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Interpolation and best simultaneous approximation ☆

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Abstract

We consider best simultaneous approximation to k continuous functions on an interval $[a, b]$ from a finite dimensional subspace of $\mathcal{C}[a, b]$, with respect to the functionals $\sum_{j=1}^{k} \psi\left(\int_a^b \phi(|f_j|)\right)$ and $\max_{1\le j\le k} \int_a^b \phi(|f_j|)$ for suitable real functions ϕ and ψ. We obtain the interpolation properties of the best simultaneous approximations. As a consequence, we extend known results in L^p-approximation over small intervals.

Keywords: Simultaneous approximation; Algebraic polynomials; L^p-norm

1. Introduction

Let $\mathcal{C} = \mathcal{C}[a, b]$ be the space of continuous real functions defined on the interval $[a, b]$, $a < b$. Let $\psi, \phi : [0, \infty) \to [0, \infty)$ be two functions such that $\psi(0) = \phi(0) = 0$, and $\psi(x) > 0, \phi(x) > 0$ if $x > 0$. Further, we assume that ψ is a strictly increasing continuously differentiable function in $(0, \infty)$ and ϕ is a convex differentiable function in $(0, \infty)$. For $h \in \mathcal{C}$ we denote

$$F_{\phi}^{[a,b]}(h) = \int_a^b \phi(|h(x)|)\mathrm{d}x \quad \text{and} \quad F_{\infty}^{[a,b]}(h) = \max_{x\in[a,b]} \{|h(x)|\}. \tag{1.1}$$

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Let $k \in \mathbb{N}$ and $h_j \in \mathcal{C}$, $1 \le j \le k$. We consider the following functionals:

$$G_{\phi,\psi}^{[a,b]}(h_1, \ldots, h_k) = \sum_{j=1}^{k} \psi(F_{\phi}^{[a,b]}(h_j)), \quad G_{\phi,\infty}^{[a,b]}(h_1, \ldots, h_k) = \max_{1 \le j \le k} F_{\phi}^{[a,b]}(h_j), \quad (1.2)$$

$$G_{\infty,\psi}^{[a,b]}(h_1, \ldots, h_k) = \sum_{j=1}^{k} \psi(F_{\infty}^{[a,b]}(h_j)), \quad G_{\infty,\infty}^{[a,b]}(h_1, \ldots, h_k) = \max_{1 \le j \le k} F_{\infty}^{[a,b]}(h_j). \quad (1.3)$$

Briefly we put $G_{\infty}^{[a,b]}$ instead of $G_{\infty,\infty}^{[a,b]}$. We shall omit $[a, b]$ in the notation of the functionals F and G when it is not necessary, or we simply shall write a instead of the interval $[-a, a]$, p instead of $\phi(x) = x^p$, $1 \le p < \infty$, and q instead of $\psi(x) = x^q$, $0 < q < \infty$.

Let $S \subset \mathcal{C}$ be a subspace of finite dimension and let $f_j \in \mathcal{C}$, $1 \le j \le k$. We say that $u_0 \in S$ is a *best simultaneous* $G_{\phi,\psi}$*-approximation to* f_j, $1 \le j \le k$, *from* S, briefly a $G_{\phi,\psi}$-b.s.a., if

$$G_{\phi,\psi}(f_1 - u_0, \ldots, f_k - u_0) = \inf_{u \in S} G_{\phi,\psi}(f_1 - u, \ldots, f_k - u). \quad (1.4)$$

We have analogous definitions for a $G_{\phi,\infty}$, $G_{\infty,\psi}$, or G_{∞}-b.s.a.

If a function h has derivative up to order n at zero we denote $T(h)$ as its Taylor polynomial of degree n. We also denote Π^n the space of algebraic polynomials of degree at most n, and we write

$$T(f_1, f_2) := \frac{T(f_1) + T(f_2)}{2}.$$

In [8] the authors prove that the best L^2-approximation to $\frac{1}{k}\sum_{j=1}^{k} f_j$ from Π^n is identical with the $G_{2,1}$-b.s.a. to f_j, $1 \le j \le k$, from Π^n. It is well known that the $G_{p,q}$-b.s.a. generally does not match with the best approximation to the mean of the functions f_j, $1 \le j \le k$, even if $p = q$ (see [9]). However, it is useful to know if they are close when we set sufficiently small intervals.

In [4] the b.s.a. from Π^n with respect to $G_{p,\infty}^{\epsilon}$, $0 < \epsilon \le 1$, $1 < p < \infty$, is considered. The authors prove that for $k = 2$, the b.s.a. to f_1 and f_2 converges to $T(f_1, f_2)$, as $\epsilon \to 0$. Best simultaneous L^2-approximation on a pairwise disjoint intervals union was also considered in that work for $k = 2$.

In [5] the authors study $G_{p,1/p}^{\epsilon}$-b.s.a., $0 < \epsilon \le 1$, $1 < p < \infty$, from Π^n. They prove that the set of cluster points of $G_{p,1/p}^{\epsilon}$-b.s.a., as $\epsilon \to 0$, is a convex and compact set and it is contained in the convex hull of the Taylor polynomials of the functions f_j, $1 \le j \le k$, at zero for the cases $p = 2$, k is arbitrary, and $k = 2$, p is arbitrary.

In this paper, we prove interpolation theorems of the $G_{\phi,\psi}$-b.s.a. and $G_{\phi,\infty}$-b.s.a. from a subspace S which is a weak Chebyshev (**WT**)-system in $\mathcal{C}$ [12]. Our theorems extend classical results of interpolation for best polynomial approximation on L^p-spaces [16], and moreover on spaces with a generalized integral norm [6]. Using the interpolation theorems we establish asymptotic results for b.s.a. on intervals $[-\epsilon, \epsilon]$ for $\epsilon \to 0$. The existence of such limits when $f_1 = f_2$, is known in the literature as the existence of best local approximation [1,16]. In addition, we extend a previous result proved in [4] for $G_{p,\infty}^{\epsilon}$-b.s.a., $1 < p < \infty$, to the cases $p = 1$ and $p = \infty$.

We remark that it is important to find the limit of the b.s.a. since as such it provides useful qualitative and approximation analytic information concerning the b.s.a. on small regions, which is difficult to obtain from a strictly numerical treatment.

2. Interpolating of best simultaneous approximations

We begin by establishing the following Lemma.

Lemma 2.1. *Let V be a linear subspace of $\mathcal{C}[a, b]$, and $f_j \in \mathcal{C}[a, b]$, $1 \le j \le k$. Suppose that $u \in V$ is a $G_{\phi,\psi}$-b.s.a. to f_j, $1 \le j \le k$, from V, and $u \ne f_j$ for all $1 \le j \le k$. Then for all $v \in V$*

$$\sum_{j=1}^{k} \beta_j \left(\int_{\{f_j \ne u\}} \phi'(|f_j - u|)\operatorname{sgn}(f_j - u)v\mathrm{d}x + \phi'(0) \int_{\{f_j = u\}} |v|\mathrm{d}x \right) \ge 0, \tag{2.1}$$

where $\beta_j = \psi'(F_\phi(f_j - u))$, $1 \le j \le k$. Here $\phi'(0)$ is the right derivative of ϕ at 0.

Proof. Let $1 \le j \le k$, $v \in V$, and $G = G_{\phi,\psi}$. Since $u \ne f_j$ on a positive measure set, $F_\phi(f_j - u) \ne 0$. For sufficiently small $t > 0$, an application of Mean Value Theorem gives us

$$\frac{\psi(F_\phi(f_j - u + tv)) - \psi(F_\phi(f_j - u))}{t} = \psi'(\eta_t)\frac{F_\phi(f_j - u + tv) - F_\phi(f_j - u)}{t}, \tag{2.2}$$

where η_t is a value between the positive real numbers $\int_a^b \phi(|f_j - u + tv|)\mathrm{d}x$ and $\int_a^b \phi(|f_j - u|)\mathrm{d}x$. Our assumptions over the functions ϕ and ψ, and (2.2) yield

$$\lim_{t\to 0^+} \frac{\psi(F_\phi(f_j - u + tv)) - \psi(F_\phi(f_j - u))}{t}$$
$$= \beta_j \left(\int_{\{f_j \ne u\}} \phi'(|f_j - u|)\operatorname{sgn}(f_j - u)v\mathrm{d}x + \phi'(0) \int_{\{f_j = u\}} |v|\mathrm{d}x \right). \tag{2.3}$$

Since u is a G-b.s.a. we have

$$0 \le \lim_{t\to 0^+} \frac{G(f_1 - u + tv, \dots, f_k - u + tv) - G(f_1 - u, \dots, f_k - u)}{t}$$
$$= \sum_{j=1}^{k} \lim_{t\to 0^+} \frac{\psi(F_\phi(f_j - u + tv)) - \psi(F_\phi(f_j - u))}{t}, \tag{2.4}$$

for all $v \in V$. The lemma immediately follows from (2.3) and (2.4). □

For $\phi(x) = x^p$, $1 < p < \infty$, we write

$$\alpha_j(p) = (\beta_j)^{\frac{1}{p-1}} \left(\sum_{1\le l\le k} \beta_l^{\frac{1}{p-1}} \right)^{-1}, \quad 1 \le j \le k. \tag{2.5}$$

Remark 2.2. If $\psi(\infty) = \infty$ and ϕ satisfies the Δ_2 condition, i.e. there is a constant $K > 0$ such that $\phi(2x) \le K\phi(x)$, $x \ge 0$, then for any finite set of functions $f_j \in \mathcal{C}$, $1 \le j \le k$, the existence of a $G_{\phi,\psi}$ or $G_{\phi,\infty}$-b.s.a. from S, follows from a standard argument of compactness. In fact, let $G = G_{\phi,\psi}$ or $G = G_{\phi,\infty}$, and let $(v_m) \subset S$ be a sequence that verifies

$$G(f_1 - v_m, \dots, f_k - v_m) \to \inf_{v\in S} G(f_1 - v, \dots, f_k - v) \quad \text{as } m \to \infty,$$

then $\{F_\phi(f_j - v_m) : 1 \le j \le k, m \in \mathbb{N}\}$ is a bounded set. Now, it is easy to show that $\{v_m : m \in \mathbb{N}\}$ is a bounded set in the Luxemburg norm induced by ϕ. See [10] for the definition and properties of this norm. Since S has finite dimension, (v_m) has a subsequence converging to a G-b.s.a.. Analogously, if $\psi(\infty) = \infty$ we can prove that there is a $G_{\infty,\psi}$ or G_∞-b.s.a.

We recall that a polynomial $P \in \Pi^n$ interpolates to a function h in the $n+1$ points, $t_0 \le t_1 \le \cdots \le t_n$, if $t_{s-1} < t_s = \cdots = t_r < t_{r+1}$, for some integer numbers s and r, $0 \le s \le r \le n$, then

$$P^{(j)}(t_s) = h^{(j)}(t_s), \quad 0 \le j \le r - s.$$

Here we put $t_{-1} = -\infty$ and $t_{n+1} = \infty$.

The next theorem for $\psi(x) = x^{1/p}$, $1 \le p < \infty$, and $S = \Pi^{n-1}$ was proved in [5] with a similar technique.

Henceforth we assume that the subspace S is a **WT**-system in $\mathcal{C}$.

Theorem 2.3. *Let $f_j \in \mathcal{C}[a,b]$, $1 \le j \le k$, and let $u \in S$ be a $G_{p,\psi}$-b.s.a. to f_j, $1 \le j \le k$, from S. Then there is j, $1 \le j \le k$, such that $u = f_j$ on a positive measure subset of $[a,b]$, or*

(a) *If $p = 2$, u interpolates to $\sum_{j=1}^{k} \alpha_j(2) f_j$ in at least n different points of $[a,b]$.*
(b) *If $k = 2$ and $1 < p < \infty$, u interpolates to $\alpha_1(p) f_1 + \alpha_2(p) f_2$, in at least n different points of $[a,b]$. For $\psi(x) = x$ we get $\alpha_1(p) = \alpha_2(p) = \frac{1}{2}$.*
(c) *If $p = 1$, there are at least n different points $x_i \in [a,b]$, $0 \le i \le n-1$, such that $\sum_{j=1}^{k} \beta_j \operatorname{sgn}(f_j - u)(x_i) = 0$. In addition, if $k = 2$ then $\beta_1 = \beta_2$, or $f_1(x_i) = f_2(x_i) = u(x_i)$, $0 \le i \le n-1$.*

Proof. If there is j, $1 \le j \le k$, such that $u = f_j$ on a positive measure subset of $[a,b]$, the theorem is obvious. Now, suppose

$$|\{x \in [a,b] : u(x) = f_j(x)\}| = 0, \quad 1 \le j \le k. \tag{2.6}$$

First we assume $p > 1$, so $\phi'(0) = 0$. By Lemma 2.1, we have

$$\int_a^b h v \,\mathrm{d}x = 0, \quad \text{for all } v \in S, \tag{2.7}$$

where

$$h := \sum_{j=1}^{k} \beta_j \phi'(|f_j - u|) \operatorname{sgn}(f_j - u). \tag{2.8}$$

If $h(x)$ has m different zeros in $[a,b]$, we shall show that $m \ge n$. Suppose that $m \le n-1$. Since S is a **WT**-system, Corollary 12 ([12], p. 204) implies that there exists $v \in S$ such that $h(x)v(x) \le 0$ on the interval $[a,b]$, and $h(x)v(x) < 0$ on a positive measure subset of $[a,b]$. It contradicts (2.7).

Henceforth we suppose $h(x_i) = 0$, $x_i \in [a,b]$, $0 \le i \le n-1$.

(a) If $p = 2$, then $\phi'(|f_j - u|)\operatorname{sgn}(f_j - u) = 2(f_j - u)$, $1 \le j \le k$. From (2.8) we get

$$u(x_i) = \sum_{j=1}^{k} \alpha_j(2) f_j(x_i), \quad 0 \le i \le n-1. \tag{2.9}$$

(b) Suppose $k = 2$, and let $x \in [a,b]$ be such that $h(x) = 0$. If $(f_1 - u)(x)(f_2 - u)(x) \ge 0$, then $f_1(x) = u(x) = f_2(x)$, while $(f_1 - u)(x)(f_2 - u)(x) < 0$ implies

$u(x) = \alpha_1(p) f_1(x) + \alpha_2(p) f_2(x)$. Therefore, in both cases we have $u(x) = \alpha_1(p) f_1(x) + \alpha_2(p) f_2(x)$. Consequently,

$$u(x_i) = \alpha_1(p) f_1(x_i) + \alpha_2(p) f_2(x_i), \quad 0 \le i \le n-1.$$

If $\psi(x) = x$, from (2.5) we get $\alpha_1(p) = \alpha_2(p) = \frac{1}{2}$.

(c) Assume $p = 1$. By Lemma 2.1 and (2.6) we get

$$\sum_{j=1}^{k} \beta_j \int_{\{f_j \neq u\}} \operatorname{sgn}(f_j - u) v \mathrm{d}x \ge 0, \tag{2.10}$$

for all $v \in S$.

From (2.10) we obtain $\int_a^b hv\mathrm{d}x \ge 0$ for all $v \in S$, where

$$h := \sum_{j=1}^{k} \beta_j \operatorname{sgn}(f_j - u). \tag{2.11}$$

As in the proof of part (a), there are at least n points x_i such that $h(x_i) = 0$, $0 \le i \le n-1$. If $k = 2$, the proof follows as in (b). □

Remark 2.4. From the proof of Theorem 2.3(b), we observe that if $f_j \notin S$, $j = 1, 2$, and we consider a strictly convex function ϕ, instead of p-power function, and ψ the identical function then (b) remains valid with $\alpha_1 = \alpha_2 = \frac{1}{2}$.

The following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.5. *Let $f_j \in \mathcal{C}[a, b]$, $1 \le j \le k$, and let $1 < p < \infty$. Let $u \in S$ be a $G_{p,\psi}$-b.s.a. to f_j, $1 \le j \le k$, from S, then there are n points in $[a, b]$, $x_0 < x_1 < \cdots < x_{n-1}$ such that u interpolates to some convex combination of f_j, $1 \le j \le k$, in those points if:* (a) $p = 2$, *k is arbitrary, or* (b) $k = 2$, *p is arbitrary.*

We need the following uniqueness result in order to establish other corollary of Theorem 2.3.

Lemma 2.6. *Suppose that $f_j \in \mathcal{C}[a, b]$, $j = 1, 2$, $1 < p < \infty$, and let V be a finite dimension subspace of $\mathcal{C}[a, b]$. Then there is a unique $G_{p,\infty}$-b.s.a. to f_j, $j = 1, 2$, from V.*

Proof. Let

$$d(f_1, f_2; \mathcal{C}) := \inf_{h \in \mathcal{C}} (G_{p,\infty}(f_1 - h, f_2 - h))^{1/p}$$

and

$$d(f_1, f_2; V) := \inf_{v \in V} (G_{p,\infty}(f_1 - v, f_2 - v))^{1/p}.$$

It is easy to see that $d(f_1, f_2; \mathcal{C}) = \frac{1}{2} G_{p,1/p}(f_1 - f_2)$. Let $v \in V$ be such that $d(f_1, f_2, V) = \max\{G_{p,1/p}(f_1 - v), G_{p,1/p}(f_2 - v)\}$. First, we assume that $f_1 \notin V$ or $f_2 \notin V$. If $d(f_1, f_2; \mathcal{C}) = d(f_1, f_2; V)$, we have that $G_{p,1/p}(f_1 - v) + G_{p,1/p}(f_2 - v) = G_{p,1/p}(f_1 - f_2)$. Since $G_{p,1/p}(.)$ is a strictly convex norm, it follows that there is $\delta \ge 0$ such that $f_1 - v = \delta(v - f_2)$, i.e.

$$v = \frac{1}{1+\delta} f_1 + \frac{\delta}{1+\delta} f_2.$$

It is the unique convex combination of f_1 and f_2 which belongs to V, otherwise we obtain that $f_1, f_2 \in V$. So, v is the unique $G_{p,\infty}$-b.s.a. to f_j, $j = 1, 2$, from V. If $d(f_1, f_2; \mathcal{C}) < d(f_1, f_2; V)$ the uniqueness of $G_{p,\infty}$-b.s.a. follows from [11], Theorem 3. Finally, we suppose that $f_1, f_2 \in V$. Then $\frac{1}{2}(f_1 + f_2)$ is the unique $G_{p,\infty}$-b.s.a. □

Since we shall establish results concerning the space Π^n, some considerations about interpolation polynomials are also necessary. We recall the Newton's divided difference formula for the interpolation polynomial of a function $h(x)$ of degree n at $x_0 \le x_1 \le \cdots \le x_n$ (see [14,2]),

$$P(x) = h(x_0) + (x - x_0)h[x_0, x_1] + \cdots + (x - x_0)\cdots(x - x_{n-1})h[x_0, \ldots, x_n]. \quad (2.12)$$

Here $h[x_0, \ldots, x_m]$ denotes the mth order Newton divided difference. If h has continuous derivatives up to order m in $[a, b]$ containing to $x_0, \ldots, x_m$, then the m-th divided difference can be expressed as

$$h[x_0, \ldots, x_m] = \frac{h^{(m)}(\xi)}{m!}, \quad (2.13)$$

for some ξ in the interval $[x_0, x_m]$. It is well known that the mth divided difference is a continuous function as a function of their arguments $x_0, \ldots, x_m$.
We also observe that the space Π^n is a **WT**-system in $\mathcal{C}$.

Corollary 2.7. *Let $f_1, f_2 \in \mathcal{C}[a, b]$ be with continuous derivatives up to order n in $[a, b]$. If $1 < p \le \infty$, and $u_p \in \Pi^n$ is the $G_{p,\infty}$-b.s.a. to f_j, $j = 1, 2$, from Π^n, then there are points in $[a, b]$, $t_0 \le t_1 \le \cdots \le t_n$, such that u_p interpolates to some convex combination of f_j, $j = 1, 2$, at the points t_i, $0 \le i \le n$.*

Proof. Let $1 < p < \infty$. For $1 < q < \infty$, we consider the functional $G_{p,q/p}$. As $(G_{p,q/p})^{1/q}$ is a strictly convex norm, we call $u_{p,q}$ to the unique $G_{p,q/p}$-b.s.a. to f_j, $j = 1, 2$, from Π^n. Theorem 2.3(b), implies that there are points in $[a, b]$, say $x_0(q) < \cdots < x_n(q)$, and real numbers, $0 \le \lambda_j(q)$, $j = 1, 2$, such that $\lambda_1(q) + \lambda_2(q) = 1$, and

$$u_{p,q}(x_i(q)) = \lambda_1(q) f_1(x_i(q)) + \lambda_2(q) f_2(x_i(q)), \quad 0 \le i \le n. \quad (2.14)$$

By Lemma 2.6 we have uniqueness of the $G_{p,\infty}$-b.s.a. to f_j, $j = 1, 2$, from Π^n. Since,

$$(|y_1|^q + |y_2|^q)^{1/q} \to \max\{|y_1|, |y_2|\}, \quad \text{as } q \to \infty, \text{ for all } y_1, y_2 \in \mathbb{R},$$

the Pólya's algorithm (see [7]) implies that $u_{p,q} \to u_p$, as $q \to \infty$. Let (q_m) be a sequence such that $q_m \to \infty$. Since the sequences $(\lambda_1(q_m), \lambda_2(q_m))$ and $(x_0(q_m), \ldots, x_n(q_m))$ are bounded, we can find convergent subsequences, which we denote other times with the same index q_m. Suppose that $\lambda_j(q_m) \to \gamma_j$, $j = 1, 2$, and $x_i(q_m) \to t_i$, $0 \le i \le n$. Note that $t_i \le t_{i+1}$, $0 \le i \le n - 1$. Using (2.12) and (2.14), and the continuity of the divided differences we get that u_p interpolates to $\gamma_1 f_1 + \gamma_2 f_2$ at the points t_i, $0 \le i \le n$.

Now we assume $p = \infty$. Let $p_m \uparrow \infty$. We proved that for each p_m there are $n + 1$ points in $[a, b]$ such that u_{p_m} interpolates to a convex combination at those points. By Pólya's algorithm u_{p_m} converges to u_∞, the unique b.s.a. respect to G_∞ (see [15], for the uniqueness). Therefore, u_∞ interpolates to a convex combination of f_1 and f_2 in at least $n + 1$ points. □

Corollary 2.8. *Let $1 \le q \le \infty$ and let $f_j \in \mathcal{C}[a, b]$, $j = 1, 2$, be with continuous derivatives up to order n in $[a, b]$. Then there exists a $G_{1,q}$-b.s.a. to f_j, $j = 1, 2$, from Π^n, say u_q, such that u_q interpolates to some convex combination of the functions f_j in at least $n + 1$ points of $[a, b]$.*

Proof. First, we assume $q < \infty$. For $1 < p < \infty$, we consider the functional $G_{p,q/p}$. Let $v_{p,q} \in \Pi^n$ be a $G_{p,q/p}$-b.s.a. to f_j, $j = 1, 2$, from Π^n. By Theorem 2.3(b), there are points $x_i = x_i(p,q) \in [a,b]$, $x_0 < x_1 < \cdots < x_n$, such that $v_{p,q}$ interpolates to some convex combination of the functions f_j at those points. On the other hand, since

$$(G_{p,q/p}(h_1,h_2))^{1/q} \to (G_{1,q}(h_1,h_2))^{1/q}, \quad \text{as } p \to 1,$$

Pólya's algorithm implies that there exists a sequence p_m, $p_m \downarrow 1$, and $u_q \in \Pi^n$ such that $v_{p_m,q} \to u_q$. Moreover, u_q is a $G_{1,q}$-b.s.a. to f_j, $j = 1, 2$, from Π^n. Now, we proceed as in the proof of Corollary 2.7 to get points $t_0(q) \le t_1(q) \le \cdots \le t_n(q) \in [a,b]$, such that u_q interpolates to some convex combination of f_j, $j = 1, 2$, at $t_i(q)$, $0 \le i \le n$.

Next, we assume $q = \infty$ and we consider the functional $G_{1,\infty}$. We take a sequence $q_m \uparrow \infty$. As a consequence of that we have above proved, we get a G_{1,q_m}-b.s.a. to f_j, $j = 1, 2$, from Π^n, say u_{q_m}, and points $t_i(q_m) \in [a,b]$, $t_0(q_m) \le t_1(q_m) \le \cdots \le t_n(q_m)$, such that u_{q_m} interpolates to some convex combination of f_j, $j = 1, 2$, at $t_i(q_m)$, $0 \le i \le n$. Since $(G_{1,q_m})^{1/q_m} \to G_{1,\infty}$, as $m \to \infty$, other applications of Pólya's algorithm give us a cluster point of the sequence u_{q_m}, say u_∞, which satisfies the theorem. □

Remark 2.9. In general, there is not uniqueness of the $G_{1,q}$-b.s.a. except if $n = 0$ and $1 < q \le \infty$ (see [13]).

3. Best simultaneous approximation in small regions

In this section using the interpolation results of Section 2, we study asymptotic behavior of the b.s.a. when the measure of the interval tends to zero.

The following theorem extends to Theorem 2.3, [5]. There, it was proved for $\psi(x) = x^{1/p}$, $1 < p < \infty$.

Theorem 3.1. *Let (ϵ_m) be a sequence such that $\epsilon_m \downarrow 0$. Suppose that $f_j \in \mathcal{C}[-\epsilon_1,\epsilon_1]$, $1 \le j \le k$, are functions with continuous derivatives up to order n in $[-\epsilon_1,\epsilon_1]$, and let u_m be a $G^{\epsilon_m}_{p,\psi}$-b.s.a. to f_j, $1 \le j \le k$, from Π^n. If $p = 2$, k is arbitrary or $1 < p < \infty$, $k = 2$, there exists a subsequence (ϵ_{m_s}) and $\gamma_j \ge 0$, $1 \le j \le k$, such that*

$$\sum_{j=1}^{k} \gamma_j = 1, \quad \text{and} \quad u_{m_s} \to \sum_{j=1}^{k} \gamma_j T(f_j), \quad \text{as } s \to \infty. \tag{3.1}$$

In addition, if $k = 2$, ϕ is a strictly convex function and $f_j \notin \Pi^n$, $j = 1, 2$, then for all net (u_ϵ) of $G^{\epsilon}_{\phi,1}$-b.s.a. to f_j, $j = 1, 2$, from Π^n, $u_\epsilon \to T(f_1, f_2)$, as $\epsilon \to 0$.

Proof. Corollary 2.5 implies that there are at least $n+1$ different points $x_i = x_i(m) \in [-\epsilon_m, \epsilon_m]$, and non-negative numbers $\lambda_j = \lambda_j(m)$, $1 \le j \le k$, $\sum_{j=1}^{k} \lambda_j = 1$, such that u_m interpolates to $g_m := \sum_{j=1}^{k} \lambda_j f_j$ at x_i. Since $\{\lambda_j, 1 \le j \le k\}$ is bounded, then there exist convergent subsequences $(\lambda_j(m_s))$, $1 \le j \le k$. Suppose that $\lambda_j(m_s) \to \gamma_j \in [0,1]$, $1 \le j \le k$, as $s \to \infty$. From (2.12) and (2.13) follows that

$$u_{m_s}(x) = g_{m_s}(x_0) + (x - x_0) g^{(1)}_{m_s}(\xi(s,1)) + \cdots + (x - x_0)\cdots(x - x_{n-1}) \frac{g^{(n)}_{m_s}(\xi(s,n))}{n!}, \tag{3.2}$$

where $\xi(s,i) \in [-\epsilon_{m_s}, \epsilon_{m_s}]$, $1 \le i \le n$, $s \in \mathbb{N}$. Taking the limit for $s \to \infty$ in (3.2), and using the continuity of the derivatives of the functions f_j we get (3.1).

Assume now $k = 2$ and ϕ is strictly convex. The second part of the theorem, immediately follows from Remark 2.4. In fact, we have $\lambda_1(m) = \lambda_2(m) = \frac{1}{2}$, for all $m \in \mathbb{N}$. □

The next theorem for $1 < q < \infty$ was proved in [5].

Theorem 3.2. *Let $1 \le q \le \infty$ and let ϵ_m be a sequence such that $\epsilon_m \downarrow 0$. Suppose that $f_j \in \mathcal{C}[-\epsilon_1, \epsilon_1]$, $j = 1, 2$, are functions with continuous derivatives up to order n in $[-\epsilon_1, \epsilon_1]$. Then there exist subsequences (ϵ_{m_s}), (u_{m_s}) of $G_{1,q}^{\epsilon_{m_s}}$-b.s.a. to f_j, $j = 1, 2$, from Π^n, and real numbers $\gamma_j \ge 0$, $j = 1, 2$, such that they verify (3.1) with $k = 2$.*

Proof. Let $1 \le q \le \infty$. From Corollary 2.8, for each ϵ_m there exists u_m a $G_{1,q}^{\epsilon_m}$-b.s.a. which interpolates a convex combination of f_j, $j = 1, 2$, with coefficients depending on m, in at least $n + 1$ points of $[-\epsilon_m, \epsilon_m]$. Now, using (2.12) and (2.13) we prove the theorem in the same way as we get Theorem 3.1. □

The next theorem extends the result established in [4] to the case $p = \infty$ and it gives a weaker version for $p = 1$. It also shows what convex combination the b.s.a. converges.

Theorem 3.3. *Let $1 < p \le \infty$ and let $f_j \in \mathcal{C}[-1, 1]$, $j = 1, 2$, be with derivatives continuous up to order $n + 1$ in $[-1, 1]$. Let u_ϵ, $0 < \epsilon \le 1$, be a $G_{p,\infty}^{\epsilon}$-b.s.a. to f_j, $j = 1, 2$, from Π^n. Then*

$$u_\epsilon \to T(f_1, f_2), \quad \text{as } \epsilon \to 0. \tag{3.3}$$

In addition, if $p = 1$, for each ϵ, $0 < \epsilon \le 1$, there exists a $G_{1,\infty}^{\epsilon}$-b.s.a., say u_ϵ, such that the net (u_ϵ) satisfies (3.3).

Proof. We assume $1 < p < \infty$. If there is some subsequence $\epsilon_m \downarrow 0$ with $G_{p,1/p}^{\epsilon_m}(f_1 - u_{\epsilon_m}) \ne G_{p,1/p}^{\epsilon_m}(f_2 - u_{\epsilon_m})$ the theorem follows from [4], Theorem 2.6. So, we can suppose that there exists $\epsilon_0 > 0$ such that

$$G_{p,1/p}^{\epsilon}(f_1 - u_\epsilon) = G_{p,1/p}^{\epsilon}(f_2 - u_\epsilon), \tag{3.4}$$

for all $0 < \epsilon \le \epsilon_0$. If $T(f_1) = T(f_2)$ the theorem follows from [4], Theorem 2.1. So, we also assume that $T(f_1) \ne T(f_2)$. By Corollary 2.7 we know that there exist $t_i = t_i(\epsilon) \in [-\epsilon, \epsilon]$, $0 \le i \le n$, such that $t_0 \le t_1 \le \cdots \le t_n$, and u_ϵ interpolates to some convex combination $h_\epsilon := \lambda_1(\epsilon) f_1 + \lambda_2(\epsilon) f_2$ at t_i. Given a sequence $\epsilon_m \downarrow 0$, clearly we can find a subsequence, which we denote in the same way, such that $\lambda_j(\epsilon_m) \to \gamma_j$, $j = 1, 2$, as $m \to \infty$. The error formula for interpolation is well known (see [2]):

$$u_{\epsilon_m}(x) = h_{\epsilon_m}(x) - (x - t_0)(x - t_1)\cdots(x - t_n)\frac{h_{\epsilon_m}^{(n+1)}(\xi_x)}{(n+1)!}, \tag{3.5}$$

where ξ_x belongs to the more small segment containing to t_i, $0 \le i \le n$, and x. On the other hand, using the error formula for the Taylor polynomial we have

$$h_{\epsilon_m}(x) = (\lambda_1(\epsilon_m)T(f_1) + \lambda_2(\epsilon_m)T(f_2))(x) + \frac{h_{\epsilon_m}^{(n+1)}(\eta_x)}{(n+1)!}x^{n+1}, \tag{3.6}$$

where η_x belongs to the segment between the points x and 0. From (3.5) and (3.6) we get

$$u_{\epsilon_m}(x) = (\lambda_1(\epsilon_m)T(f_1) + \lambda_2(\epsilon_m)T(f_2))(x)$$

$$-(x-t_0)(x-t_1)\cdots(x-t_n)\frac{h_{\epsilon_m}^{(n+1)}(\xi_x)}{(n+1)!}+\frac{h_{\epsilon_m}^{(n+1)}(\eta_x)}{(n+1)!}x^{n+1}. \tag{3.7}$$

If we put $t_i(\epsilon_m)=\epsilon_m s_i(\epsilon_m)$ with $s_i\in[-1,1]$, $0\le i\le n$, from (3.7) we have

$$\begin{aligned}&G_{p,1/p}^{\epsilon_m}\left(u_{\epsilon_m}-(\lambda_1(\epsilon_m)T(f_1)+\lambda_2(\epsilon_m)T(f_2))\right)\\&\quad\le K(G_{p,1/p}^{\epsilon_m}((x-\epsilon_m s_0)(x-\epsilon_m s_1)\cdots(x-\epsilon_m s_n))+G_{p,1/p}^{\epsilon_m}(x^{n+1}))\\&\quad=O\left(\epsilon_m^{n+1+1/p}\right),\end{aligned} \tag{3.8}$$

where $K=\max_{x\in[-1,1]}(|f_1^{(n+1)}(x)|+|f_2^{(n+1)}(x)|)$. The last equality is immediately followed by a change of variable.

On the other hand, from Lemma 2.1 in [3], there exists a constant $M>0$ such that

$$\|Q\|_{\infty,[-1,1]}\le\frac{M}{{\epsilon_m}^{n+1/p}}G_{p,1/p}^{\epsilon_m}(Q),\quad\text{for all } Q\in\Pi^n.$$

Therefore (3.8) implies

$$u_{\epsilon_m}\to\gamma_1T(f_1)+\gamma_2T(f_2),\quad\text{as } m\to\infty. \tag{3.9}$$

Next we shall prove that $\gamma_1=\gamma_2$. Since $T(f_1)\ne T(f_2)$, there exists $r\in\mathbb{Z}$, $-1\le r\le n-1$ such that $f_1^{(i)}(0)=f_2^{(i)}(0)$, $0\le i\le r$, and $f_1^{(r+1)}(0)\ne f_2^{(r+1)}(0)$. From (3.7) we also have

$$G_{p,1/p}^{\epsilon_m}(f_1-u_{\epsilon_m})=\lambda_2(\epsilon_m)G_{p,1/p}^{\epsilon_m}(T(f_1)-T(f_2))+O(\epsilon_m^{n+1+1/p}), \tag{3.10}$$

and

$$G_{p,1/p}^{\epsilon_m}(f_2-u_{\epsilon_m})=\lambda_1(\epsilon_m)G_{p,1/p}^{\epsilon_m}(T(f_1)-T(f_2))+O(\epsilon_m^{n+1+1/p}). \tag{3.11}$$

Let

$$\lambda=\frac{\left|f_1^{(r+1)}(0)-f_2^{(r+1)}(0)\right|}{(r+1)!}G_{p,1/p}^{1}(x^{r+1}). \tag{3.12}$$

Now, (3.10) and (3.11) imply

$$\lim_{m\to\infty}\frac{G_{p,1/p}^{\epsilon_m}(f_1-u_{\epsilon_m})}{\epsilon_m^{r+1+1/p}}=\gamma_2\lim_{m\to\infty}\frac{G_{p,1/p}^{\epsilon_m}(f_1-f_2)}{\epsilon_m^{r+1+1/p}}=\gamma_2\lambda, \tag{3.13}$$

and

$$\lim_{m\to\infty}\frac{G_{p,1/p}^{\epsilon_m}(f_2-u_{\epsilon_m})}{\epsilon_m^{r+1+1/p}}=\gamma_1\lim_{m\to\infty}\frac{G_{p,1/p}^{\epsilon_m}(f_1-f_2)}{\epsilon_m^{r+1+1/p}}=\gamma_1\lambda. \tag{3.14}$$

Finally, from (3.4), we observe that the first members of (3.13) and (3.14) are equals, so $\gamma_1=\gamma_2=\frac{1}{2}$.

For the proof in the case $p=\infty$, it analogously follows replacing $G_{p,1/p}^{\epsilon_m}$ by $G_{\infty}^{\epsilon_m}$. If $p=1$, the proof also analogously follows from Corollary 2.8. □

Theorem 3.4. *Let $1<p<\infty$, $1\le q<\infty$ and let $f_j\in\mathcal{C}[-1,1]$, $j=1,2$, be with derivatives continuous up to order $n+1$ in $[-1,1]$. Let u_ϵ, $0<\epsilon\le1$, be a $G_{p,q/p}^{\epsilon}$-b.s.a. to f_j, $j=1,2$, from Π^n. Then*

$$u_\epsilon\to T(f_1,f_2),\quad\text{as } \epsilon\to0. \tag{3.15}$$

Proof. Let (ϵ_m) be a sequence such that $\epsilon_m \downarrow 0$, as $m \to \infty$. Clearly

$$G^{\epsilon_m}_{p,q/p}(h_1, h_2) = (G^{\epsilon_m}_{p,1/p}(h_1))^q + (G^{\epsilon_m}_{p,1/p}(h_2))^q, \quad h_1, h_2 \in \mathcal{C}^{n+1}, \tag{3.16}$$

and

$$\begin{aligned} G^{\epsilon_m}_{p,1/p}(f_i - T(f_1, f_2)) &\le G^{\epsilon_m}_{p,1/p}(f_i - T(f_i)) + \frac{1}{2} G^{\epsilon_m}_{p,1/p}(T(f_1) - T(f_2)) \\ &= O(\epsilon_m^{n+1+1/p}) + \frac{1}{2} G^{\epsilon_m}_{p,1/p}(T(f_1) - T(f_2)) \end{aligned} \tag{3.17}$$

for $i = 1, 2$. Since $u_{\epsilon_m} \in G^{\epsilon_m}_{p,q/p}$-b.s.a., from (3.16) and (3.17) follows that

$$\begin{aligned} G^{\epsilon_m}_{p,q/p}(f_1 - u_{\epsilon_m}, f_2 - u_{\epsilon_m}) &\le G^{\epsilon_m}_{p,q/p}(f_1 - T(f_1, f_2), f_2 - T(f_1, f_2)) \\ &\le 2\left(O(\epsilon_m^{n+1+1/p}) + \frac{1}{2} G^{\epsilon_m}_{p,1/p}(T(f_1) - T(f_2))\right)^q. \end{aligned} \tag{3.18}$$

Further, we have

$$\begin{aligned} &G^{\epsilon_m}_{p,1/p}(f_1 - u_{\epsilon_m}) + G^{\epsilon_m}_{p,1/p}(f_2 - u_{\epsilon_m}) \ge G^{\epsilon_m}_{p,1/p}(f_1 - f_2) \\ &\quad = O(\epsilon_m^{n+1+1/p}) + G^{\epsilon_m}_{p,1/p}(T(f_1) - T(f_2)). \end{aligned} \tag{3.19}$$

From the inequality $(a + b)^q \le 2^{q-1}(a^q + b^q)$, $a, b \ge 0$, (3.16) and (3.19) we get

$$G^{\epsilon_m}_{p,q/p}(f_1 - u_{\epsilon_m}, f_2 - u_{\epsilon_m}) \ge \frac{\left(O(\epsilon_m^{n+1+1/p}) + G^{\epsilon_m}_{p,1/p}(T(f_1) - T(f_2))\right)^q}{2^{q-1}}. \tag{3.20}$$

We suppose $T(f_1) = T(f_2)$. Then (3.16), (3.18) and (3.20) imply $G^{\epsilon_m}_{p,1/p}(T(f_i) - u_{\epsilon_m}) = O(\epsilon_m^{n+1+1/p})$, $i = 1, 2$. Therefore, $u_{\epsilon_m} \to T(f_1, f_2)$ as $m \to \infty$.
Now, we assume $T(f_1) \neq T(f_2)$. We choose $r \in \mathbb{Z}$ as in the proof of Theorem 3.3. According to (3.18) and (3.20), we have

$$\lim_{m\to\infty} \frac{G^{\epsilon_m}_{p,q/p}(f_1 - u_{\epsilon_m}, f_2 - u_{\epsilon_m})}{\epsilon_m^{q(r+1+1/p)}} = \frac{\lambda^q}{2^{q-1}}, \tag{3.21}$$

where λ is defined in (3.12).
Analogously to the proof of Theorem 3.3 we know that there are nonnegative real numbers γ_1 and γ_2, $\gamma_1 + \gamma_2 = 1$, such that $u_{\epsilon_m} \to \gamma_1 T(f_1) + \gamma_2 T(f_2)$ as $m \to \infty$. Thus, (3.13), (3.14) and (3.16) imply

$$\lim_{m\to\infty} \frac{G^{\epsilon_m}_{p,q/p}(f_1 - u_{\epsilon_m}, f_2 - u_{\epsilon_m})}{\epsilon_m^{q(r+1+1/p)}} = (\gamma_1^q + \gamma_2^q)\lambda^q. \tag{3.22}$$

Finally, from (3.21) and (3.22) we have $\gamma_1 = \gamma_2 = \frac{1}{2}$, i.e., $u_{\epsilon_m} \to T(f_1, f_2)$ as $m \to \infty$. □

Concluding remarks. To conclude, we abstract the results obtained in this Section over convergence of the $G^{\epsilon}_{p,q}$-b.s.a. to two functions. If $1 < p < \infty, 1 \le q < \infty$ or $1 \le p \le \infty, q = \infty$ then any net of b.s.a. converges to $T(f_1, f_2)$ (see Theorems 3.1, 3.3 and 3.4). If $p = 1, 1 \le q \le \infty$, then any subsequence of b.s.a. has a subsequence which converges to a convex combination of $T(f_1)$ and $T(f_2)$ (Theorem 3.2). This convex combination can be different from $T(f_1, f_2)$ (see [5]). If $p = \infty, 1 \le q < \infty$, is an open problem.

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