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Solving a sparse system using linear algebra

César Massri [1,2]

Department of Mathematics, FCEN, University of Buenos Aires, Argentina

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ABSTRACT

We give a new theoretical tool to solve sparse systems with finitely many solutions. It is based on toric varieties and basic linear algebra; eigenvalues, eigenvectors and coefficient matrices. We adapt Eigenvalue theorem and Eigenvector theorem to work with a canonical rectangular matrix (the first Koszul map) and prove that these new theorems serve to solve overdetermined sparse systems and to count the expected number of solutions.

0. Introduction

0.1. Overview of the problem

In this article we generalize two methods to solve systems of polynomial equations using a coefficient matrix. One method is based on the eigenvalue theorem, first noticed in Lazard (1981). Another, on the eigenvector theorem, first described in Auzinger and Stetter (1988). Let us start describing them.

For simplicity, consider a generic system of n polynomial equations with finitely many solutions in $\mathbb{C}^n$, all with multiplicity one,

E-mail address: cmassri@dm.uba.ar.

[1] Address for correspondence: Department of Mathematics, FCEN, University of Buenos Aires, Argentina. Postal address: 1428. Tel.: +54 11 4576 3335.

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$$\begin{cases} f_1(x_1,\ldots,x_n) = 0 \\ \quad\vdots \\ f_n(x_1,\ldots,x_n) = 0 \end{cases}$$

where $f_1,\ldots,f_n$ are polynomials in $\mathbb{C}[x_1,\ldots,x_n]$. The quotient ring,

$$\mathcal{R} = \mathbb{C}[x_1,\ldots,x_n]/\langle f_1,\ldots,f_n\rangle,$$

is a finite-dimensional vector space and its dimension is the number of solutions (we are assuming that all the solutions have multiplicity one).

Every polynomial $f \in \mathbb{C}[x_1,\ldots,x_n]$, determines a linear map $M_f : \mathcal{R} \to \mathcal{R}$,

$$M_f(\bar{g}) = \overline{fg}, \quad g \in \mathbb{C}[x_1,\ldots,x_n],$$

where $\bar{g}$ denotes the class of the polynomial g in the quotient ring $\mathcal{R}$. The matrix of M_f is called the *multiplication matrix* associated to the polynomial f.

Theorem *(Eigenvalue Theorem). The eigenvalues of M_f are $\{f(\xi_1),\ldots,f(\xi_r)\}$, where $\{\xi_1,\ldots,\xi_r\}$ are the solutions of the system of polynomial equations. See Dickenstein and Emiris (2005, Theorem 2.1.4) for a proof.*

Theorem *(Eigenvector Theorem). Let $f = \alpha_1 x_1 + \ldots + \alpha_n x_n$ be a generic linear form and let M_f be its multiplication matrix. Assume that $B = \{1, x_1,\ldots,x_n,\ldots\}$ is a finite basis of $\mathcal{R}$ formed by monomials. Then the left eigenvectors of M_f determine all the solutions of the system of polynomial equations. Specifically, if $v = (v_0,\ldots,v_n,\ldots)$ is a left eigenvector of M_f such that $v_0 = 1$, then $(v_1,\ldots,v_n)$ is a solution of the system of polynomial equations. See Dickenstein and Emiris (2005, §2.1.3) for a proof.*

Now, let us describe the construction of the coefficient matrix (also in the case of polynomial equations).

Let $d = d_1 + \ldots + d_n - n + 1$, where $d_i = \deg(f_i)$, $1 \le i \le n$. Let S_d be the space of polynomials of degree $\le d$. Consider the following sets of monomials,

$$\begin{aligned} B_n &= \{x_1^{m_1}\ldots x_n^{m_n} \in S_d : d_n \le m_n\} \\ B_{n-1} &= \{x_1^{m_1}\ldots x_n^{m_n} \in S_d \setminus B_n : d_{n-1} \le m_{n-1}\} \\ &\vdots \\ B_1 &= \{x_1^{m_1}\ldots x_n^{m_n} \in S_d \setminus B_2 : d_1 \le m_1\} \\ B_0 &= \{x_1^{m_1}\ldots x_n^{m_n} \in S_d \setminus B_1\}. \end{aligned}$$

Using these sets, we can consider the following linear map,

$$\Psi : \langle B_0\rangle \times \ldots \times \langle B_n\rangle \to S_d, \quad \Psi(g_0,\ldots,g_n) = f_0 \cdot g_0 + \sum_{i=1}^{n} f_i \cdot g_i,$$

where the polynomial f_0 is a generic linear form and $\langle B_i\rangle$ is the vector space generated by B_i, $0 \le i \le n$. The *coefficient matrix* M is the matrix of Ψ in the monomial bases $B_0,\ldots,B_n$. It is a square matrix and can be divided into four blocks,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}.$$

The relation between the coefficient matrix and the multiplication matrix is the following,

Theorem. *For generic systems $f_1, \dots, f_n$ in n variables, the multiplication matrix associated to f_0 in $\mathcal{R}$ is the Schur complement of M_{22} in the coefficient matrix M,*

$$M_{f_0} = M_{11} - M_{12}M_{22}^{-1}M_{21}.$$

See Emiris and Rege (1994) and Mourrain and Pan (2000) for a proof.

There are several technical difficulties in order to generalize the previous constructions. For example, the choices of the sets $B_0, \dots, B_n$ and the fact that we need a generic system of n polynomial equations in n variables. The sets $B_0, \dots, B_n$ are given to assure that M_{22} is a non-degenerate matrix and that M is a square matrix. Another technical difficulty is that the system must have simple roots and that f_0 must be linear. All these difficulties may be solved to give generalizations of the constructions, not only to polynomial equations, but also to sparse systems. See in the next subsection for the existing work.

In this article, we propose a simpler approach to deal not only with polynomial equations, but also with sparse systems in general. We make a canonical choice for the map Ψ (the first Koszul map) and we make no assumption on f_0 nor on the multiplicities of the solutions. We construct a matrix $M_{11} + M_{12}F$, where F satisfies the linear equation $M_{22}F = -M_{21}$ and such that every solution ξ of the sparse system determines an eigenvalue $f_0(\xi)$ and a left eigenvector of $M_{11} + M_{12}F$. The matrix $M_{11} + M_{12}F$ can be obtained by elementary column operations on M.

Our construction can be used to solve overdetermined sparse systems and also, to count the expected number of solutions. The main problem of our matrix M is its size.

0.2. Existing work

Several classes of scientific and engineering problems are expected to reduce to algebraic systems with sparse structure. Sparse systems are typical for such a situation. For example, problems in vision (Emiris, 1997), edge detection, robot kinematics (kinematics of molecueles/mechanisms), calibration of Gough/Stewart platforms (Daney and Emiris, 2001; Mourrain, 1993), structural biology and computational chemistry (Emiris and Mourrain, 1999a).

Given a sparse system, we could ask if there exist solutions. Just as in the affine case, where the classical Hilbert Nullstellensatz is available, we can apply the Sparse Nullstellensatz to obtain an answer (Sombra, 1999, Theorem 2.13).

Theorem *(Corollary of the Sparse Nullstellensatz). If the ideal generated by $f_1, \dots, f_k$ contains the unity $1 \in \langle f_1, \dots, f_k \rangle$, then the sparse system has no solution in $(\mathbb{C} \setminus 0)^n$.*

The most common way to check the hypothesis of this theorem is using elimination theory (Jouanolou, 1991). The central object in elimination theory is the resultant, which characterizes the solvability of a sparse system with prescribed support. The resultant is a polynomial in the coefficient of the sparse system, $\{f_1, \dots, f_n\}$. It provides a necessary (and generically sufficient) condition for the existence of solutions. If the system has a solution, the resultant $R_{f_1,\dots,f_n}$ is non-zero. The most famous example of resultant is the determinant of a system of linear equations.

The first mathematicians who worked in elimination theory were Gauss, Bézout and Euler in the eighteenth century. The study of resultants, in the second half of the nineteenth century, started with Sylvester, Cayley, Macaulay and Dixon. In the last decade of the twentieth century, the theory was reborn with the pioneering work of Jouanolou in 1991 (Jouanolou, 1991). Today, the resultant may be considered, not only in affine or projective space, but also in the toric case. The foundations were laid in the work of Gel'fand, Kapranov and Zelevinsky (Gel'fand et al., 1994). Subsequence papers extended the theory into several different directions, see Kapranov et al. (1992); Pedersen and Sturmfels (1993).

In order to compute the resultant, several algorithm are given. In Canny and Emiris (1993) the authors proposed a formula for the resultant of a system of $n+1$ Laurent polynomials in n variables. They constructed a matrix whose determinant is a non-zero multiple of the resultant. This construction is closely related to that of Macaulay's, who called these matrices, *coefficient matrices* (Macaulay, 1902). In general, the construction of the coefficient matrices needs a clever choice of monomials. In Emiris and Rege (1994) the authors used a coefficient matrix to obtain a monomial basis for the coordinate ring generated by the given polynomials.

A number of methods exist for constructing matrices whose determinant is the resultant or, more generally, a non-trivial multiple of it. These matrices represent the most efficient way for computing the resultant and for solving sparse systems by means of the resultant method. For the classical resultant method see Lazard (1981); Canny (1990). For the sparse resultant see Emiris (1994), where an efficient and general algorithm is given. The author studied the complexity of the algorithm and also the numerical issues.

There are several articles that used coefficient matrices and/or multiplication matrices to compute the solutions of a system of polynomial equations. For example, in Auzinger and Stetter (1988); Bondyfalat et al. (2000) and Mourrain (2006) the authors gave an algorithm to compute the solutions of a system of polynomial equations with the same number of variables and equations. In Elkadi and Mourrain (2007, §6.2) the authors showed a generalization of the method to solve an overdetermined system of polynomial equations and in Emiris (1996, 1997, 2001); Emiris and Mourrain (1999b) and Emiris and Rege (1994) the authors gave another generalization, but to solve a sparse system with the same number of variables and equations.

In Emiris and Canny (1995), the authors gave an algorithm using a coefficient matrix that can treat an overdetermined sparse system. The authors wrote *"An important aspect of the algorithm is that it readily extends to systems of more than $n+1$ polynomials in n variables"*. They proposed a method to construct the coefficient matrix minimizing its size. This method was implemented in Emiris (1997).

As a final remark, let us mention that there exists another theory to solve a system of equations using a topological point of view. It is a called *homotopy method*. Essentially, first define a trivial system of equations to which all solutions are easily known. Then, deform the trivial system into the original system. As the system is deformed the solutions are deformed also, thereby creating paths of solutions. These paths start from each of the trivial solutions and connect to the solutions of the original system. By following these paths from the trivial system, all the solutions of the original system can be determined, see Morgan and Sommese (1987).

0.3. Main result

We propose a general framework to solve a sparse system using a rectangular coefficient matrix. Known methods require the construction of a square matrix adapted to each specific system, see for example Bondyfalat et al. (2000, 3.1) and Mourrain (1998, §3.2.3). One advantage of this new method is that the construction of the rectangular matrix is *canonical* and does not require a clever choice of the monomials for its construction. Our contribution to the theory is the exposure of the properties of the rectangular coefficient matrix M associated to the first Koszul map of a sparse system.

Given that our coefficient matrix is rectangular, it is not possible to use the previous theorems where a square matrix is required (see (Emiris, 1996) for the sparse case). Hence, we adapted them to our requirements. This means that we generalized known theorems to the case of a rectangular coefficient matrix.

Let us list the main results of this article (for definitions and notations see below). Let $f_0, \ldots, f_k$ be Laurent polynomials with Newton polytopes $\mathcal{A}_0, \ldots, \mathcal{A}_k$ respectively. Let $\mathcal{B}_i = \mathcal{A}_0 + \ldots + \widehat{\mathcal{A}_i} + \ldots + \mathcal{A}_k$, $0 \le i \le k$ and let $\mathcal{E} = \mathcal{A}_0 + \ldots + \mathcal{A}_k$.

The coefficient matrix M associated to the sparse system $\{f_1, \ldots, f_k\}$ and f_0 is the matrix of Ψ in the monomial bases $\mathcal{B}_i \cap \mathbb{Z}^n$, $0 \le i \le k$ and $\mathcal{E} \cap \mathbb{Z}^n$,

$$\Psi : S_{\mathcal{B}_0} \times \ldots \times S_{\mathcal{B}_k} \longrightarrow S_{\mathcal{E}}, \quad \Psi(g_0, \ldots, g_k) = f_0 \cdot g_0 + \sum_{i=1}^{k} f_i \cdot g_i.$$

Matrix M is rectangular and can be divided into four blocks,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad M_{11} \in \mathbb{C}^{p \times p},\ p = \dim(S_{\mathcal{B}_0}).$$

Main Hypotheses. Assume that $0 \in \mathcal{A}_0$ and f_0 is a non-constant Laurent polynomial, that the lattice polytope $\mathcal{E}$ is full dimensional and finally, that $\langle f_0, f_1, \ldots, f_k \rangle = S$. □

Using the matrix M, we can test the last assumption adapting a theorem due to Macaulay, see Macaulay (1902) or Mourrain (1998, Theorem 3.7).

Proposition *(Corollary 4). Assume that $\mathcal{E}$ is full dimensional. Then, M has full rank if and only if $\langle f_0, \ldots, f_k \rangle = S$.* □

Using the previous new proposition and as a benefit of our approach, we obtained a proof of a conjecture due to J. Canny and I. Emiris (Canny and Emiris, 2000).

Conjecture *(8.3, Sparse Effective Nullstellensatz over $\mathbb{C}$). Suppose $f_0, \ldots, f_k$ are arbitrary Laurent polynomials in $S = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with Newton polytopes $\mathcal{A}_i$, $0 \le i \le k$ such that the generated ideal is S, $\langle f_0, \ldots, f_k \rangle = S$. Then there exist Laurent polynomials $g_0, \ldots, g_k \in S$, with Newton polytopes $\mathcal{B}_i$, $0 \le i \le k$, such that*

$$1 = \sum_{i=0}^{k} f_i \cdot g_i, \quad \mathcal{B}_i \subseteq \mathcal{A}_0 + \ldots + \widehat{\mathcal{A}_i} + \ldots + \mathcal{A}_k.$$

Proof. Given that $\langle f_0, \ldots, f_k \rangle = S$, Ψ is surjective, hence $1 \in S_{\mathcal{E}}$. □

Another new result that we proved is a formula to count the number of expected solutions of a sparse system using M and also, our main theorem; an adaptation of the Eigenvalue/Eigenvector Theorem to the case of a rectangular coefficient matrix.

Theorem *(Theorem 3(a)). The sparse system $\{f_1, \ldots, f_k\}$ has a finite number of expected solutions equal to*

$$rk(M) - rk\begin{pmatrix} M_{12} \\ M_{22} \end{pmatrix} \ge 0.$$

Theorem *(Theorem 5, Proposition 6). Let F be a solution of the linear equation $M_{21} + M_{22}F = 0$. Then, every solution, ξ, of the sparse system determines a left eigenvector of $M_{11} + M_{12}F$ with eigenvalue $f_0(\xi)$. The multiplicity of $f_0(\xi)$ is greater than or equal to the multiplicity of ξ.* □

0.4. Summary

This paper is organized as follows. In Section 1 we present some preliminaries about toric varieties and lattice polytopes. In Lemma 1 we construct an irreducible projective toric variety X associated to a full dimensional lattice polytope $\mathcal{E}$ and relate $\mathbb{N}$-Minkowski summands of $\mathcal{E}$ with invertible sheaves on X generated by their global sections. In Section 2 we construct a stably twisted Koszul complex and we apply it in two different ways. Firstly, in Theorem 5, we use it to prove that every solution of the sparse system determines a left eigenvector/eigenvalue of a matrix built from this complex. Secondly, in Theorem 3(a), we use it to count the number of expected solutions of the sparse system (counted with multiplicities). In Section 3 we give an application.

1. Preliminaries

A *sparse system* is a collection of Laurent polynomials, $\{f_1, \ldots, f_k\}$,

$$f_i = \sum_{v \in \mathcal{Q}_i} c_{i,v} x_1^{v_1} \ldots x_n^{v_n}, \quad 1 \le i \le k,$$

where $\mathcal{Q}_i$ are fixed finite subsets of $\mathbb{Z}^n$. The set $\mathcal{Q}_i$ is called the *support* of f_i. The convex hull $\mathcal{A}_i$ of $\mathcal{Q}_i$,

$$\mathcal{A}_i = \mathrm{conv}(\mathcal{Q}_i) \subseteq \mathbb{R}^n,$$

is called the *Newton polytope* of f_i, denoted $N(f_i)$, $1 \le i \le k$.

Definition. A *lattice polytope* $\mathcal{A} \subseteq \mathbb{R}^n$ is the convex hull of a finite set $\mathcal{Q} \subseteq \mathbb{Z}^n$, $\mathcal{A} = \mathrm{conv}(\mathcal{Q})$.

The *dimension* of a lattice polytope $\mathcal{A} \subseteq \mathbb{R}^n$, is the dimension of the smallest affine subspace of $\mathbb{R}^n$ containing $\mathcal{A}$. We say that $\mathcal{A}$ is a *full dimensional* lattice polytope when the dimension of $\mathcal{A} \subseteq \mathbb{R}^n$ is n.

Notation. Let $S = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the algebra of Laurent polynomials. Given a lattice polytope $\mathcal{A}$, let $S_{\mathcal{A}}$ be the vector space of polynomials with Newton polytopes in $\mathcal{A}$,

$$S_{\mathcal{A}} = \{g \in S : N(g) \subseteq \mathcal{A}\}.$$

The dimension of $S_{\mathcal{A}}$ is equal to the cardinal of $\mathcal{A} \cap \mathbb{Z}^n$,

$$\dim(S_{\mathcal{A}}) = \#(\mathcal{A} \cap \mathbb{Z}^n).$$

The finite set $\mathcal{A} \cap \mathbb{Z}^n$ determines a monomial basis for $S_{\mathcal{A}}$.

Definition. Given lattice polytopes $\mathcal{B}$ and $\mathcal{E}$ in $\mathbb{R}^n$, we say that $\mathcal{B}$ is an $\mathbb{N}$-*Minkowski summand* of $\mathcal{E}$ if

$$\mathcal{B} + \mathcal{B}' = k\mathcal{E}$$

for some positive integer k and lattice polytope $\mathcal{B}' \subseteq \mathbb{R}^n$.

For example, $2\mathcal{E}$ is an $\mathbb{N}$-Minkowski summand of $\mathcal{E}$.

Remark. In the proof of the next lemma we use basic definitions from algebraic geometry that can be found in Hartshorne (1977). For example the definitions of irreducible varieties, projective varieties, complete varieties, normal varieties, invertible sheaves, Cartier divisors, Weyl divisors and basepoint free divisors.

Also, we use some definitions and concepts from toric geometry (Cox et al., 2011). For example the toric variety associated to a fan, a torus-invariant divisor, a nef divisor and finally, Demazure Vanishing. We give a precise reference where the reader can find the definitions and results about toric geometry.

Lemma 1. *Given a full dimensional lattice polytope $\mathcal{E}$, there exists an irreducible projective normal toric variety X such that every $\mathbb{N}$-Minkowski summand $\mathcal{B}$ of $\mathcal{E}$ defines an invertible sheaf $\mathcal{O}_X(D)$ with*

$$H^0(X, \mathcal{O}_X(D)) = S_{\mathcal{B}}, \quad H^p(X, \mathcal{O}_X(D)) = 0,\ p > 0.$$

Even more, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two $\mathbb{N}$-Minkowski summands of $\mathcal{E}$ and $\mathcal{O}_X(D_1)$ and $\mathcal{O}_X(D_2)$ are the corresponding invertible sheaves of $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, then the invertible sheaf associated to $\mathcal{B}_1 + \mathcal{B}_2$ is $\mathcal{O}_X(D_1 + D_2)$.

Proof. Given a full dimensional lattice polytope, we can construct a *normal fan* Σ (Cox et al., 2011, Theorem 2.3.2), and a normal toric variety X_Σ (Cox et al., 2011, Theorem 3.1.5).

The normal fan associated to a full dimensional lattice polytope is *complete* (Cox et al., 2011, Proposition 2.3.8). Then X_Σ is also a complete variety (Cox et al., 2011, Theorem 3.4.6).

There exists a more direct construction of X_Σ using a multiple of the full dimensional lattice polytope $\mathcal{E}$, but by Proposition 3.1.6 (Cox et al., 2011) both constructions agree, $X_\Sigma \cong X_{\mathcal{E}}$. The benefit of this direct construction is that $X_{\mathcal{E}}$ proves to be an irreducible projective variety. Let us call X the irreducible projective normal toric variety X_Σ.

Let $\mathcal{B}$ be an $\mathbb{N}$-Minkowski summand of $\mathcal{E}$. By Corollary 6.2.15 (Cox et al., 2011) there exists a torus invariant basepoint free Cartier divisor D on X such that

$$H^0(X, \mathcal{O}_X(D)) = S_{\mathcal{B}}.$$

This last equality follows from Proposition 4.3.3 (Cox et al., 2011) and the fact that in a normal variety, every Cartier divisor is a Weyl divisor (Cox et al., 2011, Definition 4.0.12).

Let us apply Demazure Vanishing (Cox et al., 2011, Theorem 9.2.3). By definition, the *support* of a complete fan is $\mathbb{R}^n$ (Cox et al., 2011, Definition 3.1.18). In particular, it has a *convex support of full dimension* (Cox et al., 2011, §6.1). Then the basepoint free Cartier divisor D is *nef* (Cox et al., 2011, Theorem 6.3.12). Applying Demazure Vanishing, we obtain,

$$H^p(X, \mathcal{O}_X(D)) = 0, \quad p > 0.$$

Let us prove the last paragraph of the lemma. Let D be a torus-invariant Cartier divisor on X. Then there exists a polytope $\mathcal{P}_D$ such that $H^0(X, \mathcal{O}_X(D)) = S_{\mathcal{P}_D}$ (Fulton, 1993, Lemma, p. 66). Even more, if D is the torus-invariant basepoint free Cartier divisor associated to an $\mathbb{N}$-Minkowski summand $\mathcal{B}$, then $\mathcal{P}_D = \mathcal{B}$ (Fulton, 1993, p. 68); (Cox et al., 2011, Corollary 6.2.15).

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two $\mathbb{N}$-Minkowski summands of $\mathcal{E}$ and $\mathcal{O}_X(D_1)$ and $\mathcal{O}_X(D_2)$ be the corresponding invertible sheaves associated to $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively. Given that the sheaves are generated by global sections, we obtain $\mathcal{P}_{D_1+D_2} = \mathcal{P}_{D_1} + \mathcal{P}_{D_2}$ (Fulton, 1993, Exercise, p. 69).

Let $\mathcal{O}_X(D)$ be the invertible sheaf associated to the $\mathbb{N}$-Minkowski summand $\mathcal{B}_1 + \mathcal{B}_2$ of $\mathcal{E}$. Then,

$$\mathcal{P}_D = \mathcal{B}_1 + \mathcal{B}_2 = \mathcal{P}_{D_1} + \mathcal{P}_{D_2} = \mathcal{P}_{D_1+D_2}.$$

This implies that $\mathcal{O}_X(D) \cong \mathcal{O}_X(D_1 + D_2)$. $\square$

Definition. Let $\{f_1, \dots, f_k\}$ be a sparse system in $(\mathbb{C} \setminus 0)^n$ with $r' < \infty$ solutions counted with multiplicities. The torus $(\mathbb{C} \setminus 0)^n$ is contained in the variety X of Lemma 1 as an open subset (Cox et al., 2011, Definition 3.1.1). Homogenizing every equation of the sparse system, we can consider the system in X. For the homogenization process, see Cox et al. (2011, §5.4).

Let $Z \subseteq X$ be the zero-scheme of the resulting system and let $r \geq r'$ be the number of points in Z counted with multiplicities. We say that the sparse system $\{f_1, \dots, f_k\}$ *has no solution at infinity* if $r = r'$. Otherwise, we say that it *has solution at infinity*. The number r is called the *expected number of solutions* of the sparse system.

2. Solving a sparse system

The following notations and assumptions will be used in the rest of the section.

Assumption 2. Let $f_0, \dots, f_k$ be Laurent polynomials with Newton polytopes $\mathcal{A}_0, \dots, \mathcal{A}_k$ respectively. Let $\mathcal{B}_i = \mathcal{A}_0 + \dots + \widehat{\mathcal{A}_i} + \dots + \mathcal{A}_k$, $0 \leq i \leq k$ and let $\mathcal{E} = \mathcal{A}_0 + \dots + \mathcal{A}_k$. Let $\mathcal{I} = \langle f_1, \dots, f_k \rangle \subseteq S$ be the ideal generated by the sparse system.

Assume,

- $0 \in \mathcal{A}_0$ and f_0 is a non-constant Laurent polynomial.
- The lattice polytope $\mathcal{E}$ is full dimensional.
- $\langle f_0, f_1, \dots, f_k \rangle = S$. $\square$

Remark. If $0 \notin \mathcal{A}_0$, we can divide the equation f_0 by some monomial or we can consider the convex hull of 0 and $\mathcal{A}_0$ as the new lattice polytope $\mathcal{A}_0$. These operations does not change the number of expected solutions nor the solutions in $(\mathbb{C} \setminus 0)^n$ of the sparse system. Then without loss of generality, we can assume $0 \in \mathcal{A}_0$. This assumptions implies that $\mathcal{B}_0$ is contained in $\mathcal{E} = \mathcal{A}_0 + \mathcal{B}_0$.

If $\mathcal{E}$ is not full dimensional, there exists an affine change of variables such that the variables, say $x_{s+1}, \dots, x_n$, are missing in the sparse system. This implies that we could work in $S = \mathbb{C}[x_1^{\pm 1}, \dots, x_s^{\pm 1}]$

making $\mathcal{E}$ a full dimensional lattice polytope. This change of variables involves the computation of Smith Normal Forms (Hafner and McCurley, 1991). Another remark, is that it is easy to prove that if $\mathcal{A}_0$ is full dimensional, then $\mathcal{E}$ is full dimensional. Hence, we can consider $\mathcal{A}_0$ as a full dimensional lattice polytope.

It follows from $\langle f_0, f_1, \ldots, f_k\rangle = S$ that the associated zero-scheme in X is empty. We prove in the next theorem that a sparse system satisfying the previous assumptions will have a finite number of expected solutions (possible zero). This assumption is the most important one.

Theorem 3. *Same notation as before. Suppose $f_0, \ldots, f_k$ are Laurent polynomials as in Assumption 2. Then,*

(a) *The co-rank of the following linear map is the expected number of solutions (possibly zero),*

$$\Phi : S_{\mathcal{B}_1} \times \ldots \times S_{\mathcal{B}_k} \to S_{\mathcal{E}}, \quad \Phi(g_1, \ldots, g_k) = \sum_{i=1}^{k} f_i \cdot g_i.$$

In particular, if the system has no solution at infinity, it is equal to the number of solutions in $(\mathbb{C} \setminus 0)^n$.

(b) *The lattice polytope $\mathcal{B}_0 \subseteq \mathcal{E}$ satisfies*

$$S_{\mathcal{B}_0}/(\mathrm{Im}(\Phi) \cap S_{\mathcal{B}_0}) \cong S_{\mathcal{E}}/\mathrm{Im}(\Phi).$$

(c) *The following linear map is surjective,*

$$\Psi : S_{\mathcal{B}_0} \times S_{\mathcal{B}_1} \times \ldots \times S_{\mathcal{B}_k} \longrightarrow S_{\mathcal{E}}, \quad \Psi(g_0, g_1, \ldots, g_k) = f_0 \cdot g_0 + \sum_{i=1}^{k} f_i \cdot g_i.$$

Proof. Let us work with the projective variety X of Lemma 1. For every integers $d_0, \ldots, d_k \geq 0$ consider the invertible sheaf $\mathcal{O}_X(d_0, \ldots, d_k)$ associated to the $\mathbb{N}$-Minkowski summand $d_0\mathcal{A}_0 + \ldots + d_k\mathcal{A}_k$ of $\mathcal{E}$. Then

$$H^0(X, \mathcal{O}_X(d_0, \ldots, d_k)) = S_{d_0\mathcal{A}_0+\ldots+d_k\mathcal{A}_k}, \quad H^p(X, \mathcal{O}_X(d_0, \ldots, d_k)) = 0, \quad p > 0.$$

Also, from the last paragraph of Lemma 1 we have the following property. Let d_i, d'_i be non-negative integers such that $d_i \geq d'_i \geq 0$ for all $0 \leq i \leq k$. Then,

$$\mathcal{O}_X(d'_0, \ldots, d'_k) \otimes_{\mathcal{O}_X} \mathcal{O}_X(d_0 - d'_0, \ldots, d_k - d'_k) \cong \mathcal{O}_X(d_0, \ldots, d_k) \Longrightarrow$$
$$\mathcal{O}_X(d_0 - d'_0, \ldots, d_k - d'_k) \cong \mathcal{O}_X(d_0, \ldots, d_k) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-d'_0, \ldots, -d'_k),$$

where $\mathcal{O}_X(-d'_0, \ldots, -d'_k)$ denotes the dual sheaf of $\mathcal{O}_X(d'_0, \ldots, d'_k)$.

Let $e_i \in \mathbb{Z}^{k+1}$ be the vector with 1 in the $(i+1)$-coordinate and 0 in the rest, $0 \leq i \leq k$. For example, $e_0 = (1, 0, \ldots, 0)$ and $e_k = (0, \ldots, 0, 1)$. The Laurent polynomials $\{f_1, \ldots, f_k\}$ determine a $\mathcal{O}_X$-linear map

$$\mathcal{O}_X \to \mathcal{F}, \quad \mathcal{F} = \mathcal{O}_X(e_1) \oplus \ldots \oplus \mathcal{O}_X(e_k),$$

given by $g \mapsto (f_1 g, \ldots, f_k g)$. Then, we can construct the dual Koszul complex associated to $\mathcal{F}$,

$$0 \to \bigwedge^k \mathcal{F}^\vee \to \ldots \to \bigwedge^i \mathcal{F}^\vee \to \ldots \to \mathcal{F}^\vee \to \mathcal{O}_X,$$

where $\mathcal{F}^\vee$ denotes the dual of $\mathcal{F}$ and

$$\bigwedge^s \mathcal{F}^\vee = \bigoplus_{1 \leq i_1 < \ldots < i_s \leq k} \mathcal{O}_X(-e_{i_1} - \ldots - e_{i_s}), \quad 2 \leq s \leq k.$$

Let $Z \subseteq X$ be the zero scheme of the global section $(f_1, \ldots, f_k) \in S_{\mathcal{A}_1} \oplus \ldots \oplus S_{\mathcal{A}_k} \cong H^0(X, \mathcal{F})$. Let us prove that Z is empty or of dimension 0. Assume that Z is not empty. Let $H \subseteq X$ be the hypersurface given by the zeros of the section $f_0 \in S_{\mathcal{A}_0} \cong H^0(X, \mathcal{O}_X(1, 0, \ldots, 0))$. Take an embedding of X is some $\mathbb{P}^N$ and let $\widehat{H} \subseteq \mathbb{P}^N$ be an hypersurface such that $\widehat{H} \cap X = H$. Given that the zero locus of $\{f_0, \ldots, f_k\}$ is empty in X, we have $\emptyset = Z \cap H = Z \cap (\widehat{H} \cap X) = Z \cap \widehat{H}$. Using Theorem 7.2 in Hartshorne (1977), we obtain $\dim(Z) = 0$.

Let us work with *the augmented dual Koszul complex associated to* $\mathcal{F}$,

$$0 \to \bigwedge^{k} \mathcal{F}^\vee \to \ldots \to \bigwedge^{i} \mathcal{F}^\vee \to \ldots \to \mathcal{F}^\vee \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

By §2, 1B, Proposition 1.4 (a) (Gel'fand et al., 1994) it is an exact complex.

Let $U \subseteq X$ be an affine open subset containing Z. Let T be the coordinate ring of U and let $\mathcal{J}$ be the ideal of $Z \subseteq U$. Then,

$$H^0(X, \mathcal{O}_Z(d_0, \ldots, d_k)) = H^0(U, \mathcal{O}_Z) = T/\mathcal{J}, \quad \forall d_0, \ldots, d_k \geq 0.$$

Recall from Proposition 2.9 (Hartshorne, 1977) that cohomology commutes with direct sums and from Theorem 6.0.18 and Proposition 6.0.17 (Cox et al., 2011) that invertible sheaves are locally free.

(a) The exactness of the augmented dual Koszul complex associated to $\mathcal{F}$ is preserved by twisting with the invertible sheaf $\mathcal{O}_X(1, \ldots, 1)$, and giving that each term of the resulting complex has no higher cohomology, the following complex of vector spaces is exact (Gel'fand et al., 1994, §2, 2A, Lemma 2.4),

$$S_{\mathcal{B}_1} \times \ldots \times S_{\mathcal{B}_k} \xrightarrow{\Phi} S_{\mathcal{E}} \to T/\mathcal{J} \to 0.$$

If the sparse system has no solution at infinity, we can take the torus as the open set U, then $Z \subseteq (\mathbb{C} \setminus 0)^n$ and $T/\mathcal{J} = S/\mathcal{I}$.

(b) In a similar way, twisting the augmented dual Koszul complex associated to $\mathcal{F}$ with the invertible sheaf $\mathcal{O}_X(0, 1, \ldots, 1)$, the following map is surjective,

$$S_{\mathcal{B}_0} \to T/\mathcal{J} \to 0.$$

Let K be the kernel of $S_{\mathcal{B}_0} \to T/\mathcal{J}$. Then,

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook] \arrow[d, hook, dashed] & S_{\mathcal{B}_0} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\equiv"] & T/\mathcal{J} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Im}(\Phi) \arrow[r, hook] & S_{\mathcal{E}} \arrow[r] & T/\mathcal{J} \arrow[r] & 0 \end{tikzcd}$$

The inclusion $\mathcal{B}_0 \subseteq \mathcal{E}$ follows from the assumption $0 \in \mathcal{A}_0$. Given that both rows are exact, K must be equal to $S_{\mathcal{B}_0} \cap \mathrm{Im}(\Phi)$. Then,

$$S_{\mathcal{B}_0}/(S_{\mathcal{B}_0} \cap \mathrm{Im}(\Phi)) \cong T/\mathcal{J} \cong S_{\mathcal{E}}/\mathrm{Im}(\Phi).$$

(c) Finally, given that the zero locus of $\{f_0, \ldots, f_k\}$ is empty in X, we can use similar arguments as before with the sheaf $\mathcal{F}' = \mathcal{O}_X(e_0) \oplus \ldots \oplus \mathcal{O}_X(e_k)$ to prove that the following map is surjective,

$$S_{\mathcal{B}_0} \times \ldots \times S_{\mathcal{B}_k} \xrightarrow{\Psi} S_{\mathcal{E}} \to 0.$$

Specifically, the augmented dual Koszul complex associated to $\mathcal{F}'$ is

$$0 \to \bigwedge^{k+1} \mathcal{F}'^\vee \to \ldots \to \bigwedge^{i} \mathcal{F}'^\vee \to \ldots \to \mathcal{F}'^\vee \to \mathcal{O}_X \to 0.$$

The result follows by twisting it with $\mathcal{O}_X(1, \ldots, 1)$ and taking global sections. $\square$

Remark. From the previous proof, part (c), we obtain a formula involving the number of lattice points in $\mathcal{E}$. The augmented dual Koszul complex associated to $\mathcal{F}' = \mathcal{O}_X(e_0) \oplus \ldots \oplus \mathcal{O}_X(e_k)$ twisted by $\mathcal{O}_X(1,\ldots,1)$ is exact and each term has no higher cohomology. Hence its Euler characteristic is zero,

$$\#(\mathcal{E} \cap \mathbb{Z}^n) - \sum_{i=0}^{k} \#((\mathcal{A}_0 + \ldots + \widehat{\mathcal{A}_i} + \ldots + \mathcal{A}_k) \cap \mathbb{Z}^n) + \ldots - (-1)^k \sum_{i=0}^{k} \#(\mathcal{A}_i \cap \mathbb{Z}^n) + (-1)^k = 0.$$

This formula is similar to the alternate volume formula in Bernstein (1975).

For example, consider the simplex $\Delta \subseteq \mathbb{R}^3$, $\Delta = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2$, where

$$\mathcal{A}_0 = \mathrm{conv}((0,0,0),(p,0,0)), \quad \mathcal{A}_1 = \mathrm{conv}((0,0,0),(0,q,0)),$$
$$\mathcal{A}_2 = \mathrm{conv}((0,0,0),(0,0,r))$$

and p, q, r are three positive prime numbers. Then $\#(\Delta \cap \mathbb{Z}^3)$ is equal to

$$(p+q+1) + (p+r+1) + (q+r+1) - (p+1) - (q+1) - (r+1) + 1 = p+q+r+1.$$

When $p = q = r = 1$, the standard simplex in $\mathbb{R}^3$ has 4 points in $\mathbb{Z}^3$.

For more on counting points in a lattice polytope, see De Loera (2005). □

The following corollary is an adaptation of a theorem in Macaulay (1902).

Corollary 4. *Suppose $f_0, \ldots, f_k$ are Laurent polynomials with Newton polytopes $\mathcal{A}_0, \ldots, \mathcal{A}_k$ respectively. Let $\mathcal{B}_i = \mathcal{A}_0 + \ldots + \widehat{\mathcal{A}_i} + \ldots + \mathcal{A}_k$, $0 \le i \le k$ and let $\mathcal{E} = \mathcal{A}_0 + \ldots + \mathcal{A}_k$.*

Let Ψ be the following linear map,

$$\Psi : S_{\mathcal{B}_0} \times S_{\mathcal{B}_1} \times \ldots \times S_{\mathcal{B}_k} \longrightarrow S_{\mathcal{E}}, \quad \Psi(g_0, g_1, \ldots, g_k) = f_0 \cdot g_0 + \sum_{i=1}^{k} f_i \cdot g_i.$$

Assume that $\mathcal{E}$ is full dimensional. Then,

$$rk(\Psi) = \#(\mathcal{E} \cap \mathbb{Z}^n) \iff \langle f_0, \ldots, f_k \rangle = S.$$

Proof. If $\langle f_0, \ldots, f_k \rangle = S$, by Theorem 3(c), Ψ is surjective. Analogously, if Ψ is surjective, there exists a monomial $x^m \in S_{\mathcal{E}} = \mathrm{Im}(\Psi) \subseteq \langle f_0, \ldots, f_k \rangle$. In particular, $1 \in \langle f_0, \ldots, f_k \rangle$. □

Notation. Notations and assumptions as in Assumption 2. Consider $\mathcal{E} \cap \mathbb{Z}^n$ and $\mathcal{B}_i \cap \mathbb{Z}^n$ as monomial ordered bases of $S_{\mathcal{E}}$ and $S_{\mathcal{B}_i}$ respectively, $0 \le i \le k$. Let p be the cardinal of $\mathcal{B}_0 \cap \mathbb{Z}^n$, p_i the cardinal of $\mathcal{B}_i \cap \mathbb{Z}^n$, $1 \le i \le k$ and $p + q$ be the cardinal of $\mathcal{E} \cap \mathbb{Z}^n$. Given that f_0 is not a constant in $S_{\mathcal{A}_0}$, the inclusion $\mathcal{B}_0 \subseteq \mathcal{E}$ is proper, thus $q > 0$.

$$\mathcal{B}_0 \cap \mathbb{Z}^n = \{m_1, \ldots, m_p\}, \quad \mathcal{E} \cap \mathbb{Z}^n = \{m_1, \ldots, m_p, m_{p+1}, \ldots, m_{p+q}\},$$

where m_i is a point in $\mathbb{Z}^n$, $1 \le i \le p+q$.

Let us define the coefficient matrix M associated to the sparse system $\{f_1, \ldots, f_k\}$ and f_0. Let $M \in \mathbb{C}^{(p+q)\times(p+p_1+\ldots+p_k)}$ be the rectangular matrix associated to Ψ in these bases,

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \quad M_{11} \in \mathbb{C}^{p\times p},\ M_{22} \in \mathbb{C}^{q\times(p_1+\ldots+p_k)}.$$

Then,

$$(\, x^{m_1} \ldots x^{m_p} \quad x^{m_{p+1}} \ldots x^{m_{p+q}} \,) \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = (\, f_0 x^{m_1} \ldots f_0 x^{m_p} \quad f_1 \cdot \mathcal{B}_1 \ldots f_k \cdot \mathcal{B}_k \,),$$

where $f_i \cdot \mathcal{B}_i$ is the row vector obtained by multiplying f_i with the monomials in $\mathcal{B}_i \cap \mathbb{Z}^n$, $1 \le i \le k$. We are abusing the notation; the point $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ corresponds to the monomial $x^m = x_1^{m_1} \ldots x_n^{m_n}$. $\square$

Theorem 5. *Same notation as before. Suppose $f_0, \ldots, f_k$ are Laurent polynomials as in Assumption 2. Let $F \in \mathbb{C}^{(p_1+\ldots+p_k)\times p}$ be a solution of the linear equation $M_{21} + M_{22}F = 0$.*

Then, every solution, ξ, of the sparse system determines a left eigenvector of $M_{11} + M_{12}F$. Even more, $f_0(\xi)$ is the eigenvalue of that left eigenvector.

Proof. Let us see that the hypotheses given in Assumption 2 imply that the rectangular matrix M_{22} has full rank q. The matrix M_{22} is the matrix of the composition of the following maps,

$$S_{\mathcal{B}_1} \times \ldots \times S_{\mathcal{B}_k} \xrightarrow{\Phi} S_{\mathcal{E}} \xrightarrow{\pi} S_{\mathcal{E}}/S_{\mathcal{B}_0},$$

where π is the quotient map. Then the rank of M_{22}, t, is equal to $\mathrm{rk}(\pi\,\Phi)$ and

$$\mathrm{Im}(\pi\,\Phi) = \pi(\mathrm{Im}(\Phi)) = \mathrm{Im}(\Phi)/\left(\mathrm{Im}(\Phi) \cap S_{\mathcal{B}_0}\right) \Longrightarrow$$
$$t := \mathrm{rk}(M_{22}) = \dim(\mathrm{Im}(\Phi)/\left(\mathrm{Im}(\Phi) \cap S_{\mathcal{B}_0}\right)) = \dim(\mathrm{Im}(\Phi)) - \dim(\mathrm{Im}(\Phi) \cap S_{\mathcal{B}_0}).$$

Note that the dimension of $S_{\mathcal{E}}/S_{\mathcal{B}_0}$ is equal to q,

$$\dim(S_{\mathcal{E}}/S_{\mathcal{B}_0}) = \dim(S_{\mathcal{E}}) - \dim(S_{\mathcal{B}_0}) = (p+q) - p = q.$$

Let us prove $q = t$. Using Theorem 3(b), we get

$$\dim(S_{\mathcal{E}}/\mathrm{Im}(\Phi)) = \dim(S_{\mathcal{B}_0}/\left(\mathrm{Im}(\Phi) \cap S_{\mathcal{B}_0}\right)) \Longrightarrow$$
$$\dim(S_{\mathcal{E}}) - \dim(\mathrm{Im}(\Phi)) = \dim(S_{\mathcal{B}_0}) - \dim(\mathrm{Im}(\Phi) \cap S_{\mathcal{B}_0}) \Longrightarrow$$
$$q = \dim(S_{\mathcal{E}}) - \dim(S_{\mathcal{B}_0}) = \dim(\mathrm{Im}(\Phi)) - \dim(\mathrm{Im}(\Phi) \cap S_{\mathcal{B}_0}) = t.$$

Now that we know that $\mathrm{rk}(M_{22}) = q$, it is easy to prove that there exists a matrix F such that

$$M_{22}F = -M_{21}, \quad F \in \mathbb{C}^{(p_1+\ldots+p_k)\times p}.$$

Each column of F, $c_1, \ldots, c_p$, is a solution of the linear system $M_{22}c_i = b_i$, where $b_i \in \mathbb{C}^q$ is the i-column vector of $-M_{21}$, $1 \le i \le p$.

Let $\xi \in \mathbb{C}^n$ be a solution of the sparse system, $f_1(\xi) = \ldots = f_k(\xi) = 0$. Then

$$(\,\xi^{m_1} \ldots \xi^{m_p} \quad \xi^{m_{p+1}} \ldots \xi^{m_{p+q}}\,)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = f_0(\xi)\cdot(\,\xi^{m_1} \ldots \xi^{m_p} \quad 0\,) \Longrightarrow$$
$$(\,\xi^{m_1} \ldots \xi^{m_p} \quad \xi^{m_{p+1}} \ldots \xi^{m_{p+q}}\,)\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ F & I \end{pmatrix} = f_0(\xi)\cdot(\,\xi^{m_1} \ldots \xi^{m_p} \quad 0\,)\begin{pmatrix} I & 0 \\ F & I \end{pmatrix} \Longrightarrow$$
$$(\xi^{m_1} \ldots \xi^{m_p})\,(M_{11} + M_{12}F) = f_0(\xi)\cdot(\xi^{m_1} \ldots \xi^{m_p}).$$

Then ξ determines a left eigenvector of $M_{11} + M_{12}F$ with eigenvalue $f_0(\xi)$. $\square$

Remark. In the previous theorem we proved that every solution of a sparse system as in Assumption 2 determines a left eigenvector of the square matrix $M_{11} + M_{12}F$. If the geometric multiplicity of an eigenvalue (the dimension of its left eigenspace) is greater than one, then we cannot use the computation of left eigenvectors to deduce the solutions of the sparse system. $\square$

Let us relate the multiplicity of a root ξ with the multiplicity of the eigenvalue $f_0(\xi)$ in the matrix $M_{11} + M_{12}F$.

Proposition 6. *Same notation as before. Suppose $f_0, \ldots, f_k$ are Laurent polynomials as in Assumption 2. Let $\xi \in (\mathbb{C} \setminus 0)^n$ be a solution of the sparse system with multiplicity μ. Then, the eigenvalue $f_0(\xi)$ of $M_{11} + M_{12}F$ has multiplicity greater than or equal to μ.*

Proof. The characteristic polynomial of the multiplication map $M_{f_0} : S/\mathcal{I} \to S/\mathcal{I}$ is

$$\chi(t) = (t - f_0(\xi_1))^{\mu_1} \dots (t - f_0(\xi_s))^{\mu_s},$$

where $\xi_1, \dots, \xi_s$ are the solutions of the sparse system in $(\mathbb{C} \setminus 0)^n$ and $\mu_1, \dots, \mu_s$ their respective multiplicities (see Dickenstein and Emiris (2005, 2.1.14)).

Let us relate the multiplication matrix M_{f_0} with our matrix M. Recall that the columns of the matrix of Φ are the multiples of $\{f_1, \dots, f_k\}$,

$$[\Phi] = \begin{pmatrix} M_{12} \\ M_{22} \end{pmatrix}.$$

Also, that the matrix of the map $\Psi|_{S_{\mathcal{B}_0}} : S_{\mathcal{B}_0} \to S_{\mathcal{E}}$ corresponds to the multiples of f_0,

$$\begin{pmatrix} M_{11} \\ M_{21} \end{pmatrix}.$$

In order to find the class of $f_0 \in S/\mathcal{I}$, we need to add/substract monomial multiples of $\{f_1, \dots, f_k\}$ to f_0. This process may be done by column operations in M. In particular, the matrix

$$\begin{pmatrix} M_{11} + M_{12}F & M_{12} \\ 0 & M_{22} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ F & I \end{pmatrix}$$

obtained by column operations from M, also determines the class in $S/\mathcal{I}$ of f_0. Specifically, the class of $x^{m_j} f_0$ in $S/\mathcal{I}$ is the same as the class of the j-column of M and the j-column of $M_{11} + M_{12}F$, $1 \le j \le p$,

$$x^{m_j} f_0 \equiv \sum_{i=1}^{p+q} x^{m_i} a_{ij} \equiv \sum_{i=1}^{p} x^{m_i} b_{ij} \mod \mathcal{I},$$

where $a_{ij} = M_{ij}$, $b_{ij} = (M_{11} + M_{12}F)_{ij}$ and the first p monomials are in $\mathcal{B}_0 \subseteq \mathcal{E}$ and the last q monomials are in $\mathcal{E} \setminus \mathcal{B}_0$.

Let us call σ_{f_0} the map associated to $M_{11} + M_{12}F$,

$$\sigma_{f_0} : S_{\mathcal{B}_0} \to S_{\mathcal{B}_0}, \quad \sigma_{f_0}(x^{m_j}) = \sum_{i=1}^{p} x^{m_i} b_{ij} \quad 1 \le j \le p.$$

Then, we have the following commutative diagram,

$$\begin{tikzcd} S_{\mathcal{B}_0} \arrow[r, "\pi"] \arrow[d, "\sigma_{f_0}"'] \arrow[dr, phantom, "\equiv"] & S/\mathcal{I} \arrow[d, "M_{f_0}"] \\ S_{\mathcal{B}_0} \arrow[r, "\pi"] & S/\mathcal{I} \end{tikzcd}$$

By Theorem 3(b), the map π above is an epimorphism. Hence the characteristic polynomial of M_{f_0} divides the characteristic polynomial of the matrix $M_{11} + M_{12}F$, that is, $\chi_{\sigma_{f_0}}(t) = \chi(t)P(t)$, where $P(t)$ is some polynomial (it depends on F). $\square$

To end this section, let us give an example on how to apply the previous theorems,

Example. Consider the intersection of a line with a parabola,

$$\begin{cases} f_1 = 1 + x + y = 0 \\ f_2 = 1 + x^2 + y = 0 \end{cases}$$

They intersect in $(1, -2)$ and $(0, -1) \in \mathbb{C}^2$. Note that the ideal generated by $\langle f_1, f_2 \rangle$ is radical. Let $f_0 = x - 2y$ be a linear form. The value of f_0 at each solution is $f_0(1, -2) = 5$ and $f_0(0, -1) = 2$.

Let us identify the monomial $x^n y^m$ with the point $(n, m) \in \mathbb{Z}^2$. Take the lattice polytopes associated to f_0, f_1 and f_2,

$$\mathcal{A}_0 \cap \mathbb{Z}^2 = \mathcal{A}_1 \cap \mathbb{Z}^2 = \{1, x, y\}, \quad \mathcal{A}_2 \cap \mathbb{Z}^2 = \{1, y, x, x^2\}.$$

Then,

$$\mathcal{B}_0 \cap \mathbb{Z}^2 = \{1, x^2y, x^2, y^2, x^3, xy, x, y\},$$
$$\mathcal{E} \cap \mathbb{Z}^2 = \{1, x^2y, x^2, y^2, x^3, xy, x, y, x^3y, xy^2, y^2x^2, x^4, y^3\}.$$

In these bases the coefficient matrix M is equal to,

$$\left(\begin{array}{cccccccc|cccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right).$$

Using Corollary 4, $\mathrm{rk}(M) = 13 = \#(\mathcal{E} \cap \mathbb{Z}^2)$ implies that the ideal $\langle f_0, f_1, f_2\rangle$ is S as we already knew. Also, we can recover the expected number of solutions, Theorem 3(a),

$$\#(\mathcal{E} \cap \mathbb{Z}^2) - \mathrm{rk}\begin{pmatrix} M_{12} \\ M_{22} \end{pmatrix} = 13 - 11 = 2.$$

Let us compute a matrix F and $M_{11} + M_{12}F$,

$$F = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 2 & -3 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$M_{11} + M_{12}F = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 3 & -3 & -2 & -3 & 0 & 0 \\
0 & 0 & 2 & -3 & 0 & 0 & -2 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

The characteristic polynomial of $M_{11} + M_{12}F$ is equal to $t^4(t+1)(t-2)^2(t-5)$ and its minimal polynomial is $t^3(t+1)(t-2)^2(t-5)$. The left eigenspace associated to 2 is $\langle(1,0,0,1,0,0,0,-1)\rangle$, and the left eigenspace associated to 5 is $\langle(1,-2,-2,4,1,1,1,-2)\rangle$. Then, looking at the last two coordinates (the monomials x and y), we get the two solutions $(0,-1)$ and $(1,-2)$.

Note that the characteristic polynomial and the minimal polynomial of $M_{11} + M_{12}F$ are different. Also, that the multiplicity of $2 = f_0(0,-1)$ is two.

In this example, it is possible to choose another F (without changing f_0) to get eigenvalues with multiplicity one in $M_{11} + M_{12}F$,

$$F' = \begin{pmatrix} 0 & 3 & -2 & -9 & 0 & 1 & 0 & 0 \\ 0 & 1 & 4 & 8 & 0 & -6 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 3 & 0 & 0 \\ 0 & -1 & 4 & 10 & 0 & -6 & 0 & 0 \\ 0 & 7 & -8 & -28 & 0 & 15 & 0 & 0 \\ 0 & -2 & -2 & -1 & 0 & 5 & 0 & 0 \\ 0 & -3 & -2 & 1 & 0 & 2 & 0 & 0 \\ 0 & -5 & 4 & 16 & 0 & -5 & 0 & 0 \\ 0 & -1 & -2 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -7 & 0 & 3 & 0 & 0 \\ 0 & 1 & -4 & -8 & 0 & 6 & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & -6 & 0 & 0 \\ 0 & -5 & 6 & 21 & 0 & -10 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 & 1 & 0 & 0 \end{pmatrix}. \qquad \square$$

Remark *(About the size of the matrices).* It is important to mention that our matrix construction produce an extremely large matrix. In the previous example we produced a matrix in $\mathbb{C}^{13\times 22}$, but considering different monomial bases it is possible to construct a smaller matrix in $\mathbb{C}^{6\times 7}$. The following monomials were suggested by a referee. Let $\mathcal{B}_0 \cap \mathbb{Z}^2 = \mathcal{B}_1 \cap \mathbb{Z}^2 = \{1, x, y\}$ and $\mathcal{B}_2 \cap \mathbb{Z}^2 = \{1\}$. Then, the coefficient matrix in these bases is

$$M = \left(\begin{array}{ccc|cccc} 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 1 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 \end{array}\right)$$

Applying the same procedure as before, the matrix $M_{11} + M_{12}F$ has three eigenvalues, $0, 2$ and 5 with left eigenvectors $(1,6,4)$, $(1,0,-1)$ and $(1,1,-2)$ respectively.

Let us explain briefly why this matrix works. In this example, the polytope $\mathcal{E} = \mathcal{A}_0 + \mathcal{B}_0$ satisfies $\mathcal{E} \cap \mathbb{Z}^2 = \{1, x, y, x^2, xy, y^2\}$, then the projective variety X of Lemma 1 is the projective plane $\mathbb{P}^2$. The invertible sheaf associated to $\mathcal{O}_X(d_0, d_1, d_2)$ is equal to $\mathcal{O}_{\mathbb{P}^2}(d_0 + d_1 + 2d_2)$. Then, the augmented dual Koszul complex associated to the section (f_1, f_2) is

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_Z \to 0.$$

Using Theorem III.5.1 in Hartshorne (1977), we know that the shaves $\mathcal{O}_{\mathbb{P}^2}(-1)$ and $\mathcal{O}_{\mathbb{P}^2}(-2)$ have no higher cohomology.

In order to prove that M_{22} has full rank we need to prove Theorem 3(b) and use the first part of the proof in Theorem 5. Twisting the previous complex by $\mathcal{O}_{\mathbb{P}^2}(1)$ and taking global sections, we obtain that the map $S_{\mathcal{B}_0} \to S/\mathcal{I}$ is surjective. Hence, M_{22} has full rank.

To prove that the co-rank of the column block matrix $(M_{12}; M_{22})$ is 2, we need to prove Theorem 3(a). It follows by twisting with $\mathcal{O}_{\mathbb{P}^2}(2)$ and taking global sections.

In the same way, it is easy to prove that M has full rank twisting by $\mathcal{O}_{\mathbb{P}^2}(2)$ and taking global sections the dual Koszul complex associated to (f_0, f_1, f_2)

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-4) \to \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$$
$$\to \mathcal{O}_{\mathbb{P}^2} \to 0.$$

All of our results are based on Demazure Vanishing in Lemma 1. In this particular example the sheaves $\mathcal{O}_{\mathbb{P}^2}(-1)$ and $\mathcal{O}_{\mathbb{P}^2}(-2)$ have no higher cohomology by different reasons. It is worth mentioning that the construction of M can be improved to produce smaller matrices. The size is controlled by the lattice polytope $\mathcal{E}$ and the space of global sections $H^0(X, \mathcal{F}'^{\vee}(1, \ldots, 1))$, where $X = X_{\mathcal{E}}$ is the projective variety of Lemma 1 and $\mathcal{F}'$ is the sheaf associated to the sparse system and f_0. Our approach gives a *canonical* matrix to work in general.

Let us mention the following related result on reducing the size of the coefficient matrix. In Canny and Emiris (1993), the authors work with (essentially) the same map Ψ, the first Koszul map, to produce a formula of the sparse resultant of $n+1$ Laurent polynomials in n variables. Using a *Row content function* they constructed a square coefficient matrix such that its determinant is a non-zero multiple of the sparse resultant. Continuing this work, in Emiris and Canny (1995), the authors proposed an incremental algorithm to obtain this submatrix. Finally, in Dickenstein and Emiris (2003), the authors provided a coefficient matrix of optimal size for the case of multihomogeneous systems.

3. Application

To conclude this article, let us give an application to approximate the maximum of a generic trilinear form over a product of spheres,

$$\ell : \mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \times \mathbb{R}^{s+1} \to \mathbb{R}, \quad \ell(x, y, z) = \sum_{(i,j,k)=0}^{(n,m,s)} a_{ijk} x_i y_j z_k, \quad \max_{\|x\|=\|y\|=\|z\|=1} |\ell(x, y, z)|,$$

where the norm is the usual 2-norm. This problem was studied in Massri (2013). In the literature, the maximum of ℓ over a product of spheres, is called the first singular value of ℓ (Lim, 2005, §3).

Using Lagrange method of multipliers (Apostol, 1974, §13.7) the extreme points of ℓ over a product of spheres, $\mathbb{S}^n \times \mathbb{S}^m \times \mathbb{S}^s$, satisfy

$$\begin{cases} \partial\ell/\partial x_i(x_0, \ldots, x_n, y_0, \ldots, y_m, z_0, \ldots, z_s) &= 2\alpha x_i, \quad 0 \le i \le n, \\ \partial\ell/\partial y_j(x_0, \ldots, x_n, y_0, \ldots, y_m, z_0, \ldots, z_s) &= 2\beta y_j, \quad 0 \le j \le m, \\ \partial\ell/\partial z_k(x_0, \ldots, x_n, y_0, \ldots, y_m, z_0, \ldots, z_s) &= 2\lambda z_k, \quad 0 \le k \le s, \end{cases}$$

$$\alpha, \beta, \lambda \in \mathbb{R}, \quad \|x\| = \|y\| = \|z\| = 1.$$

These equations imply that the vector $\partial\ell/\partial x(x, y, z)$ is a multiple of x. Same for y and z. In other words, considering the system in $\mathbb{P}^n \times \mathbb{P}^m \times \mathbb{P}^s$, we can hide the variables α, β and λ,

$$\begin{cases} x_j \partial\ell/\partial x_i(x, y, z) &= x_i \partial\ell/\partial x_j(x, y, z), \quad 0 \le i < j \le n, \\ y_j \partial\ell/\partial y_i(x, y, z) &= y_i \partial\ell/\partial y_j(x, y, z), \quad 0 \le i < j \le m, \\ z_j \partial\ell/\partial z_i(x, y, z) &= z_i \partial\ell/\partial z_j(x, y, z), \quad 0 \le i < j \le s. \end{cases}$$

Given that ℓ is trilinear, the expression $x_j \partial\ell/\partial x_i(x, y, z)$ is equal to $\ell(x_j e_i, y, z)$, where $e_i \in \mathbb{R}^{n+1}$ is the vector with 1 in the i-coordinate and 0 in the rest. Same for y and z. Summing up, the extreme points of ℓ satisfy the following system of equations in $\mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \times \mathbb{R}^{s+1}$,

$$\begin{cases} \ell(x_j e_i - x_i e_j, y, z) = 0, & 0 \le i < j \le n, \\ \ell(x, y_j e_i - y_i e_j, z) = 0, & 0 \le i < j \le m, \\ \ell(x, y, z_j e_i - z_i e_j) = 0, & 0 \le i < j \le s, \\ x_0^2 + \ldots + x_n^2 = 1, & \\ y_0^2 + \ldots + y_m^2 = 1, & \\ z_0^2 + \ldots + z_s^2 = 1. & \end{cases}$$

Assume that ℓ is generic and $2n, 2m, 2s \le n + m + s$ (Gel'fand et al., 1994, §14, 1.3), hence the extreme points are finite with multiplicity one. Enumerate the equations, $f_1, \ldots, f_k$. Let $\lambda_1, \ldots, \lambda_r$ be the real eigenvalues of $M_{11} + M_{12}F$ associated to solutions of the system $f_1 = \ldots = f_k = 0$ and to the generic trilinear form ℓ. If $|\lambda_1| \ge |\lambda_i|$, $2 \le i \le r$, then $|\lambda_1|$ is the maximum value of ℓ over $\mathbb{S}^n \times \mathbb{S}^m \times \mathbb{S}^s$.

Remark *(Numerical Issues).* The previous application has several numerical issues. For example, the computation of eigenvalues (Corless et al., 1997; McNamee, 2002; Oishi, 2001; Rouillier and Zimmermann, 2004; Rump, 2001), the test of the genericity of ℓ and also, the evaluation of a possible root in the equations. These last issues, may be solved using *interval arithmetics* (Ozaki et al., 2012; Rump, 1999).

In this application we did not required the computation of eigenvectors. It is a delicate numerical issue. When the matrix $M_{11} + M_{12}F$ is non-derogatory, we can apply the work in Rump and Zemke (2003). See also, Bondyfalat et al. (2000); Helmberg et al. (1993); Mayer (1994); Oishi (2001); Yamamoto (1982). In other cases, it is possible to adapt some ideas from Möller and Stetter (1995). For a method that works on general systems, we refer to Graillat and Trébuchet (2009). □

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