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Supercloseness on graded meshes for $\mathcal{Q}_1$ finite element approximation of a reaction–diffusion equation[☆]

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ABSTRACT

In this paper we analyze the standard piece-wise bilinear finite element approximation of a model reaction–diffusion problem. We prove supercloseness results when appropriate graded meshes are used. The meshes are those introduced in Durán and Lombardi (2005) [8] but with a stronger restriction on the graduation parameter. As a consequence we obtain almost optimal error estimates in the L^2-norm thus completing the error analysis given in Durán and Lombardi (2005) [8].

1. Introduction

The goal of this paper is to prove supercloseness results for the standard $\mathcal{Q}_1$ finite element approximation of a reaction–diffusion model problem when appropriate graded meshes are used. We consider the problem

$$\begin{aligned} -\varepsilon^2 \Delta u + u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where $\Omega = (0,1)^2$ and ε is a small positive parameter.

It is well known that standard finite element methods for singularly perturbed problems produce very poor results when uniform or quasi-uniform meshes are used unless they are sufficiently refined. Consequently, this kind of meshes are not useful in practical applications, and therefore, several alternatives of appropriate adapted meshes have been considered in many papers. In general, adapted meshes should be obtained by some a posteriori error control. However, in some particular cases where some information on the behavior of the solution is known, it is possible to design a priori well adapted meshes. The analysis for these simple problems helps to understand the behavior of the methods and many papers have been dedicated to obtain error estimates for different types of adapted meshes for problems with boundary layers. The best known meshes for this type of problems are the Shishkin ones (see for example [1–5]). Other well known meshes are the Bakhvalov ones (see for example [6,7,4]).

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The goal of the theoretical analysis is to show that, if the meshes are appropriately chosen, the error behaves, with respect to the number of nodes, like in the case of a problem with a smooth solution.

In [8–10] we have considered the use of graded meshes for singularly perturbed model problems. In the reaction–diffusion case we obtained in [8] an almost optimal order error estimate with a constant independent of the singular perturbation parameter ε. To recall precisely the result proved in that paper let us introduce the ε-norm

$$\|v\|_\varepsilon^2 = \varepsilon^2 \|\nabla v\|_{L^2}^2 + \|v\|_{L^2}^2,$$

then, if adequate graded meshes are used,

$$\|u - u_h\|_\varepsilon \leq C \frac{\log N}{\sqrt{N}}$$

where u_h is the finite element solution, h is a positive parameter related with the definition of the meshes, and N is the number of nodes. Here and in the rest of the paper C denotes a generic constant independent of ε and N. Observe that, up to the logarithmic factor, the order in terms of N is the same as that obtained with uniform meshes for a problem with a smooth solution.

On the other hand, if the problem involves also a convection term, we proved in [9] a similar estimate, namely,

$$\|u - u_h\|_\varepsilon \leq C \frac{\log^2(1/\varepsilon)}{\sqrt{N}},$$

but with a different kind of graded meshes.

The main difference between the two type of meshes is that those considered in [8] are independent of ε while those in [9] are ε-dependent. To have meshes independent of ε can be of interest to approximate problems with variable diffusion or systems of equations with different values of ε in each of them (such as the problem considered in [7]).

As it is usual in the theory of finite elements, we say that there is supercloseness when the difference between the approximate solution and the Lagrange interpolation of the exact solution is of higher order than the error itself. In the recent paper [10] we have proved supercloseness in the ε-norm for the case with convection. The main theorem in that paper states that, for the graded meshes introduced in [9], we have

$$\|u_h - \Pi u\|_\varepsilon \leq C \frac{\log^5(1/\varepsilon)}{N}$$

where Πu is the Lagrange interpolation of u. Similar results can be obtained in the reaction–diffusion case using graded meshes defined analogously to those in [9], indeed, this has been done in [11].

In view of these results it is natural to ask whether similar supercloseness estimates are valid for the ε-independent graded meshes considered in [8]. In this paper we give a positive answer to this question. Precisely, we prove that, if u_h is the standard $\mathcal{Q}_1$ finite element approximation to the solution u of problem (1.1) using the graded meshes introduced [8], and Πu is the Lagrange interpolation of u, then

$$\|u_h - \Pi u\|_\varepsilon \leq C \log(1/\varepsilon)^{\frac{1}{2}} \frac{(\log N)^2}{N}.$$

To obtain this result we need to prove some new weighted a priori estimates for the solution of problem (1.1). Also, we will make use of some weighted Poincaré type inequalities. Although these inequalities can be proved by known arguments, it is not easy to find them in the literature, and therefore, we will include proofs of them.

We will assume $f \in \mathcal{C}^2([0,1]^2)$ and that it satisfies the compatibility conditions

$$f(0,0) = f(1,0) = f(0,1) = f(1,1) = 0.$$

It is known that under these hypotheses $u \in \mathcal{C}^4(\Omega) \cap \mathcal{C}^2(\overline{\Omega})$. We have the following pointwise estimates for the solution u of problem (1.1) (see [12, Lemma 4.1]): if $0 \leq k \leq 4$ then

$$\left| \frac{\partial^k u}{\partial x_1^k}(x_1, x_2) \right| \leq C \left(1 + \varepsilon^{-k} e^{-x_1/\varepsilon} + \varepsilon^{-k} e^{-(1-x_1)/\varepsilon}\right) \tag{1.2}$$

$$\left| \frac{\partial^k u}{\partial x_2^k}(x_1, x_2) \right| \leq C \left(1 + \varepsilon^{-k} e^{-x_2/\varepsilon} + \varepsilon^{-k} e^{-(1-x_2)/\varepsilon}\right). \tag{1.3}$$

Given a rectangle R, $\mathcal{P}_k(R)$ and $\mathcal{Q}_k(R)$ denote the spaces of polynomials on R of total degree less than or equal k and of degree less than or equal to k in each variable, respectively. We denote with S the reference element $[0, 1]^2$.

The plan of the paper is as follows. In Section 2 we present some auxiliary results that we need in our error analysis. In Section 3 we state the weak formulation and a priori estimates for the exact solution. In Section 4 we introduce the graded meshes and some weighted interpolation results. Finally, in Section 5 we present some numerical results.

2. Auxiliary results

In this section we prove some results in weighted norms that will be key tools in our error analysis. Throughout this section we denote by S the reference element $S = [0, 1]^2$.

Lemma 2.1. *Let ℓ be one of the horizontal edges of the reference element S. Given $v \in H^1(S)$ we have, for $0 \le \alpha < 1/2$,*

$$\|v\|_{L^1(\ell)} \le C \left\{ \|v\|_{L^2(S)} + \frac{1}{(1-2\alpha)^{1/2}} \left\| x_1^\alpha \frac{\partial v}{\partial x_2} \right\|_{L^2(S)} \right\}. \tag{2.1}$$

Proof. Assume that ℓ is the edge contained in $x_2 = 0$ (the other case is, of course, analogous). We have

$$v(x_1, 0) - v(x_1, x_2) = -\int_0^{x_2} \frac{\partial v}{\partial x_2}(x_1, t)dt.$$

By integrating on $[0, 1]$ and using Cauchy–Schwarz inequality (and multiplying and dividing by x_1^α with $\alpha < 1/2$), we obtain

$$\int_0^1 |v(x_1, 0)|dx_1 \le \int_0^1 |v(x_1, x_2)|dx_1 + \int_0^1 \int_0^1 x_1^{-\alpha} x_1^\alpha \left| \frac{\partial v}{\partial x_2}(x_1, t) \right| dtdx_1.$$

Now, by integrating in the variable x_2 on $[0, 1]$, taking into account that the left hand side does not depend on x_2, we have

$$\int_0^1 |v(x_1, 0)|dx_1 \le \int_0^1 \int_0^1 |v(x_1, x_2)|dx_1\, dx_2 + \int_0^1 \int_0^1 x_1^{-\alpha} x_1^\alpha \left| \frac{\partial v}{\partial x_2}(x_1, t) \right| dtdx_1.$$

By using Cauchy–Schwarz inequality we obtain (2.1). □

Our next result is on polynomial approximation. The proof uses the well known argument based on averaged Taylor polynomials and an appropriate weighted Poincaré inequality. The proof of this inequality, in the following lemma, uses an argument given in a more general context in [13]. Actually, this argument was generalized by the authors of [13], in an unpublished paper, to prove estimates of type (2.4).

We will make use of the Hardy–Littlewood maximal function defined, for $g \in L^1_{\text{loc}}(\mathbb{R}^2)$, as

$$Mg(x) = \sup_{r>0} \frac{1}{|B(x, r)|} \int_{B(x,r)} |g(y)|\, dy.$$

It is a classic result (see for example [14]) that there exists a constant C such that

$$\|Mg\|_{L^2(\mathbb{R}^2)} \le C\|g\|_{L^2(\mathbb{R}^2)}. \tag{2.2}$$

We will also need the following result which can be found, for example, in [15]. There exists a constant C such that, for any $\delta > 0$ and any $g \in L^1_{\text{loc}}(\mathbb{R}^2)$,

$$\int_{|x-y|\le\delta} \frac{|g(y)|}{|x-y|}\, dy \le C\delta Mg(x). \tag{2.3}$$

Given $v \in L^1(S)$ we will use the weighted average defined as $\bar{v} := \int_S v\omega$, where ω is a smooth function with integral equal to one and supported in a ball B such that it expansion by two is contained in S.

Lemma 2.2. *For $v \in H^1(S)$ and $\sigma \ge 0$ we have*

$$\left\| x_1^\sigma (v - \bar{v}) \right\|_{L^2(S)} \le C \left\| x_1^{\sigma+1} \nabla v \right\|_{L^2(S)}. \tag{2.4}$$

Proof. The argument is based on the following representation formula for $v(y) - \bar{v}$. Although this formula is known (see for example [16]) we reproduce its proof for the sake of completeness.

For all $y \in S$ we have

$$v(y) - \bar{v} = \int_S G(x, y) \cdot \nabla v(x)\, dx, \tag{2.5}$$

where

$$G(x, y) = \int_0^1 \frac{(y - x)}{t^3}\, \omega\left(y + \frac{x - y}{t}\right)\, dt.$$

Indeed, for $y \in S$ and $z \in B$ we have,

$$v(y) - v(z) = \int_0^1 (y - z) \cdot \nabla v(y + t(z - y))\, dt,$$

therefore, multiplying by $\omega(z)$ and integrating in z,

$$v(y) - \bar{v} = \int_S \int_0^1 (y - z) \cdot \nabla v(y + t(z - y))\, \omega(z)\, dt\, dz.$$

Then, interchanging the order of integration and making the change of variable $x = y + t(z - y)$ we obtain (2.5).

We will use two properties of $G(x, y)$. The first one (see [17,16]) is that there exists a constant C_1 such that

$$|G(x, y)| \le \frac{C_1}{|x - y|}. \tag{2.6}$$

Indeed, $G(x, y)$ vanishes unless $y + (x - y)/t \in B \subset S$. But, if $y + (x - y)/t \in S$ and $y \in S$, the difference between them is less than or equal the diameter of S, i.e.,

$$\frac{|x - y|}{t} \le \sqrt{2}, \tag{2.7}$$

and then we have,

$$G(x, y) = \int_{|x-y|/\sqrt{2}}^1 \frac{(y - x)}{t^3}\, \omega\left(y + \frac{x - y}{t}\right) dt.$$

Therefore, using again (2.7) we obtain,

$$|G(x, y)| \le \sqrt{2}\|\omega\|_\infty \int_{|x-y|/\sqrt{2}}^1 \frac{1}{t^2}\, dt,$$

and (2.6) follows immediately from this estimate.

The second important property of $G(x, y)$, which is the key point used in [13], is that $G(x, y)$ vanishes unless

$$|x - y| \le C_2 d(x),$$

where $d(x)$ denotes the distance of x to the boundary of S and C_2 is a constant which depends only on the relation between the diameters of S and B.

To proof this property recall that $x = tz + (1 - t)y$ with $z \in B$. Then, using that the ball obtained expanding B by two is contained in S, an elementary geometric argument shows that $d(x) \ge c_3 t$, where c_3 is a positive constant which depends only on the relation between the diameters of S and B. Consequently,

$$|x - y| = t|z - y| \le \frac{\sqrt{2}}{c_3} d(x)$$

as we wanted to show.

In particular, since $d(x) \le x_1$, we have

$$\operatorname{supp} G \subset \{(x, y) \in S : |x - y| \le C_2 x_1\}. \tag{2.8}$$

Define now

$$g(x) = \begin{cases} x_1^\sigma (v(x) - \bar{v}) & \text{if } x \in S \\ 0 & \text{if } x \notin S. \end{cases}$$

Then, using (2.5) we have

$$\begin{aligned} \left\|y_1^\sigma (v - \bar{v})\right\|_{L^2(S)}^2 &= \int_S y_1^\sigma |v(y) - \bar{v}|\, |g(y)|\, dy \\ &\le \int_S \int_S y_1^\sigma |G(x, y)|\, |\nabla v(x)|\, |g(y)|\, dx\, dy. \end{aligned}$$

Therefore, interchanging the order of integration and using (2.6) and (2.8), we obtain

$$\left\|y_1^\sigma (v - \bar{v})\right\|_{L^2(S)}^2 \le C_1 \int_S \left(\int_{|x-y| \le C_2 x_1} \frac{y_1^\sigma |g(y)|}{|x - y|}\, dy \right) |\nabla v(x)|\, dx.$$

But, in the domain of integration of the interior integral we have,

$$y_1 \le |y_1 - x_1| + x_1 \le (C_2 + 1)x_1$$

and therefore,

$$\left\| y_1^\sigma (v - \bar{v}) \right\|_{L^2(S)}^2 \le C \int_S \left(\int_{|x-y| \le C_2 x_1} \frac{|g(y)|}{|x-y|}\, dy \right) x_1^\sigma |\nabla v(x)|\, dx$$

with a constant C depending on C_1, C_2 and σ. Now, (2.4) follows from this inequality using (2.3), the Schwarz inequality, and (2.2). □

Finally, another ingredient of our proofs is the following polynomial approximation result.

Lemma 2.3. *Let $\alpha \ge 0$. For the reference element $S = [0, 1]^2$ and $u \in H^3(S)$, there exists $p \in \mathcal{P}_2(S)$ such that*

$$\left\| x_1^\alpha \frac{\partial^2 (u-p)}{\partial x_1^2} \right\|_{L^2(S)} \le C \left\{ \left\| x_1^{\alpha+1} \frac{\partial^3 u}{\partial x_1^3} \right\|_{L^2(S)} + \left\| x_1^{\alpha+1} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\| \right\}_{L^2(S)} \tag{2.9}$$

$$\left\| x_1^\alpha \frac{\partial^2 (u-p)}{\partial x_1 \partial x_2} \right\|_{L^2(S)} \le C \left\{ \left\| x_1^{\alpha+1} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(S)} + \left\| x_1^{\alpha+1} \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\| \right\}_{L^2(S)}. \tag{2.10}$$

Consequently, for a general rectangle $R = [a_1, b_1] \times [a_2, b_2]$ we have

$$\left\| (x_1 - a_1)^\alpha \frac{\partial^2 (u-p)}{\partial x_1^2} \right\|_{L^2(R)} \le C \left\{ \left\| (x_1 - a_1)^{\alpha+1} \frac{\partial^3 u}{\partial x_1^3} \right\|_{L^2(R)} + h^{-1} k \left\| (x_1 - a_1)^{\alpha+1} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(R)} \right\} \tag{2.11}$$

$$\left\| (x_1 - a_1)^\alpha \frac{\partial^2 (u-p)}{\partial x_1 \partial x_2} \right\|_{L^2(R)} \le C \left\{ \left\| (x_1 - a_1)^{\alpha+1} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(R)} + h^{-1} k \left\| (x_1 - a_1)^{\alpha+1} \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L^2(R)} \right\} \tag{2.12}$$

where $h = b_1 - a_1$ and $k = b_2 - a_2$.

Proof. Let $p \in \mathcal{P}_2(S)$ be the averaged Taylor polynomial of u over S with respect to the same weight function ω used in the previous lemma (see for example [17] for the precise definition). Then, it is known that (recall that $\bar{v} := \int_S v\omega$),

$$\frac{\partial^2 p}{\partial x_1^2} = \overline{\frac{\partial^2 u}{\partial x_1^2}} \quad \text{and} \quad \frac{\partial^2 p}{\partial x_1 \partial x_2} = \overline{\frac{\partial^2 u}{\partial x \partial x_2}}$$

and therefore, it follows from Lemma 2.2 that

$$\begin{aligned} \left\| x_1^\alpha \frac{\partial^2 (u-p)}{\partial x_1^2} \right\|_{L^2(S)} &\le C \left\| x_1^{\alpha+1} \nabla \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(S)} \\ &\le C \left\{ \left\| x_1^{\alpha+1} \frac{\partial^3 u}{\partial x_1^3} \right\|_{L^2(S)} + \left\| x_1^{\alpha+1} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(S)} \right\}. \end{aligned}$$

Making a change of variables, we obtain inequality (2.11) for a general rectangle. Inequalities (2.10) and (2.12) follow in an analogous way. □

3. Weak formulation and a priori estimates

In this section, after introducing the weak formulation of problem (1.1), we show some weighted a priori estimates for the exact solution u. Those estimates are uniform in the perturbation parameter ε, and will allow us to obtain uniform (or almost uniform) finite element approximation results in the next sections.

The standard weak formulation of Problem (1.1) is given by

$$\mathcal{B}(u, v) = \int_\Omega f\, v\, dx \quad \forall v \in H_0^1(\Omega), \tag{3.1}$$

where

$$\mathcal{B}(u, v) = \int_\Omega (\varepsilon^2 \nabla u \cdot \nabla v + uv)\, dx. \tag{3.2}$$

We denote with $\|\cdot\|_\varepsilon$ the norm associated with the bilinear form $\mathcal{B}$, i.e., $\|v\|_\varepsilon^2 := \mathcal{B}(v, v)$. Let $d(t) = \min\{t, 1-t\}$ be the distance between t and the boundary of the interval $[0, 1]$. If $D(t) = t(1-t)$ then we clearly have $D(t) \le d(t) \le 2D(t)$.

Lemma 3.1. *If u is the solution of problem (1.1), there exists a constant C independent of ε such that*

(i) *if* $0 \le k \le 4,\ a+b \ge k-\frac{1}{2},\ a \ge 0,\ b > -\frac{1}{2}$ *then*

$$\varepsilon^a \left\| d(x_1)^b \frac{\partial^k u}{\partial x_1^k} \right\|_{L^2(\Omega)} \le C, \qquad \varepsilon^a \left\| d(x_2)^b \frac{\partial^k u}{\partial x_2^k} \right\|_{L^2(\Omega)} \le C,$$

(ii) *if* $a+b \ge 1,\ a \ge \frac{1}{2},\ b > -\frac{1}{2}$ *then*

$$\varepsilon^a \left\| d(x_2)^b \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\Omega)} \le C, \qquad \varepsilon^a \left\| d(x_1)^b \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(\Omega)} \le C,$$

(iii) *if* $a+b > \frac{7}{4}, b > \frac{1}{2}, c \ge \frac{3}{4}$ *then*

$$\varepsilon^a \left\| d(x_1)^b d(x_2)^c \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(\Omega)} \le C,$$

(iv) *if* $a+c \ge \frac{5}{2}, a \ge \frac{3}{4}, c > \frac{1}{2}$ *then*

$$\varepsilon^a \left\| d(x_2)^c \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L^2(\Omega)} \le C.$$

Proof. It is easy to check that the inequalities for pure derivatives follow from the pointwise estimates (1.2) and (1.3).

To prove the estimates for cross derivatives, the idea is to reduce them to known point-wise estimates for the pure derivatives by integrating by parts as many times as necessary. As an example we prove (iii), the other inequalities can be proved in an analogous way.

Clearly, it is enough to show that

$$\varepsilon^a \left\| D(x_1)^b D(x_2)^c \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(\Omega)} \le C, \quad \text{for } a+b > \frac{7}{4},\ b > \frac{1}{2},\ c \ge \frac{3}{4}.$$

We integrate by parts with respect to the variables x_1 and x_2 separately and use that $D(0) = D(1) = 0$. So for $b > 1/2$ and $c > 0$ we obtain

$$\begin{aligned} \int_0^1 \int_0^1 \left(D(x_1)^b D(x_2)^c \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right)^2 dx_1\, dx_2 = \int_0^1 \int_0^1 \Bigg[& 2b(2b-1) D(x_1)^{2b-2} D'(x_1)^2 \frac{\partial^2 u}{\partial x_1^2} + 2b D(x_1)^{2b-1} \\ & \times D''(x_1) \frac{\partial^2 u}{\partial x_1^2} + 4b D(x_1)^{2b-1} D'(x_1) \frac{\partial^3 u}{\partial x_1^3} + D(x_1)^{2b} \frac{\partial^4 u}{\partial x_1^4} \Bigg] \\ & \times \left[(-2c) D(x_2)^{2c-1} D'(x_2) \frac{\partial u}{\partial x_2} - D(x_2)^{2c} \frac{\partial^2 u}{\partial x_2^2} \right] dx_1\, dx_2. \end{aligned}$$

So, using the Cauchy–Schwarz inequality and that $|D'(t)| \le 1$ and $|D''(t)| = 2$, we obtain

$$\begin{aligned} \varepsilon^{2a} \left\| D(x_1)^b D(x_2)^c \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(\Omega)}^2 \le C \Bigg\{ \varepsilon^{2a} \Bigg[& \left\| D(x_1)^{2b-2} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\Omega)} + \left\| D(x_1)^{2b-1} \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(\Omega)} \\ & + \left\| D(x_1)^{2b-1} \frac{\partial^3 u}{\partial x_1^3} \right\|_{L^2(\Omega)} + \left\| D(x_1)^{2b} \frac{\partial^4 u}{\partial x_1^4} \right\|_{L^2(\Omega)} \Bigg] \Bigg\} \\ & \times \left[\left\| D(x_2)^{2c-1} \frac{\partial u}{\partial x_2} \right\|_{L^2(\Omega)} + \left\| D(x_2)^{2c} \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(\Omega)} \right]. \end{aligned}$$

The first factor, that involves norms of pure derivatives in x_1, is bounded if $2a+2b \ge \frac{7}{2}$, that is, $a+b \ge \frac{7}{4}$. The second factor, involving pure derivatives in x_2, is bounded if $2c \ge \frac{3}{2}$. Therefore we conclude the proof. □

The following anisotropic norms will be used in what follows to estimate the L^2-norm of interpolation errors. For $v : R \to \mathbb{R}$, where R is the rectangle $R = l_1 \times l_2$, define

$$\|v\|_{\infty\times 1,R} := \left\| \|v(x_1,\cdot)\|_{L^1(l_2)} \right\|_{L^\infty(l_1)} \qquad \|v\|_{1\times\infty,R} := \left\| \|v(\cdot,x_2)\|_{L^1(l_1)} \right\|_{L^\infty(l_2)}.$$

The next result is a straightforward consequence of the pointwise estimates (1.2) and (1.3).

Lemma 3.2. *If u is the solution of problem (1.1), there exists a constant C such that*

$$\left\| \frac{\partial u}{\partial x_1} \right\|_{1\times\infty,\Omega} \le C \quad \text{and} \quad \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty\times 1,\Omega} \le C.$$

4. Finite element approximation and error estimates

In [8] an analysis for the approximation of Problem (1.1) by standard bilinear finite elements, using appropriate graded meshes, was developed. Almost optimal order of convergence independent of ε was proved in that paper. The graded meshes used in [8], which depend on a parameter γ, with $\frac{1}{2} < \gamma < 1$, are independent of the perturbation parameter ε.

Our aim is to prove supercloseness for the same approximation considered in [8], i.e., that the difference between the finite element solution and the Lagrange interpolation of the exact solution, in the ε-weighted H^1-norm, is of higher order than the error itself. The constant in our estimate depends only weakly on the singular perturbation parameter. To do that we need further restriction on the parameter γ defining the meshes, in order to assure the validity of uniform interpolation estimates for the solution of (1.1) on the graded meshes. These restrictions are established at the end of the section.

Given a finite-dimensional subspace V_h of $H_0^1(\Omega)$, the finite element approximation $u_h \in V_h$ is given by

$$\mathcal{B}(u_h, v) = \int_\Omega f v\, dx \quad \forall v \in V_h.$$

Let us recall the definition of the graded meshes introduced in [8]. Let $h, \gamma \in (0, 1)$ be fixed. We consider the partition $\{\xi_i\}_{i=0}^M$ of the interval $[0, \frac{1}{2}]$ given by

$$\begin{cases} \xi_0 = 0 \\ \xi_1 = h^{\frac{1}{1-\gamma}} \\ \xi_{i+1} = \xi_i + h\xi_i^\gamma \quad \text{for } 1 \le i \le M-2 \\ \xi_M = 1/2 \end{cases} \tag{4.1}$$

where M is such that $\xi_{M-1} < \frac{1}{2}$ and $\xi_{M-1} + h\xi_{M-1}^\gamma \ge \frac{1}{2}$, and $\xi_M = \frac{1}{2}$. If $\frac{1}{2} - \xi_{M-1} < \xi_{M-1} - \xi_{M-2}$ we modify the definition of ξ_{M-1} by taking $\xi_{M-1} = (\frac{1}{2} + \xi_{M-2})/2$.

By symmetry, we define a partition on the interval $[\frac{1}{2}, 1]$, thus obtaining the partition $\{\xi_i\}_{i=0}^{2M}$ of the interval $[0, 1]$.

For $1 \le i, j \le 2M$ let $R_{ij} = [\xi_{i-1}, \xi_i] \times [\xi_{j-1}, \xi_j]$. Then the graded mesh is $\mathcal{T}_h = \{R_{ij}\}_{i,j=1}^{2M}$ in $\Omega = [0, 1]^2$. Also we set $h_i = \xi_i - \xi_{i-1}$.

Then, we have the finite-dimensional subspace

$$V_h = \left\{ v \in H_0^1(\Omega) : v|_{R_{ij}} \in \mathcal{Q}_1(R_{ij}),\ 1 \le i, j \le 2M \right\}.$$

Given a continuous function u, we introduce $\Pi u \in V_h$ its Lagrange interpolation. We have dropped the dependence on h in the notation Πu to simplify notation.

Our next goal is to obtain interpolation error estimates for the solution u of problem (1.1). It is clear that, by symmetry, it is enough to prove the estimates in $\widetilde{\Omega} = \left[0, \frac{1}{2}\right]^2$.

We will use the splitting of $\widetilde{\Omega}$ as $\widetilde{\Omega} = \Omega_{11} \cup \Omega_{12} \cup \Omega_{21} \cup \Omega_{22}$, where Ω_{11}, Ω_{12}, Ω_{21} and Ω_{22} are the closed sets with disjoint interiors defined by

$$\begin{aligned} \Omega_{11} &= R_{11} \\ \Omega_{12} &= \bigcup \{R_{1j}, j \ge 2\} \\ \Omega_{21} &= \bigcup \{R_{i1}, i \ge 2\} \\ \Omega_{22} &= \bigcup \{R_{ij}, i, j \ge 2\}. \end{aligned}$$

Given a rectangle R, denote by v_I the function in $\mathcal{Q}_1(R)$ which coincides with v on the vertices of R. So we have

$$\Pi v|_{R_{ij}} = (v|_{R_{ij}})_I, \quad 1 \le i, j \le 2M.$$

4.1. Interpolation error estimates

Lemma 4.1. *Let $S = [0, 1]^2$ be the reference element and $\alpha > -\frac{1}{2}$. Then for all $u \in H^2(S)$, we have*

$$\left| \frac{\partial u_I}{\partial x_1} \right| \le \left\| \frac{\partial u}{\partial x_1} \right\|_{1\times\infty,S}, \qquad \left| \frac{\partial u_I}{\partial x_2} \right| \le \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty\times 1,S} \tag{4.2}$$

$$\left\| x_1^\alpha \frac{\partial u_I}{\partial x_1} \right\|_{L^2(S)} \le \frac{1}{1+2\alpha} \left\| \frac{\partial u}{\partial x_1} \right\|_{1\times\infty,S}, \qquad \left\| x_2^\alpha \frac{\partial u_I}{\partial x_2} \right\|_{L^2(S)} \le \frac{1}{1+2\alpha} \left\| \frac{\partial u}{\partial x_2} \right\|_{\infty\times 1,S}. \tag{4.3}$$

Proof. Since u and u_I agrees at the vertices, we have

$$\begin{aligned}\frac{\partial u_I}{\partial x_1} &= x_2[u(1,1)-u(0,1)]+(1-x_2)[u(1,0)-u(0,0)]\\ &= x_2\int_0^1 \frac{\partial u}{\partial x_1}(t,1)dt+(1-x_2)\int_0^1 \frac{\partial u}{\partial x_1}(t,0)dt.\end{aligned}$$

Then

$$\begin{aligned}\left|\frac{\partial u_I}{\partial x_1}\right| &\le x_2\int_0^1 \left|\frac{\partial u}{\partial x_1}(t,1)\right|dt+(1-x_2)\int_0^1 \left|\frac{\partial u}{\partial x_1}(t,0)\right|dt\\ &\le x_2\left\|\frac{\partial u}{\partial x_1}(\cdot,1)\right\|_{L^1(l_1)}+(1-x_2)\left\|\frac{\partial u}{\partial x_1}(\cdot,0)\right\|_{L^1(l_1)}\\ &\le \left\|\left\|\frac{\partial u}{\partial x_1}(\cdot,x_2)\right\|_{L^1(l_1)}\right\|_{L^\infty(l_2)}\\ &\le \left\|\frac{\partial u}{\partial x_1}\right\|_{1\times\infty,S},\end{aligned}$$

and therefore,

$$\begin{aligned}\left\|x_1^\alpha\frac{\partial u_I}{\partial x_1}\right\|^2_{L^2(S)} &= \int_0^1\int_0^1 x_1^{2\alpha}\left|\frac{\partial u_I}{\partial x_1}\right|^2 dx_1\,dx_2\\ &\le \int_0^1\int_0^1 x_1^{2\alpha}\left\|\frac{\partial u}{\partial x_1}\right\|^2_{1\times\infty,S} dx_1\,dx_2\\ &\le \frac{1}{1+2\alpha}\left\|\frac{\partial u}{\partial x_1}\right\|^2_{1\times\infty,S}.\end{aligned}$$

Similar arguments prove the remaining inequalities. □

We have the following interpolation error estimates.

Lemma 4.2. *Let l be one of the vertical edges of the reference element S and $u\in H^1(S)$. Then, for $0\le\alpha<1/2$, it holds*

$$\|u-u_I\|^2_{L^2(S)}\le\frac{C}{1-2\alpha}\left\{\left\|x_1^\alpha\frac{\partial u}{\partial x_1}\right\|^2_{L^2(S)}+\left\|x_1^\alpha\frac{\partial u_I}{\partial x_1}\right\|^2_{L^2(S)}+\|u-u_I\|^2_{L^2(l)}\right\}. \quad (4.4)$$

In particular, if u vanishes on l, it holds

$$\|u-u_I\|^2_{L^2(S)}\le\frac{C}{1-2\alpha}\left\{\left\|x_1^\alpha\frac{\partial u}{\partial x_1}\right\|^2_{L^2(S)}+\left\|x_1^\alpha\frac{\partial u_I}{\partial x_1}\right\|^2_{L^2(S)}\right\}. \quad (4.5)$$

For a general rectangle $R_{ij}=[\xi_{i-1},\xi_i]\times[\xi_{j-1},\xi_j]$, if u vanishes on one of its vertical edges, it holds

$$\|u-u_I\|^2_{L^2(R_{ij})}\le\frac{C}{1-2\alpha}h_i^{2-2\alpha}\left\{\left\|(x_1-\xi_{i-1})^\alpha\frac{\partial u}{\partial x_1}\right\|^2_{L^2(R_{ij})}+\left\|(x_1-\xi_{i-1})^\alpha\frac{\partial u_I}{\partial x_1}\right\|^2_{L^2(R_{ij})}\right\}. \quad (4.6)$$

Proof. Suppose that $l=\{(0,x_2)\in\mathbb{R}^2:0\le x_2\le 1\}$ (clearly, the other case can be treated analogously). We have

$$\begin{aligned}\|u-u_I\|^2_{L^2(S)} &= \int_0^1\int_0^1 (u-u_I)^2(x_1,x_2)\,dx_2\,dx_1\\ &\le \int_0^1\int_0^1\left[\int_0^1\left|\frac{\partial}{\partial x_1}(u-u_I)(t,x_2)\right|dt+(u-u_I)(0,x_2)\right]^2 dx_2\,dx_1\\ &\le 2\left\{\int_0^1\int_0^1\left[\int_0^1\left|\frac{\partial}{\partial x_1}(u-u_I)(t,x_2)\right|dt\right]^2+(u-u_I)^2(0,x_2)\,dx_2\,dx_1\right\}\\ &\le 2\int_0^1\int_0^1\left[\int_0^1\left|t^{-\alpha}t^\alpha\frac{\partial}{\partial x_1}(u-u_I)(t,x_2)\right|dt\right]^2 dx_2\,dx_1+2\,\|u-u_I\|^2_{L^2(l)}\end{aligned}$$

$$\leq 2\int_0^1\int_0^1\left(\int_0^1 t^{-2\alpha}dt\right)\left(\int_0^1 t^{2\alpha}\left|\frac{\partial}{\partial x_1}(u-u_I)(t,x_2)\right|^2 dt\right)dx_2\,dx_1+2\left\|u-u_I\right\|^2_{L^2(l)}$$
$$\leq \frac{C}{1-2\alpha}\left\|x_1^\alpha\frac{\partial}{\partial x_1}(u-u_I)\right\|^2_{L^2(S)}+2\left\|u-u_I\right\|^2_{L^2(l)},$$

obtaining (4.4). If u vanishes on l, then u_I vanishes on l too, and hence $\|u-u_I\|^2_{L^2(l)}$. So we have inequality (4.5) in this case. Inequality (4.6) follows by scaling arguments. □

Lemma 4.3. *For a general rectangle $R_{ij}=[\xi_{i-1},\xi_i]\times[\xi_{j-1},\xi_j]$ and $0\leq\alpha<\frac{1}{2}$ we have*

$$\left\|\frac{\partial}{\partial x_1}(u-u_I)\right\|_{L^2(R_{ij})}\leq\frac{C}{(1-2\alpha)^{\frac{1}{2}}}\left\{h_i^{1-\alpha}\left\|(x_1-\xi_{i-1})^\alpha\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{ij})}+h_i^{-\alpha}h_j\left\|(x_1-\xi_{i-1})^\alpha\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(R_{ij})}\right\}$$
$$\left\|\frac{\partial}{\partial x_2}(u-u_I)\right\|_{L^2(R_{ij})}\leq\frac{C}{(1-2\alpha)^{\frac{1}{2}}}\left\{h_ih_j^{-\alpha}\left\|(x_2-\xi_{j-1})^\alpha\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{ij})}+h_j^{1-\alpha}\left\|(x_2-\xi_{j-1})^\alpha\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(R_{ij})}\right\}.$$

Proof. First, we consider the reference element $S=[0,1]^2$ and $u\in H^2(S)$. Let $p\in\mathcal{P}_1$ be the averaged Taylor polynomial of u with respect to the weight function ω introduced in Section 2. We have

$$\left\|\frac{\partial}{\partial x_1}(u-u_I)\right\|_{L^2(S)}\leq\left\|\frac{\partial}{\partial x_1}(u-p)\right\|_{L^2(S)}+\left\|\frac{\partial}{\partial x_1}(p-u_I)\right\|_{L^2(S)}. \tag{4.7}$$

The first term in (4.7) can be bounded using Lemma 2.2 with $\sigma=0$. Indeed, we know that $\frac{\partial p}{\partial x_1}=\overline{\frac{\partial u}{\partial x_1}}$, and therefore, it follows from that lemma that

$$\left\|\frac{\partial}{\partial x_1}(u-p)\right\|_{L^2(S)}\leq C\left\|x_1\nabla\left(\frac{\partial u}{\partial x_1}\right)\right\|_{L^2(S)}\leq C\left\|x_1^\alpha\nabla\left(\frac{\partial u}{\partial x_1}\right)\right\|_{L^2(S)}. \tag{4.8}$$

To estimate the second term of (4.7), we define $v=p-u_I$. Since $v\in\mathcal{Q}_1$ we have

$$\left\|\frac{\partial v}{\partial x_1}\right\|^2_{L^2(S)}\leq C\left\{|v(1,0)-v(0,0)|^2+|v(1,1)-v(0,1)|^2\right\}.$$

Now,

$$\begin{aligned}v(1,0)-v(0,0)&=(p-u_I)(1,0)-(p-u_I)(0,0)\\&=(p-u)(1,0)-(p-u)(0,0)\\&=\int_0^1\frac{\partial(p-u)}{\partial x_1}(t,0)\,dt,\end{aligned}$$

and then,

$$|v(1,0)-v(0,0)|\leq\left\|\frac{\partial(p-u)}{\partial x_1}\right\|_{L^1(l)}$$

where $l=\{(x_1,0)\in\mathbb{R}^2:0\leq x_1\leq 1\}$. Now, we apply Lemmas 2.1 and 2.2 (the second one with $\sigma=0$) to obtain

$$\begin{aligned}|v(1,0)-v(0,0)|&\leq C\left\{\left\|\frac{\partial(u-p)}{\partial x_1}\right\|_{L^2(S)}+\frac{1}{(1-2\alpha)^{1/2}}\left\|x_1^\alpha\frac{\partial}{\partial x_2}\frac{\partial(u-p)}{\partial x_1}\right\|_{L^2(S)}\right\}\\&\leq\frac{C}{(1-2\alpha)^{1/2}}\left\{\left\|x_1\frac{\partial^2u}{\partial x_1^2}\right\|_{L^2(S)}+\left\|x_1^\alpha\frac{\partial^2u}{\partial x_1\partial x_2}\right\|_{L^2(S)}\right\}.\end{aligned}$$

An analogous estimate holds for $|v(1,1)-v(0,1)|$, and so we have for the second term of (4.7)

$$\left\|\frac{\partial}{\partial x_1}(p-u_I)\right\|_{L^2(S)}\leq\frac{C}{(1-2\alpha)^{1/2}}\left\{\left\|x_1\frac{\partial^2u}{\partial x_1^2}\right\|_{L^2(S)}+\left\|x_1^\alpha\frac{\partial^2u}{\partial x_1\partial x_2}\right\|_{L^2(S)}\right\}. \tag{4.9}$$

Collecting inequalities (4.8) and (4.9) we have

$$\left\| \frac{\partial}{\partial x_1}(u-u_I) \right\|_{L^2(S)} \le \frac{C}{(1-2\alpha)^{1/2}} \left\{ \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(S)} + \left\| x_1^\alpha \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L^2(S)} \right\}.$$

Then, the first inequality in the statement of the lemma follows by scaling arguments. The second one can be proved analogously. □

Now, we are ready to prove the main result of this section.

Theorem 4.4. *Let u be the solution of problem (1.1), Πu be its Lagrange interpolation and suppose that $\frac{3}{4} \le \gamma < 1$. There exists a constant C, independent of ε and h, such that*

$$\|u - \Pi u\|_{L^2(\Omega)} \le C \log(1/\varepsilon)^{\frac{1}{2}} h^2.$$

Proof. It is enough to obtain the estimate replacing Ω by $\widetilde{\Omega}$. We decompose the error as

$$\|u - \Pi u\|^2_{L^2(\widetilde{\Omega})} = \|u - \Pi u\|^2_{L^2(\Omega_{11} \cup \Omega_{12})} + \|u - \Pi u\|^2_{L^2(\Omega_{21})} + \|u - \Pi u\|^2_{L^2(\Omega_{22})}$$

with Ω_{ij} as in the previous section.

For $R_{ij} \subset \Omega_{11} \cup \Omega_{12}$, since u vanishes on $l_j = \{(0, x_2), \xi_{j-1} \le x_2 \le \xi_j\}$, we use inequality (4.6) of Lemma 4.2 with $i = 1$ and $\xi_{i-1} = 0$ to obtain

$$\|u - u_I\|^2_{L^2(R_{1j})} \le C \frac{h_1^{2-2\alpha}}{1-2\alpha} \left\{ \left\| x_1^\alpha \frac{\partial u}{\partial x_1} \right\|^2_{L^2(R_{1j})} + \left\| x_1^\alpha \frac{\partial u_I}{\partial x_1} \right\|^2_{L^2(R_{1j})} \right\}$$

(recall that $\Pi u|_{R_{ij}} = (u|_{R_{ij}})_I =: u_I$). Since $h_1 = h^{\frac{1}{1-\gamma}}$, multiplying and dividing by ε^β (where β is a constant to be determined later) we have for $j \ge 1$ and $\alpha < 1/2$:

$$\begin{aligned} \|u - u_I\|^2_{L^2(R_{1j})} &\le \frac{C}{1-2\alpha} h^{2\frac{1-\alpha}{1-\gamma}} \left\{ \left\| x_1^\alpha \frac{\partial u}{\partial x_1} \right\|^2_{L^2(R_{1j})} + \left\| x_1^\alpha \frac{\partial u_I}{\partial x_1} \right\|^2_{L^2(R_{1j})} \right\} \\ &\le \frac{C}{1-2\alpha} h^{2\frac{1-\alpha}{1-\gamma}} \varepsilon^{-2\beta} \left\{ \varepsilon^{2\beta} \left\| x_1^\alpha \frac{\partial u}{\partial x_1} \right\|^2_{L^2(R_{1j})} + \varepsilon^{2\beta} \frac{1}{1+2\alpha} \left\| \frac{\partial u}{\partial x_1} \right\|^2_{1\times\infty, R_{1j}} \right\}. \end{aligned} \tag{4.10}$$

Now take $\beta > 0$ and $0 < \alpha < \frac{1}{2}$ as

$$\beta = \frac{1}{2} - \alpha = \frac{1}{\log \frac{1}{\varepsilon}}. \tag{4.11}$$

So

$$\varepsilon^{-\beta} = e, \quad \text{and} \quad \frac{1}{1-2\alpha} = \frac{1}{2}\log(1/\varepsilon),$$

and it follows from $\frac{3}{4} \le \gamma < 1$ that

$$\frac{1-\alpha}{1-\gamma} \ge 2.$$

We also have

$$2\frac{1-\alpha}{1-\gamma} \ge 4, \quad \beta + \alpha = \frac{1}{2}, \quad \text{and} \quad \beta + 2\gamma \ge \frac{3}{2}. \tag{4.12}$$

With this choice of α and β, we know from Lemma 3.1 that the first term inside the brackets in (4.10) is bounded by a constant C. The second term is also bounded in Ω, because of Lemma 3.2. Then, summing over all $R_{ij} \in \Omega_{11} \cup \Omega_{12}$ we have

$$\|u - \Pi u\|^2_{L^2(\Omega_{11} \cup \Omega_{12})} \le C \log(1/\varepsilon) h^4. \tag{4.13}$$

With an analogous argument, we estimate the error for $R_{ij} \in \Omega_{21}$.

For $R_{ij} \in \Omega_{22}$, we use the standard estimate

$$\|u - u_I\|_{L^2(R_{ij})} \le C \left\{ h_i^2 \left\| \frac{\partial^2 u}{\partial x_1^2} \right\|_{L^2(R_{ij})} + h_j^2 \left\| \frac{\partial^2 u}{\partial x_2^2} \right\|_{L^2(R_{ij})} \right\}$$

and the fact that $h_i \le h\xi_{i-1}^{\gamma}$, $h_j \le h\xi_{j-1}^{\gamma}$ over Ω_{22}. Multiplying by $\varepsilon^{\beta}\varepsilon^{-\beta}$, it follows

$$\begin{aligned}\|u-u_I\|_{L^2(R_{ij})} &\le C\left\{h^2\xi_{i-1}^{2\gamma}\left\|\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{ij})}+h^2\xi_{j-1}^{2\gamma}\left\|\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}\\ &\le C\varepsilon^{-\beta}h^2\left\{\varepsilon^{\beta}\left\|x_1^{2\gamma}\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{ij})}+\varepsilon^{\beta}\left\|x_2^{2\gamma}\frac{\partial^2 u}{\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}.\end{aligned}$$

As $\beta+2\gamma \ge \frac{3}{2}$ for $\gamma \ge \frac{3}{4}$, using the weighted inequalities from Lemma 3.1, we obtain

$$\|u-\Pi u\|_{L^2(\Omega_{22})} \le Ch^2. \tag{4.14}$$

Collecting inequalities (4.13) and (4.14) we obtain the desired estimate. □

4.2. Error estimates

Lemma 4.5. *Let u be the solution of (1.1) and suppose that $\frac{3}{4} \le \gamma < 1$. Then, there exists a constant C such that, for any $v \in V_h$,*

$$\left|\varepsilon^2\int_{\Omega}\nabla(u-\Pi u)\cdot\nabla v\,dx_1\,dx_2\right| \le C\log(1/\varepsilon)^{\frac{1}{2}}h^2\,\|v\|_{\varepsilon}.$$

Proof. As in the previous theorem it is enough to prove the estimate in $\widetilde{\Omega}$. We use again the decomposition $\widetilde{\Omega} = \Omega_{11}\cup\Omega_{12}\cup\Omega_{21}\cup\Omega_{22}$. In $\Omega_{11}=R_{11}$ we have

$$\begin{aligned}\left|\varepsilon^2\int_{R_{11}}\frac{\partial(u-u_I)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| &\le C\varepsilon^2\left\|\frac{\partial(u-u_I)}{\partial x_1}\right\|_{L^2(R_{11})}\left\|\frac{\partial v}{\partial x_1}\right\|_{L^2(R_{11})}\\ &\le C\varepsilon\left\|\frac{\partial(u-u_I)}{\partial x_1}\right\|_{L^2(R_{11})}\|v\|_{\varepsilon}.\end{aligned}$$

From Lemma 4.3, using that $h_1 = h^{\frac{1}{1-\gamma}}$, $\xi_0=0$, we have

$$\begin{aligned}\varepsilon^2\left\|\frac{\partial(u-u_I)}{\partial x_1}\right\|_{L^2(R_{11})}^2 &\le \frac{C\varepsilon^2}{1-2\alpha}\left\{h^{2\frac{1-\alpha}{1-\gamma}}\left\|x_1^{\alpha}\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{11})}^2+h^{2\frac{1-\alpha}{1-\gamma}}\left\|x_1^{\alpha}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(R_{11})}^2\right\}\\ &\le \frac{C\,h^{2\frac{1-\alpha}{1-\gamma}}\varepsilon^{-2\beta}}{1-2\alpha}\left\{\varepsilon^{2(1+\beta)}\left\|x_1^{\alpha}\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{11})}^2+\varepsilon^{2(1+\beta)}\left\|x_1^{\alpha}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(R_{11})}^2\right\}.\end{aligned}$$

Choosing α and β as in (4.11), and using the weighted inequalities from Lemma 3.1 we have

$$\begin{aligned}\left|\varepsilon^2\int_{R_{11}}\frac{\partial(u-u_I)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| &\le \frac{C\,h^{\frac{1-\alpha}{1-\gamma}}\varepsilon^{-\beta}}{(1-2\alpha)^{\frac{1}{2}}}\left\{\varepsilon^{1+\beta}\left\|x_1^{\alpha}\frac{\partial^2 u}{\partial x_1^2}\right\|_{L^2(R_{11})}+\varepsilon^{1+\beta}\left\|x_1^{\alpha}\frac{\partial^2 u}{\partial x_1\partial x_2}\right\|_{L^2(R_{11})}\right\}\|v\|_{\varepsilon}\\ &\le C\log(1/\varepsilon)^{\frac{1}{2}}h^2\,\|v\|_{\varepsilon}.\end{aligned} \tag{4.15}$$

Let $R_{ij}\in\Omega_{22}$. We have $h_i \le h\xi_{i-1}^{\gamma}$ and $h_j \le h\xi_{j-1}^{\gamma}$. We use a standard inequality (see for example [18]) and multiply and divide by ε^{β} as before to obtain

$$\begin{aligned}\left|\varepsilon^2\int_{R_{ij}}\frac{\partial(u-u_I)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| &\le C\varepsilon^2\left\{h_i^2\left\|\frac{\partial^3 u}{\partial x_1^3}\right\|_{L^2(R_{ij})}+h_ih_j\left\|\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(R_{ij})}+h_j^2\left\|\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}\left\|\frac{\partial v}{\partial x_1}\right\|_{L^2(R_{ij})}\\ &\le Ch^2\varepsilon^{-\beta}\varepsilon^{1+\beta}\left\{\left\|x_1^{2\gamma}\frac{\partial^3 u}{\partial x_1^3}\right\|_{L^2(R_{ij})}+\left\|x_1^{\gamma}x_2^{\gamma}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(R_{ij})}+\left\|x_2^{2\gamma}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}\|v\|_{\varepsilon}.\end{aligned}$$

Then, choosing β as in (4.11), and taking into account $\gamma \ge \frac{3}{4}$, we have

$$1+\beta+2\gamma \ge \frac{5}{2} \quad\text{and}\quad 1+\beta+\gamma \ge \frac{7}{4},$$

then from Lemma 3.1 we have

$$\left|\varepsilon^2 \int_{\Omega_{22}} \frac{\partial(u-\Pi u)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| \le Ch^2 \|v\|_\varepsilon . \tag{4.16}$$

For the rest of the mesh, we use an argument introduced by Zlamal in [19]. We know that for $p \in \mathcal{P}_2(R_{ij})$ and $v \in \mathcal{Q}_1(R_{ij})$,

$$\int_{R_{ij}} \frac{\partial(p-p_I)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2 = 0.$$

Then, for every $p \in \mathcal{P}_2(R_{ij})$ and $v \in \mathcal{Q}_1(R_{ij})$, we have

$$\begin{aligned}\left|\int_{R_{ij}} \frac{\partial(u-u_I)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| &= \left|\int_{R_{ij}} \frac{\partial[(u-p)-(u-p)_I]}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| \\ &\le \left\|\frac{\partial[(u-p)-(u-p)_I]}{\partial x_1}\right\|_{L^2(R_{ij})}\left\|\frac{\partial v}{\partial x_1}\right\|_{L^2(R_{ij})}.\end{aligned}$$

Now, we use Lemmas 4.3 and 2.3 to obtain, for $0<\alpha<1$,

$$\begin{aligned}\left\|\frac{\partial[(u-p)-(u-p)_I]}{\partial x_1}\right\|^2_{L^2(R_{ij})} &\le \frac{C}{1-2\alpha}\left\{h_i^{2-2\alpha}\left\|(x_1-\xi_{i-1})^\alpha \frac{\partial^2(u-p)}{\partial x_1^2}\right\|^2_{L^2(R_{ij})}\right.\\ &\quad\left.+h_i^{-2\alpha}h_j^2\left\|(x_1-\xi_{i-1})^\alpha\frac{\partial^2(u-p)}{\partial x_1\partial x_2}\right\|^2_{L^2(R_{ij})}\right\}\\ &\le \frac{C}{1-2\alpha}\left\{h_i^{2-2\alpha}\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1^3}\right\|^2_{L^2(R_{ij})}\right.\\ &\quad+h_i^{-2\alpha}h_j^2\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|^2_{L^2(R_{ij})}\\ &\quad\left.+h_i^{-2-2\alpha}h_j^4\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|^2_{L^2(R_{ij})}\right\}.\end{aligned}$$

Then for $R_{ij}\subset\Omega_{12}$ or $R_{ij}\subset\Omega_{21}$ we have

$$\begin{aligned}\left|\int_{R_{ij}} \frac{\partial(u-u_I)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| &\le \frac{C}{(1-2\alpha)^{\frac12}}\left\{h_i^{1-\alpha}\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1^3}\right\|_{L^2(R_{ij})}\right.\\ &\quad+h_i^{-\alpha}h_j\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(R_{ij})}\\ &\quad\left.+h_i^{-1-\alpha}h_j^2\left\|(x_1-\xi_{i-1})^{\alpha+1}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}\left\|\frac{\partial v}{\partial x_1}\right\|_{L^2(R_{ij})}.\end{aligned}$$

For $R_{1j}, j\ge 2$ we have $\xi_{i-1}=0$, and then,

$$\begin{aligned}\left|\varepsilon^2\int_{R_{ij}} \frac{\partial(u-u_I)}{\partial x_1}\frac{\partial v}{\partial x_1}\,dx_1\,dx_2\right| &\le \frac{C\varepsilon^2}{(1-2\alpha)^{\frac12}}\left\{h_1^{1-\alpha}\left\|x_1^{\alpha+1}\frac{\partial^3 u}{\partial x_1^3}\right\|_{L^2(R_{ij})}+h_1^{-\alpha}h_j\left\|x_1^{\alpha+1}\frac{\partial^3 u}{\partial x_1^2\partial x_2}\right\|_{L^2(R_{ij})}\right.\\ &\quad\left.+h_1^{-1-\alpha}h_j^2\left\|x_1^{\alpha+1}\frac{\partial^3 u}{\partial x_1\partial x_2^2}\right\|_{L^2(R_{ij})}\right\}\left\|\frac{\partial v}{\partial x_1}\right\|_{L^2(R_{ij})}.\end{aligned}$$

Now, we analyze each term inside the brackets in the right hand side of the previous inequality. We take α as in (4.11). From the definition of the mesh, we know that $h_1 = h^{\frac{1}{1-\gamma}}$, and then, the first term can be written as $h^{\frac{1-\alpha}{1-\gamma}}\left\|x_1^{\alpha+1}\frac{\partial^3 u}{\partial x_1^3}\right\|_{L^2(R_{ij})}$, where $\frac{1-\alpha}{1-\gamma}\ge 2$. For the second term, since $x_1 = h_1$, we can multiply and divide by h_1^s with $s>0$, use the definition of h_1

and that $h_j \le h\xi_{j-1}^{\gamma}$ to obtain

$$h_1^{-\alpha} h_j \left\| x_1^{\alpha+1} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(R_{ij})} = h_1^{s-\alpha} h_j \left\| \frac{x_1^{\alpha+1}}{h_1^s} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(R_{ij})} \le h^{\frac{s-\alpha}{1-\gamma}} h \left\| x_1^{\alpha+1-s} x_2^{\gamma} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(R_{ij})}.$$

We choose s such that

$$\frac{s-\alpha}{1-\gamma} = 1,$$

that is, $s = \alpha + 1 - \gamma$. Then the second term can be bounded by $h^2 \left\| x_1^{\gamma} x_2^{\gamma} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(R_{ij})}$. For the third term, we use that $x_1 \le h_1$ and $h_j \le h\xi_{j-1}^{\gamma}$ and so

$$h_1^{-1-\alpha} h_j^2 \left\| x_1^{\alpha+1} \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L^2(R_{ij})} \le h^2 \left\| x_2^{2\gamma} \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L^2(R_{ij})}.$$

Collecting all the estimates, we have for $R_{1j}, j \ge 2$, after multiplying and dividing by $\varepsilon^{2\beta}$

$$\left| \varepsilon^2 \int_{R_{ij}} \frac{\partial (u-u_I)}{\partial x_1} \frac{\partial v}{\partial x_1} \, dx_1 \, dx_2 \right| \le \frac{Ch^2 \varepsilon^{-2\beta}}{(1-2\alpha)^{\frac{1}{2}}} \varepsilon^{1+2\beta} \left\{ \left\| x_1^{\alpha+1} \frac{\partial^3 u}{\partial x_1^3} \right\|_{L^2(R_{ij})} + \left\| x_1^{\gamma} x_2^{\gamma} \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L^2(R_{ij})} + \left\| x_2^{2\gamma} \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L^2(R_{ij})} \right\} \|v\|_{\varepsilon}.$$

Then, choosing β as in (4.11) and using Lemma 3.1, we have

$$\left| \varepsilon^2 \int_{\Omega_{12}} \frac{\partial (u-u_I)}{\partial x_1} \frac{\partial v}{\partial x_1} \, dx_1 \, dx_2 \right| \le \frac{C}{(1-2\alpha)^{\frac{1}{2}}} h^2 \|v\|_{\varepsilon}. \tag{4.17}$$

An analogous argument works for $R_{i1}, i \ge 2$, and therefore,

$$\left| \varepsilon^2 \int_{\Omega_{21}} \frac{\partial (u-u_I)}{\partial x_1} \frac{\partial v}{\partial x_1} \, dx_1 \, dx_2 \right| \le \frac{C}{(1-2\alpha)^{\frac{1}{2}}} h^2 \|v\|_{\varepsilon}. \tag{4.18}$$

Collecting inequalities (4.15)–(4.18) we obtain

$$\left| \varepsilon^2 \int_{\tilde{\Omega}} \frac{\partial (u-u_I)}{\partial x_1} \frac{\partial v}{\partial x_1} \, dx_1 \, dx_2 \right| \le C \log(1/\varepsilon)^{\frac{1}{2}} h^2 \|v\|_{\varepsilon}$$

concluding the proof. □

Lemma 4.6. *Let u be the solution of (1.1) and suppose that $\frac{3}{4} \le \gamma < 1$. Then, there exists a constant C such that, for any $v \in V_h$,*

$$\left| \int_{\Omega} (u - \Pi u) v \, dx_1 \, dx_2 \right| \le C \log(1/\varepsilon)^{\frac{1}{2}} h^2 \|v\|_{\varepsilon}.$$

Proof. From Theorem 4.4 we know that $\|u - \Pi u\|_{L^2(\Omega)} \le C \log(1/\varepsilon)^{\frac{1}{2}} h^2$, hence

$$\begin{aligned} \left| \int_{\Omega} (u - \Pi u) v \, dx_1 \, dx_2 \right| &\le C \|u - \Pi u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\le C \log(1/\varepsilon)^{\frac{1}{2}} h^2 \|v\|_{\varepsilon}. \quad \square \end{aligned}$$

We can now state and prove our main result which says that, the ε-norm of the difference between the interpolation of the exact solution u and the finite element approximation u_h, is of higher order than the ε-norm of the error $u - u_h$.

Theorem 4.7. *Let u be the solution of (1.1), $u_h \in V_h$ its finite element approximation and $\Pi u \in V_h$ its Lagrange interpolation. Suppose that $\frac{3}{4} \le \gamma < 1$. Then, there exists a constant C such that,*

$$\|u_h - \Pi u\|_{\varepsilon} \le Ch^2 \log(1/\varepsilon)^{\frac{1}{2}}.$$

Proof. From the error equation $\mathcal{B}(u - u_h, u_h - \Pi u) = 0$, we have

$$\|u_h - \Pi u\|_\varepsilon^2 = \mathcal{B}(u_h - \Pi u, u_h - \Pi u) = \mathcal{B}(u - \Pi u, u_h - \Pi u).$$

But, from Lemmas 4.5 and 4.6, we have

$$\mathcal{B}(u - \Pi u, u_h - \Pi u) \leq C \log(1/\varepsilon)^{\frac{1}{2}} h^2 \|u_h - \Pi u\|_\varepsilon,$$

and therefore the theorem is proved. □

An immediate consequence of the last theorem combined with the interpolation result proved in Theorem 4.4 is the optimal order convergence in the L^2-norm.

Corollary 4.8. *Let u be the solution of (1.1) and $u_h \in V_h$ its finite element approximation. Suppose that $\frac{3}{4} \leq \gamma < 1$. Then, there exists a constant C such that,*

$$\|u - u_h\|_{L^2(\Omega)} \leq C \log(1/\varepsilon)^{\frac{1}{2}} h^2.$$

We end this section by stating the error estimates in terms of the number of nodes. It can be seen (see the proof of Corollary 4.5 in [8]) that there exists a constant C depending on γ such that

$$h \leq C \frac{\log N}{\sqrt{N}}.$$

Corollary 4.9. *Let u be the solution of (1.1), $u_h \in V_h$ its finite element approximation, and Πu its Lagrange interpolation. Suppose that $\frac{3}{4} \leq \gamma < 1$. Then, if N is the number of nodes in $\mathcal{T}_h$ then, there exists a constant C such that,*

$$\|u_h - \Pi u\|_\varepsilon \leq C \log(1/\varepsilon)^{\frac{1}{2}} \frac{(\log N)^2}{N}$$

and

$$\|u - u_h\|_{L^2(\Omega)} \leq C \log(1/\varepsilon)^{\frac{1}{2}} \frac{(\log N)^2}{N}.$$

Proof. The results follow from Theorem 4.7, Corollary 4.8 and the estimate

$$h \leq C \frac{\log(N)}{\sqrt{N}}$$

which was proved in [8, Corollary 4.5]. □

5. Numerical experiments

We end the paper with some numerical results. We consider the problem

$$-\varepsilon^2 \Delta u + u = f$$

where

$$f(x_1, x_2) = (-2) \frac{1 - e^{\frac{-1}{\sqrt{2}\varepsilon}}}{1 - e^{\frac{-\sqrt{2}}{\varepsilon}}} \left(e^{-\frac{x_1}{\sqrt{2}\varepsilon}} + e^{-\frac{(1-x_1)}{\sqrt{2}\varepsilon}} + e^{-\frac{x_2}{\sqrt{2}\varepsilon}} + e^{-\frac{(1-x_2)}{\sqrt{2}\varepsilon}} \right) + 4.$$

Calling

$$u_0(t) = (-2) \frac{1 - e^{\frac{-1}{\sqrt{2}\varepsilon}}}{1 - e^{\frac{-\sqrt{2}}{\varepsilon}}} \left(e^{-\frac{t}{\sqrt{2}\varepsilon}} + e^{-\frac{(1-t)}{\sqrt{2}\varepsilon}} \right) + 2.$$

The exact solution of this equation is

$$u(x_1, x_2) = u_0(x_1) u_0(x_2).$$

In Tables 1 and 2 we present the results for the graduation parameter $\gamma = 0.75$, $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-6}$ respectively. Recall that N denotes the number of nodes. The approximate orders in terms of N given in the tables are computed at each step comparing the errors between two following meshes.

Observe that the orders agree with those predicted by the theory.

Table 1
$\gamma = 0.75, \varepsilon = 10^{-2}$.

N	h	$\|u-u_h\|_{L^2}$	Order	$\|u-u_h\|_\varepsilon$	Order	$\|\Pi u-u_h\|_\varepsilon$	Order
3721	0.11	8.5422e−004	–	4.1474e−002	–	6.0837e−003	–
4489	0.10	7.1497e−004	0.94832	3.8073e−002	0.45612	5.1118e−003	0.92769
5625	0.09	5.8695e−004	0.87463	3.4604e−002	0.42341	4.2104e−003	0.85993
7225	0.08	4.7039e−004	0.88437	3.1067e−002	0.43074	3.3834e−003	0.87349
9409	0.07	3.6558e−004	0.95436	2.7459e−002	0.46743	2.6351e−003	0.94642

Table 2
$\gamma = 0.75, \varepsilon = 10^{-6}$.

N	h	$\|u-u_h\|_{L^2}$	Order	$\|u-u_h\|_\varepsilon$	Order	$\|\Pi u-u_h\|_\varepsilon$	Order
3721	0.11	5.2081e−002	–	8.2358e−002	–	3.2041e−002	–
4489	0.10	4.3042e−002	1.0159	6.8075e−002	1.0150	2.6468e−002	1.0182
5625	0.09	3.4862e−002	0.9344	5.5159e−002	0.9327	2.1416e−002	0.9389
7225	0.08	2.7471e−002	0.9519	4.3507e−002	0.9480	1.6861e−002	0.9554
9409	0.07	2.0353e−002	1.1355	3.2304e−002	1.1273	1.2688e−002	1.0766

Table 3
$\gamma = 0.60, \varepsilon = 10^{-6}$.

N	h	$\|u-u_h\|_{L^2}$	Order	$\|u-u_h\|_\varepsilon$	Order	$\|\Pi u-u_h\|_\varepsilon$	Order
1225	0.11	2.7244e−001	–	4.3076e−001	–	1.6832e−001	–
1521	0.10	2.4187e−001	0.54991	3.8243e−001	0.54991	1.4938e−001	0.55173
1849	0.09	2.1205e−001	0.67389	3.3528e−001	0.67389	1.3091e−001	0.67563
2401	0.08	1.8303e−001	0.56323	2.8940e−001	0.56323	1.1297e−001	0.56434
3025	0.07	1.5491e−001	0.72217	2.4493e−001	0.72217	9.5585e−002	0.72321

Table 4
$\|\Pi u-u_h\|_\varepsilon$ for both kinds of meshes with 9409 nodes.

ε	Graded mesh	Shishkin mesh
10^{-1}	0.004038859190331	0.003773049703335
10^{-2}	0.002635114829374	0.097598906440701
10^{-3}	0.002326352901515	0.419801187583572
10^{-4}	0.001995679630450	0.909830363131721
10^{-5}	0.003944055357608	0.554766167574668
10^{-6}	0.012687752835709	0.001683543678915

With the next numerical example we want to show that some restriction in the parameter γ is really necessary in order to have supercloseness (recall that for our proofs we needed $\frac{3}{4} \le \gamma < 1$). It is interesting to observe that for almost optimal order convergence the restriction $\gamma \ge 1/2$ was enough (see [8]). In Table 3 we present the results for $\varepsilon = 10^{-6}$ and the graduation given by $\gamma = 0.60$. It is observed that the order is deteriorated, indeed, it is close to 0.5.

Finally, we present some comparisons with the well known Shishkin meshes. An advantage of the graded meshes considered here is that they are independent of the singular perturbation parameter ε, and therefore, the same mesh can be used for different values of ε. This can be of interest, for example, in numerical approximation of systems of equations involving different order diffusion parameters. On the other hand, we have observed in numerical experiments that Shishkin meshes designed for a given value of ε do not give good approximation for larger values of ε. Indeed, this can be seen in Table 4 where we give the values of $\|\Pi u-u_h\|_\varepsilon$ for several values of ε using both kinds of meshes with the same number of nodes. The graded mesh is generated using $\gamma = 0.75$ and the Shishkin one corresponds to $\varepsilon = 10^{-6}$.

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