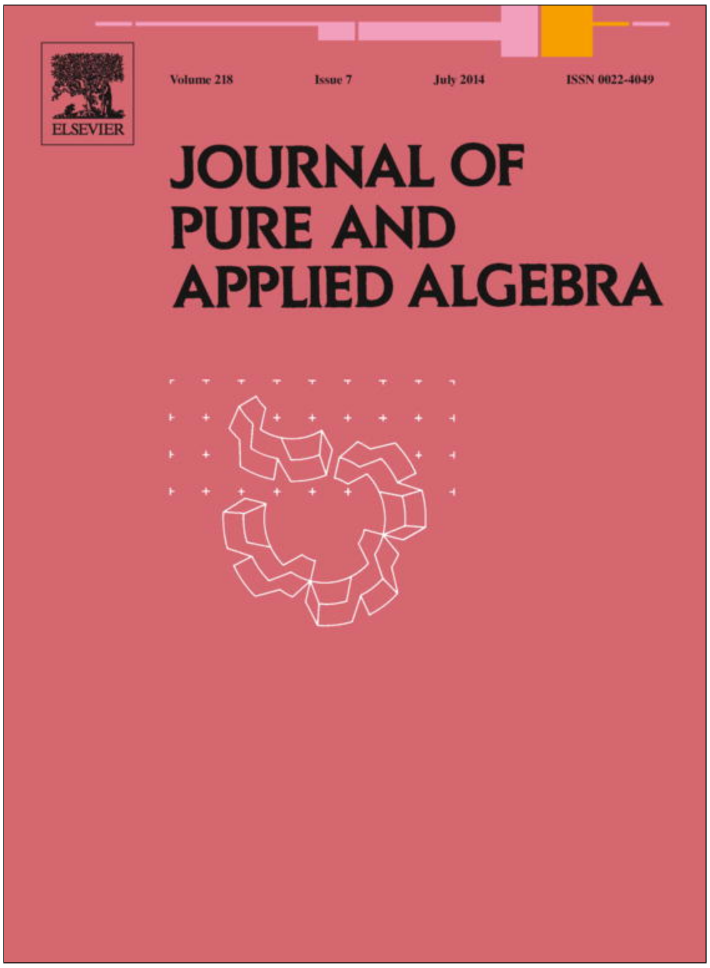

Volume 218
Issue 7
July 2014
ISSN 0022-4049
ELSEVIER
JOURNAL OF PURE AND APPLIED ALGEBRA

Journal of Pure and Applied Algebra 218 (2014) 1224–1263

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Isomorphism conjectures with proper coefficients ☆

Guillermo Cortiñas [a,*], Eugenia Ellis [b]

[a] Dep. Matemática-IMAS, FCEyN-UBA, Ciudad Universitaria Pab 1, 1428 Buenos Aires, Argentina
[b] Centro de Matemática, Facultad de Ciencias, Universidad de la República, Iguá 4225, 11400 Montevideo, Uruguay

ARTICLE INFO

Article history:
Received 22 March 2012
Received in revised form 31 July 2013
Available online 14 November 2013
Communicated by C.A. Weibel

MSC:
19D50; 19D55

ABSTRACT

Let G be a group and $\mathcal{F}$ a nonempty family of subgroups of G, closed under conjugation and under subgroups. Also let E be a functor from small $\mathbb{Z}$-linear categories to spectra, and let A be a ring with a G-action. Under mild conditions on E and A one can define an equivariant homology theory $H^G(-, E(A))$ of G-simplicial sets such that $H^G_*(G/H, E(A)) = E(A \rtimes H)$. The strong isomorphism conjecture for the quadruple $(G, \mathcal{F}, E, A)$ asserts that if $X \to Y$ is an equivariant map such that $X^H \to Y^H$ is an equivalence for all $H \in \mathcal{F}$, then

$$H^G(X, E(A)) \to H^G(Y, E(A))$$

is an equivalence. In this paper we introduce an algebraic notion of $(G, \mathcal{F})$-properness for G-rings, modeled on the analogous notion for G-C^*-algebras, and show that the strong $(G, \mathcal{F}, E, P)$ isomorphism conjecture for $(G, \mathcal{F})$-proper P is true in several cases of interest in the algebraic K-theory context.

1. Introduction

Let G be a group; a *family* of subgroups of G is a nonempty family $\mathcal{F}$ closed under conjugation and under taking subgroups. If $\mathcal{F}$ is a family of subgroups of G, then a G-simplicial set X is called a $(G, \mathcal{F})$*-complex* if the stabilizer of every simplex of X is in $\mathcal{F}$. The category of G-simplicial sets can be equipped with a closed model structure where an equivariant map $X \to Y$ is a weak equivalence (resp. a fibration) if $X^H \to Y^H$ is a weak equivalence (resp. a fibration) for every $H \in \mathcal{F}$ (see Section 2); $(G, \mathcal{F})$-complexes are the cofibrant objects in this model structure (Remark 2.5). By a general construction of Davis and Lück (see [9]) any functor E from the category $\mathbb{Z} - \mathrm{Cat}$ of small $\mathbb{Z}$-linear categories to the category Spt of spectra which sends category equivalences to equivalences of spectra gives rise to an equivariant homology theory of G-spaces $X \mapsto H^G(X, E(R))$ for each unital ring R with a G-action (unital G-ring, for short), such that if $H \subset G$ is a subgroup, then

$$H^G_*(G/H, E(H)) = E_*(R \rtimes H) \tag{1.1}$$

is just E_* evaluated at the crossed product. The *strong isomorphism conjecture* for the quadruple $(G, \mathcal{F}, E, R)$ asserts that $H^G(-, E(R))$ sends $(G, \mathcal{F})$-equivalences to weak equivalences of spectra. The strong isomorphism conjecture is equivalent to the assertion that for every G-simplicial set X the map

$$H^G(cX, E(R)) \to H^G(X, E(R)) \tag{1.2}$$

☆ Cortiñas was supported by CONICET; both authors were partially supported by the MathAmSud network U11MATH-05 and by grants UBACyT 20020100100386, and MTM2007-64704 (FEDER funds).

* Corresponding author.
E-mail addresses: gcorti@dm.uba.ar (G. Cortiñas), eugenia@cmat.edu.uy (E. Ellis).
URL: http://mate.dm.uba.ar/~gcorti (G. Cortiñas).

0022-4049/$ – see front matter
http://dx.doi.org/10.1016/j.jpaa.2013.11.016

induced by the $(G,\mathcal{F})$-cofibrant replacement $cX \to X$ is a weak equivalence. The weaker *isomorphism conjecture* is the particular case when X is a point; it asserts that if $\mathcal{E}(G,\mathcal{F}) \overset{\sim}{\twoheadrightarrow} pt$ is the cofibrant replacement then the map

$$H^G\big(\mathcal{E}(G,\mathcal{F}), E(R)\big) \to H^G\big(pt, E(R)\big) \tag{1.3}$$

called the *assembly map*, is an equivalence of spectra. This formulation of the conjecture is equivalent to that of Davis–Lück, [9] which is given in terms of topological spaces (see Proposition 2.3 and paragraph 2.6). One can more generally formulate the conjecture with coefficients in additive categories with a G-action [2], but we shall limit ourselves to rings here.

In this paper we are primarily concerned with the strong isomorphism conjecture for nonconnective algebraic K-theory – denoted K in this paper – homotopy algebraic K-theory KH, and Hochschild and cyclic homology HH and HC. Our main results are outlined in Theorem 1.4 below. First we need to explain the terms "excisive" and "proper" appearing in the theorem. Let E : Rings $\to$ Spt be a functor; we say that a not necessarily unital ring A is E*-excisive* if whenever $A \to R$ is an embedding of A as a two-sided ideal in a unital ring R, the sequence

$$E(A) \to E(R) \to E(R/A)$$

is a homotopy fibration. Unital rings are E-excisive for all functors E considered in Theorem 1.4; thus the theorem remains true if "unital" is substituted for "excisive". By a result of Weibel [34], homotopy algebraic K-theory satisfies excision; this means that every ring is KH-excisive. Wodzicki characterized excision for Hochschild and cyclic homology in terms of H-unitality [35]. By results of Suslin and Wodzicki, a ring A is excisive for rational K-theory if and only if $A \otimes \mathbb{Q}$ is H-unital (see [30] for the if part and [35] for the only if part); K-excisive rings were characterized by Suslin in [29]. Under mild assumptions on E (Standing Assumptions 3.3.2), which are satisfied by all the examples considered in Theorem 1.4, one can make sense of $H^G(-, E(A))$ for not necessarily unital, E-excisive A (see Section 3). The ring $\mathbb{Z}^{(X)}$ of polynomial functions on a locally finite simplicial set X which are supported on a finite simplicial subset, and the ring $C_{\mathrm{comp}}(|X|, \mathbb{F})$ of compactly supported continuous functions with values in $\mathbb{F} = \mathbb{R}, \mathbb{C}$ are unital if and only if X is finite, and are E-excisive for all X and all the functors E of Theorem 1.4; they are $(G,\mathcal{F})$-proper whenever X is a $(G,\mathcal{F})$-complex. In general if X is a locally finite simplicial set with a G-action and A is a G-ring, then A is called *proper* over X if it carries a $\mathbb{Z}^{(X)}$-algebra structure which is compatible with the action of G and satisfies $\mathbb{Z}^{(X)} \cdot A = A$. We say that A is $(G,\mathcal{F})$*-proper* if it is proper over a $(G,\mathcal{F})$-complex.

Theorem 1.4. *Let G be a group, $\mathcal{F}$ a family of subgroups, $E : \mathbb{Z}-\mathrm{Cat} \to \mathrm{Spt}$ a functor, and P an E-excisive, $(G,\mathcal{F})$-proper G-ring. The strong isomorphism conjecture for the quadruple $(G,\mathcal{F},E,P)$ is satisfied in each of the following cases.*

i) $E = KH$.
ii) $E = K$ *and P is proper over a 0-dimensional $(G,\mathcal{F})$-complex.*
iii) $E = K$*, $\mathcal{F}$ contains all the cyclic subgroups of G and P is a $\mathbb{Q}$-algebra.*
iv) $E = K \otimes \mathbb{Q}$ *and $\mathcal{F}$ contains all the cyclic subgroups of G.*

Theorem 13.1.1 proves that part i) of the theorem above holds for any functor $E : \mathbb{Z}-\mathrm{Cat} \to \mathrm{Spt}$ satisfying certain properties, including excision; the fact that KH satisfies them is the subject of Section 5. We prove in Theorem 11.6 that part ii) of the theorem holds for any E satisfying the standing assumptions; that they hold for K-theory is established in Proposition 4.3.1. Parts iii) and iv) are the content of Theorem 13.2.1; their proof uses the fact, established in Proposition 7.6, that cyclic homology satisfies the strong isomorphism conjecture with coefficients in arbitrary H-unital G-rings. The latter proposition generalizes a result of Lück and Reich [21], who proved it for unital rings with trivial G-action.

The concept of properness used in this article is a discrete, algebraic translation of the analogous concept of proper G-C^*-algebra. By a result of Guentner, Higson and Trout, the full C^*-crossed product version of the Baum–Connes conjecture with coefficients holds whenever the coefficient algebra is a proper G-C^*-algebra [12]. This result is a basic fact behind the Dirac-dual Dirac method that was used, for example, in the proof of the Baum–Connes conjecture for a-T-menable groups [13]. It is also at the basis of recent work of Meyer and Nest [22–24] in which the conjecture and the Dirac method are recast in terms of triangulated categories. Theorem 1.4 will be used in [5] to prove the Farrell–Jones conjecture for the K-theory of the group algebra $\mathcal{K}[G]$ with coefficients in the ring $\mathcal{K}$ of compact operators in an infinite dimensional, separable Hilbert space when the group G is a-T-menable in the sense of Gromov. We expect it can similarly be used to prove other instances of the isomorphism conjecture for (homotopy) algebraic K-theory. In the current paper, we use Theorem 1.4 to prove the following theorem, which identifies the assembly map (1.3) as the connecting map in a functorial excision sequence. The latter sequence is a pure algebraic analogue of a construction given in the book by Cuntz, Meyer and Rosenberg in the C^*-algebraic context [8, §5.3].

Theorem 1.5. *Let G be a group and $\mathcal{F}$ a family of subgroups. Then there is a functor which assigns to each G-ring A a G-ring $\mathfrak{F}^\infty A = \mathfrak{F}^\infty(\mathcal{F}, A)$ equipped with an exhaustive filtration by G-ideals $\{\mathfrak{F}^n A : n \geqslant 0\}$, and a natural transformation $A \to \mathfrak{F}^0 A$, which, if E is as in Theorem 1.4 and A is E-excisive, have the following properties.*

i) *The map* $E(A \rtimes G) \to E(\mathfrak{F}^0 A \rtimes G)$ *is an equivalence.*
ii) *The following sequence is a homotopy fibration*

$$E(\mathfrak{F}^0 A \rtimes G) \to E(\mathfrak{F}^\infty A \rtimes G) \to E((\mathfrak{F}^\infty A/\mathfrak{F}^0 A) \rtimes G).$$

In particular there is a map

$$\partial : \Omega E((\mathfrak{F}^\infty A/\mathfrak{F}^0 A) \rtimes G) \to E(\mathfrak{F}^0 A \rtimes G).$$

iii) *There is an equivalence*

$$H^G(\mathcal{E}(G,\mathcal{F}), E(A)) \xrightarrow{\sim} \Omega E((\mathfrak{F}^\infty A/\mathfrak{F}^0 A) \rtimes G)$$

which makes the following diagram commute up to homotopy

$$\begin{array}{ccc} H^G(\mathcal{E}(G,\mathcal{F}), E(A)) & \xrightarrow{\text{Assembly}} & E(A \rtimes G) \\ \downarrow \wr & & \downarrow \wr \\ \Omega E((\mathfrak{F}^\infty A/\mathfrak{F}^0 A) \rtimes G) & \xrightarrow{\partial} & E(\mathfrak{F}^0 A \rtimes G) \end{array}$$

The theorem above holds more generally for any functor satisfying certain hypothesis, listed in Assumptions 3.3.2 and 12.1; see Proposition 12.2.3 and Theorem 12.3.3.

We also prove a number of results about K-excisive and H-unital rings which are needed for the proof of the theorems above; they are summarized in the following theorem.

Theorem 1.6.

i) *If* A *is a* K*-excisive (resp.* H*-unital)* G*-ring, then* $A \rtimes G$ *is* K*-excisive (resp.* H*-unital).*
ii) *Let* $\{A_i\}$ *be a family of rings and let* $A = \bigoplus_i A_i$ *their direct sum, with coordinate-wise product. Then* A *is* K*-excisive (resp.* H*-unital) if and only if each* A_i *is.*
iii) *If* A *and* B *are* K*-excisive rings, and at least one of them is flat as a* $\mathbb{Z}$*-module, then* $A \otimes B$ *is* K*-excisive.*

Part i) of Theorem 1.6 results by combining Propositions A.6.3 and A.6.4. Part ii) follows from Propositions A.4.4 and A.4.6. Part iii) is Proposition A.5.3. The analogue of part iii) for H-unital rings is true without flatness assumptions, and was proved by Suslin and Wodzicki in [30, Theorem 7.10].

The rest of this paper is organized as follows. In Section 2 we formulate the isomorphism conjectures in terms of closed model categories (see [1] for a different homotopy-theoretic formulation). If G is a group, $\mathcal{F}$ a family of subgroups and $\mathcal{C}$ is either the category Top of topological spaces or the category $\mathbb{S}$ of simplicial sets, we introduce closed model structures on the equivariant category $\mathcal{C}^G$ in which an equivariant map $X \to Y$ is a weak equivalence (resp. a fibration) if $X^H \to Y^H$ is one for every $H \in \mathcal{F}$. We show in Proposition 2.3 that the realization and singular functors give a Quillen equivalence between $\mathbb{S}^G$ and Top^G. In Section 3 we give a list of five basic conditions for a functor $E : \mathbb{Z}-\mathrm{Cat} \to \mathrm{Spt}$, Standing Assumptions 3.3.2; most functors E considered in the paper satisfy them. All but one of these conditions refer to needed permanence properties of E-excisive rings; thus they concern only the restriction of E to Rings. The remaining condition is that for all $\mathcal{C} \in \mathbb{Z}-\mathrm{Cat}$ there must be an equivalence

$$E(\mathcal{A}(\mathcal{C})) \xrightarrow{\sim} E(\mathcal{C}). \tag{1.7}$$

Here

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{x,y \in \mathcal{C}} \hom_{\mathcal{C}}(x, y)$$

is the arrow ring. The assignment $\mathcal{C} \to \mathcal{A}(\mathcal{C})$ is functorial only for functors which are injective on objects; likewise the equivalence (1.7) is only required to be natural with respect to such functors. These conditions imply, for example, that E sends naturally equivalent functors to homotopy equivalent maps of spectra (Lemma 3.3.6), and that $H^G(X, E(-))$ maps extensions of E-excisive rings to homotopy fibrations (Proposition 3.3.9). We also discuss a fully functorial construction $\mathbb{Z}-\mathrm{Cat} \to \mathrm{Rings}$, $\mathcal{C} \to \mathcal{R}(\mathcal{C})$, inspired by a similar construction considered by M. Joachim in the C^*-algebra context [16]. It comes with a map $p : \mathcal{R}(\mathcal{C}) \to \mathcal{A}(\mathcal{C})$; we give conditions on E under which $E(p)$ is an equivalence for all $\mathcal{C}$ (Lemma 3.4.3). They apply, for example, when $E = KH$, but fail for $E = K$ (see Example 3.4.2). In Section 4 we present the model for the (nonconnective) K-theory spectrum that we use in this article – essentially borrowed from Pedersen–Weibel's paper [27] – and prove (Proposition 4.3.1) that it satisfies the standing assumptions. For this we need several properties of K-excisive rings which are proved in Appendix A (including those listed as parts i) and ii) of Theorem 1.6). Section 5 concerns Weibel's homotopy

K-theory; the fact that it satisfies the standing assumptions is proved in Proposition 5.5. We also show (Proposition 5.3) that there is a natural equivalence $KH(\mathcal{C}) \to KH(\mathcal{R}(\mathcal{C}))$ ($\mathcal{C} \in \mathbb{Z}-\mathrm{Cat}$). The basic definitions of Hochschild and cyclic homology for rings and $\mathbb{Z}$-linear categories and some of their properties (see Proposition 6.4) are reviewed in Section 6. In Section 7 we establish the strong isomorphism conjecture for the Hochschild and cyclic homology of H-unital rings (Proposition 7.6). In the next section we discuss various Chern characters connecting K-theory with cyclic homology. Of these, the relative character

$$\nu : K^{\mathrm{nil}}(\mathcal{C}) \otimes \mathbb{Q} = \mathrm{hofiber}\big(K(\mathcal{C}) \to KH(\mathcal{C})\big) \to \Omega^{-1}\big|HC(\mathcal{C})\big| \otimes \mathbb{Q}$$

(defined in (8.2.3)) plays a prominent role in the article. Here $|-|$ is the spectrum associated by the Dold–Kan correspondence. We show in Proposition 8.2.4 that its fiber

$$K^{\mathrm{ninf}}(\mathcal{C}) = \mathrm{hofiber}(\nu) \tag{1.8}$$

satisfies the standing assumptions, that in addition it is excisive and that K_*^{ninf} commutes with filtering colimits. Section 9 reviews properties of the ring $\mathbb{Z}^{(X)}$ of finitely supported, integral polynomial functions on a simplicial set X, some of which are analogous to those of the usual C^*-algebra of continuous functions on its geometric realization [7]. For example, $\mathbb{Z}^{(-)}$ is functorial for proper maps, and sends disjoint unions to direct sums (see Section 9.3). Moreover, if X is locally finite, and $Y \subset X$ is a subobject, then the restriction map $\mathbb{Z}^{(X)} \to \mathbb{Z}^{(Y)}$ is onto (Corollary 9.4.2). We also show that if X is locally finite, then $\mathbb{Z}^{(X)}$ is free as an abelian group (see Lemma 9.3.6) and that if E satisfies the standing assumptions then the ring $\mathbb{Z}^{(X)}$ is E-excisive (Proposition 9.5.1). Thus by Theorem 1.6iii), the class of K-excisive rings is closed under tensoring with $\mathbb{Z}^{(X)}$ (Proposition 9.5.3). In Section 10 we consider G-rings which are proper over a $(G,\mathcal{F})$-complex X. We establish discrete analogues of several of the properties of proper C^*-algebras discussed in [12]. For a subgroup $H \subset G$ we introduce the induction functor $\mathrm{Ind}_H^G : H-\mathrm{Rings} \to G-\mathrm{Rings}$ (Section 10.2) and show that it is an equivalence between $H-\mathrm{Rings}$ and the full subcategory of those G-rings which are proper over the 0-dimensional simplicial set G/H (Proposition 10.3.1). Next we give a discrete variant of Green's imprimitivity theorem; we show in Theorem 10.4.5 that there is an isomorphism

$$\mathrm{Ind}_H^G(A) \rtimes G \cong M_{G/H}(A \rtimes H). \tag{1.9}$$

Here $M_{G/H}$ denotes matrices indexed by $G/H \times G/H$ with finitely many nonzero coefficients. Also in this section we consider the restriction functor Res_G^H going from G-rings to H-rings and study the composites $\mathrm{Ind}_H^G \mathrm{Res}_G^H$ and $\mathrm{Res}_G^H \mathrm{Ind}_K^G$ for subgroups $K, H \subset G$ (Lemmas 10.5.1 and 10.5.4). The material in Section 10 is used in the next section to define, for a group G, a subgroup $K \subset G$, a G-simplicial set X, a functor $E : \mathbb{Z}-\mathrm{Cat} \to \mathrm{Spt}$ satisfying the standing assumptions, and a K-ring A, an induction map

$$\mathrm{Ind} : H^K\big(X, E(A)\big) \to H^G\big(X, E\big(\mathrm{Ind}_K^G(A)\big)\big).$$

We show in Proposition 11.3 that the map above is an equivalence; this establishes an algebraic analogue of [12, Proposition 12.9]. Then we use this result to prove part ii) of Theorem 1.4 for any functor satisfying Standing Assumptions 3.3.2; see Theorem 11.6. The latter theorem is applied in Section 12, where Theorem 1.5 is proved for any E satisfying Assumptions 3.3.2 and 12.1 (see Proposition 12.2.3 and Theorem 12.3.3). In Section 13 we begin by proving part i) of Theorem 1.4 for any functor E satisfying excision in addition to the hypothesis of Assumptions 3.3.2 and 12.1 (see Theorem 13.1.1). In particular, it holds when E is the functor K^{ninf} of (1.8). Parts iii) and iv) of Theorem 1.4 are the content of Theorem 13.2.1). The proof uses Proposition 7.6 and Theorem 13.1.1 applied to K^{ninf}. In Appendix A we recall the results of Suslin and Wodzicki on K-excisive and H-unital rings, and establish Theorem 1.6 (see Propositions A.4.4, A.4.6, A.5.3, A.6.3 and A.6.4).

Notation 1.10. If $\mathcal{C}$ is a (small) category, we write $\mathrm{ob}\,\mathcal{C}$ for the (small) set of objects and $\mathrm{ar}\,\mathcal{C}$ for that of arrows. We often consider a set X as a discrete category, whose only arrows are the identity maps. In particular, we do this when $X = \mathrm{ob}\,\mathcal{C}$; note that there is a faithful functor $\mathrm{ob}\,\mathcal{C} \to \mathcal{C}$.

We write $\mathbb{S}$ for the category of simplicial sets and Top for that of topological spaces. A *family* $\mathcal{F}$ of subgroups of a group G is a nonempty family closed under conjugation and under taking subgroups. We write $\mathrm{Or}_{\mathcal{F}}\, G$ for the orbit category relative to the family $\mathcal{F}$; its objects are the G-sets G/H, $H \in \mathcal{F}$; its homomorphisms are the G-equivariant maps. If C and D are categories, we write C^D for the category of functors $D \to C$, where the homomorphisms are the natural transformations. In particular Top^G and $\mathbb{S}^G$ are the categories of G-spaces and G-simplicial sets, and $\mathrm{Top}^{\mathrm{Or}_{\mathcal{F}}\, G^{op}}$ and $\mathbb{S}^{\mathrm{Or}_{\mathcal{F}}\, G^{op}}$ those of contravariant $\mathrm{Or}_{\mathcal{F}}\, G$-spaces and $\mathrm{Or}_{\mathcal{F}}\, G$-simplicial sets. If $f : C \to C'$ is a functor, we write $f_* : C^D \to C'^D$ for the functor $g \mapsto f \circ g$. Thus for example $|\ |_* : \mathbb{S}^G \to \mathrm{Top}^G$ is the equivariant geometric realization functor; this notation is used in Section 2. In the rest of the paper, if C is a chain complex of abelian groups, $|C|$ is the spectrum the Dold–Kan correspondence associates to it. Topological spaces are considered briefly in Section 2 where it is explained that we can equivalently work with simplicial sets, which is what we do in the rest of the paper. In particular – except briefly in Section 2 – a spectrum is a sequence $\{{}_nE\}$ of pointed simplicial sets and bonding maps $\Sigma_n E \to {}_{n+1}E$. If $E, F : C \to \mathrm{Spt}$ are functorial spectra, then by a *map* $f : E \to F$ we mean a zig-zag of natural maps

$$E = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \xrightarrow{f_3} \cdots Z_n = F$$

such that each right to left arrow f_i is an object-wise weak equivalence. If also the left to right arrows are object-wise weak equivalences, then we say that f is a *weak equivalence* or simply an *equivalence*. If $\{E_i\}$ is a family of spectra, we write $\bigoplus_i E_i$ for their wedge or coproduct.

Rings in this paper are not assumed unital, unless explicitly stated. We write Rings for the category of rings and ring homomorphisms, and Rings_1 for the subcategory of unital rings and unit preserving homomorphisms. We use the letters A, B for rings, and R, S for unital rings. If V is an abelian group, then the tensor algebra of V is $TV = \bigoplus_{n\geqslant 1} V^{\otimes n}$; thus for us TV is nonunital. If V is free, then TV is a free nonunital ring. If X is a set, then M_X is the ring of all matrices $(z_{x,y})_{x,y\in X\times X}$ with integer coefficients, only finitely many of which are nonzero. If A is a ring, then $M_X A = M_X \otimes A$; in particular $M_X\mathbb{Z} = M_X$. If $\{A_i\}$ is a family of rings, then $\bigoplus_i A_i$ is their direct sum as abelian groups, equipped with coordinate-wise multiplication.

2. Model category structures and assembly maps

We begin with some general considerations on model category structures for diagrams of spaces.

We consider Top and $\mathbb{S}$ with their usual, cofibrantly generated closed model structures. If $C = \mathrm{Top}, \mathbb{S}$, and I is any small category, then, by [14, Theorem 11.6.1], C^I is again a cofibrantly generated closed model category, with object-wise fibrations and weak equivalences, and where generating (trivial) cofibrations are of the form

$$\coprod_{\hom_I(\alpha,-)} f : \coprod_{\hom_I(\alpha,-)} \operatorname{dom} f \to \coprod_{\hom_I(\alpha,-)} \operatorname{cod} f$$

with $\alpha\in I$ and $f : \operatorname{dom} f \to \operatorname{cod} f$ a generating (trivial) cofibration in C. Recall that the geometric realization functor $|\ | : \mathbb{S} \to \mathrm{Top}$ and its right adjoint $\mathrm{Sing} : \mathrm{Top} \to \mathbb{S}$ form a Quillen equivalence. Hence by [14, Theorem 11.6.5], the induced functors $|-|_* : \mathbb{S}^I \rightleftarrows \mathrm{Top}^I : \mathrm{Sing}_*$ are Quillen equivalences too.

Next fix a group G and a family $\mathcal{F}$ of subgroups of G. By the previous discussion applied to the orbit category $\mathrm{Or}_{\mathcal{F}} G^{op}$, we have a Quillen equivalence

$$\mathrm{Top}^{\mathrm{Or}_{\mathcal{F}} G^{op}} \underset{|\ |_*}{\overset{\mathrm{Sing}_*}{\rightleftarrows}} \mathbb{S}^{\mathrm{Or}_{\mathcal{F}} G^{op}} \tag{2.1}$$

For $C = \mathrm{Top}, \mathbb{S}$, consider the functor

$$R : C^G \to C^{\mathrm{Or}_{\mathcal{F}} G^{op}}, \qquad R(X)(G/H) = \mathrm{map}_G(G/H, X) = X^H$$

and its left adjoint, the coend

$$L : C^{\mathrm{Or}_{\mathcal{F}} G^{op}} \to C^G, \qquad L(Y) = \int^{\mathrm{Or}\, G} Y(G/H) \times G/H.$$

For $C = \mathrm{Top}$, these functors are also considered in [9, Section 7] as $\mathrm{map}_G(-, X)$ and $\hat{Y}$. The Quillen equivalence (2.1) fits into a diagram

$$\begin{array}{ccc} \mathrm{Top}^{\mathrm{Or}_{\mathcal{F}} G^{op}} & \underset{|\ |_*}{\overset{\mathrm{Sing}_*}{\rightleftarrows}} & \mathbb{S}^{\mathrm{Or}_{\mathcal{F}} G^{op}} \\ L \downarrow \uparrow R & & L \downarrow \uparrow R \\ \mathrm{Top}^G & \underset{|\ |_*}{\overset{\mathrm{Sing}_*}{\rightleftarrows}} & \mathbb{S}^G \end{array} \tag{2.2}$$

Proposition 2.3. *Let $C = \mathrm{Top}, \mathbb{S}$. Consider $\mathcal{F}$ a family of subgroups of G.*

i) *C^G admits a closed model category structure (which depends on the family $\mathcal{F}$) where a map f is a fibration (resp. a weak equivalence) if and only if $R(f)$ is. Moreover C^G is cofibrantly generated, where the generating (trivial) cofibrations are the maps $f \times id : \operatorname{dom} f \times G/H \to \operatorname{cod} f \times G/H$, with f a generating (trivial) cofibration and $H \in \mathcal{F}$.*
ii) *Each of the pairs of functors of diagram (2.2) is a Quillen equivalence.*

Proof. One can give conditions on two sets of maps and a subcategory of a category $\mathcal{D}$ to be respectively the generating cofibrations, generating trivial cofibrations and weak equivalences in a closed model structure of $\mathcal{D}$; see M. Hovey's book

[15, Theorem 2.1.19]. It is straightforward that those conditions are satisfied in our case, for $\mathcal{D} = C^G$. This proves i). The top pair of functors in diagram (2.2) is a Quillen equivalence by the discussion above the proposition. By definition of fibrations and weak equivalences in C^G, these are both preserved and reflected by R. In particular (L, R) is a Quillen pair. To show that it is an equivalence, it suffices, by [15, Corollary 1.3.16], to show that if $X \in C^{\mathrm{Or}_{\mathcal{F}} G^{op}}$ is cofibrant, then the unit map

$$X \to RLX \tag{2.4}$$

is a weak equivalence. In the case $C = \mathrm{Top}$, this is proved in [9, Theorem 7.4 part 2)]; the proof for $C = \mathbb{S}$ is similar. □

Remark 2.5. An object of a cofibrantly generated category is cofibrant if and only if it is a retract of a cellular complex built from generating cofibrant cells. In the case of $\mathbb{S}^G$, every object is built from cells of the form $\Delta^n \times G/H$ for $H \subset G$ a subgroup; it is cofibrant for the model structure of Proposition 2.3 if and only if all such cells have $H \in \mathcal{F}$. Thus the cofibrant cell complexes exhaust the class of cofibrant objects. Observe also that they can be characterized as those objects $X \in \mathbb{S}^G$ such that $X^H = \emptyset$ for $H \notin \mathcal{F}$.

Equivariant homology 2.6. For the model structures of Proposition 2.3, the functorial cofibrant replacement in Top^G of the point space $*$ is a model for the classifying space of G with respect to $\mathcal{F}$ and the cofibrant replacement of $*$ in $\mathbb{S}^G$ is a simplicial version. Moreover because $|-|_* : \mathbb{S}^G \to \mathrm{Top}^G$ is a Quillen equivalence, it takes the simplicial version to the topological one. In particular if E is a functor from Top^G to spectra and $\pi : \mathcal{E}(G, \mathcal{F}) \to *$ is the cofibrant replacement in $\mathbb{S}^G$, then we have a map

$$E(\pi) : E\big(\big|\mathcal{E}(G, \mathcal{F})\big|\big) \to E(*). \tag{2.7}$$

If

$$E(X) = F_{\%}(X) = R(X) \otimes_{\mathrm{Or}\, G} F := \int^{\mathrm{Or}\, G} X_+^H \wedge F(G/H) \tag{2.8}$$

is the Davis–Lück construction [9, §5.1] of some functor $F : \mathrm{Or}\, G \to \mathrm{Spt}$, (2.7) is the assembly map of [9]. In case $F = |F'|$ is the geometric realization of a functorial spectrum in the simplicial set sense, we have further

$$\big|F'\big|_{\%}\big(|X|\big) = \left| \int^{\mathrm{Or}\, G} X_+^H \wedge F'(G/H) \right| = \big|F'_{\%}(X)\big|$$

and the assembly map for F is the geometric realization of that of F'. Hence we can equivalently work with assembly maps in the topological or the simplicial setting; we choose to do the latter. In particular all spectra considered henceforth are simplicial. If C is a chain complex, we will write $|C|$ for the spectrum associated to it by the Dold–Kan correspondence; since topological spaces will occur only rarely from now on, and since we will not use $|\ |$ to indicate realization, this should cause no confusion.

3. Rings and categories

3.1. Crossed products and equivariant homology

A *groupoid* is a small category where all arrows are isomorphisms. Let $\mathcal{G}$ be a groupoid, and let R be a unital ring. An *action* of $\mathcal{G}$ on R is a functor $\rho : \mathcal{G} \to \mathrm{Rings}_1$ such that $\rho(x) = R$ for all $x \in \mathrm{ob}\,\mathcal{G}$. For example we may take $\rho(g) = id_R$ for all arrows $g \in \mathrm{ar}\mathcal{G}$; this is called the *trivial* action. Whenever ρ is fixed, we omit it from our notation, and write

$$g(r) = \rho(g)(r)$$

for $g \in \mathrm{ar}\mathcal{G}$ and $r \in R$. Given a triple $(\mathcal{G}, \rho, R)$, we consider a small $\mathbb{Z}$-linear category $R \rtimes \mathcal{G}$. The objects of $R \rtimes \mathcal{G}$ are those of $\mathcal{G}$, and

$$\hom_{R \rtimes \mathcal{G}}(x, y) = R \otimes \mathbb{Z}\big[\hom_{\mathcal{G}}(x, y)\big].$$

If $s \in R$ and $g \in \hom_{\mathcal{G}}(x, y)$, we write $s \rtimes g$ for $s \otimes g$. Composition is defined by the rule

$$(r \rtimes f) \cdot (s \rtimes g) = rf(s) \rtimes fg \tag{3.1.1}$$

here $r, s \in R$, and f and g are composable arrows in $\mathcal{G}$. In case the action of $\mathcal{G}$ on R is trivial, we also write $R[\mathcal{G}]$ for $R \rtimes \mathcal{G}$.

Let G be a group; consider the functor $\mathcal{G}^G : G - \mathfrak{Sets} \to \mathfrak{Gpd}$ which sends a G-set S to its *transport groupoid*. By definition $\mathrm{ob}\,\mathcal{G}^G(S) = S$, and $\hom_{\mathcal{G}^G(S)}(s, t) = \{g \in G :\ g \cdot s = t\}$.

Notation 3.1.2. If E is a functor from $\mathbb{Z}$-linear categories to spectra, R a unital G-ring, and X a G-space, we put

$$H^G\big(X, E(R)\big) := E\big(R \rtimes \mathcal{G}^G(?)\big)_\%(X)$$

for the Davis–Lück construction (2.8).

3.2. The ring $\mathcal{A}(\mathcal{C})$

Let $\mathcal{C}$ be a small $\mathbb{Z}$-linear category. Put

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{a,b\in \operatorname{ob}\mathcal{C}} \hom_{\mathcal{C}}(a,b). \tag{3.2.1}$$

If $f \in \mathcal{A}(\mathcal{C})$ write $f_{a,b}$ for the component in $\hom_{\mathcal{C}}(b,a)$. The following multiplication law

$$(fg)_{a,b} = \sum_{c\in \operatorname{ob}\mathcal{C}} f_{a,c}g_{c,b} \tag{3.2.2}$$

makes $\mathcal{A}(\mathcal{C})$ into an associative ring, which is unital if and only if $\operatorname{ob}\mathcal{C}$ is finite. Whatever the cardinal of $\operatorname{ob}\mathcal{C}$ is, $\mathcal{A}(\mathcal{C})$ is always a ring with *local units*, i.e. a filtering colimit of unital rings.

$\mathcal{A}(?)$ *and tensor products*. The *tensor product* of two $\mathbb{Z}$-linear categories $\mathcal{C}$ and $\mathcal{D}$ is the $\mathbb{Z}$-linear category $\mathcal{C}\otimes\mathcal{D}$ with $\operatorname{ob}(\mathcal{C}\otimes\mathcal{D}) = \operatorname{ob}(\mathcal{C})\times\operatorname{ob}(\mathcal{D})$ and

$$\hom_{\mathcal{C}\otimes\mathcal{D}}\big((c_1,d_1),(c_2,d_2)\big) = \hom_{\mathcal{C}}(c_1,c_2)\otimes\hom_{\mathcal{D}}(d_1,d_2).$$

We have

$$\mathcal{A}(\mathcal{C}\otimes\mathcal{D}) = \mathcal{A}(\mathcal{C})\otimes\mathcal{A}(\mathcal{D}).$$

Example 3.2.3. If $\mathcal{G}$ is a groupoid acting trivially on a unital ring R, then

$$\mathcal{A}\big(R[\mathcal{G}]\big) = \mathcal{A}\big(R\otimes\mathbb{Z}[\mathcal{G}]\big) = R\otimes\mathcal{A}\big(\mathbb{Z}[\mathcal{G}]\big).$$

$\mathcal{A}(?)$ *and crossed products*. If A is any, not necessarily unital ring, and $\mathcal{G}$ is a groupoid acting on A, we put

$$\mathcal{A}(A\rtimes\mathcal{G}) = \bigoplus_{x,y\in\operatorname{ob}\mathcal{G}} A\otimes\mathbb{Z}\big[\hom_{\mathcal{G}}(x,y)\big].$$

The rules (3.1.1) and (3.2.2) make $\mathcal{A}(A\rtimes\mathcal{G})$ into a ring, which in general is nonunital and does not have local units. The ring $\mathcal{A}(A\rtimes\mathcal{G})$ may also be described in terms of the *unitalization* $\tilde{A}$ of A. By definition, $\tilde{A} = A\oplus\mathbb{Z}$ equipped with the trivial $\mathcal{G}$-action on the $\mathbb{Z}$-summand and the following multiplication

$$(a,\lambda)(b,\mu) = (ab+\lambda b+a\mu, \lambda\mu). \tag{3.2.4}$$

We have

$$\mathcal{A}(A\rtimes\mathcal{G}) = \ker\big(\mathcal{A}(\tilde{A}\rtimes\mathcal{G})\to\mathcal{A}\big(\mathbb{Z}[\mathcal{G}]\big)\big). \tag{3.2.5}$$

Note that $\mathcal{A}(A\rtimes\mathcal{G})$ is defined, even though $A\rtimes\mathcal{G}$ is not. One can actually define $A\rtimes\mathcal{G}$ as a nonunital category, i.e. a category without identity morphisms, but we do not go into that in this paper.

Next we fix a group G and a subgroup $H\subset G$ and consider the ring $\mathcal{A}(A\rtimes\mathcal{G}^G(G/H))$ associated to the crossed product by the transport groupoid. Note that

$$\hom_{\mathcal{G}^G(G/H)}(H,H) = H = \hom_{\mathcal{G}^H(H/H)}(H,H)$$

thus there is a fully faithful functor $\mathcal{G}^H(H/H)\to\mathcal{G}^G(G/H)$. This functor induces a ring homomorphism

$$\jmath : A\rtimes H = \mathcal{A}\big(A\rtimes\mathcal{G}^H(H/H)\big)\subset\mathcal{A}\big(A\rtimes\mathcal{G}^G(G/H)\big).$$

The next lemma compares the map $\jmath$ with the canonical inclusion

$$\iota : A\rtimes H\to M_{G/H}(A\rtimes H), \qquad x\mapsto e_{H,H}\otimes x.$$

In the following lemma and elsewhere, we make use of a section $s: G/H\to G$ of the canonical projection onto the quotient by a subgroup $H\subset G$. We say that the section s is *pointed* if it is a map of pointed sets, that is, if it maps the class of H to the element $1\in G$.

Lemma 3.2.6. *Let A be a ring, G a group acting on A, and $H \subset G$ a subgroup. Then there is an isomorphism $\alpha : \mathcal{A}(A \rtimes \mathcal{G}^G(G/H)) \xrightarrow{\cong} M_{G/H}(A \rtimes H)$ making the following diagram commute*:

$$\begin{array}{ccc} A \rtimes H & \xrightarrow{\jmath} & \mathcal{A}(A \rtimes \mathcal{G}^G(G/H)) \\ & \searrow^{\iota} & \downarrow{\wr}^{\alpha} \\ & & M_{G/H}(A \rtimes H) \end{array}$$

The isomorphism α is natural in A but not in the pair (G, H), as it depends on a choice of pointed section $s : G/H \to G$ of the projection $\pi : G \to G/H$.

Proof. Let s be as in the lemma; put $\hat{g} = s(\pi(g))$ $(g \in G)$. The isomorphism $\alpha : \mathcal{A}(A \rtimes \mathcal{G}^G(G/H)) \xrightarrow{\cong} M_{G/H}(A \rtimes H)$ is defined as follows. For $b \in A$, $s, t \in G$, and $g \in \hom_{\mathcal{G}^G(G/H)}(sH, tH)$, put

$$\alpha(b \rtimes g) = e_{tH,sH} \otimes \hat{t}^{-1}(b) \rtimes \left(\hat{t}^{-1} g \hat{s}\right).$$

It is straightforward to check that α is an isomorphism and that $\alpha\jmath = \iota$. □

Functoriality of $\mathcal{A}(?)$. If $F : \mathcal{C} \mapsto \mathcal{D}$ is a $\mathbb{Z}$-linear functor which is injective on objects, then it defines a homomorphism $\mathcal{A}(F) : \mathcal{A}(\mathcal{C}) \to \mathcal{A}(\mathcal{D})$ by the rule $\alpha \mapsto F(\alpha)$. Hence we may regard $\mathcal{A}$ as a functor

$$\mathcal{A} : \mathrm{inj} - \mathbb{Z} - \mathrm{Cat} \to \mathrm{Rings} \tag{3.2.7}$$

from the category of $\mathbb{Z}$-linear categories and functors which are injective on objects, to the category of rings. However $\mathcal{A}(F)$ is not defined for general $\mathbb{Z}$-linear F.

Remark 3.2.8. The use of the prefix inj here differs from that in [9]. Indeed, here inj indicates that functors are injective on objects, whereas in [9], it refers to functors which are injective on arrows.

3.3. *The nonunital case*

A *Milnor square* is a pullback square of rings

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \downarrow & & \downarrow{f} \\ S' & \xrightarrow[g]{} & S \end{array} \tag{3.3.1}$$

such that either f or g is surjective. Below we shall assume f is surjective. Let $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ be a functor. If A is a not necessarily unital ring, embedded as an ideal in a unital ring R, we write $E(R : A) = \mathrm{hofiber}(E(R) \to E(R/A))$. The functor E is said to satisfy *excision* for the Milnor square (3.3.1) if

$$\begin{array}{ccc} E(R') & \longrightarrow & E(R) \\ \downarrow & & \downarrow{E(f)} \\ E(S') & \longrightarrow & E(S) \end{array}$$

is homotopy Cartesian. If $\ker f \cong A$, then E satisfies excision on (3.3.1) if and only if

$$E\left(R', R : A\right) = \mathrm{hofiber}\left(E\left(R' : A\right) \to E(R : A)\right)$$

is weakly contractible. We say that the ring A is E-*excisive* if E satisfies excision on every Milnor square (3.3.1) with $\ker f \cong A$. Assume unital rings are E-excisive; if A is any, not necessarily E-excisive ring, we consider its unitalization $\tilde{A}$, defined in (3.2.4) above. Put

$$E(A) = \mathrm{hofiber}\left(E(\tilde{A}) \to E(\mathbb{Z})\right).$$

Because of our assumption that unital rings are E-excisive, if A happens to be unital, the two definitions of $E(A)$ are naturally homotopy equivalent. Note that if

$$0 \to A' \to A \to A'' \to 0$$

is an exact sequence of rings and A' is E-excisive, then

$$E(A') \to E(A) \to E(A'')$$

is a homotopy fibration. We say that E is *excisive* or that it *satisfies excision*, if every ring is E-excisive.

Standing Assumptions 3.3.2. *From now on, we shall be primarily concerned with functors $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ that satisfy the following*:

i) *Every ring with local units is E-excisive.*
ii) *If H is a group and A an E-excisive H-ring, then $A \rtimes H$ is E-excisive.*
iii) *If A is E-excisive, X a set and $x \in X$, then $M_X A$ is E-excisive, and E sends the map $A \to M_X A$, $a \mapsto e_{x,x}a$ to a weak equivalence.*
iv) *There is a natural weak equivalence $E(\mathcal{A}(\mathcal{C})) \xrightarrow{\sim} E(\mathcal{C})$ of functors $\mathrm{inj} - \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$.*
v) *Let $\{A_i : i \in I\}$ be a family of rings, and let $A = \bigoplus_{i \in I} A_i$ be their direct sum, with coordinate-wise multiplication. Then A is E-excisive if and only if each A_i is. Moreover if these equivalent conditions are satisfied, then the map $\bigoplus_i E(A_i) \to E(A)$ is an equivalence.*

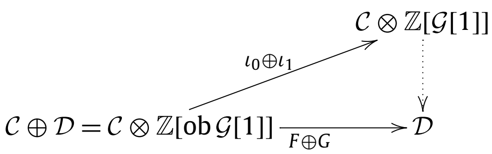
Remark 3.3.3. Observe that standing assumptions i)–iii) and v) are only concerned with the restriction of E to the full subcategory $\mathrm{Rings} \subset \mathbb{Z} - \mathrm{Cat}$, and that assumption iv) says that $E_{|\mathrm{Rings}}$ determines the whole functor up to weak equivalence. However the assumptions are enough to prove for instance that E maps category equivalences to equivalences of spectra; see Remark 3.3.7. Note also that the equivalence of iv) is natural only with respect to functors which are injective on objects, because $\mathcal{A}(-)$ is only functorial on $\mathrm{inj} - \mathbb{Z} - \mathrm{Cat}$. One could ask whether it is possible to extend a functor $E : \mathrm{Rings} \to \mathrm{Spt}$ satisfying i)–iii) and v) to all of $\mathbb{Z} - \mathrm{Cat}$ in such a way that iv) is satisfied. In the next subsection we introduce a functor $\mathcal{R} : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Rings}$ which restricts to the identity on Rings and a natural transformation $p : \mathcal{R} \to \mathcal{A}$ of functors $\mathrm{inj} - \mathbb{Z} - \mathrm{Cat} \to \mathrm{Rings}$ and discuss conditions on E under which $E(p)$ is an equivalence.

Remark 3.3.4. The examples we are primarily interested in – such as K-theory – satisfy a stronger version of property i). Indeed, they not only satisfy excision for rings with local units, but also for (flat) s-unital rings. A ring A is called *s-unital* if for every finite collection $a_1, \dots, a_n \in A$ there exists an element $e \in A$ such that $a_i e = e a_i = a_i$. Note that if we add the requirement that e be idempotent we recover the notion of ring with local units. As is explained in Appendix A (Example A.3.5) every s-unital ring is H-unital, and every s-unital ring which is flat as an abelian group is K-excisive.

Remark 3.3.5. If E satisfies excision, then assumptions i) and ii) hold automatically, and assumptions iii) and v) hold if and only if they hold for unital rings.

Lemma 3.3.6. *Let $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ be a functor satisfying the standing assumptions above. If $F_i : \mathcal{C} \to \mathcal{D}$, $i = 0, 1$ are naturally isomorphic linear functors, then $E(F_0)$ and $E(F_1)$ are homotopic.*

Proof. Let $\mathcal{G}[1] = \{0 \leftrightarrows 1\}$ be the groupoid with two objects and exactly one isomorphism between any two given (equal or distinct) objects. The linear functors $F, G : \mathcal{C} \to \mathcal{D}$ are equivalent if the dotted arrow in the following diagram of $\mathbb{Z}$-linear functors exists and makes it commute



Hence it suffices to show that $E(\iota_0) \cong E(\iota_1)$. By assumption iv), we are reduced to showing that $E(\mathcal{A}(\iota_0)) \cong E(\mathcal{A}(\iota_1))$. Observe that $\mathcal{G}[1]$ is the transport groupoid of $\mathbb{Z}/2\mathbb{Z}$ considered as a $\mathbb{Z}/2\mathbb{Z}$-set. Hence we may apply Lemma 3.2.6 to show that $\mathcal{A}(\mathcal{C} \otimes \mathbb{Z}[\mathcal{G}[1]]) = M_2(\mathcal{A}(\mathcal{C}))$ and that the ι_i induce the two canonical inclusions $x \mapsto x \otimes e_{1,1}$, $x \otimes e_{2,2}$. Hence we are done by assumption iii) (see [4, Lemma 2.2.4]). $\square$

Remark 3.3.7. It follows from Lemma 3.3.6 that E sends category equivalences to equivalences of spectra.

Let G be a group. Assume E satisfies the standing assumptions above. For A an E-excisive G-ring, consider the $\mathrm{Or}\, G$-spectrum

$$G/H \mapsto E\big(A \rtimes \mathcal{G}^G(G/H)\big) = \mathrm{hofiber}\big(E\big(\tilde{A} \rtimes \mathcal{G}^G(G/H)\big) \to E\big(\mathbb{Z}\big[\mathcal{G}^G(G/H)\big]\big)\big). \tag{3.3.8}$$

Applying $(?)_\%$ to (3.3.8) defines an equivariant homology theory of G-simplicial sets, which we denote $H^G(-, E(A))$. Moreover, for each fixed G-simplicial set X, $H^G(X, E(?))$ is a functor of E-excisive rings. Observe that, for unital A, we have two definitions of $E(A \rtimes \mathcal{G}^G(-))$ and two definitions of $H^G(-, E(A))$; the next proposition says that the two definitions are equivalent.

Proposition 3.3.9. *Let $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ be a functor and G a group. Assume that E satisfies Standing Assumptions 3.3.2 above.*

a) *If R is a unital G-ring, then the two definitions of $E(R \rtimes \mathcal{G}^G(-))$ and the two definitions of $H^G(-, E(R))$ are equivalent.*
b) *If*

$$0 \to A' \to A \to A'' \to 0$$

is an exact sequence of E-excisive G-rings, and X is a G-simplicial set, then

$$E\big(A' \rtimes \mathcal{G}^G(-)\big) \to E\big(A \rtimes \mathcal{G}^G(-)\big) \to E\big(A'' \rtimes \mathcal{G}^G(-)\big)$$

and

$$H^G\big(X, E\big(A'\big)\big) \to H^G\big(X, E(A)\big) \to H^G\big(X, E\big(A''\big)\big)$$

are homotopy fibrations.

Proof. If A is E-excisive and $H \subset G$ is a subgroup, then conditions ii) and iii) together with Lemma 3.2.6 imply that $\mathcal{A}(A \rtimes \mathcal{G}^G(G/H))$ is E-excisive. Hence, by condition iv), the spectrum in (3.3.8) is equivalent to $E(\mathcal{A}(A \rtimes \mathcal{G}^G(G/H)))$. In particular, by i), $\mathcal{A}(R \rtimes \mathcal{G}^G(G/H))$ is E-excisive for unital R, and the map

$$\mathrm{hofiber}\big(E\big(\tilde{R} \rtimes \mathcal{G}^G(G/H)\big) \to E\big(\mathbb{Z}\big[\mathcal{G}^G(G/H)\big]\big)\big) \to E\big(R \rtimes \mathcal{G}^G(G/H)\big)$$

induced by the projection $\tilde{R} \cong R \times \mathbb{Z} \to R$ is an equivalence. This proves a). Moreover, because $\mathcal{A}(? \rtimes \mathcal{G}^G(G/H))$ preserves exact sequences, applying (3.3.8) to the exact sequence of part b) yields an object-wise homotopy fibration of $\mathrm{Or}\, G$-spectra, which is the first homotopy fibration of b). Applying $(?)_\%$ we obtain the second one. □

Remark 3.3.10. Let $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ and let A be any, not necessarily E-excisive G-ring, equivariantly embedded as an ideal in a unital G-ring R. Consider the $\mathrm{Or}\, G$-spectrum

$$E\big(R \rtimes \mathcal{G}^G(-) : A \rtimes \mathcal{G}^G(-)\big) = \mathrm{hofiber}\big(E\big(R \rtimes \mathcal{G}^G(-)\big) \to E\big((R/A) \rtimes \mathcal{G}^G(-)\big)\big).$$

Put

$$H^G\big(X, E(R : A)\big) = E\big(R \rtimes \mathcal{G}^G(-) : A \rtimes \mathcal{G}^G(-)\big)_\%(X).$$

A $(G, \mathcal{F})$-cofibrant replacement $cX \to X$ gives rise to a map of homotopy fibrations

$$\begin{array}{ccccc} H^G(cX, E(R:A)) & \longrightarrow & H^G(cX, E(R)) & \longrightarrow & H^G(cX, E(R/A)) \\ \downarrow & & \downarrow & & \downarrow \\ H^G(X, E(R:A)) & \longrightarrow & H^G(X, E(R)) & \longrightarrow & H^G(X, E(R/A)) \end{array}$$

If $H^G(cX, E(S)) \to H^G(X, E(S))$ is an equivalence for all unital S, then both the middle and right hand side vertical maps are equivalences; it follows that the same is true of the map on the left. We record a particular case of this in the following corollary.

Corollary 3.3.11. *Let $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ be a functor; assume E satisfies Standing Assumptions 3.3.2. Further let G be a group, $X \in \mathbb{S}^G$, $\mathcal{F}$ a family of subgroups, $cX \to X$ and $(G, \mathcal{F})$-cofibrant replacement. Assume that the assembly map $H^G(cX, E(R)) \to H^G(X, E(R))$ is an equivalence for every unital ring R. Then $H^G(cX, E(A)) \to H^G(X, E(A))$ is an equivalence for every E-excisive ring A.*

Proposition 3.3.12. *Let $A \triangleleft R$ be an ideal in a unital G-ring, closed under the action of G. Let $E : \mathrm{Rings} \to \mathrm{Spt}$ be a functor satisfying the standing assumptions. If A is E-excisive then*

$$E\big(A \rtimes \mathcal{G}^G(-)\big) \to E\big(R \rtimes \mathcal{G}^G(-) : A \rtimes \mathcal{G}^G(-)\big)$$

is an object-wise weak equivalence of $\mathrm{Or}\, G$-spectra.

Proof. The proof follows from Lemma 3.2.6, using assumptions ii), iii) and iv). □

3.4. The ring $\mathcal{R}(\mathcal{C})$

Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category. Imitating a construction used by M. Joachim [16] in the C^*-algebra context, we shall associate to $\mathcal{C}$ a ring $\mathcal{R}(\mathcal{C})$ which is a quotient of the tensor algebra of $\mathcal{A}(\mathcal{C})$; first we need some notation. If M is an abelian group, we write $T(M)=\bigoplus_{n\geqslant 1} M^{\otimes n}$ for the (unaugmented) tensor algebra. Put

$$\mathcal{R}(\mathcal{C})=T\big(\mathcal{A}(\mathcal{C})\big)/\big\langle\big\{g\otimes f-g\circ f\colon\ f\in \hom_{\mathcal{C}}(a,b),\, g\in\hom_{\mathcal{C}}(b,c),\ a,b,c\in \operatorname{ob}\mathcal{C}\big\}\big\rangle.$$

Note that any $\mathbb{Z}$-linear functor $\mathcal{C}\to\mathcal{D}\in\mathbb{Z}-\mathrm{Cat}$ defines a homomorphism $\mathcal{R}(\mathcal{C})\to\mathcal{R}(\mathcal{D})$. Thus we may regard $\mathcal{R}$ as a functor

$$\mathcal{R}:\mathbb{Z}-\mathrm{Cat}\to \mathrm{Rings},\qquad \mathcal{C}\mapsto\mathcal{R}(\mathcal{C}).$$

Observe that the canonical surjection $T(\mathcal{A}(\mathcal{C}))\to\mathcal{A}(\mathcal{C})$ factors through a map

$$p:\mathcal{R}(\mathcal{C})\twoheadrightarrow\mathcal{A}(\mathcal{C}) \tag{3.4.1}$$

whose kernel is the ideal generated by the elements $g\otimes f$ for non-composable g and f. For example if $\mathcal{C}$ has only one object, then p is the identity. In particular any functor $E:\mathrm{Rings}\to\mathrm{Spt}$ can be extended to $\mathbb{Z}-\mathrm{Cat}$ via $E(\mathcal{C})=E(\mathcal{R}(\mathcal{C}))$, and p induces a natural transformation $E(p):E(\mathcal{C})\to E(\mathcal{A}(\mathcal{C}))$ of functors of $\mathrm{inj}-\mathbb{Z}-\mathrm{Cat}$.

Example 3.4.2. Let R,S be unital rings, and let $\mathcal{C}$ be the $\mathbb{Z}$-linear category with two objects a and b such that $\hom_{\mathcal{C}}(a,b)=\hom_{\mathcal{C}}(b,a)=0$, $\hom_{\mathcal{C}}(a,a)=R$ and $\hom_{\mathcal{C}}(b,b)=S$. Then $\mathcal{A}(\mathcal{C})=R\oplus S$ and $\mathcal{R}(\mathcal{C})=R\coprod S$ is the nonunital coproduct. We shall see in Proposition 4.3.1 that K-theory satisfies the standing assumptions; however in general $K_*(R\coprod S)\neq K_*(R)\oplus K_*(S)$.

In Lemma 3.4.3 we give conditions on E which guarantee that it sends the map (3.4.1) to a weak equivalence. First we need some notation. If B is a ring, we write $\mathrm{ev}_i:B[t]\to B$, $i=0,1$ for the evaluation maps. If $f,g:A\to B$ are ring homomorphisms, then a (polynomial) *elementary homotopy* between f and g is a map $H:A\to B[t]$ such that $\mathrm{ev}_0H=f$ and $\mathrm{ev}_1H=g$. A *homotopy* from f to g is a sequence of homomorphisms $f=h_0,\dots,h_n=g$ and elementary homotopies $H_i:A\to B[t]$ from h_i to h_{i+1}. The functor E is *invariant under polynomial homotopy* if for every ring A, E sends the inclusion $A\subset A[t]$ to a weak equivalence. Because the composite $\mathrm{inc}\circ\mathrm{ev}_0:A[t]\to A[t]$ is homotopic to the identity, if E is invariant under polynomial homotopy, and f and g are homotopic ring homomorphisms, then $E(f)$ and $E(g)$ define the same map in HoSpt.

Lemma 3.4.3. *Let $E:\mathrm{Rings}\to\mathrm{Spt}$ be a functor. Assume that E satisfies standing assumptions* i) *and* iii)*. Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category such that $\mathcal{R}(\mathcal{C})$ is E-excisive. Then E_* sends* (3.4.1) *to a naturally split surjection. Assume in addition that E is invariant under polynomial homotopy. Then E sends* (3.4.1) *to a weak equivalence.*

Proof. Let $\mathrm{ob}_+\mathcal{C}=\operatorname{ob}\mathcal{C}\coprod\{+\}$ be the set of objects of $\mathcal{C}$ with a base point added. Consider the homomorphism

$$j:\mathcal{A}(\mathcal{C})\to M_{\mathrm{ob}_+\mathcal{C}}\mathcal{R}(\mathcal{C}),\quad j(f)=f\otimes e_{b,a}\ \big(f\in\hom_{\mathcal{C}}(a,b)\big).$$

Write p for the map (3.4.1). Consider the matrices

$$V=\sum_{a\in\operatorname{ob}\mathcal{C}}1_a\otimes e_{a,+},\qquad W=\sum_{a\in\operatorname{ob}\mathcal{C}}1_a\otimes e_{+,a}.$$

The composite $q=M_{\mathrm{ob}_+\mathcal{C}}(p)\circ j$ sends $f\in\mathcal{A}(\mathcal{C})$ to

$$q(f)=Wf\otimes e_{+,+}V.$$

Observe that left multiplication by W and right multiplication by V leave the subring $M_{\mathrm{ob}_+\mathcal{C}}\mathcal{A}(\mathcal{C})$ stable, and that $aVWa'=aa'$ for all $a,a'\in M_{\mathrm{ob}_+\mathcal{C}}\mathcal{A}(\mathcal{C})$. By the argument of [4, 2.2.6], all this together with matrix invariance imply that $E_*(q)=E_*(?\otimes e_{+,+})$ is an isomorphism. This proves the first assertion of the lemma. To prove the second, it suffices to show that $r=j\circ p$ is homotopic to the inclusion $\iota(a)=a\otimes e_{+,+}$. If $f\in\hom_{\mathcal{C}}(a,b)$, write $H(f)\in M_{\mathrm{ob}_+\mathcal{C}}(\mathcal{R}(\mathcal{C}))[t]$ for

$$H(f)=f\otimes\big(-t(t^3-2t)e_{+,+}+t(t^2-1)e_{+,a}+\big(1-t^2\big)\big(t^3-2t\big)e_{b,+}+\big(1-t^2\big)^2e_{b,a}\big).$$

Note that $\mathrm{ev}_0H(f)=r(f)$, $\mathrm{ev}_1H(f)=\iota(f)$. Further, one checks that if $g\in\hom_{\mathcal{C}}(b,c)$, then $H(gf)=H(g)H(f)$. Thus H induces a homomorphism $\mathcal{R}(\mathcal{C})\to M_{\mathrm{ob}_+\mathcal{C}}(\mathcal{R}(\mathcal{C}))[t]$ which is a homotopy from r to ι. This concludes the proof. □

Example 3.4.4. If $E:\mathrm{Rings}\to\mathrm{Spt}$ is excisive and homotopy invariant and satisfies standing assumptions iii) and v), then its extension $E\circ\mathcal{R}:\mathbb{Z}-\mathrm{Cat}\to\mathrm{Spt}$ satisfies all the standing assumptions and agrees with E on Rings. If F is another extension of E which also satisfies the standing assumptions, then composing $E(\mathcal{R}(\mathcal{C}))\to E(\mathcal{A}(\mathcal{C}))$ with the map of Assumption 3.3.2iv), we get an equivalence $E(\mathcal{R}(\mathcal{C}))\to F(\mathcal{C})$ which is natural with respect to functors which are injective on objects.

4. *K*-theory

4.1. The *K*-theory spectrum

Given a $\mathbb{Z}$-linear category $\mathcal{C}$, we denote by $\mathcal{C}_\oplus$ the $\mathbb{Z}$-linear category whose objects are finite sequences of objects of $\mathcal{C}$, and whose morphisms are matrices of morphisms in $\mathcal{C}$ with the obvious matrix product as composition. Concatenation of sequences yields a sum $\oplus$ and hence we obtain, functorially, an additive category; write $\operatorname{Idem}\mathcal{C}_\oplus$ for its idempotent completion. We shall also need *Karoubi's cone* $\Gamma(\mathcal{C})$ [18, p. 270]. The objects of $\Gamma(\mathcal{C})$ are the sequences $x=(x_1,x_2,\ldots)$ of objects of $\mathcal{C}$ such that the set

$$F(x)=\{c\in\mathcal{C}:\ (\exists n)\ x_n=c\} \tag{4.1.1}$$

is finite. A map $x\to y$ in $\Gamma(\mathcal{C})$ is a matrix $f=(f_{i,j})$ of homomorphisms $f_{i,j}:x_j\to y_i$ such that

(1) There exists an N such that every row and every column of f has at most N nonzero entries.
(2) The set $\{f_{i,j}:\ i,j\in\mathbb{N}\}$ is finite.

Interspersing of sequences defines a symmetric monoidal operation $\boxplus:\Gamma(\mathcal{C})\times\Gamma(\mathcal{C})\to\Gamma(\mathcal{C})$ and there is an endofunctor τ such that $1\boxplus\tau\cong\tau$ (see [17, §III]). If $\mathcal{C}$ has finite direct sums, e.g. if $\mathcal{C}=\mathcal{D}_\oplus$ for some $\mathbb{Z}$-linear category $\mathcal{D}$, then the interspersing operation is naturally equivalent to the induced sum $(x\oplus y)_i=x_i\oplus y_i$ [17, Lemme 3.3]. In particular, if $\mathcal{C}$ is additive, then $\Gamma\mathcal{C}$ is a *flasque* additive category; that is, there is an additive endofunctor $\tau:\mathcal{C}\to\mathcal{C}$ such that $\tau\oplus 1\cong\tau$. A morphism f in $\Gamma(\mathcal{C})$ is *finite* if $f_{ij}=0$ for all but finitely many (i,j). Finite morphisms form an ideal, and we write $\Sigma(\mathcal{C})$ for the category with the same objects as $\Gamma(\mathcal{C})$, and morphisms taken modulo the ideal of finite morphisms. The category $\Sigma(\mathcal{C})$ is Karoubi's *suspension* of $\mathcal{C}$. By [27, Theorem 5.3], if $\mathcal{C}$ is additive, we have a homotopy fibration sequence

$$K^Q(\operatorname{Idem}\mathcal{C})\to K^Q\big(\Gamma(\operatorname{Idem}\mathcal{C})\big)\to K^Q\big(\Sigma(\operatorname{Idem}\mathcal{C})\big). \tag{4.1.2}$$

Here each of the categories is regarded as a semisimple exact category, and K^Q denotes the fibrant simplicial set for its algebraic K-theory according to Quillen [28]. Because $\Gamma(\operatorname{Idem}\mathcal{C})$ is flasque, $K^Q(\Gamma(\operatorname{Idem}\mathcal{C}))$ is contractible, whence $K^Q(\operatorname{Idem}\mathcal{C})\cong\Omega K^Q(\Sigma(\operatorname{Idem}\mathcal{C}))$. Now let $\mathcal{C}$ be any small $\mathbb{Z}$-linear category, possibly without direct sums. Consider the sequence of categories

$$\mathcal{C}^{(0)}=\operatorname{Idem}(\mathcal{C}_\oplus),\qquad \mathcal{C}^{(n+1)}=\operatorname{Idem}\big(\Sigma\mathcal{C}^{(n)}\big). \tag{4.1.3}$$

Then we have a spectrum $K(\mathcal{C})=\{{}_nK(\mathcal{C})\}$, with

$${}_nK(\mathcal{C})\cong K^Q\big(\mathcal{C}^{(n)}\big). \tag{4.1.4}$$

Remark 4.1.5. If R is a unital ring, then by [18, Proposition 1.6], we have category equivalences

$$\operatorname{Idem}\big(\Gamma\big(\operatorname{proj}(R)\big)\big)\cong\operatorname{proj}\big(\Gamma(R)\big)\quad\text{and}\quad\operatorname{Idem}\big(\Sigma\big(\operatorname{proj}(R)\big)\big)\cong\operatorname{proj}\big(\Sigma(R)\big). \tag{4.1.6}$$

Hence the spectrum $K(R)$ defined above is equivalent to the usual K-theory spectrum of Gersten–Karoubi–Wagoner.

4.2. Comparing $K(\mathcal{C})$ with $K(\mathcal{A}(\mathcal{C}))$

The operation $\lozenge$. Let X be a set and let $\mathcal{C}$ and $\mathcal{D}$ be $\mathbb{Z}$-linear categories with $\operatorname{ob}\mathcal{C}=\operatorname{ob}\mathcal{D}=X$. Consider the category $\mathcal{C}\lozenge\mathcal{D}$ with set of objects $\operatorname{ob}(\mathcal{C}\lozenge\mathcal{D})=X$, homomorphisms

$$\hom_{\mathcal{C}\lozenge\mathcal{D}}(x,y)=\hom_{\mathcal{C}}(x,y)\oplus\hom_{\mathcal{D}}(x,y)$$

and coordinate-wise composition. If $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ are $\mathbb{Z}$-linear categories, we have

$$(\mathcal{C}\lozenge\mathcal{D})_\oplus=\mathcal{C}_\oplus\lozenge\mathcal{D}_\oplus,\qquad \operatorname{Idem}\big((\mathcal{C}\lozenge\mathcal{D})_\oplus\big)=\operatorname{Idem}\mathcal{C}_\oplus\times\operatorname{Idem}\mathcal{D}_\oplus, \tag{4.2.1}$$

$$(\mathcal{C}\lozenge\mathcal{D})\otimes\mathcal{E}=(\mathcal{C}\otimes\mathcal{E})\lozenge(\mathcal{D}\otimes\mathcal{E}). \tag{4.2.2}$$

Unitalization. We have already recalled the definition of the unitalization $\tilde{A}$ of a not necessarily unital ring A. Now we need a version of unitalization for $\mathbb{Z}$-linear categories; this can be more generally defined for nonunital $\mathbb{Z}$-categories, but we will have no occasion for that. Let $\mathcal{C}\in\mathbb{Z}-\mathrm{Cat}$; write $\tilde{\mathcal{C}}$ for the category with $\operatorname{ob}\tilde{\mathcal{C}}=\operatorname{ob}\mathcal{C}$ and with homomorphisms given by

$$\hom_{\tilde{\mathcal{C}}}(x,y)=\hom_{\mathcal{C}}(x,y)\oplus\delta_{x,y}\mathbb{Z}=\begin{cases}\hom_{\mathcal{C}}(x,y) & x\neq y,\\ \hom_{\mathcal{C}}(x,x)\oplus\mathbb{Z} & x=y.\end{cases}$$

Composition between $(f, \delta_{x,y}n) \in \hom_{\tilde{\mathcal{C}}}(x, y)$ and $(g, \delta_{y,z}m) \in \hom_{\tilde{\mathcal{C}}}(y, z)$ is defined by the formula

$$(g, \delta_{y,z}m) \circ (f, \delta_{x,y}n) = (gf + \delta_{y,z}mf + \delta_{x,y}gn, \delta_{x,y}\delta_{y,z}mn).$$

Observe that if R is a ring, considered as a $\mathbb{Z}$-linear category with one object, then

$$\tilde{R} \to R \times \mathbb{Z} = R \lozenge \mathbb{Z}, \qquad (r, n) \mapsto (r + n \cdot 1, n)$$

is an isomorphism. This isomorphism generalizes to $\mathbb{Z}$-categories as follows. Let $\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle \in \mathbb{Z} - \mathrm{Cat}$, be the $\mathbb{Z}$-linear category with the same objects as $\mathcal{C}$ and homomorphisms given by

$$\hom_{\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle}(x, y) = \delta_{x,y}\mathbb{Z}.$$

We have an isomorphism of linear categories

$$\mathcal{C} \lozenge \mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle \overset{\cong}{\to} \tilde{\mathcal{C}} \tag{4.2.3}$$

which is the identity on objects, as well as on $\hom_{\mathcal{C}\lozenge\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle}(x, y)$ for $x \neq y$, and which sends

$$\hom_{\mathcal{C}\lozenge\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle}(x, x) \ni (f, n) \mapsto (f - n1_x, n) \in \hom_{\tilde{\mathcal{C}}}(x, x).$$

The map $K(\mathcal{C}) \to K(\mathcal{A}(\mathcal{C}))$. If $\mathcal{C}$ is a $\mathbb{Z}$-linear category, and $x, y \in \mathrm{ob}\,\mathcal{C}$, then by definition of $\mathcal{A}(\mathcal{C})$,

$$\hom_{\mathcal{C}}(x, y) \subset \mathcal{A}(\mathcal{C}) \tag{4.2.4}$$

and the inclusion is compatible with composition. We also have an inclusion

$$\hom_{\tilde{\mathcal{C}}}(x, x) \ni (f, n) \mapsto (f, n) \in \widetilde{\mathcal{A}(\mathcal{C})}. \tag{4.2.5}$$

The inclusions (4.2.4) and (4.2.5) together with the only map $\mathrm{ob}\,\tilde{\mathcal{C}} \to \mathrm{ob}\,\widetilde{\mathcal{A}(\mathcal{C})} = \{\bullet\}$ define a functor

$$\phi : \tilde{\mathcal{C}} \to \widetilde{\mathcal{A}(\mathcal{C})}. \tag{4.2.6}$$

Observe that $\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle \subset \tilde{\mathcal{C}}$ and that $\phi(\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle) \subset \mathbb{Z} \subset \widetilde{\mathcal{A}(\mathcal{C})}$. We have a commutative diagram

$$\begin{array}{ccc}
\tilde{\mathcal{C}} & \xrightarrow{\phi} & \widetilde{\mathcal{A}(\mathcal{C})} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\
\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle & \longrightarrow & \mathbb{Z}
\end{array}$$

Here the vertical maps are the obvious projections. By (4.2.3) and (4.2.1) we have an equivalence

$$K(\tilde{\mathcal{C}}) \overset{\sim}{\to} K(\mathcal{C}) \times K\big(\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle\big).$$

Under this equivalence the map induced by π_1 becomes the canonical projection; hence its fiber is $K(\mathcal{C})$. On the other hand, by definition, $K(\mathcal{A}(\mathcal{C}))$ is the fiber of $K(\pi_2)$. Hence ϕ induces a map

$$\varphi : K(\mathcal{C}) \to K\big(\mathcal{A}(\mathcal{C})\big). \tag{4.2.7}$$

Proposition 4.2.8. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category. Then the map (4.2.7) is a weak equivalence, natural in $\mathcal{C}$.*

Proof. Because both the source and the target of (4.2.7) commute with filtering colimits, we may assume that $\mathcal{C}$ has finitely many objects. Then $\mathcal{A}(\mathcal{C})$ is unital, and thus we have an isomorphism $\widetilde{\mathcal{A}(\mathcal{C})} \cong \mathcal{A}(\mathcal{C}) \times \mathbb{Z}$. Recall that the idempotent completion of an additive category $\mathfrak{A}$ is the category whose objects are the idempotent endomorphisms in $\mathfrak{A}$ and where a map $f : e_1 \to e_2$ is an element of $\hom_{\mathfrak{A}}(\mathrm{dom}\, e_1, \mathrm{dom}\, e_2)$ such that $f = e_2 f e_1$. One checks that the composite

$$\begin{aligned}
\mathcal{C}_\oplus &\to \mathrm{Idem}\,\mathcal{C}_\oplus \overset{1\times 0}{\to} \mathrm{Idem}\,\mathcal{C}_\oplus \times \mathrm{Idem}\,\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle \\
&\cong \mathrm{Idem}(\tilde{\mathcal{C}}_\oplus) \overset{\phi}{\to} \mathrm{Idem}\big(\widetilde{\mathcal{A}(\mathcal{C})}_\oplus\big) \cong \mathrm{Idem}\big(\mathcal{A}(\mathcal{C})_\oplus\big) \times \mathrm{Idem}(\mathbb{Z}_\oplus) \to \mathrm{Idem}\big(\mathcal{A}(\mathcal{C})_\oplus\big)
\end{aligned}$$

is the functor ψ which sends an object $(c_1, \dots, c_n)$ to the idempotent

$$\mathrm{diag}(1_{c_1}, \dots, 1_{c_n})$$

and a map $f = (f_{i,j}) : (c_1, \dots, c_n) \to (d_1, \dots, d_m)$ to the corresponding matrix $(f_{i,j}) \in \hom_{\mathcal{A}(\mathcal{C})_\oplus}(\bullet^n, \bullet^m)$. Because ψ is fully faithful and cofinal, it induces an equivalence $K(\mathcal{C}) \to K(\mathcal{A}(\mathcal{C}))$. It follows that (4.2.7) is an equivalence. □

4.3. *K-theory and the standing assumptions*

Proposition 4.3.1. *The functor* $K:\mathbb{Z}-\mathrm{Cat}\to\mathrm{Spt}$ *satisfies the standing assumptions.*

Proof. Assumption iv) was proved in Proposition 4.2.8 above. The remaining assumptions are either proved in Appendix A or follow from results therein. By Example A.1.1, rings with local units are K-excisive; hence K-theory satisfies i). Assumption ii) holds by Proposition A.6.3. If A is K-excisive and X is a set, then M_XA is K-excisive, by Proposition A.5.3. Assumption iii) follows from this and the fact that K-theory is matrix stable on unital rings. Assumption v) is proved in Proposition A.4.4. □

5. Homotopy K-theory

If $\mathcal{C}$ is a $\mathbb{Z}$-linear category, then we write $\mathcal{C}^{\Delta^\bullet}$ for the simplicial $\mathbb{Z}$-linear category

$$\mathcal{C}^{\Delta^\bullet}:[n]\mapsto\mathcal{C}^{\Delta^n}=\mathcal{C}\otimes\mathbb{Z}[t_0,\dots,t_n]/\langle t_0+\cdots+t_n-1\rangle. \tag{5.1}$$

Applying the functor K dimensionwise we get a simplicial spectrum whose total spectrum is the *homotopy K-theory* spectrum $KH(\mathcal{C})$. In particular if R is a unital ring, then $KH(R)$ was defined by Weibel in [34]. The following theorem was proved in [34]; see also [4, §5].

Theorem 5.2 *(Weibel). The functor* $KH:\mathrm{Rings}\to\mathrm{Spt}$ *is excisive, matrix invariant, and invariant under polynomial homotopy.*

Proposition 5.3. *There is a natural weak equivalence* $KH(\mathcal{C})\xrightarrow{\sim}KH(\mathcal{R}(\mathcal{C}))$.

Proof. We begin by observing that the inclusions (4.2.4) and (4.2.5) lift to inclusions $\hom_{\mathcal{C}}(x,y)\subset\mathcal{R}(\mathcal{C})$ and $\hom_{\tilde{\mathcal{C}}}(x,x)\subset\widetilde{\mathcal{R}(\mathcal{C})}$. Thus we have a functor

$$\phi':\tilde{\mathcal{C}}\to\widetilde{\mathcal{R}(\mathcal{C})}.$$

Composing it with

$$\tilde{p}:\widetilde{\mathcal{R}(\mathcal{C})}\to\widetilde{\mathcal{A}(\mathcal{C})}$$

we obtain the map

$$\phi:\tilde{\mathcal{C}}\to\widetilde{\mathcal{A}(\mathcal{C})}$$

of (4.2.6) above. Tensoring with $\mathbb{Z}^{\Delta^\bullet}$ and applying $K(-)$ we obtain a commutative diagram

$$\begin{array}{ccc} KH(\tilde{\mathcal{C}}) & \xrightarrow{\phi'} & KH(\widetilde{\mathcal{R}(\mathcal{C})}) \\ & \searrow^{\phi} & \downarrow p \\ & & KH(\widetilde{\mathcal{A}(\mathcal{C})}) \end{array}$$

The diagram above maps to the diagram

$$\begin{array}{ccc} KH(\mathbb{Z}\langle\mathrm{ob}\,\mathcal{C}\rangle) & \xrightarrow{\phi} & KH(\mathbb{Z}) \\ & \searrow^{\phi} & \| \\ & & KH(\mathbb{Z}) \end{array}$$

Taking fibers and using (4.2.1), (4.2.2) and (4.2.3), we obtain a homotopy commutative diagram

$$\begin{array}{ccc} KH(\mathcal{C}) & \xrightarrow{\varphi''} & KH(\mathcal{R}(\mathcal{C})) \\ & \searrow^{\varphi'} & \downarrow p \\ & & KH(\mathcal{A}(\mathcal{C})) \end{array}$$

Here φ' comes from a map of simplicial spectra

$$\varphi^\bullet:K\big(\mathcal{C}\otimes\mathbb{Z}^{\Delta^\bullet}\big)\xrightarrow{\sim}K\big(\tilde{\mathcal{C}}\otimes\mathbb{Z}^{\Delta^\bullet}:\mathcal{C}\otimes\mathbb{Z}^{\Delta^\bullet}\big)\to K\big(\widetilde{\mathcal{A}(\mathcal{C})}\otimes\mathbb{Z}^{\Delta^\bullet}:\mathcal{A}(\mathcal{C})\otimes\mathbb{Z}^{\Delta^\bullet}\big)\xleftarrow{\sim}K\big(\mathcal{A}(\mathcal{C})\otimes\mathbb{Z}^{\Delta^\bullet}\big), \tag{5.4}$$

and $\varphi^0 = \varphi$ is the map (4.2.7), which is an equivalence by Proposition 4.2.8. The same argument of the proof of Proposition 4.2.8 shows that φ^n is an equivalence for every n. On the other hand, by Theorem 5.2 and Lemma 3.4.3, the map $p : KH(\mathcal{R}(\mathcal{C})) \to KH(\mathcal{A}(\mathcal{C}))$ is an equivalence. It follows that φ'' is an equivalence too. □

Proposition 5.5. *The functor* $KH : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ *satisfies the standing assumptions.*

Proof. All assumptions except iv) follow from Theorem 5.2. Assumption iv) follows from the proof of Proposition 5.3, and also from combining the statement of that proposition with Lemma 3.4.3. □

6. Cyclic homology

Let A be a ring, and M an A-bimodule. If $a \in A$ and $m \in M$, write $[a, m] = am - ma$ and

$$[A, M] = \left\{ \sum_i [a_i, m_i]:\ a_i \in A, m_i \in M \right\}, \qquad M_\natural = M/[A, M].$$

Let B be another ring. We say that B is an algebra over A if B is equipped with an A-bimodule structure such that the multiplication $B \otimes B \to B$ factors through an A-bimodule map $B \otimes_A B \to B$. Consider the graded abelian group given in degree n by the $n+1$ tensor power modulo A-bimodule commutators:

$$T(B/A)_n = \left(B^{\otimes_A n+1}\right)_\natural.$$

Note $T(B/A)$ is a quotient of $T(B/\mathbb{Z})$. Recall from [19, §2.5] that a *cyclic module* over a commutative ring k is a contravariant functor from Connes' cyclic category to k-modules; in this paper all cyclic modules are over $\mathbb{Z}$. If B is unital, then $T(B/\mathbb{Z})$ carries a canonical cyclic module structure [33, Section 9.6]; if A is unital also, and the A-bimodule structure on B comes from a unital homomorphism $A \to B$, then the structure passes down to the quotient; we write $C(B/A)$ for $T(B/A)$ equipped with this cyclic module structure. The cyclic theory of B/A, which includes Hochschild, cyclic, negative cyclic and periodic cyclic homology, is that of $C(B/A)$. If A is unital but B is not, one can unitalize B as an A-algebra by $\tilde{B}_A = B \oplus A$, $(b, a)(b', a') = (bb' + ba' + ab', aa')$; the cyclic theory of B/A is that of the cyclic module $C(\tilde{B}_A : B/A) = \ker(C(\tilde{B}_A/A) \to C(A/A))$. In the unital case, there is a natural quasi-isomorphism $C(B/A) \to C(\tilde{B}_A : B/A)$. In the general case, when neither A nor B is assumed to be unital, then B has a canonical $\tilde{A}$-algebra structure, and the cyclic theory of B as an A-algebra is that of B as an $\tilde{A}$-algebra; we put $M(B/A) = C(\tilde{B}_A : B/\tilde{A})$. Note that if A is unital, then $M(B/A) = C(\tilde{B}_A : B/A)$, whence there is no ambiguity. We use the following notation for homology; we write $HH(B/A) = (M(B/A), b)$ for the Hochschild complex, $HH(B/A)_n$ for its degree n-summand, and $HH_n(B/A)$ for its nth homology group. We use the same convention with cyclic, negative cyclic and periodic cyclic homology, which we denote HC, HN and HP.

Let ℓ be a commutative unital ring and R a unital ℓ-algebra. Recall that R is called *separable* over ℓ if R is projective as an $R \otimes_\ell R^{op}$-module. Recall also that a map of mixed complexes is called a *quasi-isomorphism* if it induces an isomorphism in Hochschild homology; this automatically implies that it also induces an isomorphism in cyclic, negative cyclic, and periodic cyclic homologies.

Lemma 6.1. *Let I be a filtering poset, $I \to \mathrm{ar}(\mathrm{Rings})$, $i \mapsto \{A_i \to B_i\}$ a functor to the category of ring homomorphisms, and $A \to B = \mathrm{colim}_i(A_i \to B_i)$. Assume that $A_i \to B_i$ is unital for all i. Put $C(B/A) = \mathrm{colim}_i\, C(B_i/A_i)$. Then $C(B/A) \to M(B/A)$ is a quasi-isomorphism. If furthermore each A_i is separable over $\mathbb{Z}$, then also $C(B) = C(B/\mathbb{Z}) \to C(B/A)$ is a quasi-isomorphism.*

Proof. The first assertion follows from the fact that both C and M commute with filtering colimits, and that the map is a quasi-isomorphism in the unital case [19, Theorem 1.2.13]. The second assertion follows similarly from the unital case. □

Example 6.2. Let $\mathcal{C}$ be a small $\mathbb{Z}$-linear category. We have an injective functor $\mathbb{Z}\langle \mathrm{ob}\,\mathcal{C}\rangle \to \mathcal{C}$, and thus a homomorphism $\mathcal{A}(\langle \mathrm{ob}\,\mathcal{C}\rangle) = \mathbb{Z}^{(\mathrm{ob}\,\mathcal{C})} \to \mathcal{A}(\mathcal{C})$, which is the filtering colimit over the finite subsets $X \subset \mathrm{ob}\,\mathcal{C}$, of the functor $X \mapsto (\mathcal{A}(X) \to \mathcal{A}(\mathcal{C}_X))$. Here $\mathcal{C}_X \subset \mathcal{C}$ is the full subcategory whose objects are the elements of X. Since $\mathcal{A}(X)$ is separable, the natural maps $C(\mathcal{A}(\mathcal{C})) \to C(\mathcal{A}(\mathcal{C})/\mathbb{Z}^{(\mathrm{ob}\,\mathcal{C})}) \to M(\mathcal{A}(\mathcal{C})/\mathbb{Z}^{(\mathrm{ob}\,\mathcal{C})})$ are quasi-isomorphisms, by Lemma 6.1. Put

$$C(\mathcal{C}) = C\left(\mathcal{A}(\mathcal{C})/\mathbb{Z}^{(\mathrm{ob}\,\mathcal{C})}\right).$$

Note that this cyclic module is functorial on $\mathbb{Z} - \mathrm{Cat}$, even though as we have seen in (3.2.7), $\mathcal{A}(-)$ is only functorial on $\mathrm{inj} - \mathbb{Z} - \mathrm{Cat}$. The cyclic module $C(\mathcal{C})$ is often called the *$\mathbb{Z}$-linear cyclic nerve* of $\mathcal{C}$ [21, §4.2]. The cyclic theory of a $\mathbb{Z}$-linear category $\mathcal{C}$ is that of $C(\mathcal{C})$. Note that if R is a unital ring considered as a $\mathbb{Z}$-linear category with one object, then $C(R)$ is the same cyclic module that was defined above.

Remark 6.3. As explained in Example 6.2 above, the projection

$$C\big(\mathcal{A}(\mathcal{C})\big) \to C\big(\mathcal{A}(\mathcal{C})/\mathbb{Z}^{(\operatorname{ob}\mathcal{C})}\big) = C(\mathcal{C})$$

is a quasi-isomorphism. This map has a left inverse $C(\mathcal{C}) \to C(\mathcal{A}(\mathcal{C}))$; namely the inclusion

$$C(\mathcal{C})_n = \bigoplus_{(c_0,\dots,c_n)\in\operatorname{ob}\mathcal{C}^{n+1}} \hom_{\mathcal{C}}(c_1,c_0)\otimes\cdots\otimes\hom_{\mathcal{C}}(c_0,c_n) \subset \mathcal{A}(\mathcal{C})^{\otimes n+1} = C\big(\mathcal{A}(\mathcal{C})\big)_n.$$

This inclusion is a quasi-isomorphism, and is compatible with the map (4.2.6); indeed they both form part of a map of distinguished triangles:

$$\begin{array}{ccccc} C(\mathcal{C}) & \longrightarrow & C(\tilde{\mathcal{C}}) & \longrightarrow & C(\mathbb{Z}\langle\operatorname{ob}\mathcal{C}\rangle) \\ \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle \phi} & & \downarrow \\ C(\mathcal{A}(\mathcal{C})) & \longrightarrow & C(\widetilde{\mathcal{A}(\mathcal{C})}) & \longrightarrow & C(\mathbb{Z}) \end{array}$$

M. Wodzicki [35] characterized the rings A such that whenever

$$0 \to A \to B \to C \to 0$$

is a pure exact sequence of rings – the definition of pure is recalled in (A.3.2) – the sequence

$$HH(A) \to HH(B) \to HH(C)$$

is a distinguished triangle. He showed that those rings are the same as those for which the analogous diagram with HC substituted for HH is a triangle, and that they are precisely the H-unital rings, whose definition is recalled in Appendix A (see Appendix A.3).

Proposition 6.4. *Hochschild and cyclic homology both satisfy the standing assumption* iv)*. The remaining assumptions hold with H-unital substituted for E-excisive.*

Proof. The first assertion follows from Remark 6.3. Rings with local units, and more generally s-unital rings are H-unital by [35, Corollary 4.5]. By Proposition A.6.4, $A \rtimes G$ is H-unital for every H-unital G-ring A. It is clear from the definition of H-unitality that H-unital rings are closed under filtering colimits. Thus it suffices to verify standing assumption iii) for finite X, and this is [35, Corollary 9.8]. Finally assumption v) is proved in Proposition A.4.6. □

Remark 6.5. The properties established in Section 3.3 for functors satisfying the standing assumptions carry over mutatis mutandis to HH and HC. Lemma 3.3.6 and part a) of Proposition 3.3.9 hold verbatim. Part b) of Proposition 3.3.9 holds with H-unital substituted for E-excisive, under the requirement that the exact sequence therein be pure. Corollary 3.3.11 and Proposition 3.3.12 hold with H-unital substituted for E-excisive.

7. Assembly for Hochschild and cyclic homology

Let G be a group, S a G-set and R a unital G-ring. We have a direct sum decomposition

$$C\big(R \rtimes \mathcal{G}^G(S)\big) = \bigoplus_{(g)\in\operatorname{con}(G)} C_{(g)}\big(R \rtimes \mathcal{G}^G(S)\big) \tag{7.1}$$

here $\operatorname{con}(G)$ is the set of conjugacy classes and $C_{(g)}(R \rtimes \mathcal{G}^G(S))_n$ is generated by those elementary tensors $x_0 \rtimes g_0 \otimes \cdots \otimes x_n \rtimes g_n$ with $g_0 \cdots g_n \in (g)$. If $g \in G$, we write R_g for R considered as a bimodule over itself with the usual left multiplication and the right multiplication given by $x \cdot r = xg(r)$. In Proposition 7.5, we shall need the absolute Hochschild homology of R with coefficients in R_g. In general if M is any R-bimodule, we write $HH(R, M)$ for the Hochschild complex with coefficients in M [33, §9.1.1].

Proposition 7.5 below computes the G-equivariant homology of a G-simplicial set X with coefficients in $HH(R)$ for an arbitrary unital G-ring R. The case when G acts trivially on R was obtained by Lück and Reich in [21]. The case when X is a point may be regarded as a transport groupoid version of Lorenz' computation of $HH(R \rtimes G)$ [20]; $HC(R \rtimes G)$ was computed by Feĭgin and Tsygan in [10]. Our proof uses ideas from each of the three cited articles.

Lemma 7.2. *Let G be a group, S a G-set, $g \in G$, and $Z_g \subset G$ the centralizer of g. Write $\mathcal{E}Z_g := \mathcal{E}(Z_g, \{1\})$. Then there is a natural weak equivalence of simplicial abelian groups*

$$\mathbb{Z}[\mathcal{E}Z_g]\otimes_{\mathbb{Z}[Z_g]}\big(\mathbb{Z}[S^g]\otimes HH(R,R_g)\big)\xrightarrow{\sim} HH_{(g)}\big(R\rtimes\mathcal{G}^G(S)\big). \tag{7.3}$$

Taking homotopy groups one obtains the relative Tor *groups* [33, 8.7.5]:

$$\pi_* HH_{(g)}\big(R\rtimes\mathcal{G}^G(S)\big)=\operatorname{Tor}_*^{[(R\otimes R^{op})\rtimes Z_g]/\mathbb{Z}}(R,R_g).$$

Proof. Note that

$$\big[\mathbb{Z}[\mathcal{E}Z_g]\otimes_{\mathbb{Z}[Z_g]}\big(\mathbb{Z}[S^g]\otimes HH(R,R_g)\big)\big]_n=\mathbb{Z}[Z_g]^{\otimes n}\otimes\mathbb{Z}[S^g]\otimes R^{\otimes n+1}.$$

Define a map

$$\begin{aligned}
&\alpha:\mathbb{Z}[\mathcal{E}Z_g]\otimes_{\mathbb{Z}[Z_g]}\big(\mathbb{Z}[S^g]\otimes HH(R,R_g)\big)\to HH_{(g)}\big(R\rtimes\mathcal{G}^G(S)\big),\\
&\alpha(z_1\otimes\cdots\otimes z_n\otimes s\otimes x_0\otimes\cdots\otimes x_n)\\
&\quad= x_0\rtimes(z_1\cdots z_n)^{-1}g\otimes(z_1\cdots z_n)(x_1)\rtimes z_1\otimes(z_2\cdots z_n)(x_2)\rtimes z_2\otimes\cdots\otimes z_n(x_n)\rtimes z_n\\
&\quad\in \hom_{R\rtimes\mathcal{G}^G(S)}(z_1\cdots z_n s,s)\otimes\cdots\otimes\hom_{R\rtimes\mathcal{G}^G(S)}(s,z_n s).
\end{aligned}$$

One checks that α is a simplicial homomorphism. Write $U=(R\otimes R^{op})\rtimes Z_g$. To prove that α is a weak equivalence, and also that its domain and codomain both compute $\operatorname{Tor}_*^{U/\mathbb{Z}}(R,R_g)$, it suffices to find simplicial resolutions $P\xrightarrow{\sim}R_g$ and $Q\xrightarrow{\sim}R_g$ by relatively projective U-modules and a simplicial module homomorphism $\hat{\alpha}:P\to Q$ covering the identity of R_g and such that $R\otimes_U\hat{\alpha}=\alpha$. We need some notation. Write $\mathrm{E}(Z_g,M)$ for the simplicial $\mathbb{Z}[Z_g]$-module resolution of a left Z_g-module M associated to the cotriple $N\mapsto\mathbb{Z}[Z_g]\otimes N$ [33, 8.6.11]. Let $C^{bar}(R,R_g)$ be the bar resolution (Appendix A.2); Z_g acts diagonally on $\mathbb{Z}[S^g]\otimes C^{bar}(R,R_g)$. Write $P=\mathrm{E}(Z_g,\mathbb{Z}[S^g]\otimes C^{bar}(R,R_g))$ for the diagonal of the bisimplicial module $([p],[q])\mapsto \mathrm{E}_p(Z_g,\mathbb{Z}[S^g]\otimes C^{bar}(R,R_g)_q)$. By construction, $P\xrightarrow{\sim}R_g$ is a simplicial U-module resolution, and every U-module P_n is extended from $\mathbb{Z}$, whence relatively projective. Next, given $k\in G$, consider the simplicial submodule $V(k)\subset C^{bar}(R\rtimes\mathcal{G}^G(S))$ generated by the elementary tensors

$$x_0\rtimes h_0\otimes\cdots\otimes x_{n+1}\rtimes h_{n+1}\in\hom_{R\rtimes\mathcal{G}^G(S)}(h_1\cdots h_{n+1}s,ks)\otimes\cdots\otimes\hom_{R\rtimes\mathcal{G}^G(S)}(s,h_{n+1}s)$$

with $s\in S$ and $h_0\cdots h_{n+1}=k$ $(n\geqslant 0)$. Put $Q=V(g)$; note Q is stable under multiplication by elements of the form $a\rtimes z\otimes b\rtimes z^{-1}\in(R\rtimes G)\otimes(R\rtimes G)^{op}$ with $z\in Z_g$. We have a ring homomorphism

$$\iota:U\to(R\rtimes G)\otimes(R\rtimes G)^{op}a\otimes b\rtimes z\mapsto a\rtimes z\otimes b^z\rtimes z^{-1}.$$

Thus Q is a simplicial U-module. We have an isomorphism of graded U-modules

$$\theta:\bigoplus_{\bar{h}\in G/Z_g}\bigoplus_{k\in G}U\otimes V(k)\to Q\,\theta\big((a\otimes b\rtimes z)\otimes v\big)=\iota(a\otimes b\rtimes z)\cdot\big(1\rtimes h\otimes v\otimes 1\rtimes(hk)^{-1}g\big).$$

In particular each U-module Q_n is extended from $\mathbb{Z}$. Next observe that the augmentation of $C^{bar}(R\rtimes G)$ restricts to an augmentation

$$Q\to R\rtimes g\cong R_g \tag{7.4}$$

and that the canonical contracting chain homotopy $x\mapsto 1\otimes x$ induces a contracting homotopy for (7.4). Thus (7.4) is a simplicial resolution by relatively projective U-modules. Consider the map

$$\begin{aligned}
&\hat{\alpha}:P\to Q,\\
&\hat{\alpha}(z_0\otimes\cdots\otimes z_n\otimes s\otimes x_0\otimes\cdots\otimes x_{n+1})\\
&\quad=(z_0\cdots z_n)(x_0)\rtimes z_0\otimes\cdots\otimes z_n(x_n)\rtimes z_n\otimes x_{n+1}\rtimes(z_0\cdots z_n)^{-1}g\\
&\quad\in\hom_{R\rtimes\mathcal{G}^G(S)}(z_0^{-1}s,s)\otimes\cdots\otimes\hom_{R\rtimes\mathcal{G}^G(S)}\big(s,(z_0\cdots z_n)^{-1}s\big).
\end{aligned}$$

One checks that $\hat{\alpha}$ is a simplicial U-module homomorphism covering the identity of R_g and that $R\otimes\hat{\alpha}=\alpha$, concluding the proof. □

Proposition 7.5. *(Compare [21, 9.16].) Let G be a group, $X\in\mathbb{S}^G$. For each $\xi\in\operatorname{con}(G)$ choose a representative g_ξ. Then there is an isomorphism*

$$\bigoplus_{\xi\in\operatorname{con}(G)}\mathbb{H}_*\big(Z_{g_\xi},\mathbb{Z}[X^{g_\xi}]\otimes HH(R,R_{g_\xi})\big)\xrightarrow{\cong}H_*^G\big(X,HH(R)\big)$$

natural in X and R, which depends on the choice of representatives $\{g_\xi:\ \xi\in\operatorname{con}(G)\}$. Here $\mathbb{H}(Z_g,-)$ is hyperhomology of complexes of Z_g-modules, and the tensor product is equipped with the diagonal action.

Proof. By (7.1) we have

$$H_*^G(X, HH(R)) = \bigoplus_{\xi \in \mathrm{con}(G)} H_*^G(X, HH_{(\xi)}(R))$$

By Lemma 7.2 and the definition of equivariant homology, if $g \in \xi$, then

$$\begin{aligned} H^G(X, HH_\xi(R)) &= \int^{\mathrm{Or}\, G} HH_\xi(R \rtimes \mathcal{G}^G(G/H)) \otimes \mathbb{Z}[\mathrm{map}(G/H, X)] \\ &\xleftarrow{\sim} \int^{\mathrm{Or}\, G} (\mathbb{Z}[\mathcal{E}Z_g] \otimes_{\mathbb{Z}[Z_g]} [\mathbb{Z}[\mathrm{map}(G/\langle g\rangle, G/H)] \otimes HH(R, R_g)]) \otimes \mathbb{Z}[\mathrm{map}(G/H, X)] \\ &= \mathbb{Z}[\mathcal{E}Z_g] \otimes_{\mathbb{Z}[Z_g]} (\mathbb{Z}[X^g] \otimes HH(R, R_g)) = \mathbb{H}(Z_g, \mathbb{Z}[X^g] \otimes HH(R, R_g)). \qquad \square \end{aligned}$$

Proposition 7.6. *Let G be a group, $\mathcal{F}$ a family of subgroups of G and A an H-unital G-ring. Assume that $\mathcal{F}$ contains all cyclic subgroups of G. Then the functor $H^G(-, HH(A))$ preserves $(G, \mathcal{F})$-weak equivalences. In particular, the assembly map*

$$H_*^G(\mathcal{E}(G, \mathcal{F}), HH(A)) \to HH_*(A \rtimes G)$$

is an isomorphism. The analogue statements for cyclic homology also hold.

Proof. In view of Corollary 3.3.11 and Remark 6.5, it suffices to prove the proposition with a unital ring R substituted for A. The first statement about Hochschild homology follows from 7.5, and the fact that if K is a group, then $\mathbb{H}(K, -)$ preserves quasi-isomorphisms. The second follows from the first and the fact that $\mathcal{E}(G, \mathcal{F}) \to *$ is an equivalence. Next, given a cyclic module M, consider the subcomplex

$$\mathcal{HC}^n(M) = \ker(S^n : HC(M) \to HC(M)[-2n]).$$

Note that

$$0 = \mathcal{HC}^0(M) \subset HH(M) = \mathcal{HC}^1(M) \subset \mathcal{HC}^2(M) \subset \cdots \subset \bigcup_n \mathcal{HC}^n(M) = HC(M)$$

is an exhaustive filtration. Hence, because $H^G(X, -)$ preserves filtering colimits ($X \in \mathbb{S}^G$), to prove the statement of the lemma for cyclic homology, it is sufficient to show that for each n, $H^G(-, \mathcal{HC}^n(R))$ preserves $(G, \mathcal{F})$-equivalences of G-simplicial sets. Observe that if M is a cyclic module, then we have an exact sequence

$$0 \to \mathcal{HC}^n(M) \to \mathcal{HC}^{n+1}(M) \to HH(M)[-2n] \to 0.$$

Using the sequence above and what we have already proved, one shows by induction that $H^G(-, \mathcal{HC}^n(R))$ preserves $(G, \mathcal{F})$-equivalences. This finishes the proof. $\square$

8. The Chern character and infinitesimal K-theory

8.1. Nonconnective Chern character

Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category. By results of Randy McCarthy [25, §3.3 and §4.4] we have a Chern character

$$K^Q(\mathrm{Idem}\, \mathcal{C}_\oplus) \to |\tau_{\geq 0} HN(\mathrm{Idem}\, \mathcal{C}_\oplus)| \tag{8.1.1}$$

going from the K-theory simplicial set to the simplicial set obtained via the Dold–Kan correspondence from the good truncation of the negative cyclic complex without negative terms. In this section we use McCarthy's construction to obtain a map

$$K(\mathcal{C}) \to |HN(\mathcal{C})|$$

going from the nonconnective K-theory spectrum of Section 4 to the spectrum obtained from the negative cyclic complex via Dold–Kan correspondence. We shall need the following result of [25].

Proposition 8.1.2. *(See [25, Theorem 2.3.4].) Let $\mathcal{D}$ be a $\mathbb{Z}$-linear category and $\mathcal{C} \subset \mathcal{D}$ a full subcategory. Assume that for every object $d \in \mathcal{D}$ there exists an $n = n(d)$, a finite sequence $c_1, \ldots, c_n$ of objects of $\mathcal{C}$, and morphisms $\phi_i : c_i \to d$ and $\psi_i : d \to c_i$ such that $\sum_i \phi_i \psi_i = 1_d$. Then the inclusion functor $\mathcal{C} \to \mathcal{D}$ induces a quasi-isomorphism $C(\mathcal{C}) \to C(\mathcal{D})$.*

Lemma 8.1.3. *Let $\mathcal{C}$ be an additive category, and let $\bullet$ be the only object of $\Gamma(\mathbb{Z})$. Consider the functor*

$$\mu : \Gamma\mathbb{Z} \otimes \mathcal{C} \to \Gamma(\mathcal{C}), \qquad \mu(\bullet, c) = (c, c, \ldots), \qquad \mu(f \otimes \alpha)_{ij} = f_{ij}\alpha.$$

Then

i) *The functor μ is fully faithful.*
ii) *Let $F(-)$ be as in (4.1.1). For every object $x \in \Gamma(\mathcal{C})$ there exist morphisms $\phi_c : \mu(\bullet, c) \to x$ and $\psi_c : x \to \mu(\bullet, c)$, $c \in F(x)$ such that $\sum_{c\in F(x)} \phi_c\psi_c = 1_x$.*
iii) *The functor μ induces a fully faithful functor $\bar{\mu} : \Sigma \otimes \mathcal{C} \to \Sigma(\mathcal{C})$.*

Proof. Part i) is proved in [6, Lemma 4.7.1] for the case when $\mathcal{C}$ has only one object; the same argument applies in general. To prove ii), let $x \in \Gamma(\mathcal{C})$ be an object. If $c \in F(x)$, write $I(c) = \{n \in \mathbb{N}\colon\ x_n = c\}$, and let $\chi_{I(c)}$ be the characteristic function. Put

$$\phi_c : \mu(\bullet, c) \to x, \qquad \psi_c : x \to \mu(\bullet, c), \qquad (\phi_c)_{i,j} = (\psi_c)_{i,j} = \delta_{i,j}\chi_{I(c)}(j)1_c.$$

One checks that

$$\sum_{c\in F(x)} \phi_c\psi_c = 1_x.$$

This proves ii). Next, consider the exact sequence

$$0 \to M_\infty\mathbb{Z} \to \Gamma\mathbb{Z} \xrightarrow{\pi} \Sigma\mathbb{Z} \to 0.$$

As is explained in [6, p. 92], it follows from results of Nöbeling [26] that the sequence above is split as a sequence of abelian groups. Hence if $c, d \in \mathcal{C}$, then

$$\ker\big(\pi \otimes 1 : \hom_{\Gamma\mathbb{Z}\otimes\mathcal{C}}\big((\bullet, c), (\bullet, d)\big) \to \hom_{\Sigma\mathbb{Z}\otimes\mathcal{C}}\big((\bullet, c), (\bullet, d)\big)\big) = M_\infty\mathbb{Z} \otimes \hom_{\mathcal{C}}(c, d).$$

Next observe that if $\alpha \in \hom_{\mathcal{C}}(c, d)$ and $f \in M_\infty\mathbb{Z}$, then $\mu(f \otimes \alpha)$ is a finite morphism. Hence μ passes to the quotient, inducing a functor $\bar{\mu} : \Sigma\mathbb{Z} \otimes \mathcal{C} \to \Sigma(\mathcal{C})$. If $c, d \in \operatorname{ob}\mathcal{C}$ and we put $x = \mu(\bullet, c)$, $y = \mu(\bullet, d)$ then we have a map of exact sequences

$$\begin{array}{ccccc}
0 \to M_\infty\mathbb{Z} \otimes \hom_{\mathcal{C}}(c, d) & \longrightarrow & \Gamma\mathbb{Z} \otimes \hom_{\mathcal{C}}(c, d) & \longrightarrow & \Sigma\mathbb{Z} \otimes \hom_{\mathcal{C}}(c, d) \to 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \hom_{\mathrm{Fin}(\mathcal{C})}(x, y) & \longrightarrow & \hom_{\Gamma(\mathcal{C})}(x, y) & \longrightarrow & \hom_{\Sigma(\mathcal{C})}(x, y) \to 0
\end{array}$$

Here $\mathrm{Fin}(\mathcal{C}) \subset \Gamma(\mathcal{C})$ is the subcategory of finite morphisms. The second vertical map is an isomorphism by part i). In particular the first map is injective; furthermore, one checks that it is onto. It follows that the third vertical map is an isomorphism; this proves iii). $\square$

Proposition 8.1.4. *Let $\mathcal{C}$ be a $\mathbb{Z}$-linear category. Then*:

i) *$C(\mathcal{C}) \to C(\mathcal{C}_\oplus)$ is a quasi-isomorphism.*
ii) *If $\mathcal{C}$ is additive, then $C(\mathcal{C}) \to C(\operatorname{Idem}\mathcal{C})$ is a quasi-isomorphism.*
iii) *The maps $C(\Gamma(\mathbb{Z}) \otimes \mathcal{C}) \to C(\Gamma(\mathcal{C}))$, $C(\Gamma(\mathcal{C})) \to 0$ and $C(\Sigma(\mathbb{Z}) \otimes \mathcal{C}) \to C(\Sigma(\mathcal{C}))$ are quasi-isomorphisms.*
iv) *The sequence*

$$\operatorname{Idem}\mathcal{C}_\oplus \to \Gamma\mathcal{C}_\oplus \to \Sigma\mathcal{C}_\oplus$$

induces a distinguished triangle of Hochschild, cyclic, negative cyclic and periodic cyclic complexes.

Proof. The first two assertions are straightforward applications of Proposition 8.1.2. That $C(\Gamma(\mathbb{Z}) \otimes \mathcal{C}) \to C(\Gamma(\mathcal{C}))$ and $C(\Sigma(\mathbb{Z}) \otimes \mathcal{C}) \to C(\Sigma(\mathcal{C}))$ are quasi-isomorphisms follows from Proposition 8.1.2 and Lemma 8.1.3. In particular we have quasi-isomorphisms

$$\begin{array}{ccccc}
C(\Gamma(\mathcal{C})) & \xleftarrow{\sim} & C(\mathcal{A}(\Gamma\mathbb{Z} \otimes \mathcal{C})/\mathcal{A}(\operatorname{ob}(\Gamma\mathbb{Z} \otimes \mathcal{C})) & \xleftarrow{\sim} & C(\mathcal{A}(\Gamma\mathbb{Z} \otimes \mathcal{C})) \\
 & & & & \| \\
 & & HH(\Gamma\mathcal{A}(\mathcal{C})) & \xleftarrow[\sim]{} & C(\Gamma\mathcal{A}(\mathcal{C}))
\end{array}$$

But because $\mathcal{A}(\mathcal{C})$ is H-unital, $HH(\Gamma\mathcal{A}(\mathcal{C}))$ is acyclic by [35, Theorem 10.1]. To prove iv), consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \Gamma\mathcal{C} & \longrightarrow & \Sigma\mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Idem}\mathcal{C}_\oplus & \longrightarrow & \Gamma\mathcal{C}_\oplus & \longrightarrow & \Sigma\mathcal{C}_\oplus \end{array} \tag{8.1.5}$$

By i) and ii), the first vertical map induces quasi-isomorphisms of cyclic modules. If R is a unital ring flat as a $\mathbb{Z}$-module, then the quasi-isomorphism $C(\mathcal{C}) \to C(\mathcal{C}_\oplus)$ of i) induces a quasi-isomorphism $C(R\otimes\mathcal{C}) = C(R)\otimes C(\mathcal{C}) \to C(R\otimes\mathcal{C}_\oplus)$. In particular this applies when $R = \Gamma\mathbb{Z}, \Sigma\mathbb{Z}$. Hence the second and third vertical maps in (8.1.5) are quasi-isomorphisms as well, by iii). By Lemma 6.1 the cyclic modules of the top row are quasi-isomorphic to the cyclic modules of their associated rings; thus iv) reduces to the fact, proved in [35, §10], that the sequence

$$C(\mathcal{A}(\mathcal{C})) \to C(\Gamma\mathcal{A}(\mathcal{C})) \to C(\Sigma\mathcal{A}(\mathcal{C}))$$

induces distinguished triangles for HH, HC, HN and HP. □

Let $\mathcal{C}^{(n)}$ be as in (4.1.3). Observe that by Proposition 8.1.4, we have an equivalence $|\tau_{\geqslant 0}HN(\mathcal{C}^{(n)})| \xrightarrow{\sim} |\tau_{\geqslant 0}HN(\mathcal{C})[+n]|$. Composing with the map $K^{\mathbb{Q}}(\mathcal{C}^{(n)}) \to |\tau_{\geqslant 0}HN(\mathcal{C}^{(n)})|$ we obtain a sequence $K^{\mathbb{Q}}(\mathcal{C}^{(n)}) \to \tau_{\geqslant 0}HN(\mathcal{C})[+n]$ which induces a map of nonconnective spectra

$$ch : K(\mathcal{C}) \to \big|HN(\mathcal{C})\big|. \tag{8.1.6}$$

Remark 8.1.7. If $\mathcal{C}$ has only one object, then the Chern character (8.1.6) agrees with the usual one. This follows from (4.1.6) and the ring analogue of Proposition 8.1.4, part iv), proved in [35, §10]. Furthermore, for any $\mathbb{Z}$-linear category $\mathcal{C}$, the character (8.1.6) agrees with that of $\mathcal{A}(\mathcal{C})$. Indeed, $K(\mathcal{A}(\mathcal{C})) \xrightarrow{\sim} K(\mathcal{C})$ by Proposition 4.2.8, and the proof of Proposition 8.1.4 makes clear that the homology sequences of iv) are equivalent to the corresponding sequences for $\mathcal{A}(\mathcal{C})$.

8.2. K^{nil} and the relative Chern character

Let $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ be a functor and $\mathcal{C} \in \mathbb{Z} - \mathrm{Cat}$. Consider the homotopy fiber

$$E^{\mathrm{nil}}(\mathcal{C}) = \mathrm{hofiber}\big(E(\mathcal{C}) \to E\big(\mathcal{C}\otimes\mathbb{Z}^{\Delta^\bullet}\big)\big).$$

Write

$$ch^{\Delta} : KH(\mathcal{C}) = K\big(\mathcal{C}\otimes\mathbb{Z}^{\Delta^\bullet}\big) \to HN\big(\mathcal{C}\otimes\mathbb{Z}^{\Delta^\bullet}\big)$$

for the result of applying the map $K \to HN$ dimensionwise. We have a map of spectra $ch^{\mathrm{nil}} : K^{\mathrm{nil}}(\mathcal{C}) \to HN^{\mathrm{nil}}(\mathcal{C})$ which fits into a map of homotopy fibrations

$$\begin{array}{ccccc} K^{\mathrm{nil}}(\mathcal{C}) & \longrightarrow & K(\mathcal{C}) & \longrightarrow & KH(\mathcal{C}) \\ \downarrow{\scriptstyle ch^{\mathrm{nil}}} & & \downarrow{\scriptstyle ch} & & \downarrow{\scriptstyle ch^{\Delta}} \\ |HN^{\mathrm{nil}}(\mathcal{C})| & \longrightarrow & |HN(\mathcal{C})| & \longrightarrow & |HN(\mathcal{C}\otimes\mathbb{Z}^{\Delta^\bullet})| \end{array}$$

Lemma 8.2.1. *Let $\mathcal{C}$ be a $\mathbb{Q}$-linear category. Then there is a homotopy commutative diagram with vertical weak equivalences*

$$\begin{array}{ccccc} |HN^{\mathrm{nil}}(\mathcal{C})| & \longrightarrow & |HN(\mathcal{C})| & \longrightarrow & |HN(\mathcal{C}\otimes\mathbb{Z}^{\Delta^\bullet})| \\ \downarrow{\scriptstyle \iota} & & \| & & \downarrow{\scriptstyle \wr} \\ \Omega^{-1}|HC(\mathcal{C})| & \longrightarrow & |HN(\mathcal{C})| & \longrightarrow & HP(\mathcal{C}) \end{array}$$

Proof. By Example 6.2, this is a statement about the $\mathbb{Q}$-algebra $\mathcal{A}(\mathcal{C})$. The latter is proved in [11, Theorem 4.1]. □

By [34, Proposition 1.6], if A is a $\mathbb{Q}$-algebra the groups $K_*^{\mathrm{nil}}(A)$ are $\mathbb{Q}$-vectorspaces. Hence for every ring A we have a map

$$q : K^{\mathrm{nil}}(A)\otimes\mathbb{Q} \to K^{\mathrm{nil}}(A\otimes\mathbb{Q}) \tag{8.2.2}$$

which is an equivalence if A is a $\mathbb{Q}$-algebra. We write

$$\nu = \iota ch^{\mathrm{nil}}(-\otimes\mathbb{Q})q : K^{\mathrm{nil}}(\mathcal{C})\otimes\mathbb{Q} \to \Omega^{-1}\big|HC(\mathcal{C}\otimes\mathbb{Q})\big| \xleftarrow{\sim} \Omega^{-1}\big|HC(\mathcal{C})\big|\otimes\mathbb{Q}, \qquad K^{\mathrm{ninf}}(\mathcal{C}) = \mathrm{hofiber}(\nu). \tag{8.2.3}$$

We remark that ν is a variant of the relative character introduced by Weibel in [32].

Proposition 8.2.4. $K^{\mathrm{ninf}} : \mathbb{Z}-\mathrm{Cat} \to \mathrm{Spt}$ *satisfies the standing assumptions. In addition, it is excisive and* K_*^{ninf} *commutes with filtering colimits.*

Proof. It is proved in [3] that

$$K^{\mathrm{inf},\mathbb{Q}} := \mathrm{hofiber}\big(ch^{\mathbb{Q}} : K(-)\otimes\mathbb{Q} \to HN(-\otimes\mathbb{Q})\big) \tag{8.2.5}$$

is excisive; it follows that K^{ninf} is excisive too. Next observe that K^{ninf} satisfies iii) and v) of Standing Assumptions 3.3.2 for unital rings, and iv) for all $\mathcal{C}\in\mathbb{Z}-\mathrm{Cat}$, since both K^{nil} and HC do. Because K^{ninf} is excisive, this implies that it satisfies all standing assumptions, by Remark 3.3.5. Finally K_*^{ninf} commutes with filtering colimits because both K_*^{nil} and HC_* do. □

9. Rings of polynomial functions on a simplicial set

In this section we introduce $\mathbb{Z}^{(X)}$ the ring of polynomial functions on a simplicial set X. We prove an algebraic analogue result from [7] and excision properties needed in the rest of the paper.

9.1. Finiteness

An object K in a category $\mathfrak{A}$ is *small* if $hom_{\mathfrak{A}}(K,-)$ preserves colimits. If $\mathfrak{A}=\mathbb{S}$, then X is small if and only if it has only a finite number of nondegenerate simplices, or, equivalently, if there exists a finite set of nonnegative integers $n_1,\dots,n_r$ and a surjection

$$\coprod_{i=1}^{r}\Delta^{n_i} \twoheadrightarrow X.$$

Small simplicial sets are called *finite*. Similarly, a G-simplicial set is small if there are $n_1,\dots,n_r\geqslant 0$ and a G-equivariant surjection

$$\coprod_{i=1}^{r}\Delta^{n_i}\times G \twoheadrightarrow X.$$

Let $\mathcal{F}$ be a family of subgroups of G. A *finite* $(G,\mathcal{F})$-complex is G-simplicial set obtained by attaching finitely many cells of the form $\Delta^n\times G/H$ with $H\in\mathcal{F}$. A G-finite simplicial set is a finite $(G,\mathcal{A}ll)$-complex. The concept of G-finiteness is the simplicial set version of the concept of G-compactness. Indeed one checks that a G-simplicial set X is G-finite if and only if X/G is finite as a simplicial set.

9.2. Locally finite simplicial sets

If X is a simplicial set and $\sigma\in X$ is a simplex, we write $\langle\sigma\rangle\subset X$ for the simplicial subset generated by σ. We have

$$\langle\sigma\rangle_n = \big\{\alpha^*(\sigma):\ \alpha\in\mathrm{hom}\big([n],[\dim\sigma]\big)\big\}.$$

The *star* of σ is the following set of simplices of X:

$$\mathrm{St}(\sigma)=\mathrm{St}_X(\sigma)=\big\{\tau\in X:\ \langle\tau\rangle\cap\langle\sigma\rangle\neq\emptyset\big\}.$$

The *closed star* is the simplicial subset $\overline{\mathrm{St}}(\sigma)=\langle\mathrm{St}(\sigma)\rangle$ generated by $\mathrm{St}(\sigma)$. If M is a set of simplices of X we put $\mathrm{St}_X(M)=\bigcup_{\sigma\in M}\mathrm{St}_X(\sigma)$, $\overline{\mathrm{St}}_X(M)=\langle\mathrm{St}_X(M)\rangle$. We also define the *link* of M as $\mathrm{Link}(M)=\overline{\mathrm{St}}_X(M)\backslash\mathrm{St}_X(M)$.

Lemma 9.2.1. *Let X be a simplicial set; write NX for the set of nondegenerate simplices. The following are equivalent.*

i) $(\forall\sigma\in X)\{\tau\in NX:\ \langle\tau\rangle\supset\langle\sigma\rangle\}$ *is a finite set.*
ii) *For every $\sigma\in X$, $\overline{\mathrm{St}}_X(\sigma)$ is a finite simplicial set.*

Proof. If $\sigma\in X$, then $\langle\sigma\rangle$ has finitely many nondegenerate simplices, and thus the set $\{\langle\tau\rangle\cap\langle\sigma\rangle:\ \tau\in X\}$ is finite. Hence if i) holds, there are finitely many $\tau\in NX$ such that $\langle\tau\rangle\cap\langle\sigma\rangle\neq\emptyset$; in other words, $NX\cap\mathrm{St}_X(\sigma)$ is a finite set, and therefore $\overline{\mathrm{St}}_X(\sigma)$ is a finite simplicial set. Thus i) ⇒ ii). Next note that $\langle\tau\rangle\supset\langle\sigma\rangle$ implies $\tau\in\mathrm{St}_X(\sigma)$, whence ii) ⇒ i). □

We say that X is *locally finite* if it satisfies the equivalent conditions of the lemma above.

9.3. Rings of polynomial functions on a simplicial set

If X is a simplicial set and A is a ring, we put

$$A^X = \hom_{\mathbb{S}}\big(X, A^{\Delta^\bullet}\big).$$

The simplicial ring $A^{\Delta^\bullet} = A \otimes \mathbb{Z}^{\Delta^\bullet}$ is defined as in (5.1). Note $X \mapsto A^X$, $f \mapsto f^*$ gives a functor $\mathbb{S}^{op} \to \text{Rings}$. By its very definition, the functor A^- sends colimits to limits; if I is a small category and $X : I \to \mathbb{S}$ is a functor, then

$$A^{\operatorname{colim}_i X_i} = \lim_i A^{X_i}.$$

Example 9.3.1. Any simplicial set X is the union of the subobjects generated by each of its nondegenerate simplices; in symbols

$$X = \operatorname*{colim}_{\sigma \in NX} \langle\sigma\rangle.$$

Thus we obtain

$$A^X = \lim_{\sigma \in NX} A^{\langle\sigma\rangle} = \Big\{\phi \in \prod_{\sigma \in NX} A^{\langle\sigma\rangle}\colon \phi(\sigma)_{|\langle\sigma\rangle\cap\langle\tau\rangle} = \phi(\tau)_{|\langle\sigma\rangle\cap\langle\tau\rangle}, \sigma, \tau \in NX\Big\}.$$

If $\phi \in A^X$, then its *support* is

$$\operatorname{supp}(\phi) = \big\langle\big\{\sigma \in X\colon \phi(\sigma) \neq 0\big\}\big\rangle.$$

Note that if $\phi, \psi \in A^X$ and $f : X \to Y$ is a simplicial map, then

$$\operatorname{supp}(\phi \cdot \psi) \subset \operatorname{supp}(\phi) \cap \operatorname{supp}(\psi), \qquad \operatorname{supp}\big(f^*(\phi)\big) \subset f^{-1}\big(\operatorname{supp}(\phi)\big). \tag{9.3.2}$$

We say that ϕ is *finitely supported* if $\operatorname{supp}(\phi)$ is a finite simplicial set. Note ϕ is finitely supported if and only if there is only a finite number of nondegenerate simplices σ such that $\phi(\sigma) \neq 0$. Put

$$A^{(X)} = \big\{f \in A^X\colon \operatorname{supp}(f) \text{ is finite}\big\}.$$

If X is finite, then clearly $A^X = A^{(X)}$. In general, $A^{(X)} \subset A^X$ is a two-sided ideal, by (9.3.2). We remark that if $f : X \to Y$ is an arbitrary map of simplicial sets, then the associated ring homomorphism $f^* : A^Y \to A^X$ does not necessarily send $A^{(Y)}$ into $A^{(X)}$. However, if f happens to be *proper*, i.e. if $f^{-1}(K)$ is finite for every finite $K \subset Y$, then $f^*(A^{(Y)}) \subset A^{(X)}$, by (9.3.2). Hence $A^{(-)}$ is a functor on the category of simplicial sets and proper maps. Next we consider the behavior of this functor with respect to colimits. First of all, if $\{X_i\}$ is a family of simplicial sets, then we have

$$A^{(\coprod X_i)} = \bigoplus_i A^{(X_i)}. \tag{9.3.3}$$

Here $\bigoplus$ indicates the direct sum of abelian groups, equipped with coordinate-wise multiplication. Second, $A^{(-)}$ maps co-equalizers of proper maps to equalizers; if $\{f_j : X \to Y\}$ is a family of proper maps, then

$$A^{(\operatorname{coeq}_j\{f_j : X \to Y\})} = \operatorname{eq}_j\big\{f_j^* : A^{(Y)} \to A^{(X)}\big\}. \tag{9.3.4}$$

Next recall that if I is a small category and $X : I \to \mathbb{S}$ is a functor, then the colimit of X can be computed as a coequalizer:

$$\operatorname*{colim}_i X_i = \operatorname{coeq}\Big(\coprod_{\alpha \in \operatorname{Ar}(I)} X_{s(\alpha)} \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \coprod_{i \in \operatorname{Ob}(I)} X_i\Big).$$

Here $\operatorname{Ob}(I)$ and $\operatorname{Ar}(I)$ are respectively the sets of objects and of arrows of I, and if $\alpha \in \operatorname{Ar}(I)$ then $s(\alpha) \in \operatorname{Ob}(I)$ is its source; we also write $r(\alpha)$ for the range of α. The maps ∂_0 and ∂_1 are defined as follows. The restriction of ∂_i to the copy of $X_{s(\alpha)}$ indexed by α is the inclusion $X_{s(\alpha)} \subset \coprod_j X_j$ if $i = 0$ and the composite of $X(\alpha)$ followed by the inclusion $X_{r(\alpha)} \subset \coprod_j X_j$ if $i = 1$. The conditions that ∂_0 and ∂_1 be proper are equivalent to the following

∂_0) Each object of I is the source of finitely many arrows.
∂_1) Each object of I is the range of finitely many arrows, and X sends each map of I to a proper map.

Example 9.3.5. For example the functor $\sigma \mapsto \langle\sigma\rangle$ from the set of nondegenerate simplices of X, ordered by $\sigma \leqslant \tau$ if $\langle\sigma\rangle \subset \langle\tau\rangle$, always satisfies ∂_1; condition ∂_0 is precisely condition i) of Lemma 9.2.1. Hence ∂_0 is satisfied if and only if X is locally finite, and in that case we have

$$A^{(X)} = \mathrm{eq}\left(\bigoplus_{\sigma\in NX} A^{\langle\sigma\rangle} \overset{\partial_0^*}{\underset{\partial_1^*}{\rightrightarrows}} \bigoplus_{\substack{\langle\tau\rangle\subset\langle\sigma\rangle,\\ \sigma,\tau\in NX}} A^{\langle\tau\rangle} \right).$$

Lemma 9.3.6. *If X is a locally finite simplicial set, then $\mathbb{Z}^{(X)}$ is a free abelian group.*

Proof. By [6, 3.1.3] the lemma is true when X is finite. Hence if X is any simplicial set, and $\sigma \in X$ is a simplex, then $\mathbb{Z}^{\langle\sigma\rangle}$ is free. If X locally finite, then by Example 9.3.5, $\mathbb{Z}^{(X)}$ is a subgroup of a free group, and therefore it is free. □

9.4. Extending polynomial functions

Theorem 9.4.1. *Let X be a simplicial set, $Y \subset X$ a simplicial subset and A a ring. Let $\phi \in A^Y$ and $K = \operatorname{supp}\phi$. Then there exists $\psi \in A^X$ with $\operatorname{supp}\psi \subset \overline{\mathrm{St}}_X K$ such that $\psi_{|\mathrm{Link}_X(K)} = 0$ and $\psi_{|Y} = \phi$.*

Proof. We have $K \subset \mathrm{St}_Y K \subset \overline{\mathrm{St}}_Y K$, whence $\phi_{|\mathrm{Link}_Y(K)} = 0$. Note $\mathrm{St}_X K \cap Y = \mathrm{St}_Y K$; thus ϕ vanishes on $\mathrm{Link}_X(K) \cap Y$. Hence we may extend ϕ to a map $\phi' : Y' = Y \cup \mathrm{Link}_X(K) \to A^{\Delta^\bullet}$ by $\phi'_{|\mathrm{Link}_X(K)} = 0$. Put $Y'' = Y \cup \overline{\mathrm{St}}_X K$. Because $Y' \subset Y''$ is a cofibration and $A^{\Delta^\bullet} \twoheadrightarrow 0$ is a trivial fibration, we may further extend ϕ' to a map $\phi'' : Y'' \to A^{\Delta^\bullet}$. By construction, $\{\sigma \in X : \phi''(\sigma) \neq 0\} \subset \mathrm{St}_X K$, and ϕ'' vanishes on $\mathrm{Link}_X K$. Hence we may finally extend ϕ'' to a map $\psi : X \to A^{\Delta^\bullet}$, by letting $\psi(\sigma) = 0$ if $\sigma \notin \overline{\mathrm{St}}_X K$. This concludes the proof. □

Corollary 9.4.2. *If X is locally finite and $Y \subset X$ is a simplicial subset, then the restriction map $A^{(X)} \to A^{(Y)}$ is surjective.*

Proof. It follows from Theorem 9.4.1, using Lemma 9.2.1. □

9.5. Excision properties

Proposition 9.5.1. *If X is a locally finite simplicial set, then $\mathbb{Z}^{(X)}$ is s-unital.*

Proof. Let $\phi_1, \dots, \phi_n \in \mathbb{Z}^{(X)}$, and let $K = \bigcup_i \operatorname{supp}(\phi_i)$. By Theorem 9.4.1 there is $\mu \in \mathbb{Z}^{(X)}$ such that $\mu_{|K} = 1$ is the constant map. Thus

$$\phi_i = \phi_i \mu \quad (\forall i). \qquad \square \tag{9.5.2}$$

Proposition 9.5.3. *If A is K-excisive and X is locally finite, then $\mathbb{Z}^{(X)} \otimes A$ is K-excisive.*

Proof. Follows from Lemma 9.3.6 and Propositions 9.5.1 and A.5.3. □

Remark 9.5.4. If A is a ring and X a locally finite simplicial set, then there is a natural map

$$\mathbb{Z}^{(X)} \otimes A \to A^{(X)}.$$

It was proved in [6, 3.1.3] that this map is an isomorphism if X is finite.

10. Proper G-rings, induction, compression and restriction

10.1. Proper rings over a G-simplicial set

Fix a group G and consider rings equipped with an action of G by ring automorphisms. We write $G - \mathrm{Rings}$ for the category of such rings and equivariant ring homomorphisms. If $C \in G - \mathrm{Rings}$ is commutative but not necessarily unital and $A \in G - \mathrm{Rings}$, then by a *compatible* (G, C)*-algebra structure* on A we understand a C-bimodule structure on A such that the following identities hold for $a, b \in A$, $c \in C$, and $g \in G$:

$$c \cdot a = a \cdot c, \qquad c \cdot (ab) = (c \cdot a)b = a(c \cdot b), \qquad g(c \cdot a) = g(c) \cdot g(a). \tag{10.1.1}$$

If X is a G-simplicial set and $A \in G - \mathrm{Rings}$, then we say that A is *proper* over X if it carries a compatible $(G, \mathbb{Z}^{(X)})$-algebra structure such that

$$\mathbb{Z}^{(X)} \cdot A = A. \tag{10.1.2}$$

If $\mathcal{F}$ is a family of subgroups of G, we say that A is $(G, \mathcal{F})$-*proper* if it is proper over some $(G, \mathcal{F})$-complex X.

Example 10.1.3. Fix a group G, a family of subgroups $\mathcal{F}$ and a $(G, \mathcal{F})$-complex X. By Proposition 9.5.1, we have $\mathbb{Z}^{(X)} \cdot \mathbb{Z}^{(X)} = \mathbb{Z}^{(X)}$; thus $\mathbb{Z}^{(X)}$ is proper over X. Hence if A is a G-ring with a compatible $(G, \mathbb{Z}^{(X)})$-action, then $\mathbb{Z}^{(X)} \cdot A$ is proper over X. If A is proper over X, and B is any ring, then $A \otimes B$ is proper over X. In particular, $\mathbb{Z}^{(X)} \otimes B$ is proper. If $T \in \mathrm{Top}$ is the geometric realization of X, and $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, then the $\mathbb{F}$-algebra $P = C_{\mathrm{comp}}(T)$ of compactly supported continuous functions $T \to \mathbb{F}$ is proper over X. To check that $\mathbb{Z}^{(X)} \cdot P = P$, observe that if $f \in P$ then its support meets finitely many maximal simplices; write $K \subset X$ for their union. By Corollary 9.4.2, there exists $\phi \in \mathbb{Z}^{(X)}$ which is constantly equal to 1 on K; thus $f = \phi \cdot f \in \mathbb{Z}^{(X)} \cdot P$.

Let X be a locally finite simplicial set, and $Y \subset X$ a subobject. Put

$$I(Y) = \{\phi\colon \operatorname{supp}\phi \subset Y\} \lhd \mathbb{Z}^{(X)}.$$

Note that if $\psi \in \mathbb{Z}^{(Y)}$ and $\hat{\psi} \in \mathbb{Z}^{(X)}$ restricts to ψ, then the product

$$\psi \cdot \phi := \hat{\psi}\phi$$

depends only on ψ. This defines a compatible action of $\mathbb{Z}^{(Y)}$ on $I(Y)$ which makes the latter ring proper over Y. More generally, if $A \in \mathrm{Rings}$ has a compatible $(G, \mathbb{Z}^{(X)})$-structure, we put

$$A(Y) = I(Y) \cdot A \lhd A. \tag{10.1.4}$$

Observe that $A(Y)$ is an ideal of A, proper over Y. In particular if X is a $(G, \mathcal{F})$-complex, then $A(Y)$ is $(G, \mathcal{F})$-proper for all $Y \subset X$.

Lemma 10.1.5. *Let A be a G-ring. Assume that A is $(G, \mathcal{F})$-proper. Then A has an exhaustive filtration $\{A(K)\}$ by ideals such that each $A(K)$ is proper over a finite $(G, \mathcal{F})$-complex K.*

Proof. By hypothesis, there exists a $(G, \mathcal{F})$-complex X such that A is proper over X. Consider the filtration $\{A(K)\}$ where $A(K)$ is defined in (10.1.4) and K runs among the G-finite simplicial subsets of X. By the discussion above, $A(K) \subset A$ is an ideal, proper over K. It is clear that $\{I(K)\}$ and $\{A(K)\}$ are filtering systems and that $\bigcup_K I(K) = \mathbb{Z}^{(X)}$. We claim furthermore that $A = \bigcup_K A(K)$. By definition of $\mathbb{Z}^{(X)}$-algebra, $A = \mathbb{Z}^{(X)} \cdot A$. Hence if $a \in A$, then there exist $\phi_1, \dots, \phi_n \in \mathbb{Z}^{(X)}$ and $a_1, \dots, a_n \in A$ such that $a = \sum_i \phi_i a_i$. Hence $a \in A(K)$ for $K = \bigcup_i G \cdot \operatorname{supp}(\phi_i)$. $\square$

Lemma 10.1.6. *(See [12, p. 51].) Let $A \in G - \mathrm{Rings}$ be proper over a locally finite G-simplicial set X, and let $f : X \to Y$ be an equivariant map with Y locally finite. Then the map $f^* : \mathbb{Z}^Y \to \mathbb{Z}^X$ induces a compatible $(G, \mathbb{Z}^{(Y)})$-algebra structure on A which makes it proper over Y.*

Proof. We begin by showing that the compatible $(G, \mathbb{Z}^{(X)})$-algebra structure on A extends to a compatible $(G, \mathbb{Z}^X)$-module structure. By the lemma above, if $a \in A$ then there exists a finite simplicial subset $K \subset X$ such that $a \in A(K) = I(K) \cdot A$. By Theorem 9.4.1 there exists $\mu_K \in Z^X$, with $\operatorname{supp}(\mu_K) \subset \overline{\mathrm{St}}(K)$ such that

$$\mu_K a = a \quad \forall a \in A(K). \tag{10.1.7}$$

Because X is locally finite, $\overline{\mathrm{St}}(K)$ is finite and $\mu_K \in \mathbb{Z}^{(X)}$. Thus we have a map $A(K) \to I(\overline{\mathrm{St}}(K)) \otimes A(K)$, $a \mapsto \mu_K \otimes a$. Now $I(\overline{\mathrm{St}}(K))$ is an ideal in $\mathbb{Z}^X$ by (9.3.2); using the multiplication of $\mathbb{Z}^X$ we obtain a map

$$\mathbb{Z}^X \otimes A(K) \to A\big(\overline{\mathrm{St}}(K)\big), \qquad \phi \otimes a \mapsto (\phi \cdot \mu_K)a. \tag{10.1.8}$$

If $L \supset K$, and we choose an element μ_L as above, then for $a \in A(K)$ and $\phi \in \mathbb{Z}^X$ we have:

$$(\phi \cdot \mu_L) \cdot a = (\phi \cdot \mu_L) \cdot (\mu_K \cdot a) = (\phi \cdot \mu_K)a.$$

This shows that (10.1.8) is independent of the choice of the element μ_K of (10.1.7), and that we have a well-defined action $\mathbb{Z}^X \otimes A \to A$. Compatibility with the G-action follows from the fact that $g \cdot \mu_K$ is the identity on $g \cdot K$. The remaining compatibility conditions are immediate. Now A becomes a $\mathbb{Z}^{(Y)}$-module through f^*. If $K \subset X$ is a finite simplicial subset, then $L = f(K) \subset Y$ is finite, and since Y is locally finite, there is a $\mu_L \in \mathbb{Z}^{(Y)}$ which is the identity on L, and thus $f^*(\mu_L)$ is the identity on K. It follows that the action of $\mathbb{Z}^{(Y)}$ on A satisfies (10.1.2). The remaining $(G, \mathbb{Z}^{(Y)})$-compatibility conditions of (10.1.1) are straightforward. $\square$

10.2. Induction

Let G be a group, $H \subset G$ a subgroup and A an H-ring. Consider

$$\operatorname{Big}\operatorname{Ind}_H^G(A) = \{f : G \to A\colon\ f(gh) = h^{-1} f(g)\}.$$

Note that $\operatorname{Big}\operatorname{Ind}_H^G(A)$ is a G-ring with operations defined pointwise, and where G acts by left multiplication. If $f \in \operatorname{Big}\operatorname{Ind}_H^G(A)$ and $x = sH \in G/H$, then the value of f at any $g \in x$ determines f on the whole x; in particular,

$$\operatorname{supp}(f) \cap sH \neq \emptyset \Rightarrow sH \subset \operatorname{supp}(f) \quad (sH \in G/H).$$

Hence

$$\operatorname{supp}(f) = \coprod_{sH \cap \operatorname{supp}(f) \neq \emptyset} sH.$$

Consider the projection $\pi : G \to G/H$. Put

$$\operatorname{Ind}_H^G(A) = \{f \in \operatorname{Big}\operatorname{Ind}_H^G(A)\colon\ \#\pi(\operatorname{supp}(f)) < \infty\}.$$

One checks that $\operatorname{Ind}_H^G(A) \subset \operatorname{Big}\operatorname{Ind}_H^G(A)$ is a subring; we shall presently introduce some of its typical elements. If $s \in G$, we write $\chi_s : G \to \mathbb{Z}$ for the characteristic function. If $a \in A$ and $s \in G$, then

$$\xi_H(s, a) = \sum_{h \in H} h^{-1}(a)\chi_{sh} \in \operatorname{Ind}_H^G(A).$$

Let $r : G/H \to G$ be a pointed section and $\mathcal{R} = r(G/H)$. Every element $\phi \in \operatorname{Big}\operatorname{Ind}_H^G(A)$ can be written as a formal sum

$$\phi = \sum_{s \in \mathcal{R}} \xi_H(s, \phi(s)). \tag{10.2.1}$$

Note that $\phi \in \operatorname{Ind}_H^G(A)$ if and only if the sum above is finite. In particular

$$\operatorname{Ind}_H^G(A) = \sum_{s \in G, a \in A} \mathbb{Z}\xi_H(s, a) \subset \operatorname{Big}\operatorname{Ind}_H^G(A).$$

Next observe that, for each fixed $s \in G$, the map

$$\xi_H(s, -) : A \to \operatorname{Big}\operatorname{Ind}_H^G(A)$$

is a ring homomorphism. Moreover, we have the following relations

$$g\xi_H(s, a) = \xi_H(gs, a), \tag{10.2.2}$$

$$\xi_H(sh, a) = \xi_H(s, ha), \tag{10.2.3}$$

$$\xi_H(s, a)\xi_H(t, b) = \begin{cases} 0 & \text{if } sH \neq tH, \\ \xi_H(s, ab) & \text{if } s = t. \end{cases} \tag{10.2.4}$$

It follows that $(s, a) \mapsto \xi_H(s, a)$ gives a G-equivariant map

$$G \times_H A \to \operatorname{Ind}_H^G(A).$$

Here $G \times_H A = G \times A/\sim$, where $(g_1, a_1) \sim (g_2, a_2) \Leftrightarrow h = g_1^{-1} g_2 \in H$ and $a_1 = ha_2$. Extending by linearity we obtain an isomorphism of left G-modules

$$\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} A \to \operatorname{Ind}_H^G(A).$$

Thus we may think of $\operatorname{Ind}_H^G(A)$ as the G-module induced from the H-module A equipped with a ring structure compatible with that of A. In fact (10.2.4) implies that if $r : G/H \to G$ is a section as above, then

$$\mathbb{Z}^{(G/H)} \otimes A \to \operatorname{Ind}_H^G(A), \qquad \chi_x \otimes a \mapsto \xi_H(r(x), a) \tag{10.2.5}$$

is a (nonequivariant) ring isomorphism.

Lemma 10.2.6. *Let X be an H-simplicial set; put*

$$\operatorname{Ind}_H^G(X) = G \times_H X.$$

There is a natural, G-equivariant isomorphism $\mathbb{Z}^{(\operatorname{Ind}_H^G(X))} \cong \operatorname{Ind}_H^G(\mathbb{Z}^{(X)})$.

Proof. Let $\pi : G \times X \to \operatorname{Ind}_H^G(X)$ be the projection. We have a G-ring isomorphism

$$\theta : \operatorname{Big}\operatorname{Ind}_H^G(\mathbb{Z}^X) \to \mathbb{Z}^{\operatorname{Ind}_H^G(X)}, \qquad \theta(f)\big(\pi(g,x)\big) = f(g)(x).$$

For $s \in G$ and $\phi \in \mathbb{Z}^X$,

$$\theta\big(\xi_H(s,\phi)\big)\pi(g,x) = \begin{cases} \phi(s^{-1}gx) & \text{if } g \in sH, \\ 0 & \text{else.} \end{cases}$$

In particular, for $\theta(\xi_H(s,\phi))$ not to vanish on $\pi(g,x)$, we must have $g = sh$ and $x \in h^{-1}\{\phi \neq 0\}$ for some $h \in H$. Hence $\operatorname{supp}(\theta(\xi_H(s,\phi))) \subset \pi(\{s\} \times \operatorname{supp}(\phi))$ which is a finite simplicial set if $\phi \in \mathbb{Z}^{(X)}$. Therefore θ maps $\operatorname{Ind}_H^G(\mathbb{Z}^{(X)})$ inside $\mathbb{Z}^{(\operatorname{Ind}_H^G(X))}$. It remains to show that $\theta^{-1}(\mathbb{Z}^{(\operatorname{Ind}_H^G(X))}) \subset \operatorname{Ind}_H^G(\mathbb{Z}^{(X)})$. Let $\{g_i\} \subset G$ be a full set of representatives of G/H. Every element of $G \times_H X$ can be written uniquely as $\pi(g_i, x)$ for some i and some $x \in X$. Hence as a simplicial set, $\operatorname{Ind}_H^G(X)$ is the disjoint union of the $Y_i = \pi(\{g_i\} \times X)$. In particular if $\phi \in \mathbb{Z}^{(\operatorname{Ind}_H^G(X))}$, then its support meets finitely many of the Y_i, and $\operatorname{supp}(\phi) \cap Y_i$ is a finite simplicial set. Thus there is a finite number of i such that $\psi = \theta^{-1}(\phi)$ is nonzero on $g_i H$, and its restriction to each of these subsets takes values in $\mathbb{Z}^{(X)}$. By (10.2.1), this implies that $\psi \in \operatorname{Ind}_H^G(\mathbb{Z}^{(X)})$, as we had to prove. □

If $C, A \in H - \text{Rings}$ with C commutative and we have a compatible (H, C)-algebra structure on A, then $\operatorname{Ind}_H^G(A)$ carries a compatible $(G, \operatorname{Ind}_H^G(C))$-algebra structure, given by

$$\xi_H(s,c) \cdot \xi_H(t,a) = \begin{cases} \xi_H(s, c \cdot a) & s = t, \\ 0 & sH \neq tH. \end{cases}$$

If moreover $C \cdot A = A$, then $\operatorname{Ind}_H^G(C) \cdot \operatorname{Ind}_H^G(A) = \operatorname{Ind}_H^G(A)$. We record a particular case of this in the following

Lemma 10.2.7. *If $A \in H - \text{Rings}$ is proper over an H-simplicial set X, then the G-ring $\operatorname{Ind}_H^G(A)$ is proper over $\operatorname{Ind}_H^G(X)$.*

Proof. It follows from Lemma 10.2.6 and the discussion above. □

10.3. Compression

Let $A \in G - \text{Rings}$, and $H \subset G$ a subgroup. Assume that A is proper over G/H. Let $\chi_H \in \mathbb{Z}^{(G/H)}$ be the characteristic function of H. The *compression* of A over H is the subring

$$\operatorname{Comp}_H^G(A) = \chi_H \cdot A.$$

Note the action of G on A restricts to an action of H on $\operatorname{Comp}_H^G(A)$, which makes it into an object of $H - \text{Rings}$.

Proposition 10.3.1. *(Compare [12, Lemma 12.3, and paragraph after 12.4].)*

i) *If $B \in H - \text{Rings}$, then $\operatorname{Ind}_H^G(B)$ is proper over G/H, and*

$$B \to \operatorname{Comp}_H^G \operatorname{Ind}_H^G B, \qquad b \mapsto \xi_H(1, b)$$

is an H-equivariant isomorphism.

ii) *If $A \in G - \text{Rings}$ is proper over G/H, then*

$$\operatorname{Ind}_H^G \operatorname{Comp}_H^G(A) \to A, \qquad \xi_H(s, \chi_H a) \mapsto \chi_{sH} s(a)$$

is a G-equivariant isomorphism.

Proof. Any $B \in H - \text{Rings}$ is proper over the 1-point space $*$. Hence $\operatorname{Ind}_H^G(B)$ is proper over $\operatorname{Ind}_H^G(*) = G/H$, by Lemma 10.2.7. The proof that the maps of i) and ii) are isomorphisms is straightforward; to show equivariance, one uses (10.2.2) and (10.2.3). □

10.4. A discrete variant of Green's imprimitivity theorem

Let G be a group, $H \subset G$ a subgroup and A an H-ring. Observe that, by definition, the G-ring $\mathrm{Ind}_H^G(A)$ is a G-subring of the ring $\mathrm{map}(G, A^{\Delta^\bullet}) = \mathrm{map}(G, A) = A^G$ (note that this is not the same as the subring of G-invariants of A). Since $A^{(G)} \lhd A^G$ is a G-ideal, we may regard $A^{(G)}$ as a left $\mathrm{Ind}_H^G(A)$-module via left multiplication in A^G, and moreover, this action is compatible with that of G, in the sense that the two together define a left $\mathrm{Ind}_H^G(A) \rtimes G$-module structure on $A^{(G)}$. We may also regard $A^{(G)}$ as a right module over $A \rtimes H$, via

$$\big[\phi \cdot (a \rtimes h)\big](g) = h^{-1}\big(\phi\big(gh^{-1}\big)a\big).$$

One checks that these two actions satisfy

$$(f \rtimes g) \cdot \big[\phi \cdot (a \rtimes h)\big] = \big[(f \rtimes g) \cdot \phi\big] \cdot (a \rtimes h).$$

Hence they make $A^{(G)}$ into an $(\mathrm{Ind}_H^G(A) \rtimes G, A \rtimes H)$-bimodule. In particular left multiplication by elements of $\mathrm{Ind}_H^G(A) \rtimes G$ induces a ring homomorphism

$$\mathrm{Ind}_H^G(A) \rtimes G \to \mathrm{End}_{A \rtimes H}\big(A^{(G)}\big). \tag{10.4.1}$$

Observe that the decomposition $G = \coprod_{x \in G/H} x$ induces

$$A^{(G)} = \bigoplus_{x \in G/H} A^{(x)} \tag{10.4.2}$$

and that $A^{(x)} \cdot (A \rtimes H) \subset A^{(x)}$. Hence (10.4.2) is a direct sum of right $A \rtimes H$-modules. Thus we may think of an element $T \in \mathrm{End}_{A \rtimes H}(A^{(G)})$ as a matrix $T = [T_{x,y}]_{x,y \in G/H}$, where $T_{x,y} : A^{(y)} \to A^{(x)}$ is a homomorphism of right $A \rtimes H$-modules, and is such that for each $v \in A^{(y)}$, $T_{x,y}(v) = 0$ for all but a finite number of x. Moreover

$$A \rtimes H \to A^{(gH)}, \qquad a \rtimes h \mapsto \chi_g \cdot (a \rtimes h) = \chi_{gh} h^{-1}(a)$$

is an isomorphism of right $A \rtimes H$-modules. Fix a full set of representatives $\mathcal{R}$ of G/H, with $1 \in \mathcal{R}$, write $M_{\mathcal{R}} \in \mathbb{Z} - \mathrm{Rings}$ for the ring of $\mathcal{R} \times \mathcal{R}$-matrices with finitely many nonzero coefficients in $\mathbb{Z}$, and put $M_{\mathcal{R}}(A \rtimes H) = M_{\mathcal{R}} \otimes (A \rtimes H)$. We have a ring homomorphism

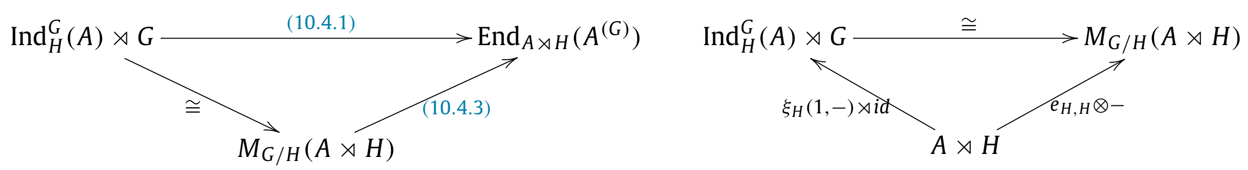
$$M_{\mathcal{R}}(A \rtimes H) \to \mathrm{End}_{A \rtimes H}\big(A^{(G)}\big), \qquad M \mapsto \left(\sum_{y \in \mathcal{R}} \chi_y \cdot \alpha_y \mapsto \sum_{x \in \mathcal{R}} \chi_x \sum_{y \in \mathcal{R}} m_{x,y} \alpha_y\right).$$

Furthermore, we have a map $G \to \mathcal{R}$, which sends each $s \in G$ to the representative $\hat{s} \in \mathcal{R}$ of sH. Using this map we obtain an isomorphism $M_{G/H} \cong M_{\mathcal{R}}$ which sends the matrix unit $E_{sH,tH}$ to $E_{\hat{s},\hat{t}}$. By composition, we obtain a ring homomorphism

$$M_{G/H}(A \rtimes H) \to \mathrm{End}_{A \rtimes H}\big(A^{(G)}\big) \cong \mathrm{End}_{A \rtimes H}\big((A \rtimes H)^{(G/H)}\big). \tag{10.4.3}$$

Remark 10.4.4. If A happens to be unital, then both (10.4.1) and (10.4.3) are injective.

Theorem 10.4.5. *Let G be a group, $H \subset G$ a subgroup, and $A \in H - \mathrm{Rings}$. Then there is an isomorphism $\mathrm{Ind}_H^G(A) \rtimes G \cong M_{G/H}(A \rtimes H)$ such that the following diagrams commute*



Proof. We use the notation introduced in the paragraph preceding the theorem. If $s \in G$, put $\phi(s) = \hat{s}^{-1}s \in H$. Note that $\phi(sh) = \phi(s)h$ $(s \in G, h \in H)$. One checks that the following map is a well-defined, bijective ring homomorphism with the required properties

$$\alpha : \mathrm{Ind}_H^G(A) \rtimes G \to M_{G/H}(A \rtimes H), \qquad \alpha\big(\xi_H(s,a) \rtimes g\big) = e_{sH, g^{-1}sH} \otimes \phi(s)(a) \rtimes \phi(s)\phi\big(g^{-1}s\big)^{-1}. \qquad \square$$

Remark 10.4.6. The isomorphism of the theorem above is natural in A, but not in the pair (G, H), as it depends on a choice of a full set of representatives $\mathcal{R}$ of G/H, or what is the same, of a choice of pointed section $G/H \to G$ of the canonical projection.

10.5. Restriction

Let B be a G-ring, $H \subset G$ a subgroup. Write $\mathrm{Res}_G^H B$ for the H-ring obtained by restriction to H of the action of G on B.

Lemma 10.5.1. *If B is a G-ring, then $\mathrm{Ind}_H^G \mathrm{Res}_G^H B \to \mathbb{Z}^{(G/H)} \otimes B$, $\xi_H(s,b) \mapsto \chi_{sH} \otimes s(b)$ is a G-ring isomorphism.*

Proof. Straightforward. □

Now suppose $K \subset G$ is another subgroup. Let $x \in H\backslash G/K$. Put

$$\mathrm{Res}_G^H \mathrm{Ind}_K^G(A)[x] = \left\{ f \in \mathrm{Ind}_K^G(A) : \mathrm{supp}(f) \subset x \right\} \in H - \mathrm{Rings}. \tag{10.5.2}$$

We have

$$\mathrm{Res}_G^H \mathrm{Ind}_K^G(A) = \bigoplus_{x \in H\backslash G/K} \mathrm{Res}_G^H \mathrm{Ind}_K^G(A)[x]. \tag{10.5.3}$$

Write $x = H\theta K$ for some $\theta \in G$. Consider the subgroup

$$H \supset H_\theta = H \cap \theta K \theta^{-1}.$$

We shall see presently that the H-ring (10.5.2) is proper over H/H_θ. Consider the subgroup

$$K \supset K_{\theta^{-1}} = \theta^{-1} H \theta \cap K.$$

Conjugation by θ^{-1} defines an isomorphism

$$c_{\theta^{-1}} : H_\theta \to K_{\theta^{-1}}, \qquad c_{\theta^{-1}}(h) = \theta^{-1} h \theta.$$

Hence we may view $\mathrm{Res}_K^{K_{\theta^{-1}}} A$ as an H_θ-ring via $c_{\theta^{-1}}$; we write $c^*_{\theta^{-1}}(\mathrm{Res}_K^{K_{\theta^{-1}}} A)$ for the resulting H_θ-ring.

Lemma 10.5.4. *The map*

$$\alpha : \mathrm{Res}_G^H \mathrm{Ind}_K^G(A)[H\theta K] \to \mathrm{Ind}_{H_\theta}^H\left(c^*_{\theta^{-1}}\left(\mathrm{Res}_K^{K_{\theta^{-1}}}(A)\right)\right), \qquad \alpha(f)(h) = f(h\theta)$$

is an isomorphism of H-rings.

Proof. One checks that if $m \in H_\theta$, then $\alpha(f)(hm) = m^{-1}\alpha(f)(h)$. It is clear that α is H-equivariant. A calculation shows that $\alpha(\xi_K(h\theta, a)) = \xi_{H_\theta}(h, a)$. It follows that α is an isomorphism. □

11. Induction and equivariant homology

Lemma 11.1. *Let G be a group, $K \subset G$ a subgroup, A a K-ring, and $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ a functor satisfying the standing assumptions. Then A is E-excisive if and only if $\mathrm{Ind}_K^G(A)$ is E-excisive.*

Proof. The map (10.2.5) gives a nonequivariant isomorphism

$$\mathrm{Ind}_K^G(A) \cong \mathbb{Z}^{(G/K)} \otimes A = \bigoplus_{x \in G/K} A.$$

The equivalence of the lemma follows from standing assumption v). □

Let G, K and A be as in Lemma 11.1, and let X be a G-simplicial set. If A is unital, then for each subgroup $S \subset K$ we have a functor

$$A \rtimes \mathcal{G}^K(K/S) \to \mathrm{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/S), \qquad kS \mapsto kS, \qquad a \rtimes k \mapsto \xi_K(1,a) \rtimes k.$$

If A is any E-excisive ring, the map above is defined for the unitalization $\tilde{A}$; applying E, taking fibers relative to the augmentation $\tilde{A} \to \mathbb{Z}$, and using the standing assumptions, we get a map $E(A \rtimes \mathcal{G}^K(K/S)) \to E(\mathrm{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/S))$. The maps

$$X_+^S \wedge E\left(A \rtimes \mathcal{G}^K(K/S)\right) \to X_+^S \wedge E\left(\mathrm{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/S)\right) \to H^G\left(X, E\left(\mathrm{Ind}_K^G A\right)\right)$$

assemble to

$$\mathrm{Ind} : H^K\left(X, E(A)\right) \to H^G\left(X, E\left(\mathrm{Ind}_K^G(A)\right)\right). \tag{11.2}$$

Proposition 11.3. *(Compare [12, Proposition 12.9].) Let A be an E-excisive G-ring. Then the map (11.2) is an equivalence.*

Proof. As a functor of G-simplicial sets, equivariant homology satisfies excision and commutes with filtering colimits (see [9]). Because of this, and because X is obtained by gluing together cells of the form $\mathrm{Ind}_H^G(\Delta^n)$, $H \in \mathcal{A}ll$, it suffices to prove the proposition for $X = \mathrm{Ind}_H^G(T)$ where H acts trivially on T. Let $\mathcal{R}$ be a full set of representatives of $K\backslash G/H$. We have

$$\mathrm{Ind}_H^G(T) = T \times G/H = \coprod_{\theta\in\mathcal{R}} T \times K\theta H \cong \coprod_{\theta\in\mathcal{R}} T \times K/K_\theta.$$

Here as in Section 10.5, $K_\theta = c_\theta(H) \cap K$. Thus

$$H^K\big(\mathrm{Ind}_H^G(T), E(A)\big) = T_+ \wedge \bigvee_{\theta\in\mathcal{R}} E\big(A \rtimes \mathcal{G}^K(K/K_\theta)\big).$$

On the other hand,

$$H^G(\mathrm{Ind}_H^G(T), E\big(\mathrm{Ind}_K^G(A)\big) = T_+ \wedge E\big(\mathrm{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/H)\big).$$

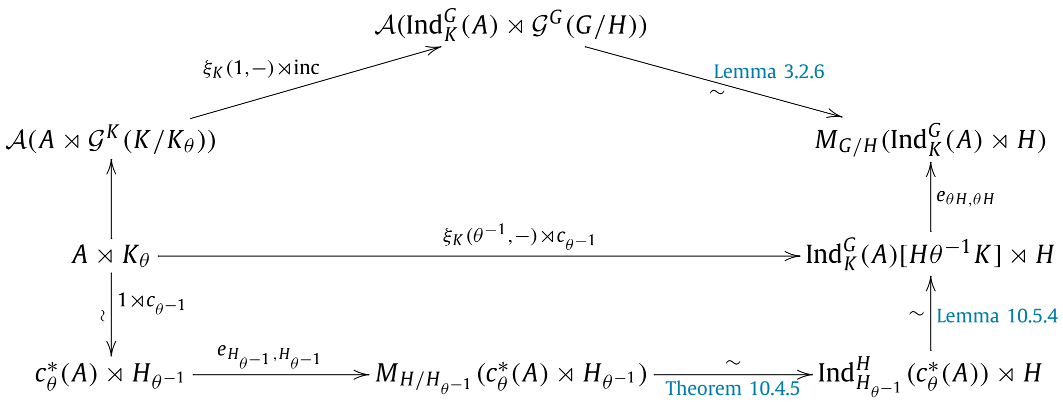
We have to show that

$$\bigvee_{\theta\in\mathcal{R}} E\big(A \rtimes \mathcal{G}^K(K/K_\theta)\big) \to E\big(\mathrm{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/H)\big)$$

is an equivalence. By standing assumptions iv) and v) we may replace the map above by that induced by the corresponding ring homomorphism

$$\bigoplus_{\theta\in\mathcal{R}} \mathcal{A}\big(A \rtimes \mathcal{G}^K(K/K_\theta)\big) \to \mathcal{A}\big(\mathrm{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/H)\big). \tag{11.4}$$

Here $\mathcal{A}(A \rtimes \mathcal{G}^K(K/K_\theta)) \to \mathcal{A}(\mathrm{Ind}_K^G(A) \rtimes \mathcal{G}^G(G/H))$ is induced by $\xi_K(1,-) : A \to \mathrm{Ind}_K^G(A)$ and by the inclusions $K \subset G$ and $K/K_\theta \to G/H$, $kK_\theta \mapsto k\theta H$. One checks that the following diagram commutes



Because the lower rectangle commutes, $E(A \rtimes K_\theta \to \mathrm{Ind}_K^G(A)[H\theta^{-1}K] \rtimes H)$ is an equivalence, by matrix stability. Again by matrix stability and by Lemma 3.2.6, applying E to the top left vertical arrow is an equivalence. Hence to prove that E applied to (11.4) is an equivalence, it suffices to show that E applied to

$$\mathrm{Ind}_K^G(A) \rtimes H = \bigoplus_{\theta\in\mathcal{R}} \mathrm{Ind}_K^G(A)[H\theta K] \rtimes H \xrightarrow{\sum_\theta e_{\theta H,\theta H}} M_{G/H}(\mathrm{Ind}_K^G(A) \rtimes H) \tag{11.5}$$

is one. But another application of matrix stability (using [4, Proposition 2.2.6]) shows that E applied to (11.5) gives the same map in HoSpt as E applied to the inclusion

$$e_{H,H} : \mathrm{Ind}_K^G(A) \rtimes H \to M_{G/H}\big(\mathrm{Ind}_K^G(A) \rtimes H\big).$$

This concludes the proof. □

Theorem 11.6. *Let $E : \mathbb{Z}-\mathrm{Cat} \to \mathrm{Spt}$ be a functor satisfying Standing Assumptions 3.3.2. Also let G be a group, $\mathcal{F}$ a family of subgroups of G and B an E-excisive ring, proper over a 0-dimensional $(G,\mathcal{F})$-complex X. Then $H^G(-, E(B))$ maps $(G,\mathcal{F})$-equivalences to equivalences. In particular, the assembly map*

$$H^G\big(\mathcal{E}(G,\mathcal{F}), E(B)\big) \to E(B \rtimes G)$$

is an equivalence.

Proof. We have $X = \coprod_i G/K_i$ for some $K_i \in \mathcal{F}$, and $\mathbb{Z}^{(X)} = \bigoplus_i \mathbb{Z}^{(G/K_i)}$. The ring $B_i = \mathbb{Z}^{(G/K_i)} \cdot B$ is proper over G/K_i, and is excisive by standing assumption v). Again by standing assumption v), it suffices to prove the assertion of the theorem individually for each B_i; in other words, we may assume $X = G/K$ for some $K \in \mathcal{F}$. Hence for $A = \mathrm{Comp}_G^K B$ we have $B = \mathrm{Ind}_K^G A$, by Proposition 10.3.1. Moreover, by Lemma 11.1, A is E-excisive. Let $Y \to Z$ be a $(G, \mathcal{F})$-equivalence. We have a commutative diagram

$$\begin{array}{ccc} H^G(Y, E(B)) & \longrightarrow & H^G(Z, E(B)) \\ \uparrow{\scriptstyle \mathrm{Ind}} & & \uparrow{\scriptstyle \mathrm{Ind}} \\ H^K(Y, E(A)) & \longrightarrow & H^K(Z, E(A)) \end{array}$$

The bottom horizontal arrow is an equivalence because $K \in \mathcal{F}$. The two vertical arrows are equivalences by Proposition 11.3. It follows that the top horizontal arrow is an equivalence too. □

12. Assembly as a connecting map

Throughout this section, we consider a fixed functor $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$, and – except when otherwise stated – we assume that, in addition to the standing assumptions, it satisfies the following:

Sectional Assumptions 12.1.

vi) *E_* commutes with filtering colimits.*
vii) *If A is E-excisive and L has local units and is flat as a $\mathbb{Z}$-module, then $L \otimes A$ is E-excisive.*

12.1. Preliminaries

Mapping cones. Let $f : A \to B$ be a ring homomorphism; the *mapping cone* of f is defined as the pullback

$$\begin{array}{ccc} \Gamma_f & \longrightarrow & \Gamma B \\ \downarrow & & \downarrow \\ \Sigma A & \xrightarrow{\Sigma f} & \Sigma B \end{array}$$

Lemma 12.1.1. *Let $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ be a functor satisfying both the standing and the sectional assumptions, and $f : A \to B$ a homomorphism of E-excisive rings. Then*

i) *$E(\Gamma B)$ is weakly contractible.*
ii) *$E(\Sigma B) \xrightarrow{\sim} \Sigma E(B)$.*
iii) *The following is a distinguished triangle in* HoSpt

$$E(B) \to E(\Gamma_f) \to \Sigma E(A) \xrightarrow{\Sigma E(f)} \Sigma E(B).$$

Proof. By Lemma 8.1.3, $\Gamma B = \Gamma\mathbb{Z} \otimes B$, whence it is E-excisive, by Sectional Assumption 12.1vii). Part i) follows from matrix stability and the fact that $\Gamma\mathbb{Z}$ is a ring with infinite sums (see e.g. [4, Proposition 2.3.1]). Parts ii) and iii) follow from i) and excision. □

Matrix rings and group actions.

Lemma 12.1.2. *Let G be a group, A a G-ring and X a G-set. Write $M_{\underline{X}}$ for the ring M_X equipped with the G-action*

$$g(e_{x,y}) = e_{gx,gy}.$$

The map

$$(M_{\underline{X}} A) \rtimes G \to M_X(A \rtimes G),\ (e_{x,y} \otimes a) \rtimes g \mapsto e_{x,g^{-1}y} \otimes (a \rtimes g)$$

is a G-equivariant isomorphism of rings.

12.2. Dirac extensions

Let G be a group, $\mathcal{F}$ a family of subgroups, $E:\mathbb{Z}-\mathrm{Cat}\to\mathrm{Spt}$ a functor satisfying the standing assumptions, and A an E-excisive ring. A *Dirac extension* for $(G,\mathcal{F},A,E)$ consists of an extension of E-excisive G-rings

$$0\to B\to Q\to P\to 0 \tag{12.2.1}$$

together with a zig-zag

$$A=Z_0\xrightarrow{f_0}Z_1\xleftarrow{f_2}Z_2\xrightarrow{f_3}\cdots Z_n=B$$

such that

a) $E(f_i\rtimes H)$ is an equivalence for every subgroup $H\subset G$.
b) $E_*(Q\rtimes H)=0$ for every $H\in\mathcal{F}$.
c) $H^G(-,E(P))$ sends $(G,\mathcal{F})$-equivalences to equivalences.

Remark 12.2.2. Condition a) together with standing assumptions iii) and iv) and Lemma 3.2.6 imply that the zig-zag $f=\{f_i\}$ induces an equivalence

$$H^G\big(X,E(A)\big)\xrightarrow{\sim}H^G\big(X,E(B)\big)$$

for every G-space X. Similarly, it follows from condition b) that $H^G_*(Y,E(Q))=0$ for every $(G,\mathcal{F})$-complex Y.

Proposition 12.2.3. *Let $E:\mathbb{Z}-\mathrm{Cat}\to\mathrm{Spt}$ be a functor satisfying the standing assumptions, G a group, $\mathcal{F}$ a family of subgroups of G, and A a G-ring. Let (12.2.1) be a Dirac extension for $(G,\mathcal{F},A,E)$. Then there are an exact sequence*

$$E_{*+1}(A\rtimes G)\to E_{*+1}(Q\rtimes G)\to E_{*+1}(P\rtimes G)\xrightarrow{\partial}E_*(A\rtimes G)$$

an isomorphism $H^G_*(\mathcal{E}(G,\mathcal{F}),E(A))\cong E_{*+1}(P\rtimes G)$, and a commutative diagram

$$\begin{array}{ccc}H^G_*(\mathcal{E}(G,\mathcal{F}),E(A)) & \xrightarrow{\text{Assembly}} & E_*(A\rtimes G)\\ & \searrow^{\cong}\qquad\nearrow_{\partial} & \\ & E_{*+1}(P\rtimes G) & \end{array}$$

Proof. By Proposition 3.3.9 and Remark 12.2.2 we have a distinguished triangle

$$H^G(X,E(A))\to H^G(X,E(Q))\to H^G(X,E(P))\xrightarrow{\partial^X}\Sigma H^G(X,E(A)) \tag{12.2.4}$$

for every G-simplicial set X. The proposition follows by comparison of the long exact sequence of homotopy associated to the triangles for $X=\mathcal{E}(G,\mathcal{F})$, and $X=*$, using that, again by Remark 12.2.2, we have $H^G_*(\mathcal{E}(G,\mathcal{F}),E(Q))=0$. $\square$

Remark 12.2.5. If $X\in\mathbb{S}^G$ and $cX\xrightarrow{\sim}X$ is a $(G,\mathcal{F})$-cofibrant replacement, then the same argument as that of the proof of Proposition 12.2.3 shows that the map $H^G(cX,E(A))\to H^G(X,E(A))$ is an equivalence if and only if the boundary map ∂^X in the sequence (12.2.4) is an equivalence.

12.3. A canonical Dirac extension

Let G be a group and $\mathcal{F}$ a family of subgroups. Consider the discrete G-simplicial sets

$$X=X_{\mathcal{F}}=\coprod_{H\in\mathcal{F}}G/H,\qquad Y=G/G\coprod X.$$

The group G acts on Y and thus on the ring M_Y of $Y\times Y$-matrices with finitely many nonzero integral coefficients. The point y_0 corresponding to the unique orbit of G/G is fixed by G, whence the map $\iota:\mathbb{Z}\to M_{\underline{Y}}$, $\lambda\to\lambda E_{y_0,y_0}$ is G-equivariant. In particular we have a directed system of G-rings $\{id\otimes\iota:(M_\infty M_{\underline{Y}})^{\otimes n}\to(M_\infty M_{\underline{Y}})^{\otimes n+1}\}_n$. Put

$$\mathfrak{F}^0=\operatorname*{colim}_n(M_\infty M_{\underline{Y}})^{\otimes n}.$$

Since X is discrete, the ring of finitely supported functions breaks up into a sum

$$\mathbb{Z}^{(X)} = \bigoplus_{x\in X} k\chi_x.$$

Multiplication by an element of $M_{\underline{Y}}$ gives an $\mathbb{Z}$-linear endomorphism of $\mathbb{Z}^{(Y)}$. This defines an equivariant monomorphism

$$M_{\underline{Y}} \to \mathrm{End}_{\mathbb{Z}}\big(\mathbb{Z}^{(Y)}\big)$$

whose image consists of those linear transformations T such that the matrix of T with respect to the basis $\{\chi_y\colon\ y\in Y\}$ has finitely many nonzero entries. Note that multiplication by χ_x in $\mathbb{Z}^{(X)}\subset\mathbb{Z}^{(Y)}$ is in this image. Thus we have an equivariant injective ring homomorphism

$$\rho:\mathbb{Z}^{(X)}\to M_{\underline{Y}}.$$

For each $n\geqslant 1$, consider the G-ring

$$\mathfrak{F}^n = \left(\bigotimes_{i=1}^{n}\Gamma_\rho\right)\otimes\mathfrak{F}^0.$$

The inclusion $M_\infty M_{\underline{Y}}\to\Gamma_\rho$ induces an inclusion $\mathfrak{F}^n\subset\mathfrak{F}^{n+1}$ for each $n\geqslant 0$. Put

$$\mathfrak{F}^\infty = \bigcup_{n\geqslant 0}\mathfrak{F}^n.$$

If $A\in\mathrm{Rings}$, we also write $\mathfrak{F}^nA=\mathfrak{F}^n\otimes A$ $(n\geqslant 0)$. We have

Lemma 12.3.1.

i) *$\mathfrak{F}^n\subset\mathfrak{F}^\infty$ is an ideal $(n<\infty)$.*
ii) *For each $n\geqslant 0$, $\mathfrak{F}^n$ and $\mathfrak{F}^{n+1}/\mathfrak{F}^n\cong\Sigma\mathbb{Z}^{(X)}\otimes\mathfrak{F}^n$ have local units and are flat as abelian groups. Moreover $\mathfrak{F}^{n+1}/\mathfrak{F}^n$ is $(G,\mathcal{F})$-proper.*
iii) *If $H\in\mathcal{F}$, $\chi_H\in\mathbb{Z}^{(G/H)}\subset\mathbb{Z}^{(X)}$ is the characteristic function, and A is a G-ring, we have a commutative diagram*

$$\begin{array}{ccc}
(\mathbb{Z}^{(G/H)}\otimes\mathfrak{F}^nA)\rtimes H\subset(\mathbb{Z}^{(X)}\otimes\mathfrak{F}^nA)\rtimes H & \xrightarrow{(\rho\otimes 1)\rtimes id} & (M_{\underline{Y}}\mathfrak{F}^nA)\rtimes H\\
\uparrow{\scriptstyle \chi_H\otimes 1} & & \downarrow{\scriptstyle \wr\ (12.1.2)}\\
\mathfrak{F}^nA\rtimes H & \xrightarrow[e_{H,H}\otimes -]{} & M_Y(\mathfrak{F}^nA\rtimes H)
\end{array}$$

Proof. Part i) is clear. Observe that

$$\mathfrak{F}^{n+1}/\mathfrak{F}^n=\Sigma\mathbb{Z}^{(X)}\otimes\mathfrak{F}^n \tag{12.3.2}$$

therefore $\mathfrak{F}^{n+1}/\mathfrak{F}^n$ is proper. That $\mathfrak{F}^n$ is flat is clear for $n=0$; the general case follows by induction, using (12.3.2). The ring $\mathfrak{F}^0$ has local units because $M_{\underline{Y}}$ and M_∞ do. To prove that $\mathfrak{F}^n$ has local units for $n\geqslant 1$, it suffices to show that Γ_ρ does. We may and do identify Γ_ρ with the inverse image of $\Sigma(\rho(\mathbb{Z}^{(X)}))$ under the projection $\pi:\Gamma M_{\underline{Y}}\to\Sigma M_{\underline{Y}}$; thus

$$\Gamma_\rho=\Gamma\rho\big(\mathbb{Z}^{(X)}\big)+M_\infty M_{\underline{Y}}\subset\Gamma M_{\underline{Y}}.$$

One checks that if $\phi_1,\dots,\phi_r\in\Gamma_\rho$, then there are finite subsets $F_1\subset X$ and $F_2\subset\mathbb{N}$ such that for $y_0=G/G\in Y$, the element

$$e=1\otimes\sum_{x\in F_1}e_{x,x}+\sum_{p\in F_2}e_{p,p}\otimes e_{y_0,y_0}\in\Gamma_\rho$$

satisfies $e^2=e$ and $e\phi_i=\phi_ie=\phi_i$ for all $i=1,\dots,r$. This proves part ii); part iii) is straightforward. □

Theorem 12.3.3. *(Compare [8, Theorem 5.18].) Let $E:\mathbb{Z}-\mathrm{Cat}\to\mathrm{Spt}$ be a functor satisfying both the standing and the sectional assumptions. Let G a group, $\mathcal{F}$ a family of subgroups, and A an E-excisive G-ring. Then*

$$\mathfrak{F}^0A\to\mathfrak{F}^\infty A\to\mathfrak{F}^\infty A/\mathfrak{F}^0A$$

is a Dirac extension for $(G,\mathcal{F},E,A)$.

Proof. The three rings in the extension of the theorem are E-excisive, by Lemma 12.3.1ii) and Sectional Assumption 12.1vii). The map $E(A \rtimes H) \to E(\mathfrak{F}^0 A \rtimes H)$ is an equivalence for all subgroups $H \subset G$ by Lemma 12.1.2, standing assumptions ii) and iii) and sectional assumption vi). Next we prove that if $cX \to X$ is a cofibrant replacement, then $H^G(cX, E(\mathfrak{F}^\infty A/\mathfrak{F}^0 A)) \to H^G(X, E(\mathfrak{F}^\infty A/\mathfrak{F}^0 A))$ is an equivalence. By excision and sectional assumption vi), it suffices to show that

$$H^G\big(cX, E\big(\mathfrak{F}^n A/\mathfrak{F}^0 A\big)\big) \to H^G\big(X, E\big(\mathfrak{F}^n A/\mathfrak{F}^0 A\big)\big) \quad (n \geqslant 1) \tag{12.3.4}$$

is an equivalence. Consider the extension

$$0 \to \mathfrak{F}^n A/\mathfrak{F}^0 A \to \mathfrak{F}^{n+1} A/\mathfrak{F}^0 A \to \mathfrak{F}^{n+1} A/\mathfrak{F}^n A \to 0.$$

By Proposition 3.3.9, $cX \to X$ gives a map of homotopy fibration sequences

$$\begin{array}{ccc}
H^G(cX, E(\mathfrak{F}^n A/\mathfrak{F}^0 A)) & \longrightarrow & H^G(X, E(\mathfrak{F}^n A/\mathfrak{F}^0 A)) \\
\downarrow & & \downarrow \\
H^G(cX, E(\mathfrak{F}^{n+1} A/\mathfrak{F}^0 A)) & \longrightarrow & H^G(X, E(\mathfrak{F}^{n+1} A/\mathfrak{F}^0 A)) \\
\downarrow & & \downarrow \\
H^G(cX, E(\mathfrak{F}^{n+1} A/\mathfrak{F}^n A)) & \longrightarrow & H^G(X, E(\mathfrak{F}^{n+1} A/\mathfrak{F}^n A))
\end{array}$$

By Lemma 12.3.1 and Theorem 11.6, the bottom horizontal map is an equivalence. Hence (12.3.4) is an equivalence for each n, by induction. It remains to show that $E_*(\mathfrak{F}^\infty A \rtimes H) = 0$ for each $H \in \mathcal{F}$. Because E_* preserves filtering colimits by assumption, we may further restrict ourselves to proving that the map $j_n : E_*(\mathfrak{F}^n A \rtimes H) \to E_*(\mathfrak{F}^{n+1} A \rtimes H)$ induced by inclusion is zero for all n. By Lemma 12.1.1 we have a long exact sequence ($q \in \mathbb{Z}$)

$$\begin{array}{ccccc}
E_q(\mathfrak{F}^n A \rtimes H) & \xrightarrow{j_n} & E_q(\mathfrak{F}^{n+1} A \rtimes H) & \longrightarrow & E_{q-1}(\mathbb{Z}^{(X)} \otimes \mathfrak{F}^n A \rtimes H) \\
 & & & & \downarrow{\scriptstyle \partial} \\
 & & & & E_{q-1}(\mathfrak{F}^n A \rtimes H)
\end{array}$$

where $\partial = E_{q-1}(\rho \otimes 1 \rtimes 1)$. By Lemma 12.3.1, part iii), ∂ is a split surjection. It follows that $j_n = 0$; this concludes the proof. $\square$

Example 12.3.5. The hypothesis of Theorem 12.3.3 are satisfied, for example, by the functorial spectra K, K^{ninf} and KH.

13. Isomorphism conjectures with proper coefficients

13.1. The excisive case

Theorem 13.1.1. *Let $E : \mathbb{Z} - \mathrm{Cat} \to \mathrm{Spt}$ be a functor. Assume that E satisfies Standing Assumptions 3.3.2, that it is excisive and that E_* commutes with filtering colimits. Let A be a $(G, \mathcal{F})$-proper G-ring. Then the functor $H^G(-, E(A))$ sends $(G, \mathcal{F})$-equivalences to equivalences. In particular the assembly map*

$$H^G\big(\mathcal{E}(G, \mathcal{F}), E(A)\big) \to E(A \rtimes G)$$

is an equivalence.

Proof. By definition of properness, there is a locally finite $(G, \mathcal{F})$-complex X such that A is proper over X. We consider first the case when X is finite dimensional. If $\dim X = 0$, the theorem follows from Theorem 11.6. Let $n > 0$ and assume the theorem true in dimensions $< n$. If $\dim X = n$, and $Y \subset X$ is the $n - 1$-skeleton, we have a pushout diagram

$$\begin{array}{ccc}
\coprod_i \mathrm{Ind}_{H_i}^G(\Delta^n) & \longrightarrow & X \\
\uparrow & & \uparrow \\
\coprod_i \mathrm{Ind}_{H_i}^G(\partial \Delta^n) & \longrightarrow & Y
\end{array}$$

Here $H_i \in \mathcal{F}$ and the horizontal arrows are proper, since X is assumed locally finite. Hence we obtain a pullback diagram

$$\begin{CD} \bigoplus_i \mathbb{Z}^{(\Delta^n)} \otimes \mathbb{Z}^{(G/H_i)} @<<< \mathbb{Z}^{(X)} \\ @VVV @VVV \\ \bigoplus_i \mathbb{Z}^{(\partial\Delta^n)} \otimes \mathbb{Z}^{(G/H_i)} @<<< \mathbb{Z}^{(Y)} \end{CD} \tag{13.1.2}$$

Let $I = \ker(\mathbb{Z}^{(X)} \to \mathbb{Z}^{(Y)})$ be the kernel of the restriction map; because the diagram above is Cartesian, $I \cong \bigoplus_i \ker(\mathbb{Z}^{(\Delta^n)} \otimes \mathbb{Z}^{(G/H_i)} \to \bigoplus_i \mathbb{Z}^{(\partial\Delta^n)} \otimes \mathbb{Z}^{(G/H_i)})$. The quotient $A/I \cdot A$ is proper over Y, and $I \cdot A$ is proper over $\coprod_i \operatorname{Ind}_{H_i}^G(\Delta^n)$, whence also over the zero-dimensional $\coprod_i G/H_i$, by Lemma 10.1.6. Thus the theorem is true for both $A/I \cdot A$ and $I \cdot A$; because E is excisive by hypothesis, this implies that the theorem is also true for A. This proves the theorem for X-finite dimensional. The general case follows from this using Lemma 10.1.5 and the hypothesis that E_* commutes with filtering colimits. □

Example 13.1.3. Both KH and K^{ninf} satisfy the hypothesis of Theorem 13.1.1.

Remark 13.1.4. The proof of Theorem 13.1.1 makes clear that if the hypothesis that E_* commutes with filtering colimits is dropped, then the theorem remains true when A is proper over a finite dimensional $(G, \mathcal{F})$-complex. On the other hand, the hypothesis that E be excisive is key, since the standing assumptions alone do not guarantee that the excision arguments of the proof go through, not even for $A = \mathbb{Z}$. The argument uses that the common kernel of the vertical maps of (13.1.2) be E-excisive; by Standing Assumption 3.3.2v) this is equivalent to saying that $I_n = \ker(\mathbb{Z}^{\Delta^n} \to \mathbb{Z}^{\partial\Delta^n})$ is E-excisive. However I_n is not K-excisive, because $\operatorname{Tor}_1^{\tilde{I}_n}(\mathbb{Z}, I_n) = I_n/I_n^2 \neq 0$ (see Section A.1).

13.2. The K-theory isomorphism conjecture with proper coefficients

Theorem 13.2.1. *Let G be a group, $\mathcal{F}$ a family of subgroups of G, and A a G-ring. Assume that $\mathcal{F}$ contains all the cyclic subgroups, and that A is proper over a locally finite $(G, \mathcal{F})$-complex. Also assume that $A \otimes \mathbb{Q}$ is K-excisive. Then $H^G(-, K(A))$ sends $(G, \mathcal{F})$-equivalences to rational equivalences. If moreover A is a $\mathbb{Q}$-algebra, then $H^G(-, K(A))$ sends $(G, \mathcal{F})$-equivalences to integral equivalences. In particular the assembly map*

$$H_*^G\big(\mathcal{E}(G, \mathcal{F}), K(A)\big) \to K_*(A \rtimes G)$$

is a rational isomorphism if A is a $(G, \mathcal{F})$-proper ring, and an integral isomorphism if in addition A is a $\mathbb{Q}$-algebra.

Proof. By Theorem 13.1.1, $H^G(-, KH(A))$ maps $(G, \mathcal{F})$-equivalences to equivalences. Hence using the fibration

$$K^{\text{nil}} \to K \to KH$$

we see that it suffices to show that the statement of the theorem is true with K^{nil} substituted for K. Because the map (8.2.2) is an equivalence for $\mathbb{Q}$-algebras, it suffices to prove that if A is a $(G, \mathcal{F})$-proper ring, then $H^G(-, K^{\text{nil}}(A))$ sends $(G, \mathcal{F})$-equivalences to rational equivalences. Consider the fibration

$$K^{\text{ninf}} \to K^{\text{nil}} \otimes \mathbb{Q} \to \Omega^{-1}\big|HC(- \otimes \mathbb{Q})\big|.$$

Because $\mathcal{F}$ contains all cyclic subgroups and $A \otimes \mathbb{Q}$ is H-unital, $H^G(-, HC(A \otimes \mathbb{Q}))$ sends $(G, \mathcal{F})$-equivalences to equivalences, by Proposition 7.6 and Corollary 3.3.11. Similarly, $H^G(-, K^{\text{ninf}}A)$ sends $(G, \mathcal{F})$-equivalences to equivalences, by Theorem 13.1.1 and Proposition 8.2.4. It follows that the same is true of $H^G(-, K^{\text{nil}}(A) \otimes \mathbb{Q})$. This completes the proof. □

Example 13.2.2. If X is a $(G, \mathcal{F})$-complex locally finite as a simplicial set and B is K-excisive, then $\mathbb{Z}^{(X)} \otimes B$ is $(G, \mathcal{F})$-proper by Example 10.1.3 and is K-excisive by Proposition 9.5.3. If T is the geometric realization of X and $\mathbb{F} = \mathbb{R}, \mathbb{C}$, then the ring $\mathcal{C}_{\text{comp}}(T)$ of $\mathbb{F}$-valued compactly supported continuous functions is proper over X, again by Example 10.1.3, and therefore it $(G, \mathcal{F})$-proper. In fact the argument given in 10.1.3 to show that $\mathbb{Z}^{(X)} \cdot \mathcal{C}_{\text{comp}}(T) = \mathcal{C}_{\text{comp}}(T)$ shows that $\mathcal{C}_{\text{comp}}(T)$ is s-unital and therefore K-excisive, by Example A.3.5. Hence $\mathcal{C}_{\text{comp}}(T) \otimes B$ is K-excisive if B is, by Proposition A.5.3.

Appendix A. *K*-excisive and *H*-unital rings

A.1. The groups $\operatorname{Tor}_*^{\tilde{A}}(-, A)$

Let $M = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}, \mathbb{Q}$. Theorems of Suslin [29] (for $M = \mathbb{Z}, \mathbb{Z}/n\mathbb{Z}$) and Suslin–Wodzicki [30] (for $M = \mathbb{Q}$) establish that a ring A is excisive for K-theory with coefficients in M if and only if

$$\operatorname{Tor}_*^{\tilde{A}}(M, A) = 0.$$

Example A.1.1. A ring A is said to have the (right) *triple factorization property* if for every finite family $a_1, \ldots, a_n \in A$ there exist $b_1, \ldots, b_n, c, d \in A$ such that

$$a_i = b_i cd \quad \text{and} \quad \{a \in A\colon ad = 0\} = \{a \in A\colon acd = 0\}.$$

It was proved in [30, Theorem C] that rings having the triple factorization property are K-excisive. In particular, rings with local units are K-excisive.

Let M be an abelian group; regard M as an $\tilde{A}$-module through the augmentation $\tilde{A} \to \mathbb{Z}$. We shall introduce a functorial abelian group $\bar{Q}(A, M)$ which computes $\mathrm{Tor}_*^{\tilde{A}}(M, A)$. Consider the functor $\perp\colon \tilde{A} - mod \to \tilde{A} - mod$,

$$\perp N = \bigoplus_{x \in N} \tilde{A}.$$

The functor $\perp$ is the free $\tilde{A}$-module cotriple [33, 8.6.6]. Let $Q(A) \to A$ be the canonical simplicial resolution by free $\tilde{A}$-modules associated to $\perp$ [33, 8.7.2]; by definition, its n-th term is $Q_n(A) = \perp^{n+1} A$. Put

$$\bar{Q}(A, M) = M \otimes_{\tilde{A}} Q(A).$$

We have

$$\pi_*\big(\bar{Q}(A, M)\big) = \mathrm{Tor}_*^{\tilde{A}}(M, A).$$

We abbreviate $\bar{Q}(A) = \bar{Q}(A, \mathbb{Z})$. Note that

$$\bar{Q}(A, M) = M \otimes \bar{Q}(A).$$

We have

$$\bar{Q}_0(A) = \mathbb{Z}[A], \qquad \bar{Q}_{n+1}(A) = \mathbb{Z}\big[Q_n(A)\big].$$

Lemma A.1.2. *Let $F \overset{\sim}{\twoheadrightarrow} A$ be a simplicial resolution in* Rings *and M an abelian group. Let* diag $\bar{Q}(F)$ *be the diagonal of the bisimplicial abelian group $\bar{Q}(F)$. Then*

$$\mathrm{Tor}_*^{\tilde{A}}(M, A) = \pi_*\big(M \otimes \operatorname{diag} \bar{Q}(F)\big).$$

Proof. Because $F \to A$ is a simplicial resolution in Rings, $\bar{Q}_0(F) = \mathbb{Z}[F] \to \mathbb{Z}[A] = \bar{Q}_0(A)$ is a free simplicial resolution in $\mathfrak{A}b$ of the free abelian group $\mathbb{Z}[A]$. Observe that if $G \to N$ is a free resolution of a free abelian group N, then $\tilde{A} \otimes G \to \tilde{A} \otimes N$ is a free simplicial $\tilde{A}$-module resolution, and $\mathbb{Z}[\tilde{A} \otimes G] \to \mathbb{Z}[\tilde{A} \otimes N]$ is a free simplicial $\mathbb{Z}$-module resolution. Thus for each n, $\bar{Q}_n(F) \to \bar{Q}_n(A)$ is a free resolution of the free abelian group $\bar{Q}_n(A)$, and thus it remains a resolution after tensoring by M. It follows that $M \otimes \operatorname{diag} \bar{Q}(F)$ computes $\mathrm{Tor}_*^{\tilde{A}}(M, A)$. $\square$

Proposition A.1.3. *Let $F \overset{\sim}{\twoheadrightarrow} A$ be a simplicial resolution and M an abelian group. Then there is a first quadrant spectral sequence*

$$E^2_{p,q} = \pi_q\big(\mathrm{Tor}_p^{\tilde{F}}(M, F)\big) \Rightarrow \mathrm{Tor}_{p+q}^{\tilde{A}}(M, A).$$

Proof. This is just the spectral sequence of the bisimplicial abelian group $([p], [q]) \mapsto \bar{Q}_p(F_q, M)$. $\square$

Corollary A.1.4. *Let $F \overset{\sim}{\twoheadrightarrow} A$ be free simplicial resolution in* Rings. *Then*

$$\pi_*\big(M \otimes \big(F/F^2\big)\big) = \mathrm{Tor}_*^{\tilde{A}}(M, A).$$

Proof. In view of the previous proposition, and of the fact that $\mathrm{Tor}_0^{\tilde{B}}(M, B) = M \otimes B/B^2$ for every ring B, it suffices to show that if V is a free abelian group, and TV the tensor algebra, then $\mathrm{Tor}_n^{\widetilde{TV}}(M, TV) = 0$ for $n \geqslant 1$. But this is clear, since TV is free as a $\widetilde{TV}$-module; indeed, the multiplication map $\widetilde{TV} \otimes V \to TV$ is an isomorphism. $\square$

A.2. Bar complex

Let A be a ring. Consider the complex $P(A)$ given by $P_n(A)=\tilde{A}\otimes A^{\otimes n+1}$ $(n\geqslant 0)$, with boundary map

$$b''(a_{-1}\otimes a_0\otimes a_1\otimes\cdots\otimes a_n)=\sum_{i=-1}^{n-1}(-1)^i a_{-1}\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n.$$

The multiplication map $\mu:P_0(A)=\tilde{A}\otimes A\to A$ gives a surjective quasi-isomorphism $\mu:P(A)\twoheadrightarrow A$ [33, 8.6.12]. A canonical $\mathbb{Z}$-linear section of μ is $j=1\otimes -:A\to\tilde{A}\otimes A$. Let $\epsilon:\tilde{A}\to A$, $\epsilon(a,n)=a$. A $\mathbb{Z}$-linear homotopy $j\mu\to 1$ is defined by

$$s:P_n(A)\to P_{n+1}(A),\qquad s(a_{-1}\otimes\cdots\otimes a_n)=1\otimes\epsilon(a_{-1})\otimes a_0\otimes\cdots\otimes a_n.$$

Thus $P(A)$ is a resolution of A by $\tilde{A}$-modules, and moreover these $\tilde{A}$-modules are scalar extensions of $\mathbb{Z}$-modules. Put

$$C^{bar}(A)=\mathbb{Z}\otimes_{\tilde{A}}P(A),\qquad b'=\mathbb{Z}\otimes_{\tilde{A}}b''.$$

If A is flat as $\mathbb{Z}$-module, then $C^{bar}(A)$ computes $\mathrm{Tor}_*^{\tilde{A}}(\mathbb{Z},A)$ and $C^{bar}(A,M)=M\otimes C^{bar}(A)$ computes $\mathrm{Tor}_*^{\tilde{A}}(M,A)$. In general, the homology of $C^{bar}(A)$ can be interpreted as the Tor groups relative to the extension $\mathbb{Z}\to\tilde{A}$. For an arbitrary ring A, one can use the natural homotopy s to give a natural map

$$Q(A)\to P(A).$$

The induced map $M\otimes\bar{Q}(A)\to M\otimes C^{bar}(A)$ is a quasi-homomorphism if A is flat as a $\mathbb{Z}$-module. In particular, we have the following.

Lemma A.2.1. *Let $F\overset{\sim}{\twoheadrightarrow}A$ be a simplicial resolution by flat rings, and M an abelian group. Then*

$$\mathrm{Tor}_*^{\tilde{A}}(M,A)=H_*\big(\mathrm{Tot}\big(M\otimes C^{bar}(F)\big)\big).$$

A.3. H-unital rings

A ring A is called *H-unital* if for every abelian group V, the complex $C^{bar}(A)\otimes V$ is acyclic.

Remark A.3.1. Note that for A flat as a $\mathbb{Z}$-module, H-unitality is equivalent to the acyclicity of $C^{bar}(A)$, that is, to the vanishing of the groups $\mathrm{Tor}_*^{\tilde{A}}(\mathbb{Z},A)$. Thus for a flat ring H-unitality equals K excessiveness.

Pure exact sequences. Let

$$0\to A\to B\to C\to 0 \tag{A.3.2}$$

be an exact sequence of rings. We say that (A.3.2) is *pure* if for every abelian group V, the sequence of abelian groups

$$0\to A\otimes V\to B\otimes V\to C\otimes V\to 0$$

is exact. Pure injective and pure surjective maps, and pure acyclic complexes are defined in the obvious way. If $X(-)$ is a functorial chain complex, then we say that A is *pure X-excisive* if for every pure exact sequence (A.3.2),

$$X(A)\otimes V\to X(B)\otimes V\to X(C)\otimes V$$

is a distinguished triangle for every abelian group V. The following theorem was proved by M. Wodzicki in [35].

Theorem A.3.3 *(Wodzicki). The following conditions are equivalent for a ring A.*

i) *A is H-unital.*
ii) *A is pure C^{bar}-excisive.*
iii) *A is pure HH-excisive.*
iv) *A is pure HC-excisive.*

Example A.3.4. Any linearly split sequence (A.3.2) is pure. In particular, any sequence (A.3.2) with A a $\mathbb{Q}$-algebra is pure, since any $\mathbb{Q}$-vectorspace is injective as an abelian group. Thus for a $\mathbb{Q}$-algebra A, Wodzicki's theorem remains valid if we omit the word "pure" everywhere. Furthermore, by the Suslin–Wodzicki theorem cited above, for A a $\mathbb{Q}$-algebra the conditions of Theorem A.3.3 are also equivalent to A being $K^{\mathbb{Q}}$-excisive. In fact it is well-known that for a $\mathbb{Q}$-algebra A, being $K^{\mathbb{Q}}$-excisive is equivalent to being K-excisive; as explained in [3, Lemma 4.1] this well-known fact follows from the main result of [31]. See [30, Lemma 1.9] for a different proof.

Example A.3.5. Each s-unital ring is H-unital, by [35, Corollary 4.5]. Thus any s-unital ring which is flat as a $\mathbb{Z}$-module is K-excisive, by Remark A.3.1.

A.4. Colimits

The bar complex manifestly commutes with filtering colimits, and thus H-unital rings are closed under them. The next proposition establishes the analogue of this property for K-excisive rings.

Proposition A.4.1. *Let $\{A_i\}$ be a filtering system of rings, and let M be an abelian group. Write $A = \operatorname{colim} A_i$. Then*

$$\operatorname{Tor}_*^{\tilde{A}}(M, A) = \operatorname*{colim}_i \operatorname{Tor}_*^{\tilde{A}_i}(M, A_i).$$

Proof. Write $\perp : \mathrm{Rings} \to \mathrm{Rings}$, $\perp B = T(\mathbb{Z}[B])$ for the cotriple associated with the forgetful functor $\mathrm{Rings} \to \mathfrak{S}ets$ and its adjoint. Write $F(A) \overset{\sim}{\twoheadrightarrow} A$ for the cotriple resolution $F(A)_n = \perp^{n+1} A$ [33, §8/6]. We have $F(A) = \operatorname{colim}_i F(A_i)$. Thus $\operatorname{Tot}(M \otimes C^{bar} F(A)) = \operatorname{colim}_i M \otimes C^{bar} F(A_i)$. Hence we are done by Lemma A.2.1. □

Corollary A.4.2. *K-excisive rings are closed under filtering colimits.*

Let M^0 and M^1 be chain complexes of abelian groups, and let $f \in [1]^n$. Put

$$T^f\big(M^0, M^1\big) = M^{f(1)} \otimes \cdots \otimes M^{f(n)}.$$

Let

$$M^0 \star M^1 = \bigoplus_{n \geqslant 0} \bigoplus_{f \in \operatorname{map}([n],[1])} T^f\big(M^0, M^1\big).$$

Lemma A.4.3. *Let A and B be rings. Then*

$$C^{bar}(A \oplus B) = \big(C^{bar}(A)[-1] \star C^{bar}(B)[-1]\big)[+1].$$

Proof. If D is a ring then $C^{bar}(D) = T(D[-1])[+1]$ as graded abelian groups. Hence for $\coprod$ the coproduct of rings, we have

$$C^{bar}(A \oplus B) = T\big(A[-1] \oplus B[-1]\big)[+1] = \Big(T\big(A[-1]\big) \coprod T\big(B[-1]\big)\Big)[+1] = \big(C^{bar}(A)[-1] \star C^{bar}(B)[-1]\big)[+1].$$

It is straightforward to check that the identifications above are compatible with boundary maps. □

Proposition A.4.4. *Let $\{A_i\}$ be a family of rings and $A = \bigoplus_i A_i$. Then A is K-excisive if and only if each A_i is, and in that case $\bigoplus_i K(A_i) \to K(A)$ is an equivalence.*

Proof. Let B and C be rings, and let $F \to B$ and $G \to C$ be free simplicial resolutions in Rings. Then $F \oplus G \to B \oplus C$ is a flat simplicial resolution. Fix $q \geqslant 0$, and put $C^0 = C^{bar}(F_q)$, $C^1 = C^{bar}(G_q)$. Let $p \geqslant 1$, and $f \in [1]^p$. Then by the Künneth formula

$$H_n\big(T^f\big(C^0[-1], C^1[-1]\big)[+1]\big) = T^f\big(H_*\big(C^0\big), H_*\big(C^1\big)\big)_{n+1} = \begin{cases} T^f\big(F_q/F_q^2, G_q/G_q^2\big) & p = n+1, \\ 0 & p \neq n+1. \end{cases}$$

Hence the second page of the spectral sequence for the double complex of Lemma A.2.1 is

$$E^2_{p,q} = \bigoplus_{f \in [1]^{p+1}} \pi_q\big(T^f\big(F/F^2, G/G^2\big)\big).$$

If B and C are K-excisive, we have $E^2 = 0$, by the Eilenberg–Zilber theorem and the Künneth formula, and thus $B \oplus C$ is again K-excisive. It follows from this and from Proposition A.4.1 that if $\{A_i\}$ is a family of K-excisive rings as in the proposition, then A is K-excisive. If B and C are arbitrary, then

$$E^2_{0,q} = \operatorname{Tor}_q^{\tilde{B}}(\mathbb{Z}, B) \oplus \operatorname{Tor}_q^{\tilde{C}}(\mathbb{Z}, C), \qquad E^2_{p,0} = \bigoplus_{f \in [1]^{p+1}} T^f\big(B/B^2, C/C^2\big).$$

Hence if $B \oplus C$ is excisive, $E^2_{*,0} = 0$. It follows that $E^2_{0,1} = 0$, and therefore $\pi_1(T^f(F/F^2, G/G^2))$ involves direct summands of tensor products of the form $E^2_{p,0} \otimes E^2_{0,1}$ and its symmetric, and both of these are zero. Thus $E^2_{*,1} = 0$. A recursive argument shows that $E^2 = 0$, whence both B and C are K-excisive. If now A and $\{A_i\}$ are as in the proposition, A is excisive, and

$j \in I$, then setting $B = A_j$ and $C = \bigoplus_{i \neq j} A_i$ above, we obtain that A_j is K-excisive. The last assertion of the proposition is well-known if each A_i is unital. More generally, assume all A_i are K-excisive, and consider the exact sequence

$$0 \to A \to \bigoplus_i \tilde{A}_i \to \bigoplus_i \mathbb{Z} \to 0. \tag{A.4.5}$$

We have a commutative diagram with homotopy fibration rows

$$\begin{array}{ccccc}
\bigoplus_i K(A_i) & \longrightarrow & \bigoplus_i K(\tilde{A}_i) & \longrightarrow & \bigoplus_i K(\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
K(A) & \longrightarrow & K(\bigoplus_i \tilde{A}_i) & \longrightarrow & K(\bigoplus_i \mathbb{Z})
\end{array}$$

Because the middle and right vertical arrows are equivalences, it follows that the left one is an equivalence too. □

Proposition A.4.6. *Let $\{A_i\}$ be a family of rings and $A = \bigoplus_i A_i$. Then A is H-unital if and only if each A_i is, and in that case $\bigoplus_i HH(A_i) \to HH(A)$ and $\bigoplus_i HC(A_i) \to HC(A)$ are quasi-isomorphisms.*

Proof. The last assertion is proved by the same argument as its K-theoretic counterpart. By Theorem A.3.3 and Lemma A.4.3, if B and C are rings and B is H-unital, then $C^{bar}(B \oplus C) \otimes V \to C^{bar}(C) \otimes V$ is a quasi-isomorphism for every abelian group V. Thus if also C is H-unital, then so is $B \oplus C$. Using this and the fact that H-unitality is preserved under filtering colimits, it follows that if $\{A_i\}$ is a family of H-unital rings, then $A = \bigoplus_i A_i$ is H-unital. Suppose conversely that A is H-unital, and consider the pure extension (A.4.5). A similar argument as that of the proof of Proposition A.4.4 shows that $\bigoplus_i HH(A_i) \to HH(A)$ is a quasi-isomorphism. Next fix an index j and let

$$0 \to A_j \to B \to C \to 0$$

be a pure extension. Then

$$0 \to A \to \bigoplus_{i \neq j} A_i \oplus \tilde{B} \to \bigoplus_{i \neq j} A_i \oplus \tilde{C} \to 0$$

is a pure extension. Applying HH yields a distinguished triangle quasi-isomorphic to

$$\bigoplus_i HH(A_i) \to \bigoplus_{i \neq j} HH(A_i) \oplus HH(B) \oplus HH(\mathbb{Z}) \to \bigoplus_{i \neq j} HH(A_i) \oplus HH(C) \oplus HH(\mathbb{Z}).$$

Removing summands, we obtain a triangle

$$HH(A_j) \to HH(B) \to HH(C).$$

We have shown that A_j satisfies excision for pure extensions in Hochschild homology; by Theorem A.3.3, this implies that A_j is H-unital. □

A.5. Tensor products

It was proved by Suslin and Wodzicki [30, Theorem 7.10] that the tensor product of H-unital rings is H-unital. Here we establish a weak analogue of this property for K-excisive rings.

Let A be a ring. Put

$$L_{-1}A = A, \qquad L_{n+1}A = \ker\big(A \otimes L_n(A) \xrightarrow{\mu} L_n(A)\big) \quad (n \geqslant -1).$$

Here μ is the multiplication map.

Lemma A.5.1. *Let A be a K-excisive ring, and V an abelian group. Assume both A and V are flat over $\mathbb{Z}$. Then $L_{n-1}A$ is flat as an abelian group and*

$$\operatorname{Tor}_n^{\widetilde{A \otimes TV}}(\mathbb{Z}, A \otimes TV) = L_{n-1}A \otimes V^{\otimes n+1} \quad (n \geqslant 0).$$

Proof. If M is a left A-module such that

$$A \cdot M = M, \tag{A.5.2}$$

and $L(M) = \ker(A \otimes M \to M)$ is the kernel of the multiplication map, then we have a short exact sequence

$$0 \to L(M) \otimes T^{\geqslant n+1}V \to \widetilde{A \otimes TV} \otimes M \otimes V^{\otimes n} \to M \otimes T^{\geqslant n}V \to 0.$$

By definition, $L_n A = L^{n+1}A$. By [30, Theorem 7.8 and Lemma 7.6], $M = L_n A$ satisfies (A.5.2) for all n, and moreover, it is a flat abelian group, by induction. Thus for $n \geqslant 1$, the sequence

$$0 \to L_{n-1}(M) \otimes T^{\geqslant n+1}V \to \widetilde{A \otimes TV} \otimes L_{n-2}M \otimes V^{\otimes n} \to L_{n-2}M \otimes T^{\geqslant n}V \to 0$$

is exact. Hence

$$\mathrm{Tor}_i^{\widetilde{A\otimes TV}}(\mathbb{Z}, A \otimes TV) = \mathrm{Tor}_i^{\widetilde{A\otimes TV}}\big(\mathbb{Z}, L_{-1}A \otimes T^{\geqslant 1}V\big) = \mathrm{Tor}_0^{\widetilde{A\otimes TV}}\big(\mathbb{Z}, L_{i-1}A \otimes T^{\geqslant i+1}V\big) = L_{i-1}A \otimes V^{\otimes i+1}. \qquad \square$$

Proposition A.5.3. *Let A and B be K-excisive rings, at least one of them flat as a $\mathbb{Z}$-module. Then $A \otimes B$ is K-excisive.*

Proof. Assume A is flat. Let $F \overset{\sim}{\twoheadrightarrow} B$ be a simplicial resolution by free rings. Then $A \otimes F \overset{\sim}{\twoheadrightarrow} A \otimes B$ is a resolution by flat rings. By Lemma A.5.1, the second page of the spectral sequence of Proposition A.1.3 is

$$E^2_{p,q} = \pi_q\big(L_{p-1}A \otimes \big(F/F^2\big)^{\otimes p+1}\big) = L_{p-1}A \otimes \pi_q\big(\big(F/F^2\big)^{\otimes p+1}\big)$$

which equals zero by Corollary A.1.4 and the Künneth formula, since B is K-excisive by assumption, and $L_{p-1}A$ is flat by Lemma A.5.1. $\square$

A.6. Crossed products

Let G be a group and $\pi : \mathbb{Z}[G] \to \mathbb{Z}$ the augmentation $g \mapsto 1$. Put

$$JG = \ker \pi.$$

Lemma A.6.1. *Let V be a $\mathbb{Z}[G]$-module, free as an abelian group. Then*

$$\mathrm{Tor}_n^{\widetilde{TV \rtimes G}}(\mathbb{Z}, TV \rtimes G) = V^{\otimes n+1} \otimes JG^{\otimes n} \otimes \mathbb{Z}[G], \quad n \geqslant 0.$$

Proof. Note that the subset

$$V^{\otimes n} \oplus TV^{\geqslant n+1} \rtimes G \subset TV \rtimes G$$

is a left ideal, and that the map

$$\widetilde{TV \rtimes G} \otimes V^{\otimes n} \to V^{\otimes n} \oplus TV^{\geqslant n+1} \rtimes G, \qquad 1 \otimes y \mapsto y, \qquad x \rtimes g \otimes y \mapsto xg(y) \rtimes g \tag{A.6.2}$$

is a $\widetilde{TV \rtimes G}$-module isomorphism. Let M be a $\mathbb{Z}[G]$-module. Consider the map

$$V^{\otimes n} \otimes M \oplus \big(TV^{\geqslant n+1} \rtimes G\big) \otimes M \to TV^{\geqslant n}V \otimes M, \qquad \big(x, (y \rtimes g) \otimes m\big) \mapsto x + y \otimes gm.$$

Tensoring the isomorphism (A.6.2) with M and composing, we obtain a $\mathbb{Z}$-split surjective homomorphism of $\widetilde{TV \rtimes G}$-modules

$$\widetilde{TV \rtimes G} \otimes V^{\otimes n} \otimes M \twoheadrightarrow TV^{\geqslant n} \otimes M.$$

This map fits in an exact sequence

$$0 \to T^{\geqslant n+1}V \otimes JG \otimes M \to \widetilde{TV \rtimes G} \otimes V^{\otimes n} \otimes M \to T^{\geqslant n}V \otimes M \to 0.$$

If M is flat as an abelian group, then the middle term in the exact sequence above is a flat $\widetilde{TV \rtimes G}$-module. Applying this successively, starting with $M = \mathbb{Z}[G]$, we obtain

$$\mathrm{Tor}_n^{\widetilde{TV \rtimes G}}(\mathbb{Z}, TV \rtimes G) = \mathrm{Tor}_0^{\widetilde{TV \rtimes G}}\big(\mathbb{Z}, TV^{\geqslant n+1} \otimes JG^{\otimes n} \otimes \mathbb{Z}[G]\big) = V^{\otimes n+1} \otimes JG^{\otimes n} \otimes \mathbb{Z}[G]. \qquad \square$$

Proposition A.6.3. *Let G be a group and $A \in G-\mathrm{Rings}$. Assume A is K-excisive. Then $A \rtimes G$ is K-excisive.*

Proof. Note that the forgetful functor from $G-\mathrm{Rings}$ to sets has a left adjoint; namely $X \mapsto T(\mathbb{Z}[G \times X])$. Hence A admits a free resolution $F \overset{\sim}{\twoheadrightarrow} A$ such that each F_n is a G-ring; for example we may take the cotriple resolution associated to the adjoint pair just described. Since F is a simplicial G-ring, we can take its crossed product with G, to obtain a $\mathbb{Z}$-flat resolution $F \rtimes G \overset{\sim}{\twoheadrightarrow} A \rtimes G$. Now proceed as in the proof of Proposition A.5.3, using Lemma A.6.1. $\square$

Proposition A.6.4. *Let G be a group and $A \in G-\mathrm{Rings}$. Assume A is H-unital. Then $A \rtimes G$ is H-unital.*

Proof. The bar resolution $E(G, M)$ [33, §6.5] is functorial on the G-module M. Applying it dimensionwise to $C^{bar}(A)$, we obtain a simplicial chain complex $E(G, C^{bar}(A))$. We may view the latter as a double chain complex with $A^{\otimes q+1} \otimes \mathbb{Z}[G^{p+1}]$ in the (p, q) spot. Removing the first row and the first column yields a double complex whose total chain complex we shall call $M[-1]$. Note M is a chain complex of $A \rtimes G$-modules and homomorphisms. We have $M_0 \cong (A \rtimes G)^{\otimes 2}$, and the multiplication map $(A \rtimes G)^{\otimes 2} \to A \rtimes G$ induces a surjection onto the kernel L of the augmentation $A \rtimes G \to A$, $a \rtimes g \to a$. Note that the hypothesis that A is H-unital implies that the augmented complex

$$\cdots \to M_1 \to M_0 \to L \tag{A.6.5}$$

is pure acyclic. Since each M_n is extended, (A.6.5) is a pure pseudo-free resolution in the terminology of [30, 7.7]. On the other hand, because A is H-unital, the multiplication map $\mu : A^{\otimes 2} \to A$ is pure surjective; thus $\mu \circ (id \otimes g)$ is pure surjective for each $g \in G$. It follows from this that the multiplication map $(A \rtimes G)^{\otimes 2} \to A \rtimes G$ is pure surjective. We have shown that $A \rtimes G$ satisfies condition d) of [30, Theorem 7.8], which by [30] implies that $A \rtimes G$ is H-unital. □

References

[1] P. Balmer, M. Matthey, Model theoretic reformulation of the Baum–Connes and Farrell–Jones conjectures, Adv. Math. 189 (2004) 495–500.
[2] A. Bartels, H. Reich, Coefficients for the Farrell–Jones conjecture, Adv. Math. 209 (2007) 337–362.
[3] G. Cortiñas, The obstruction to excision in K-theory and in cyclic homology, Invent. Math. 164 (2006) 143–173.
[4] G. Cortiñas, Algebraic v. topological K-theory: a friendly match, in: Topics in Algebraic and Topological K-Theory, in: Lect. Notes Math., vol. 2008, Springer, Berlin, 2011, pp. 103–165.
[5] G. Cortiñas, G. Tartaglia, Compact operators and algebraic K-theory for groups which act properly and isometrically on Hilbert space, Preprint, 2013.
[6] G. Cortiñas, A. Thom, Bivariant algebraic K-theory, J. Reine Angew. Math. 610 (2007) 71–123 (Crelle's Journal).
[7] J. Cuntz, Noncommutative simplicial complexes and the Baum–Connes conjecture, Geom. Funct. Anal. 12 (2002) 307–329.
[8] J. Cuntz, R. Meyer, J. Rosenberg, Topological and Bivariant K-Theory, Oberwolfach Semin., vol. 36, Birkhäuser, 2007, 262 pp.
[9] J. Davis, W. Lück, Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory, K-Theory 15 (1998) 241–291.
[10] B.L. Feĭgin, B.L. Tsygan, Additive K-theory, in: K-Theory, Arithmetic and Geometry, Moscow, 1984–1986, in: Lect. Notes Math., vol. 1289, Springer, Berlin, 1987, pp. 67–209.
[11] S. Geller, C. Weibel, Hochschild and cyclic homology are far from being homotopy functors, Proc. Am. Math. Soc. 106 (1) (1989) 49–57.
[12] E. Guentner, N. Higson, J. Trout, Equivariant E-theory for C^*-algebras, Mem. Am. Math. Soc. 148 (703) (2000), viii+86 pp.
[13] N. Higson, G. Kasparov, E-theory and KK-theory for groups which act properly and isometrically on Hilbert space, Invent. Math. 144 (1) (2001) 23–74.
[14] P. Hirschhorn, Model Categories and Their Localizations, Math. Surv. Monogr., vol. 99, Amer. Math. Soc., Providence, RI, 2003.
[15] M. Hovey, Model Categories, Math. Surv. Monogr., vol. 63, Amer. Math. Soc., Providence, RI, 1999.
[16] M. Joachim, K-homology of C^*-categories and symmetric spectra representing K-homology, Math. Ann. 327 (2003) 641–670.
[17] M. Karoubi, Foncteurs derivées et K-théorie, in: Séminaire Heidelberg-Saarbrucken-Strasbourg, in: Lect. Notes Math., vol. 136, 1967/68, pp. 107–186, 1970.
[18] M. Karoubi, O.E. Villamayor, K-théorie algébrique et K-théorie topologique. I, Math. Scand. 28 (1971) 265–307, 1972.
[19] J.L. Loday, Cyclic Homology, Grundlehren Math. Wiss., vol. 301, Springer-Verlag, Berlin, 1992.
[20] M. Lorenz, On the homology of graded algebras, Commun. Algebra 20 (1992) 489–507.
[21] W. Lück, H. Reich, Detecting K-theory by cyclic homology, Proc. Lond. Math. Soc. 93 (3) (2006) 593–634.
[22] R. Meyer, Universal coefficient theorems and assembly maps in KK-theory, in: Topics in Algebraic and Topological K-Theory, in: Lect. Notes Math., vol. 2008, Springer, Berlin, 2011, pp. 45–102.
[23] R. Meyer, R. Nest, The Baum–Connes conjecture via localization of categories, Lett. Math. Phys. 69 (2004) 237–263.
[24] R. Meyer, R. Nest, The Baum–Connes conjecture via localisation of categories, Topology 45 (2) (2006) 209–259.
[25] R. McCarthy, The cyclic homology of an exact category, J. Pure Appl. Algebra 93 (3) (1994) 251–296.
[26] G. Nöbeling, Verallgemeinerung eines Satzes von Herrn E. Specker, Invent. Math. 6 (1968) 41–55.
[27] E. Pedersen, C. Weibel, K-theory homology of spaces, in: Algebraic Topology, Arcata, CA, 1986, in: Lect. Notes Math., vol. 1370, Springer, Berlin, 1989, pp. 346–361.
[28] D. Quillen, Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories, in: Proc. Conf., Battelle Memorial Inst., Seattle, WA 1972, in: Lect. Notes Math., vol. 341, Springer, Berlin, 1973, pp. 85–147.
[29] A. Suslin, Excision in the integral algebraic K-theory, Proc. Steklov Inst. Math. 208 (1995) 255–279.
[30] A. Suslin, M. Wodzicki, Excision in algebraic K-theory, Ann. Math. 136 (1992) 51–122.
[31] C. Weibel, Mayer–Vietoris sequences and mod p K-theory, Springer Lect. Notes Math. 966 (1982) 390–406.
[32] C. Weibel, Nil K-theory maps to cyclic homology, Trans. Am. Math. Soc. 303 (2) (1987) 541–558.
[33] C. Weibel, An Introduction to Homological Algebra, Cambridge Univ. Press, 1994.
[34] C. Weibel, Homotopy algebraic K-theory, Contemp. Math. 83 (1989) 461–488.
[35] M. Wodzicki, Excision in cyclic homology and in rational algebraic K-theory, Ann. Math. 129 (1989) 591–639.