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Generalizing Entanglement via Informational Invariance for Arbitrary Statistical Theories

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Abstract:
Given an arbitrary statistical theory, different from quantum mechanics, how to decide which are the nonclassical correlations? We present a formal framework which allows for a definition of nonclassical correlations in such theories, alternative to the current one. This enables one to formulate extrapolations of some important quantum mechanical features via adequate extensions of "reciprocal" maps relating states of a system with states of its subsystems. These extended maps permit one to generalize i) separability measures to any arbitrary statistical model as well as ii) previous entanglement criteria. The standard definition of entanglement becomes just a particular case of the ensuing, more general notion.

Keywords:
Entanglement; Quantum-separability; Convex-sets

1. INTRODUCTION

Quantum mechanics can be regarded as an extension of the classical probability calculus that allows for random variables that are not simultaneously measurable [1]. Working from this peculiar perspective, it can be shown that many phenomena usually considered as typically quantal, like quantum no-cloning and no-broadcasting theorems, the trade-off between state disturbance and measurement, and the existence and basic properties of entangled states, are in fact generic features of non-classical probabilistic theories that verify a basic *non-signaling* constraint [1]. This is the point of departure of our present considerations.

In particular, entanglement [2] is conventionally viewed as the most emblematic expression of non-classicality. Schrödinger is widely quoted stating that "entanglement is *the* characteristic trait of quantum mechanics" [3–5]. Indeed, characterizing entanglement has become one of the most important current tasks of physics [6], with a host of possible technological applications. An entanglement criterion based on *geometrical* properties of entanglement has been recently presented in [7]. These geometrical features of entanglement will be employed here to extrapolate many entanglement's properties to *arbitrary probabilistic theories*. This is done by recourse to an essential mathematical ingredient, the so-called Convex Operational Model (COM) approach. The COM approach is founded on geometrical properties of a special convex set, that containing all the states of an arbitrary statistical theory [8, 22] (see also

[9–12, 23, 38, 39, 50]).

The COM approach has its roots in operational theories and has been shown to be useful to generalize many quantum mechanical notions mentioned above, such as teleportation protocols, no broadcasting, and no cloning theorems [8, 22, 23]. The geometrical approach based on convex sets can also be seen as a framework in which non-linear theories which generalize quantum mechanics, can be included, studied, and compared with it [24–26]. It is also important to remark that an axiomatization independent (and equivalent to) the von Neumann formalism can be given using the geometrical-operational approach [24–26].

The importance of entanglement as a resource for measuring classicality of a state has been highlighted in [27]. Other measures of non-classicality exist, of course. One of the most important is the negativity of the Wigner function [29]. Another important measure of non-classicality -often found in quantum optics- has to do with the properties of coherent states, i.e., a state will be considered classical if it can be written as a mixture of coherent sates (which satisfy a minimal violation of Heisenberg's uncertainty principle). More recently, quantum discord (QD) [31–35, 37] has became another measure of non-classicality. QD refers to important manifestations of the quantumness of correlations in composite systems that are different from those of entanglement-origin and may be relevant in quantum information technologies [31–33, 36].

In this work, we restrict ourselves to entanglement (see [38] for the QD case) and provide a characterization of it using maps. We show that this characterization can be generalized to arbitrary statistical models. The issue has been studied, for example, in [38, 39]. Our entanglement-extension (based in [7]) allows for an alternative approach, which provides a quite general characterization of non classical correlations in arbitrary statistical models, *leaving the standard treatment as a particular case*.

More explicitly, our characterization of entanglement is based on the maps that relate states of the system with states of its subsystems. In particular, following the generalization presented in [40], we define generalized partial traces by imposing conditions on (1) *morphisms between extensions of convexity models* and (2) a *special map* which allows one to create the set of separable states given the available states of two parties. Differently from the standard approach [38, 39], in our proposal the characterization of non-classical correlations is based on maps.

The interlink between these two items is investigated, and, via appeal to constructions presented in [7], we concoct a geometrical characterization of entanglement in arbitrary COM's. Specifically, we generalize the notion of *informational invariance*, advanced in [7]. It is shown that this characterization of entanglement lies at the heart of the separability problem in any statistical theory, providing i) an alternative visualization of it and ii) enriching the convex/operational approach to QM [24–26, 40] (as well as to other statistical theories).

The alternative perspective presented in this work will allow us to obtain, for a canonical family of separability measures (based on the Schlienz-Mahler one [45]), its most general form. As a result, we will be able to construct a general quantitative (and in many cases computable) measure of non-classicality for arbitrary statistical theories, including non linear generalizations of QM. This general characterization of a vast family of entanglement measures will permit one to compare the behavior of measures of non-classical correlations in different theories, and thus, to single out specifical features of QM. Why is this of importance? The answer is given in, for example, [11] and [12]. Several possible applications were envisaged in [13–20].

Since our constructions and their implications are formulated in the geometrical setting of the COM approach, they could become applicable to many physical theories of interest. An example of such theories are "Popescu-Rohrlich" boxes [9]. Our construction could also be applied to quantum mechanics with a limited set of allowed measurements, general C^*-algebraic theories, theories derived by relaxing uncertainty relations, etc.

In principle, the scope of the generalization given by the COM approach is not constrained to physical theories. It also includes mathematical models of any statistical theory, provided these theories satisfy very general requirements. Thus, the generalization of quantum mechanical notions –*and specially measures of non-classical correlations*– to arbitrary statistical theories via the COM approach is a useful alternative tool for extrapolating such notions to different fields of research. For example, the influence of quantum effects and entanglement in evolution was studied in a toy biological model based on a Chaitin's idea [30]. The study of more realistic models may require rather sophisticated mathematical frameworks for which the COM approach and the kind of generalization presented in this work (as well as in others, for example [23]), can be useful.

In Section 2 we briefly recapitulate the notion of quantum effects and in section 3 we review the COM approach. Next, in section 4, we write in a convenient form the main details of the geometrical structure that underlies entanglement, as advanced in [7]. By following [40] we build in section 5 a geometrical generalization of the relevant structures, and discuss its application to the development of generalized entanglement measures. Finally, in section 6 some conclusions are drawn. An Appendix on quantal effects is also provided.

2. QUANTAL EFFECTS

An algebraic structure called an effect algebra has been introduced for investigations in the foundations of quantum mechanics [46]. The elements of an effect algebra $\mathscr{E}$ are called quantum effects and are very important indeed for quantum statistics and quantum measurement theory [47]. One may regard a quantum effect as an elementary yes-no measurement that may be un-sharp or imprecise.

Quantum effects are used to construct generalized quantum measurements (or observables). The structure of an effect algebra is given by a partially defined binary operation $\bigoplus$ that is used to form a combination $a \bigoplus b$ of effects $a, b \in \mathscr{E}$. The element $a \bigoplus b$ represents a statistical combination of a and b whose probability of occurrence equals the sum of the probabilities that a and b occur individually. Usually, effect algebras possess a convex structure. For example, if a is a quantum effect and $\lambda \in [0,1]$, then λa represents the effect a attenuated by a factor of λ. Then, $\lambda a \bigoplus (1-\lambda)b$ is a generalized convex combination that can be constructed in practice. If a quantum system $\mathscr{S}$ is represented by a Hilbert space $\mathscr{H}$, then a self-adjoint operator $\hat{A}$ such that $0 \leq \hat{A} \leq 1$ corresponds to an effect for $\mathscr{S}$ [46]. For more details, see Appendix A.

3. COM'S PRELIMINARIES

Following [23], we now review elementary COM-notions. The aim of this formalism is to model general statistical or operational theories. Any statistical theory has a set of states $\omega \in \Omega$ and a set of observables.

It is reasonable to postulate that the set Ω is convex, because the mixture of two states in any statistical theory ought to yield a new state. For the convex set Ω one should then associate probabilities to any observable a. This entails that one must define a probability $a(\omega) \in [0,1]$ for any state $\omega \in \Omega$. Usually, any observable is an affine functional belonging to a space $A(\Omega)$ (the space of all affine functionals). It is also assumed that there exists a unitary observable u such that $u(\omega) = 1$ for all $\omega \in \Omega$ and (in analogy with the quantum case, in which they form an ordered space), the set of all quantum effects (the reader not familiarized with the concept is advised to look at Appendix A) will be encountered in the interval $[0,u]$. A measurement will be represented by a set of effects $\{a_i\}$ such that $\sum_i a_i = u$.

Ω is then naturally embedded ($\omega \mapsto \hat{\omega}$) in the dual space $A(\Omega)^*$ as follows: $\hat{\omega}(a) := a(\omega)$. Call $V(\Omega)$ the linear span of Ω in $A(\Omega)^*$. Ω will be considered finite dimensional if and only if $V(\Omega)$ is finite dimensional, and we restrict ourselves to such situation (and to compact spaces). This implies that Ω will be the convex hull of its extreme points, called pure states (for details see, for example, [8, 22, 39]). In a finite dimension d *a system will be classical if and only if it is a simplex, i.e., the convex hull of* $d+1$ *linearly independent pure states*. It is a well known fact that in a simplex a point may be expressed as a unique convex combination of its extreme points, *a characteristic feature of classical theories that no longer holds in a quantum one*.

Summing up, a COM may be regarded as a triplet $(\mathbf{A}, \mathbf{A}^*, u_{\mathbf{A}})$, where $\mathbf{A}$ is a finite dimensional vector space, $\mathbf{A}^*$ its dual and $u_{\mathbf{A}} \in \mathbf{A}$ is a unit functional.

For compound systems, if its components have state spaces Ω_A and Ω_B, let Ω_{AB} denote the joint state space. Under reasonable assumptions, it turns out [23] that Ω_{AB} may be identified with a linear span of $(V(\Omega_A) \otimes V(\Omega_B))$. A maximal tensor product state space $\Omega_A \otimes_{max} \Omega_B$ can be defined as the one which contains all bilinear functionals $\varphi : A(\Omega_A) \times A(\Omega_B) \longrightarrow \mathbb{R}$ such that $\varphi(a,b) \geq 0$ for all effects a and b and $\varphi(u_A, u_B) = 1$. The maximal tensor product state space has the property of being the biggest set of states in $(A(\Omega_A) \otimes A(\Omega_B))^*$ which assigns probabilities to all product- measurements.

On the other hand, the minimal tensor product state space $\Omega_A \otimes_{min} \Omega_B$ is defined as the one which is formed by the convex hull of all product states. A product state is a state of the form $\omega_A \otimes \omega_B$ such that $\omega_A \otimes \omega_B(a,b) = \omega_A(a)\omega_B(b)$ for all pairs $(a,b) \in A(\Omega_A) \times A(\Omega_B)$. The actual set of states Ω_{AB} (to be called $\Omega_A \otimes \Omega_B$ from now on) of a particular system will satisfy $\Omega_A \otimes_{min} \Omega_B \subseteq \Omega_A \otimes \Omega_B \subseteq \Omega_A \otimes_{max} \Omega_B$. For the classical case (A and B classical) we will have $\Omega_A \otimes_{min} \Omega_B = \Omega_A \otimes_{max} \Omega_B$. For the quantum case we have the strict inclusions $\Omega_A \otimes_{min} \Omega_B \subset \Omega_A \otimes \Omega_B \subset \Omega_A \otimes_{max} \Omega_B$.

One can reasonably conceive of a separable state in an arbitrary COM as one which may be written as a convex combination of product states [38, 39], i.e.

Definition 3.1.
A state $\omega \in \Omega_A \otimes \Omega_B$ will be called *separable* if there exist p_i, $\omega_A^i \in \Omega_A$ and $\omega_B^i \in \Omega_B$ such that

$$\omega = \sum_i p_i \omega_A^i \otimes \omega_B^i \tag{1}$$

If $\omega \in \Omega_A \otimes \Omega_B$ but it is not separable, we will call it *entangled*. Entangled states exist only if $\Omega_A \otimes \Omega_B$ is strictly greater than $\Omega_A \otimes_{min} \Omega_B$.

Using these constructions, marginal states can be defined as follows [23]. Given a state $\omega \in \Omega_A \otimes \Omega_B$, define

$$\omega_A(a) := \omega(a \otimes u_B) \tag{2a}$$

$$\omega_B(b) := \omega(u_A \otimes b) \tag{2b}$$

It is possible to show that the marginals of an entangled state are necessarily mixed, while those of an unentangled pure state are necessarily pure.

These definitions are sufficient for a generalization of entanglement to arbitrary COM's. In the following section we review a geometrical construction whose generalization yields an alternative conceptualization of the entanglement-notion. The new view turns out to be more general than the one summarized above.

4. GEOMETRICAL CHARACTERIZATION OF ENTANGLEMENT USING MAPS

Let us now focus attention on quantum mechanics for the time being. For a compound system represented by a Hilbert space $\mathscr{H}$ (we restrict ourselves in what follows to a finite dimension), $\mathscr{S}(\mathscr{H})$ is the convex hull of the set of all product states. Let $\mathscr{C}$ be the convex set of quantum states and $\mathscr{L}_{\mathscr{C}}$ the set of all convex subsets of $\mathscr{C}$ (with analogous definitions of $\mathscr{C}_i$ and $\mathscr{L}_{\mathscr{C}_i}$ for its subsystems, $i = 1,2$).

4.1 Canonical Maps

We focus attention now in the specially important map Π

Definition 4.1.

$$\Pi : \mathscr{C} \longrightarrow \mathscr{C}$$
$$\rho \mapsto \rho^A \otimes \rho^B.$$

It is of the essence that product states $\rho = \rho^A \otimes \rho^B$ not only satisfy

$$\Pi(\rho^A \otimes \rho^B) = \rho^A \otimes \rho^B, \tag{3}$$

but *are the only states which do satisfy (3)*. Partial traces are particular maps defined between $\mathscr{C}$, $\mathscr{C}_1$, and $\mathscr{C}_2$:

$$\begin{aligned} \mathrm{tr}_i : \mathscr{C} &\longrightarrow \mathscr{C}_j \\ \rho &\mapsto \mathrm{tr}_i(\rho) \end{aligned} \quad , \tag{4}$$

from which we can construct the induced maps τ_i, also very important for our present purposes, on $\mathscr{L}_{\mathscr{C}}$, via the image of any subset $C \subseteq \mathscr{C}$ under tr_i

$$\begin{aligned} \tau_i : \mathscr{L}_{\mathscr{C}} &\longrightarrow \mathscr{L}_{\mathscr{C}_i} \\ C &\mapsto \mathrm{tr}_j(C) \end{aligned} \quad , \tag{5}$$

where for $i = 1$ we take the partial trace with $j = 2$ and vice versa. In turn, we can define the product map

$$\begin{aligned} \tau : \mathscr{L}_{\mathscr{C}} &\longrightarrow \mathscr{L}_{\mathscr{C}_1} \times \mathscr{L}_{\mathscr{C}_2} \\ C &\mapsto (\tau_1(C), \tau_2(C)) \end{aligned} \quad . \tag{6}$$

This map generalizes partial traces to convex subsets of $\mathscr{C}$.

Given the convex subsets $C_1 \subseteq \mathscr{C}_1$ and $C_2 \subseteq \mathscr{C}_2$ it is possible to define a product

Definition 4.2.
Given the convex subsets $C_1 \subseteq \mathscr{C}_1$ and $C_2 \subseteq \mathscr{C}_2$ we introduce now

$$C_1 \otimes C_2 := \{\rho_1 \otimes \rho_2 \,|\, \rho_1 \in C_1, \rho_2 \in C_2\} \tag{7}$$

Using this, we define the (for us all-important) map Λ:

Definition 4.3.

$$\Lambda : \mathscr{L}_{\mathscr{C}_1} \times \mathscr{L}_{\mathscr{C}_2} \longrightarrow \mathscr{L}_{\mathscr{C}}$$
$$(C_1, C_2) \mapsto Conv(C_1 \otimes C_2)$$

where $Conv(\cdots)$ stands for *convex hull* of a given set. Applying Λ to the particular case of the quantum sets of states of the subsystems ($\mathscr{C}_1$ and $\mathscr{C}_2$), one sees that Definitions 4.2 and 4.3 entail

$$\Lambda(\mathscr{C}_1, \mathscr{C}_2) = Conv(\mathscr{C}_1 \otimes \mathscr{C}_2) \tag{8}$$

and this is nothing but

$$\Lambda(\mathscr{C}_1, \mathscr{C}_2) = \mathscr{S}(\mathscr{H}), \tag{9}$$

because $\mathscr{S}(\mathscr{H})$ is by definition (for finite dimension) the convex hull of the set of all product states.

4.2 Informational Invariance

The map Λ gives a precise mathematical expression to the operation of making tensor products and mixing, which has a clear physical meaning.

Let us elaborate: if it is possible to prepare in the laboratory A a given set of states C_1, it is reasonable to assume that C_1 is convex, because if it is not, it is possible to make it convex by recourse to classical algorithms (for example, by tossing a biased coin, preparing one state or the other according to the outcome, and then forgetting the outcome). Same for the set C_2 in laboratory B. Then, it is possible (without any recourse to non-classical interactions) to prepare all product states of the form $\rho_1 \otimes \rho_2$ with $\rho_1 \in C_1$ $\rho_2 \in C_2$. Also, it is possible to prepare all possible mixtures of such product states using a classical algorithm of the type mentioned above. Now, this new set of states is nothing but $\Lambda(C_1, C_2)$. Thus, $\Lambda(C_1, C_2)$ is the maximal set of states which can be generated without using non-classical correlations, given that the set of states C_1 is available at laboratory A, and C_2 is available in B.

In particular, equation (9) entails that *the set of all separable states of* $\mathscr{C}$ *is the image of the pair* $(\mathscr{C}_1, \mathscr{C}_2)$ *under the map* Λ, i.e., all possible products and their mixtures for the whole sets of states $\mathscr{C}_1$ and $\mathscr{C}_2$.

Let us now turn to the function $\Lambda \circ \tau$ (the composition of τ with Λ) [7]. For the special case of a convex set formed by only one "matrix" (point) $\{\rho\}$ we have

$$\Lambda \circ \tau(\{\rho\}) = \{\rho^A \otimes \rho^B\} \tag{10}$$

which is completely equivalent to Π (see Definition 4.1), and thus satisfies an analogue of Equation (3). Using this function it is possible to derive a separability criterium in terms of properties of convex sets that are polytopes [7]:

Proposition 4.1.
$\rho \in \mathscr{S}(\mathscr{H})$ if and only if there exists a polytope S_ρ such that $\rho \in S_\rho$ and $\Lambda \circ \tau(S_\rho) = S_\rho$.

Let us consider now the separability of pure states. Its characterization in the bipartite instance is quite simple. We assert that $\rho = |\psi\rangle\langle\psi|$ will be separable if and only if it is a product of pure reduced states, i.e., if and only if there exist $|\phi_2\rangle \in \mathscr{H}_1$ and $|\phi_2\rangle \in \mathscr{H}_2$ such that $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$. In mathematical terms, this can be written as

$$|\psi\rangle\langle\psi| \in \mathscr{S}(\mathscr{H}) \Leftrightarrow \Lambda \circ \tau(\{|\psi\rangle\langle\psi|\}) = \{|\psi\rangle\langle\psi|\}$$
$$(\Leftrightarrow \Pi(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|). \tag{11}$$

Equation (11) tells us that a *pure* state is separable, if and only if it remains invariant under the function $\Lambda \circ \tau$ (or equivalently, invariant under Π).

While this criterium is no longer valid for general mixed states, the more general criterium 4.1 is available for this case: a general *mixed* state ρ is separable if and only if there exists a convex subset S_ρ invariant under $\Lambda \circ \tau$. It is clear that the criterium 4.1 is analogous to (11), being a generalization of it to convex subsets of $\mathscr{C}$, with $\Lambda \circ \tau$ playing the role of the generalization of Π. Thus, a generalization of the notion of product state for convex sets can now be defined [7]

Definition 4.4.
A convex subset $C \subseteq \mathscr{C}$ such that $\Lambda \circ \tau(C) = C$ is called a *convex separable subset* (CSS) of $\mathscr{C}$.

Product states are limit cases of convex separable subsets (they constitute the special case when the CSS has only one point) [7]. CSS have the property of being *informational invariants* in the sense that the information that they contain as probability spaces [28] may be recovered via tensor products and mixing of their (induced) reduced sub-states.

Let us turn now to a distinctive property of Π. It is possible to prove that if Π is applied twice is seen to be idempotent, i.e.,

$$\Pi^2 = \Pi. \tag{12}$$

and the same holds for $\Lambda \circ \tau$

$$(\Lambda \circ \tau)^2 = \Lambda \circ \tau. \tag{13}$$

Consequently, the generalization of Π satisfies an equality equivalent to (12).

An important remark is to be made. It is easy to show that if we apply τ_i to $\mathscr{C}$, we obtain $\mathscr{C}_i$. Thus, using Equation (9), we obtain

$$\Lambda \circ \tau(\mathscr{S}(\mathscr{H})) = \mathscr{S}(\mathscr{H}) \tag{14}$$

and thus, *$\mathscr{S}(\mathscr{H})$ is itself an informational invariant (a CSS), and in fact, the largest one*. As we shall see in the following Sections, this fact can be gainfully used to define separability and generalize the geometrical structure of entanglement to arbitrary statistical theories. We shall also see that the generalization of the properties of the functions Λ, τ and $\Lambda \circ \tau$, allow us to see how to define a huge family of entanglement measures in arbitrary COM's.

5. ENTANGLEMENT AND SEPARABILITY IN ARBITRARY CONVEXITY MODELS

In [40], a general study of extensions of convex operational models is presented. This general framework includes compound systems. We will follow that paper's approach to advance our entanglement-generalization, applicable to arbitrary extensions of convexity models.

5.1 Extensions of Convexity Models

Given two arbitrary convex operational models **A** and **B** (see Section 3) representing two systems (they not necessarily possess the same underlying theory), a morphism between them will be given by an affine map $\phi : \Omega_A \rightarrow \Omega_B$ such that the affine dual map φ^* -defined by the functional $\varphi^*(b) := b \circ \varphi$ (where "$\circ$" denotes composition)- maps the effects of **B** into effects of **A** [40].

An affine map, is intuitively understood as the canonical mathematical expression of a map preserving the convex structure, which is the structure underlying all statistical theories.

A link between (or process from)) **A** - **B** will be represented by a morphism $\phi : \mathbf{A} \rightarrow \mathbf{B}$ such that, for every state $\alpha \in \Omega_{\mathbf{A}}$, $u_{\mathbf{B}}(\phi(\alpha)) \leq 1$ (this is a normalization condition). If we want to study processes, $u_{\mathbf{B}}(\phi(\alpha))$ will represent the probability that the process represented by ϕ takes place. In this way, morphisms can be used to represent links between systems (see next paragraph), as for example, "being a subsystem of", as well as processes understood as general evolutions in time, continuous or not.

COM-extensions are studied in [40]. Let us remember therefrom the definition of the "extension"-notion.

A COM **C** will be said to be an extension of **A** if there exists a morphism $\phi : \Omega_{\mathbf{C}} \rightarrow \Omega_{\mathbf{A}}$ which is surjective.

We emphasize the great generality of this formulation: in the above definition of "extension", almost all possible conceivable cases are contained. A subsystem of a classical or (quantal) system constitutes an example of an extension in the above sense (it is the morphism of the canonical set-theoretical projection in the classical case, and of partial trace in the quantum instance). Not only subsystems of a compound system are captured by this notion of extension. Also limits between theories, or coarse grained versions of a given theory, may be considered –under this characterization– as extensions.

5.2 General Formal Setting

In order to look for a generalization of entanglement which captures the results of previous Sections, we must look at triads of COM's **C**, $\mathbf{C_1}$ and $\mathbf{C_2}$, with states spaces $\Omega_{\mathbf{C}}$, $\Omega_{\mathbf{C}_1}$, and $\Omega_{\mathbf{C}_2}$, such that there exist two morphisms (extension maps) ϕ_1 and ϕ_2 in such a way that **C** be an extension of both $\mathbf{C_1}$ and $\mathbf{C_2}$.

It is clear that the product map $\phi = (\phi_1, \phi_2)$ may be considered as the best candidate for a generalization of the map τ (see Equation (6)). But in order to have adequate generalizations of partial traces, i.e., in order to obtain equivalence with the marginal states defined in 2), we need an additional condition: for any product state $a = a_1 \otimes a_2$, we should have $\phi(a) = (\phi_1(a_1), \phi_2(a_2)) = (a_1, a_2)$, i.e., the extension maps, when applied to a product state, must yield the corresponding factors of the product, as partial traces do. Thus, we give the following definition:

Definition 5.1.

An extension map $\phi = (\phi_1, \phi_2)$ will be called *a generalized partial trace* between COM's $\mathbf{C}$, $\mathbf{C}_1$ and $\mathbf{C}_2$ if it satisfies

- ϕ_1 and ϕ_2 are surjective morphisms between $\Omega_{\mathbf{C}}$, $\Omega_{\mathbf{C}_1}$, and $\Omega_{\mathbf{C}_2}$.
- For any product state $a = a_1 \otimes a_2$, $\phi(a) = (a_1, a_2)$.

and this is how the notion of marginal state defined in 2 can be recovered using extensions maps, a much more general notion, in the sense that a particular extension ϕ needs not to be a generalized partial trace as defined above.

If we want an analogue of Λ (definition IV.3), we must demand additional requirements as well. We will denote the sets of convex subsets of $\Omega_{\mathbf{C}}$ and $\Omega_{\mathbf{C}_i}$ $(i = 1,2)$ by $\mathscr{L}_{\mathbf{C}}$ and $\mathscr{L}_{\mathbf{C}_i}$, respectively. We are looking for a map Ψ with the following property. Once the extension maps ϕ_i are fixed, Ψ should map any pair of non-empty convex subsets (C_1, C_2) of $\mathscr{L}_{\mathbf{C}_1} \times \mathscr{L}_{\mathbf{C}_2}$ into a non-empty convex subset C of $\mathbf{C}$ with the following *compatibility property*: for any $c \in C := \Psi(C_1, C_2)$, the extension maps must satisfy $\phi_1(c) \in C_1$ and $\phi_2(c) \in C_2$. This condition means that the image of (C_1, C_2) under the map Ψ is compatible with the sub-states assigned by the extension maps ϕ_1 and ϕ_2.

As the maps ϕ_i are morphisms, it is possible to use them to define canonically induced functions on convex subsets, and then to map convex subsets of $\Omega_{\mathbf{C}}$ into convex subsets of $\Omega_{\mathbf{C}_i}$, i.e., between $\mathscr{L}_{\mathbf{C}}$ and $\mathscr{L}_{\mathbf{C}_i}$ (there is an analogy with the earlier language involving τ_i's and partial traces: we can make similar definitions as those of Equations (5) and (6)). With some abuse of notation we will keep calling these maps $\phi_i' s$, without undue harm. Summing up, we will use the following definition:

Definition 5.2.
A triad $\mathbf{C}$, $\mathbf{C}_1$, and $\mathbf{C}_2$ will be called a *compound system endowed with a pre-informational invariance-structure* if

1. There exist morphisms ϕ_1 and ϕ_2 such that $\mathbf{C}$ is an extension of $\mathbf{C}_1$ and $\mathbf{C}_2$.
2. There exists also a map $\Psi : \mathscr{L}_{\mathbf{C}_1} \times \mathscr{L}_{\mathbf{C}_2} \to \mathscr{L}_{\mathbf{C}}$ which maps a pair of non-empty convex subsets $(C_1, C_2) \in \mathscr{L}_{\mathbf{C}_1} \times \mathscr{L}_{\mathbf{C}_2}$ into a nonempty convex subset $C \in \mathscr{L}_{\mathbf{C}}$, such that for every $c \in C$, $\phi(c) = (\phi_1(c), \phi_2(c)) \in C_1 \times C_2$.

Notice (again) that the morphisms ϕ_i may not be, necessarily, generalized partial traces. Most physical systems of interest satisfy these requirements. As we shall see below, all essential features of entanglement can be recovered using these canonical maps between state spaces.

The function Λ (defined in 4.3) can be naturally generalized to an arbitrary compound system as follows. Given operational models $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, let $\mathscr{L}_{\mathbf{A}}$, $\mathscr{L}_{\mathbf{B}}$, and $\mathscr{L}_{\mathbf{C}}$ be the sets of convex subsets of $\Omega_{\mathbf{A}}$ $\Omega_{\mathbf{B}}$, and $\Omega_{\mathbf{C}}$, respectively, one defines

Definition 5.3.

$$\widetilde{\Lambda} : \mathscr{L}_{\Omega_A} \times \mathscr{L}_{\Omega_B} \longrightarrow \mathscr{L}_{\Omega_C}$$

$$\widetilde{\Lambda}(C_1, C_2) \mapsto Conv(C_1 \otimes C_2).$$

where $C_1 \otimes C_2$ is defined as in 4.2 and $Conv(\ldots)$ stands again for convex closure. It is easy to check that the function defined by 5.3 represents a particular case of a function of the type Ψ (Definition 5.2). Notice that the functions Ψ may include more general examples, i.e, there are several forms of going up from the subsystems to the system. For example, we may take

$$\Psi(C_1, C_2) = \phi_1^{-1}(C_1) \cap \phi_2^{-1}(C_2), \tag{15}$$

(which in the quantum realm would correspond to $\Psi(C_1,C_2) = \mathrm{tr}_1^{-1}(C_1)\cap\mathrm{tr}_2^{-1}(C_2)$). If $C_1 = \{\rho_1\}$ $C_2 = \{\rho_2\}$, the function Ψ thus defined yields a convex set of states which may be global ones, compatible with given reduced states ρ_1 and ρ_2. It should also be clear that a function Ψ different from $\widetilde{\Lambda}$ will arise in a model in which the extension contains a third system (apart from $\mathbf{C}_1$ and $\mathbf{C}_2$).

Thus, we see that the definitions involved in 5.2 are much more general than partial traces and the Λ-map. In this sense, any new construction that we define below which uses such functions, contains the usual examples as particular cases.

Before going on, remark that the constructions presented here represent a general setting for COM's. In this setting, systems are represented as COM's with a given geometry and the theory may depend critically on the specific choice of the maps ϕ and Ψ. This choice may represent i) a structural feature of the theory, as is, for example, the case of partial traces in *QM* (which link states of the system with states of the subsystems), or ii) a theoretical aspect that we want investigate in some detail (as for example, the problem of which global states are compatible with two given reduced states of the subsystems mentioned above). Once these maps and the geometry of the convex sets of states (and observables) are specified, the formal setting is ready for defining "entanglement", informational invariance, and entanglement measures.

5.3 Generalized Entanglement

The extension $\widetilde{\Lambda}$ of the function Λ to arbitrary statistical models, together with the notion of generalized partial traces, allow for the extension of the notions of informational invariance and CSS to any COM

Definition 5.4.
A convex subset C of the set of states Ω of a compound statistical system $\mathbf{C}$ consisting of $\mathbf{C}_1 - \mathbf{C}_2$ and endowed with i) a generalized partial trace ϕ and ii) the up-function $\widetilde{\Lambda}$ will be called a CSS if it satisfies

$$\widetilde{\Lambda}\circ\phi(C) = C. \tag{16}$$

For finite dimension, using Carathéodory's theorem it is also possible to show that if ϕ is a generalized partial trace, a state ρ of an arbitrary physical system may be appropriately called separable, in the sense of definition 3.1, if and only if there exists a CSS C (e.g., such that $\widetilde{\Lambda}\circ\phi(C) = C$) such that $\rho \in C$. The demonstration of this fact is analogous to that of 4.1 [7]. Note that in order that an equivalence with definition 3.1 may hold we must use $\widetilde{\Lambda}\circ\phi$ in the definition of informational invariance (and not the more general $\Psi\circ\phi$) with ϕ a generalized partial trace. With these constructions at hand, let us restrict ourselves, for the sake of simplicity, to compound systems with only two subsystems and *look for a generalization of the entanglement and separability notions*.

It should now be clear that the analogues of the maps Λ and τ are $\widetilde{\Lambda}$ and ϕ, respectively. An important remark needs to be stated at this point. If we have a classical compound $\mathbf{C}$ system, with subsystems $\mathbf{C}_1$ and $\mathbf{C}_2$, then it is easy to show that *the whole set of states Ω is an informational invariant*. This means that we have the following proposition

Proposition 5.1.
If a system $\mathbf{C}$ with state space Ω, formed by subsystems $\mathbf{C}_1$ and $\mathbf{C}_2$ is classical, then $\widetilde{\Lambda}\circ\phi(\Omega) = \Omega$.

This proposition allows us to characterize classicality as a special case of informational invariance.

Note that informational invariance does not imply classicality: the state space could be a CSS but not a simplex.

Any system for which its state space is not information-invariant will exhibit entanglement.

In general, it will be reasonable to define the set of separable states as the largest informational invariant subset. In particular, if separability is defined as in 3.1, any state ω which does not belongs to this maximally invariant subset (which is the set of separable states as defined in 3.1), will satisfy $\widetilde{\Lambda}\circ\tau(\omega)\neq\omega$. But it is important to remark that a more general notion of non separability will be given by the condition $\Psi\circ\tau(\omega)\neq\omega$.

Thus, given a system which is an extension of two other systems, an alternative definition/axiomatization of an entanglement structure can be given by imposing conditions on the maps ϕ and Ψ as follows:

Definition 5.5.

Given a *two component compound system endowed with a pre-informational invariance structure* $\mathbf{C}$, formed by $\mathbf{C}_1$, and $\mathbf{C}_2$, with up-map Ψ and a down-map ϕ, then

1. A state $c\in\mathbf{C}$ will be called a *non-product state* if $\Psi\circ\phi(\{c\})\neq\{c\}$. Otherwise, it will be called a *product state*.
2. For an *invariant convex subset* C one has $C\in\mathscr{L}_{\mathbf{C}}$, such that $\Psi\circ\phi(C)=C$.
3. If there exist a largest (in the sense of inclusion) invariant subset, we will denote it by $\mathscr{S}(\mathbf{C})$.
4. A *two-components compound system* for which
   - there exists $\mathscr{S}(\mathbf{C})$ and
   - strict inclusion in $\mathbf{C}$ is guaranteed,

   will be said to be an *entanglement operational model*.
5. In an *entanglement operational model* a state c which satisfies $c\notin\mathscr{S}(\mathbf{C})$ will be said to be entangled.

It is clear that using these constructions we can export the quantum entanglement structure to a wide class of COM's, and for that reason, to many new statistical physics' systems. And this is done by imposing conditions on very general notions, such as maps between operational models.

If in the above definition we take Ψ to be $\widetilde{\Lambda}$ and ϕ a generalized partial trace, entanglement is thus defined in terms of informational invariance. It should be clear also that quantum mechanics is the best example for entanglement, and that all states in classical mechanics are separable. Remark that the properties of a two-components system will depend, in a strong sense, on the choice of the functions Ψ and ϕ. These should be selected as the canonical ones, i.e., the ones which are somehow natural for the physics of the problem under study.

Nevertheless, we remark that nothing prevents us from making more general choices for practical purposes. Then, we can also "postulate" a generalized separability criterium (having a different "content" than the one which uses $\widetilde{\Lambda}$) that is not necessarily equivalent to the one of definition 3.1) and contains it as a special case:

Definition 5.6.

A state $c\in\mathbf{C}$ in an *entanglement operational model* is said to be separable iff there exists $C\subseteq\mathscr{S}(\mathbf{C})$ containing c such that $\Psi\circ\phi(C)=C$.

Note that any general definition of the convex invariant subsets can be formulated via the particular choice of the all-important functions ϕ and Ψ. These constructions may be useful to develop and search for generalizations/corrections of/to quantum mechanics and for the study of quantum entanglement in theories of a more general character than quantum mechanics. Our constructions constitute a valid alternative to others that one can find in the literature. An interesting open problem would be that of finding the way in which we can express the violation of Bell's inequalities using our present approach.

5.4 Generalized Entanglement Measures

The constructions erected in previous sections give us a point of view that suggests in clear fashion just how to generalize a certain family of entanglement measures analogous to the Schlienz-Mahler ones [7, 45]. Given that a state c will be entangled iff $\Psi\circ\phi(\{c\})\neq\{c\}$, it is tempting to regard the difference between $\Psi\circ\phi(\{c\})$ and $\{c\}$ as a measure of entanglement. For the simple case in which $\Psi\circ\phi(\rho)$ has only one element (as is the case if $\Psi=\widetilde{\Lambda}$), we define (with some abuse of notation in avoiding the set theoretical "$\{\ldots\}$" symbols):

$$G(\rho):=\|H(\Psi\circ\phi(\rho)-\rho)\|, \tag{17}$$

with H and $\|\ldots\|$ a convenient function and norm, respectively. Thus, our construction includes a generalization of a family of quantitative measures of entanglement for arbitrary statistical models. One of the main advantages of this approach is that it provides a completely geometrical formulation of entanglement measures. For the quantum case, and taking $\Psi=\Lambda$ and $\phi=(\mathrm{tr}_1(\ldots),\mathrm{tr}_2(\ldots))$ the family (17) adopts the form

$$SM(\rho)=\|F(\rho^A\otimes\rho^B-\rho)\| \tag{18}$$

with F and $\|\ldots\|$ a convenient function and norm, respectively. It can be shown that they are computable and if F and $\|\ldots\|$ are suitably chosen, they provide entanglement criteria as strong as the celebrated *Partial Transpose* one (one of the strongest computable ones) [41–44].

Equation 18 may be reexpressed as follows:

Given a state ρ, make the tensor product of its partial traces (i.e., apply the map Π defined in 4.1), compute an specified function of their difference, and take the norm.

6. CONCLUSIONS

We have worked out our generalizations of some important quantum mechanics' features via the "reciprocal" maps

- $$\begin{aligned}\tau:\mathscr{L}_{\mathscr{C}}&\longrightarrow\mathscr{L}_{\mathscr{C}_1}\times\mathscr{L}_{\mathscr{C}_2}\\ C&\mapsto(\tau_1(C),\tau_2(C))\end{aligned} \tag{19}$$

  which generalizes partial traces to convex subsets of $\mathscr{C}$.

- $$\Lambda : \mathcal{L}_{\mathscr{C}_1} \times \mathcal{L}_{\mathscr{C}_2} \longrightarrow \mathcal{L}_{\mathscr{C}}$$
  $$(C_1, C_2) \mapsto Conv(C_1 \otimes C_2),$$
  where $Conv(\cdots)$ stands for *convex hull*. Applying Λ to the particular case of ordinary quantum sets of states of two subsystems ($\mathscr{C}_1$ and $\mathscr{C}_2$), one sees that
  $$\Lambda(\mathscr{C}_1, \mathscr{C}_2) = \mathscr{S}(\mathscr{H}), \tag{20}$$
  the set of all separable states, i.e., for finite dimension, the convex hull of the set of all product states.

We can summarize our results as follows:

- We provided a generalization of some geometrical properties of entanglement to any statistical theory via the COM approach. This is done by generalizing a previously discovered geometrical structure (see [7]). The generalization is achieved by imposing conditions between very general maps defined between convex operational models, enriching the approach presented in [40]. Although there is a standard way in which entanglement may be generalized (provided by definition 3.1), our approach is different and poses the emphasis on the maps mentioned above. Our present framework possess the advantage of being describable in purely geometrical terms. Because of the great generality of the COM approach, these constructions hold for all statistical theories.
- In particular, we presented the extension of the maps Λ and τ (Ψ and ϕ, respectively) to *arbitrary* statistical models. We showed that it is possible to generalize Λ in any COM with the map $\widetilde{\Lambda}$ [Cf. Definition((5.3))].
- The alternative perspective provided by these generalizations allows us to define

  (1) new families of entanglement measures, valid for arbitrary statistical models [Cf. Eq. (17)] which are based on the Schlienz-Mahler one [45], and
  (2) also yields appropriate extensions of the notions of informational invariance and convex separable subsets (CSS) to any arbitrary COM.

APPENDIX: QUANTAL EFFECTS

In modeling probabilistic operational theories one associates to any probabilistic system a triplet (X, Σ, p), where

1. Σ represents the set of states of the system,
2. X is the set of possible measurement outcomes, and
3. $p : X \times \Sigma \mapsto [0,1]$ assigns to each outcome $x \in X$ and state $s \in \Sigma$ a probability $p(x,s)$ of x to occur if the system is in the state s.
4. If we fix s we obtain the mapping $s \mapsto p(\cdot, s)$ from $\Sigma \to [0,1]^X$.

Note that

- This identifies all the states of Σ with maps.

- Considering their closed convex hull, we obtain the set Ω of possible probabilistic mixtures (represented mathematically by convex combinations) of states in Σ.
- In this way one also obtains, for any outcome $x \in X$, an affine evaluation-functional $f_x : \Omega \to [0,1]$, given by $f_x(\alpha) = \alpha(x)$ for all $\alpha \in \Omega$.
- More generally, any affine functional $f : \Omega \to [0,1]$ may be regarded as representing a measurement outcome and thus use $f(\alpha)$ to represent the probability for that outcome in state α.

For the special case of quantum mechanics, the set of all affine functionals so-defined are called effects. They form an algebra (known as the *effect algebra*) and represent generalized measurements (unsharp, as opposed to sharp measures defined by projection valued measures). The specifical form of an effect in quantum mechanics is as follows. A generalized observable or *positive operator valued measure* (POVM) will be represented by a mapping

$$E : B(\mathscr{R}) \to \mathscr{B}(\mathscr{H}) \tag{21a}$$

such that

$$E(\mathscr{R}) = \mathbf{1} \tag{21b}$$

$$E(B) \geq 0, \text{ for any } B \in B(\mathscr{R}) \tag{21c}$$

and for any disjoint family $\{B_j\}$

$$E(\cup_j(B_j)) = \sum_j E(B_j). \tag{21d}$$

The first condition means that E is normalized to unity, the second one that E maps any Borel set B to a positive operator, and the third one that E is σ-additive with respect to the weak operator topology. In this way, a generalized POVM can be used to define a family of affine functionals on the state space $\mathscr{C}$ (which corresponds to Ω in the general probabilistic setting) of quantum mechanics as follows

$$E(B) : \mathscr{C} \to [0,1] \tag{22a}$$

$$\rho \mapsto \mathrm{tr}(E\rho) \tag{22b}$$

Positive operators $E(B)$ which satisfy $0 \leq E \leq \mathbf{1}$ are called effects (which form an *effect algebra*. Let us denote by $\mathrm{E}(\mathscr{H})$ the set of all effects.

Indeed, a POVM is a measure whose values are non-negative self-adjoint operators on a Hilbert space. It is the most general formulation of a measurement in the theory of quantum physics.

A rough analogy would consider that a POVM is to a projective measurement what a density matrix is to a pure state. Density matrices can describe part of a larger system that is in a pure state (purification of quantum state); analogously, POVMs on a physical system can describe the effect of a projective measurement performed on a larger system. Another, slightly different way to define them is as follows:

Let (X,M) be measurable space; i.e., M is a $\sigma-$algebra of subsets of X. A POVM is a function F defined on M whose values are bounded non-negative self-adjoint operators on a Hilbert space $\mathscr{H}$ such that $F(X) = I_H$ (identity) and for every i) $\xi \in \mathscr{H}$ and ii) projector $P = |\psi\rangle\langle\psi|$; $|\psi\rangle \in \mathscr{H}$, $P \to \langle F(P)\xi|\xi\rangle$ is a non-negative countably additive measure on M. This definition should be contrasted with that for the projection-valued measure, which is very similar, except that, in the projection-valued measure, the Fs are required to be projection operators.

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