of order two

We wish to evaluate the partial sums $\Phi_M(N)$ defined in (2.5), but it is not possible to deal directly with them. We first consider, for $M \leq \infty$, the sums of order two of coefficients $c_M(n)$ of the Dirichlet series $F_M(s)$, namely

$$\Psi_M(T) := \sum_{n \leq T} c_M(n)(T - n)$$

which can be evaluated with the Perron formula (2.7) as

$$\Psi_M(T) = \frac{1}{2\pi i} \int_{L-i\infty}^{L+i\infty} F_M(s) \frac{T^{2s+1}}{s(2s+1)} \, ds$$

with $L > 1$.

Property US states that $F_M(s)$ has a meromorphic extension to $\Re s \geq 1 - \gamma$ for some positive γ, with a unique simple pole at $s = \sigma_M$ for all $M \geq M_0$. With Property US (ii), it is possible to deform the integration contour, and Cauchy formula implies the equality

$$\Psi_M(T) := R_M \frac{T^{2\sigma_M + 1}}{\sigma_M(2\sigma_M + 1)} + I_M(T),$$

with

$$R_M := \mathrm{Res}_{s=\sigma_M} F_M(s), \qquad I_M(T) = \frac{1}{2\pi i} \int_{1-\gamma-i\infty}^{1-\gamma+i\infty} F_M(s) \frac{T^{2s+1}}{s(2s+1)} \, ds.$$

Thanks to the US Property (ii), the integral $I_M(T)$ is uniformly bounded for all $M \geq M_0$, more precisely $|I_M(T)| \leq C\, T^{3-2\gamma}$ with C independent of M. The residue R_M was evaluated in Theorem 4 and we obtain, for $M \leq \infty$,

$$R_M = \frac{-1}{\lambda'_M(\sigma_M)} \mathbf{F}_{M,\sigma_M}[f_{M,\sigma_M}](0) \nu_{M,\sigma_M}[f].$$

The dominant spectral objects of $\mathbf{H}_{M,\sigma_M}$ converge to the dominant spectral objects of $\mathbf{H}_1$, uniformly with respect to M. This convergence implies that R_M and σ_M are uniformly bounded in M from above and below. With the definition of α as $\alpha := \min\{\sigma_M - (1-\gamma); M \geq M_0\}$, the equality

$$\begin{aligned}\Psi_M(T) &= R_M \frac{T^{2\sigma_M+1}}{\sigma_M(2\sigma_M+1)}\left[1 + O\left(T^{2(1-\sigma_M-\gamma)}\right)\right] \\ &= R_M \frac{T^{2\sigma_M+1}}{\sigma_M(2\sigma_M+1)}\left[1 + O\left(T^{-2\alpha}\right)\right]\end{aligned}$$

holds, with the constants involved in the O-term uniform with respect to M.

5.2 Transfer of estimates

In order to exploit the above estimates, and transform them into estimates on $\Phi_M(T)$, we use a simplified version of Lemma 10 from [2].

Lemma 6 *Assume that* $\Psi_M(T) := \sum_{n \leq T} c_M(n)(T-n)$ *satisfies*

$$\Psi_M(T) = F_M(T)\left[1 + O\left(T^{-2\alpha}\right)\right], \quad T \to \infty,$$

where the O*-term is uniform with respect to* M*, and* $\alpha < 1/2$*. Denote by* $T^- := T - \lfloor T^{1-\alpha} \rfloor$, $T^+ := T + \lfloor T^{1-\alpha} \rfloor$*. One has:*

$$\frac{1}{T-T^-}\left[\Psi_M(T) - \Psi_M\left(T^-\right)\right] = F'_M(T)\left[1 + O\left(T^{-\alpha}\right)\right], \tag{5.1}$$

$$\frac{1}{T^+-T}\left[\Psi_M\left(T^+\right) - \Psi_M(T)\right] = F'_M(T)\left[1 + O\left(T^{-\alpha}\right)\right], \tag{5.2}$$

where the constants in the O*-terms are uniform with respect to* M*.*

Since the Dirichlet series $F_M(s)$ has positive coefficients, there exist relations between the sums $\Psi_M(T)$ (of order two), and the sums $\Phi_M(T)$ (of order one, which are the sums of interest), namely

$$\frac{1}{T-T^-}\left[\Psi_M(T) - \Psi_M\left(T^-\right)\right] \leq \Phi_M(T) \leq \frac{1}{T^+-T}\left[\Psi_M\left(T^+\right) - \Psi_M(T)\right].$$

Then, for $M_0 \leq M \leq \infty$, the following estimate holds for $\Phi_M(T)$,

$$\Phi_M(T) = R_M \frac{T^{2\sigma_M}}{\sigma_M}\left[1 + O\left(T^{-\alpha}\right)\right].$$

This finally provides the estimate of Theorem 1 for the probability of the subset $\mathcal{O}_N^{[M]}$, namely

$$\mathbb{P}_{N,f}\left[\mathcal{O}_N^{[M]}\right] = C_M N^{2(\sigma_M-1)}\left[1 + O\left(N^{-\alpha}\right)\right] \quad \text{with } C_M := \frac{R_M}{\sigma_M R_\infty}.$$

Then Theorem 4 provides an asymptotic expansion of $\sigma_M - 1$ together with the estimate $C_M = 1 + O(\log M/M)$. This concludes the proof of Theorem 1.

6 Conclusions, conjectures, and generalisations

This paper precisely studies the probability that a rational with denominator at most N has all its continued fraction digits smaller than M; it considers all the possible pairs (M, N), the only restriction being that M must be greater than some M_0. This result improves previous results due to Cusick, Hensley and Vallée [4, 9, 18] (described in Theorem B), and enlarges the family of sequences $M(N)$ covered by Hensley's previous result [11] described in Theorem C.

The Dynamical Analysis paradigm used in this paper also provides a machinery that allows extending the main result of the paper (Theorem 1) to a larger class of constraints, as we now explain.

As in [16, 18], we consider constraints on the continued fraction digits associated to an infinite subset $\mathcal{A}$ of $\mathbb{N}$. We say that the number $x \in \mathcal{I}$ is $\mathcal{A}$-constrained iff any digit of its *CFE*-expansion belongs to $\mathcal{A}$. We relate to the constraint $\mathcal{A}$ the constrained Riemann zeta function $\zeta_{\mathcal{A}}$ defined as

$$\zeta_{\mathcal{A}}(s) := \sum_{m \in \mathcal{A}} \frac{1}{m^s}.$$

The constraint $\mathcal{A}$ is said to be open if the intersection of the convergence domain of $\zeta_{\mathcal{A}}$ with the real axis is an open interval $\Sigma_{\mathcal{A}} :=]p_{\mathcal{A}}, +\infty[$. In this case, the Hausdorff dimension of reals whose all *CFE*-digits belong to $\mathcal{A}$ exists and is denoted by $\sigma_{\mathcal{A}}$. It is proved that $\mathcal{A} \mapsto \sigma_{\mathcal{A}}$ is strictly increasing.

To an infinite constraint $\mathcal{A}$, we associate the family of constraints $\mathcal{A}(M)$ defined as $\mathcal{A}(M) := \mathcal{A} \cap]M, \infty[$. In the same vein, the Hausdorff dimension of reals whose all digits belong to $\mathcal{A}(M)$ exists and is denoted by $\sigma_{\mathcal{A}(M)}$. In [16, 18], Theorem B is proven to hold in the case of an open constraint $\mathcal{A}$. In this case, for any $M \leq \infty$, the probability that a rational of denominator less than N has all its digits in $\mathcal{A}(M)$ satisfies

$$\mathbb{P}_N\big[\mathcal{O}_N^{\mathcal{A}(M)}\big] = C_{\mathcal{A}(M)} N^{2(\sigma_{\mathcal{A}(M)}-1)}\big[1 + \epsilon_{\mathcal{A}(M)}(N)\big].$$

Now, in the same vein as previously, we ask the following question: *Is it is possible to make precise the remainder term $\epsilon_{\mathcal{A}(M)}(N)$?* We can answer the question in the case where the constraint $\mathcal{A}$ is both "smooth" and "large".

We say that the infinite constraint $\mathcal{A}$ is smooth if the Riemann zeta function $\zeta_{\mathcal{A}(M)}$ associated to $\mathcal{A}(M)$ and defined as

$$\zeta_{\mathcal{A}(M)}(s) := \sum_{\substack{m \in \mathcal{A} \\ m \leq M}} \frac{1}{m^s}$$

admits the following estimate

$$\zeta_{\mathcal{A}(M)}(2\sigma) = \Theta\big(M^{1-g_{\mathcal{A}}(\sigma)}\big) \quad \text{(with a } \Theta \text{ uniform for } M \to \infty,\ \sigma \in]p_{\mathcal{A}}, 1]),$$

where the function $g_{\mathcal{A}}$ tends to 1 when σ tends to $(1/2)p_{\mathcal{A}}$. Then, the following is true: (i) the function $\sigma \mapsto g_{\mathcal{A}}(\sigma)$ is strictly increasing, so that the inequality $g_{\mathcal{A}}(\sigma_{\mathcal{A}}) > 1$ holds; (ii) the map $\mathcal{A} \mapsto g_{\mathcal{A}}$ is decreasing, and thus satisfies $g_{\mathcal{A}}(\sigma) \geq 2\sigma$.

For constraints $\mathcal{A}$ which are both open and smooth, a (weak) version of Theorem A holds: the speed of convergence of $\sigma_{\mathcal{A}(M)}$ towards $\sigma_{\mathcal{A}}$ is of order $M^{1-g_{\mathcal{A}}(\sigma_{\mathcal{A}})}$. The paper [18] introduces and studies classes of constraints which provide natural instances of constraints which are both open and smooth: these are modular constraints of the form

$$\mathcal{A}_{\langle B,d\rangle} = \{m \in \mathbb{N}; m \bmod d \in B\}, \quad B \subset \{0, 1, \ldots, d-1\}, \tag{6.1}$$

or co-finite constraints, related to some finite subset B of $\mathbb{N}$, of the form

$$\mathcal{A}_{\langle \notin B\rangle} = \{m \in \mathbb{N}; m \notin B\}, \quad B \text{ finite}. \tag{6.2}$$

Such constraints are open and smooth, with $p_{\mathcal{A}} = 1/2$ and $g_{\mathcal{A}}(\sigma) = 2\sigma$.

An open constraint $\mathcal{A}$ is large if the real $\sigma_{\mathcal{A}}$ belongs to the (maximal) US-strip of the operator $\mathbf{H}_s$. In this case, there exists an integer $M_{\mathcal{A}}$ for which, for any $M \geq M_{\mathcal{A}}$, the remainder term is of the form

$$\epsilon_{\mathcal{A}(M)}(N) = O\big(N^{-\alpha_{\mathcal{A}}}\big),$$

where the O-term is uniform with respect to $M \geq M_{\mathcal{A}}$ and $\alpha_{\mathcal{A}}$ is related to the position of $\sigma_{\mathcal{A}}$ inside the US-strip. (More precisely, $2\alpha_{\mathcal{A}}$ equals the distance of $\sigma_{\mathcal{A}}$ to the left line of the US vertical strip.) Then, for a constraint $\mathcal{A}$ which is open, smooth and large, the probability that an $\mathcal{A}$-constrained rational with a denominator at most N has all its digits less than M is of the form

$$\mathbb{P}_{N,f}\big[\mathcal{O}_N^{\mathcal{A}(M)} \big| \mathcal{O}_N^{\mathcal{A}}\big] = \frac{C_{\mathcal{A}(M)}}{C_{\mathcal{A}}} N^{2(\sigma_{\mathcal{A}(M)} - \sigma_{\mathcal{A}})}\big[1 + O\big(N^{-\alpha_{\mathcal{A}}}\big)\big].$$

We then obtain, in the same vein as in our main Theorem 1, a threshold phenomenon, depending on the relative order of $\sigma_{\mathcal{A}(M)} - \sigma_{\mathcal{A}}$ (of order $M^{1-g_{\mathcal{A}}(\sigma_{\mathcal{A}})}$) with respect to $n := \log N$. Consider

$$R_{M,n} = \frac{M}{n^{1/(g_{\mathcal{A}(\sigma_{\mathcal{A}})}-1)}},$$

(a) If $R(M,n) \to +\infty$, then, almost everywhere, any rational of $\mathcal{O}_N^{\mathcal{A}}$ has all its *CFE*-digits less than M.
(b) If $R(M,n) \to 0$, then, almost everywhere, any rational of $\mathcal{O}_N^{\mathcal{A}}$ has at least one of its *CFE*-digits greater than M.

This result applies in particular to the case of modular constraints where the exponent $g_{\mathcal{A}(\sigma_{\mathcal{A}})} - 1$ equals $2\sigma_{\mathcal{A}} - 1$. The Hausdorff dimension $\sigma_{\mathcal{A}}$ is strictly greater than $1/2$ and can be computed with principles described in [18] and proved later by Lhote [14]. If our conjecture about the US-strip holds (see Sect. 1, Subsection Our Results), then our result applies to all the particular constraints $\mathcal{A}$ previously described, the modular ones described in (6.1) or the co-finite ones, defined in (6.2).

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