Quasi stationary distributions and Fleming Viot processes

Pablo A. Ferrari*

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**Universidad de Buenos Aires, Universidade de São Paulo

In 1947 Yaglom [2] studied the subcritical Galton Watson process conditioned to survival until time t, and proved that in the limit as $t \to \infty$ the distribution of the conditioned law converges to a law ν on $\mathbb{N}$.

Denote ν_n the conditioned law at time n,

$$\nu_n(y) = \frac{\sum_{x\in\mathbb{N}} \nu(x) Q^n(x,y)}{\sum_{x\in\mathbb{N}} \nu(x)(1 - Q^n(x,0))}\,,$$

where Q is the transition matrix of the Markov chain; in this case the Galton Watson branching process. Since the process is absorbed at 0 we have $Q(0,0) = 1$. Yaglom shows that the limit ν is a fixed point for the above evolution: $\nu_0 = \nu$ implies $\nu_n = \nu$. That is, starting with ν and conditioned that at time n the process is not extinct, the distribution is still ν. For this reason this measure was then called *quasi stationary distribution*.

A simple algebra shows that the above evolution is equivalent to

$$\nu_n(y) = \sum_{x\in\mathbb{N}} \nu_{n-1}(x)[Q(x,y) + \nu_{n-1}(y)Q(x,0)]$$

This can be interpreted as a dynamics where the mass escaping to zero at time n, given by $\sum_{x\in\mathbb{N}} \nu_{n-1}(x)Q(x,0)$, returns to the system instantaneously to be distributed proportionally to the existing mass at each $y \in \mathbb{N}$.

The Fleming Viot process is an attempt to approximate this refeeding procedure by a linear dynamics represented by N interacting Markov chains evolving independently until one or more of them are absorbed at the origin. When this happens, the absorbed particles come back to the system according to the empirical distribution of the particles just before the absorption.

Denoting by $\eta_n^N(x)$ the number of particles at site x at time n,

$$E\eta_n^N(y) = \sum_x E\Big[\eta_{n-1}^N(x)\Big(Q(x,y) + Q(x,0)\frac{\eta_{n-1}^N(y)}{N}\Big)\Big]\ ,$$

so that, after dividing both sides by N, we conclude that $E\eta_n^N/N$ and ν_n satisfy the same equations in the limit as $N \to \infty$ provided the two pair correlations of η_n^N/N factorize in the limit. This N-particle system is called *Flemming Viot process*, and was proposed by Burdzy, Holyst, Ingerman and March [1] for Brownian motion conditioned to stay inside a bounded domain.

While the problem of existence of the Fleming-Viot process is in general easy to treat in a discrete state space, it becomes an issue in the Brownian motion setup. The paper of Grigorescu and Kang shows the existence of the process for a generalization of the Fleming Viot process for diffusions which includes [1] and many others. Villemonais proves non explosion for another generalization of the process, in particular the time inhomogeneous case, and establish conditions that guarantee tightness of the empirical measures of particles as N goes to infinity.

The paper of Groisman and Jonckheere reviews some central results in the theory of quasi stationary distributions in a discrete state space and relate them to the Fleming Viot process. In particular they prove that the motion of a tagged particle converges as N grows to infinity to a time inhomogeneous Markov process where the refeeding distribution at time t is the conditioned one-particle law. In the limit as $t \to \infty$ the refeeding law is the quasi stationary measure ν and the process becomes time homogeneous. The limiting process is Markov and the invariant measure is the same as the refeeding measure.

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References

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[2] Yaglom, A. M. *Certain limit theorems of the theory of branching random processes.* Doklady Akad. Nauk SSSR (N.S.) 56, (1947). 795–798.