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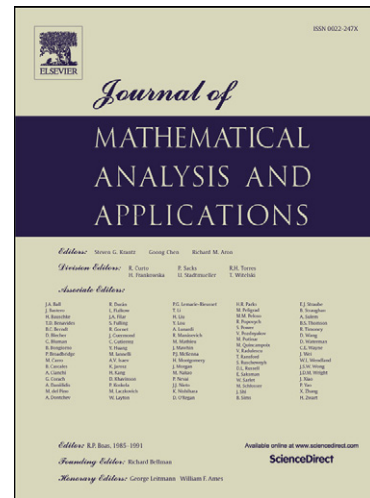
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**AN OPTIMIZATION PROBLEM FOR NONLINEAR STEKLOV  
EIGENVALUES WITH A BOUNDARY POTENTIAL**

JULIÁN FERNÁNDEZ BONDER, GRACIELA O. GIUBERGIA,  
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ABSTRACT. In this paper, we analyze an optimization problem for the first (nonlinear) Steklov eigenvalue plus a boundary potential with respect to the potential function which is assumed to be uniformly bounded and with fixed  $L^1$ -norm.

1. INTRODUCTION

In recent years a great deal of attention has been putted in optimal design problems for eigenvalues (both linear and nonlinear) due to many interesting applications. For a comprehensive description of the current developments in the field in the case of linear eigenvalues and very interesting open problems, we refer to [12]. In the nonlinear setting, we refer to the recent research papers [3, 4, 5, 7, 8, 11] and references therein.

To be precise, the eigenvalue problem that we are interested in is the following

$$(1.1) \quad \begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \mathbf{n}} + \sigma \phi |u|^{p-2}u = \lambda |u|^{p-2}u & \text{in } \partial\Omega. \end{cases}$$

Here  $\Omega \subset \mathbb{R}^n$  is a bounded smooth domain,  $\Delta_p u$  is the usual  $p$ -Laplace operator defined as  $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ ,  $\mathbf{n}$  denotes the outer unit normal vector to  $\partial\Omega$ ,  $\phi \in L^\infty(\partial\Omega)$  is a nonnegative boundary potential and  $\sigma > 0$  is a real parameter.

Under these hypotheses, the functional associated to (1.1) is trivially coercive, that is

$$I(u, \phi) := \int_{\Omega} |\nabla u|^p + |u|^p dx + \sigma \int_{\partial\Omega} \phi |u|^p d\mathcal{H}^{n-1} \geq \|u\|_{W^{1,p}(\Omega)}^p.$$

This functional is associated to (1.1) in the sense that eigenvalues  $\lambda$  of (1.1) are critical values of  $I$  restricted to the manifold  $\|u\|_{L^p(\partial\Omega)} = 1$ . See [9].

In particular, It is easy to see that the minimum value of  $I$

$$(1.2) \quad \lambda(\sigma, \phi) := \inf \{ I(u, \phi) : u \in W^{1,p}(\Omega), \|u\|_{L^p(\partial\Omega)} = 1 \}$$

is the first (lowest) eigenvalue of (1.1). Therefore, the existence of the first eigenvalue and the corresponding eigenfunction  $u$  follows from the compact embedding  $W^{1,p}(\Omega) \subset L^p(\partial\Omega)$ .

In this work, we are interested in the minimization problem for  $\lambda(\sigma, \phi)$  with respect to different configurations for the boundary potential  $\phi$ . That is, given certain class of admissible potentials  $\mathcal{A}$ , we look for the minimum possible value of  $\lambda(\sigma, \phi)$  when  $\phi \in \mathcal{A}$ .

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This study complements the ones started in [7]. In that paper, the authors analyzed the Steklov problem but with an interior potential and show the connections of that problem with the one considered in [11].

In this opportunity, we consider the class of uniformly bounded potentials, i.e.

$$\mathcal{A} := \{\phi \in L^\infty(\partial\Omega) : 0 \leq \phi \leq 1\}.$$

Observe that  $\mathcal{A}$  is the closure of the characteristic functions in the weak\* topology.

Clearly, the minimization problem in the whole class  $\mathcal{A}$  has no sense since the infimum is realized with  $\phi \equiv 0$ . The relevant problem here is to consider the minimization among those potentials in  $\mathcal{A}$  that has fixed  $L^1$ -norm. That is

$$(1.3) \quad \Lambda(\sigma, a) := \inf \left\{ \lambda(\sigma, \phi) : \phi \in \mathcal{A}, \int_{\partial\Omega} \phi d\mathcal{H}^{n-1} = a \right\}$$

The first result in this paper is the existence of an optimal potential for  $\Lambda(\sigma, a)$  and, moreover, it is shown that this optimal potential can be taken as the characteristic function a sub-level set  $D_\sigma$  of the corresponding eigenfunction. See [1, 2] for related results.

As another application we investigate the connection with the optimization problem considered in [6]. That is, given  $E \subset \partial\Omega$ , consider the equation

$$(1.4) \quad \begin{cases} -\Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega \\ u = 0 & \text{in } E \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{in } \partial\Omega \setminus E \end{cases}$$

whose first eigenvalue is given by

$$(1.5) \quad \lambda(\infty, E) := \inf \left\{ \|u\|_{W^{1,p}(\Omega)}^p : \|u\|_{L^p(\partial\Omega)} = 1, u = 0, \mathcal{H}^{n-1} \text{ a.e. in } E \right\},$$

Associated to (1.5) we have the optimal configuration problem

$$(1.6) \quad \Lambda(\infty, a) := \inf \left\{ \lambda(\infty, E) : \mathcal{H}^{n-1}(E) = a \right\}.$$

Our second result shows that  $\Lambda(\sigma, a) \rightarrow \Lambda(\infty, a)$  as  $\sigma \rightarrow \infty$  and, moreover, the optimal configuration  $\phi_\sigma = \chi_{D_\sigma}$  of  $\Lambda(\sigma, a)$  converges (in the topology of  $L^1$ -convergence of the characteristic functions) to an optimal configuration of the limit problem  $\Lambda(\infty, a)$ .

The remaining of the paper is devoted to analyze qualitative properties of optimal configurations for  $\Lambda(\sigma, a)$ .

First, we consider the spherical symmetric case, that is when  $\Omega$  is a ball, and in this simple case by means of symmetrization arguments we can give a full description of the optimal configurations.

Finally, we address the general problem and study the behavior of  $\lambda(\sigma, \chi_D)$  for regular deformations of the set  $D$ . We employ the so-called method of Hadamard and prove differentiability of  $\lambda(\sigma, \chi_D)$  with respect to regular deformations and provide a simple formula for the derivative of the eigenvalue. The main novelty of this formula is that it involves a  $(n-2)$ -dimensional integral along the boundary of  $D$  relative to  $\partial\Omega$ . Up to our knowledge, this is the first time that this type of lower-dimensional integrals were observed in this type of computations for quasilinear problems. In the case of linear equations similar terms in the Hadamard variation formula was found in [15] but the authors make full use of the linearity in their approach.

We want to remark that the results in this work are new even in the linear setting,  $p = 2$ .

## 2. PRELIMINARY REMARKS

A simple modification of the arguments in [10] shows that, given  $\phi \in \mathcal{A}$  and  $\sigma > 0$ , the first eigenvalue  $\lambda(\sigma, \phi)$  is simple. i.e. any two eigenfunctions are multiple of each other. Therefore, there exists a unique nonnegative, normalized eigenfunction  $u$  (normalized means that  $\|u\|_{L^p(\partial\Omega)} = 1$ ).

The purpose of this very short section is to recall some regularity properties of this eigenfunction.

First, we note that by [16], there exists  $\alpha > 0$  such that  $u \in C_{loc}^{1,\alpha}(\Omega)$ . Now, by an usual argument, we have that  $|u|$  is an eigenfunction associated to  $\lambda(\sigma, \phi)$ . Hence, the Harnack inequality, c.f. [16], implies that any first eigenfunction  $u$  has constant sign and, moreover, that  $u > 0$  in  $\Omega$ .

Next, by the results of [14], an eigenfunction of (1.1) is continuous up to the boundary. In fact,  $u \in C^\beta(\bar{\Omega})$  for some  $\beta > 0$ .

Summing up, we have

**Proposition 2.1.** *Given  $\phi \in \mathcal{A}$  and  $\sigma > 0$ , there exists a unique nonnegative eigenfunction  $u \in W^{1,p}(\Omega)$  of (1.1) associated to  $\lambda(\sigma, \phi)$ . Moreover, this eigenfunction  $u$  verifies that  $u \in C_{loc}^{1,\alpha}(\Omega) \cap C^\beta(\bar{\Omega})$  for some  $\alpha, \beta > 0$ . Finally,  $u > 0$  in  $\Omega$ .*

## 3. EXISTENCE OF OPTIMAL CONFIGURATIONS

In this section we first establish the existence of optimal configurations for  $\Lambda(\sigma, a)$ . Then we analyze the limit  $\sigma \rightarrow \infty$  and show the convergence to the problem  $\Lambda(\infty, a)$ .

Let us begin with the existence result.

**Theorem 3.1.** *For any  $\sigma > 0$  and  $0 \leq a \leq \mathcal{H}^{n-1}(\partial\Omega)$  there exist an optimal pair  $(u, \phi) \in W^{1,p}(\Omega) \times \mathcal{A}$ , which has the following properties*

- (1)  $u \in C_{loc}^{1,\alpha}(\Omega) \cap C(\bar{\Omega})$
- (2)  $\phi = \chi_D$  where, for some  $s$ ,  $\{u < s\} \subset D \subset \{u \leq s\}$ ,  $\mathcal{H}^{n-1}(D) = a$

*Proof.* We consider a minimizing sequence  $\{\phi_k\}_{k \in \mathbb{N}} \subset \mathcal{A}$  of (1.3) and their associated normalized eigenfunctions  $\{u_k\}_{k \in \mathbb{N}} \subset W^{1,p}(\Omega)$ .

From the reflexivity of the Sobolev space  $W^{1,p}(\Omega)$ , the compactness of the embeddings  $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$  and  $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$  and  $L^\infty(\partial\Omega)$  being a dual space, we obtain a subsequence (again denoted  $\{u_k, \phi_k\}$ ) and  $u \in W^{1,p}(\Omega)$ ,  $\phi \in L^\infty(\partial\Omega)$  such that

$$(3.1) \quad u_k \rightharpoonup u \quad \text{in } W^{1,p}(\Omega)$$

$$(3.2) \quad u_k \rightarrow u \quad \text{in } L^p(\partial\Omega)$$

$$(3.3) \quad u_k \rightarrow u \quad \text{in } L^p(\Omega)$$

$$(3.4) \quad \phi_k \xrightarrow{*} \phi \quad \text{in } L^\infty(\partial\Omega)$$

From the admissibility of  $\phi_k$  and (3.4), we get  $0 \leq \phi \leq 1$  and  $\int_{\partial\Omega} \phi d\mathcal{H}^{n-1} = a$ . Using (3.2), we get  $\|u\|_{L^p(\partial\Omega)} = 1$ . As a consequence of the lower semicontinuity of the norm  $\|\cdot\|_{W^{1,p}(\Omega)}$  with respect to weak convergence, we obtain

$$(3.5) \quad \int_{\Omega} |\nabla u|^p + |u|^p dx \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla u_k|^p + |u_k|^p dx$$

Using (3.2), we can see that  $|u_k|^p \rightarrow |u|^p$  in  $L^1(\partial\Omega)$ . Therefore, taking into account (3.4) we obtain

$$(3.6) \quad \int_{\partial\Omega} \phi_k |u_k|^p d\mathcal{H}^{n-1} \rightarrow \int_{\partial\Omega} \phi |u|^p d\mathcal{H}^{n-1}$$

From (3.5) and (3.6), we have  $(u, \phi)$  is an optimal pair for (1.3).

By an elementary variation of the *Bathtub Principle* ([14, Pag. 28]), we can prove that the minimization problem

$$\inf_{\int_{\partial\Omega} \phi d\mathcal{H}^{n-1} = a} \int_{\partial\Omega} \phi |u|^p d\mathcal{H}^{n-1},$$

has a solution of the form  $\phi = \chi_D$ , where  $\{u < s\} \subset D \subset \{u \leq s\}$  and  $\mathcal{H}^{n-1}(D) = a$  and therefore  $(\chi_D, u)$  is an optimal pair for  $\Lambda(\sigma, a)$ .  $\square$

Now we prove a Lemma about the continuity of the eigenvalues and eigenfunctions with respect to the potential  $\phi$  in the weak \* topology.

**Lemma 3.2.** *Let  $\phi_j, \phi \in L^\infty(\partial\Omega)$  be such that  $\phi_j \xrightarrow{*} \phi$  in  $L^\infty(\partial\Omega)$ . Let  $\lambda_j = \lambda(\sigma, \phi_j)$  and  $\lambda = \lambda(\sigma, \phi)$  the eigenvalues defined by (1.2) and let  $u_j, u \in W^{1,p}(\Omega)$  be the positive normalized eigenfunctions associated to  $\lambda_j$  and  $\lambda$  respectively.*

*Then  $\lambda_j \rightarrow \lambda$  and  $u_j \rightarrow u$  strongly in  $W^{1,p}(\Omega)$  as  $j \rightarrow \infty$ .*

*Proof.* First, define  $v \equiv \mathcal{H}^{n-1}(\partial\Omega)^{-1/p}$  and from (1.2) we get

$$\lambda_j \leq I(v, \phi_j) = \frac{|\Omega| + \int_{\Omega} \phi_j}{\mathcal{H}^{n-1}(\partial\Omega)} \leq C$$

for every  $j \in \mathbb{N}$ . Therefore, since  $\|u_j\|_{W^{1,p}(\Omega)} \leq \lambda_j$  (recall that the eigenfunctions  $u_j$  are normalized) it follows that  $\{u_j\}_{j \in \mathbb{N}}$  is bounded in  $W^{1,p}(\Omega)$ .

From these, we obtain the existence of a function  $w \in W^{1,p}(\Omega)$  such that, for a subsequence,

$$\begin{aligned} u_j &\rightharpoonup w && \text{weakly in } W^{1,p}(\Omega) \\ u_j &\rightarrow w && \text{strongly in } L^p(\Omega) \\ u_j &\rightarrow w && \text{strongly in } L^p(\partial\Omega) \end{aligned}$$

It then follows that  $w \geq 0$  and that  $\|w\|_{L^p(\partial\Omega)} = 1$ .

Now, from the weakly sequentially lower semicontinuity it holds

$$(3.7) \quad \lambda \leq I(w, \phi) \leq \liminf I(u_j, \phi) = \liminf I(u_j, \phi_j) + \sigma \int_{\partial\Omega} (\phi - \phi_j) |u_j|^p d\mathcal{H}^{n-1}.$$

Since  $|u_j|^p \rightarrow |u|^p$  strongly in  $L^1(\partial\Omega)$ , it easily follows that

$$\lambda \leq \liminf \lambda_j.$$

For the reverse inequality, we proceed in a similar fashion. In fact, from (1.2)

$$\lambda_j \leq I(u, \phi_j).$$

Therefore

$$\limsup \lambda_j \leq \lim I(u, \phi_j) = I(u, \phi) = \lambda,$$

so  $\lambda_j \rightarrow \lambda$ .

Finally, from (3.7), one obtains that  $I(w, \phi) = \lambda$  and since there exists a unique nonnegative normalized eigenfunction associated to  $\lambda$  it follows that  $w = u$ . Moreover, again from (3.7) it is easily seen that  $\|u_j\|_{W^{1,p}(\Omega)} \rightarrow \|u\|_{W^{1,p}(\Omega)}$  and so  $u_j \rightarrow u$

strongly in  $W^{1,p}(\Omega)$  and, since the limit is uniquely determined, the whole sequence  $\{u_j\}_{j \in \mathbb{N}}$  is convergent.  $\square$

The next Lemma, that was proved in [6] gives the strict monotonicity of the quantity  $\Lambda(\infty, a)$  with respect to  $a$  and will be helpful in showing the behavior of  $\Lambda(\sigma, a)$  for  $\sigma \rightarrow \infty$ .

**Lemma 3.3** (Corollary 3.7, [6]). *The function  $\Lambda(\infty, \cdot)$  is strictly monotonic.*

Now we are ready to prove the convergence of  $\Lambda(\sigma, a)$  to  $\Lambda(\infty, a)$  as  $\sigma \rightarrow \infty$ .

**Theorem 3.4.** *If  $\sigma_j$  is a sequence tending to  $\infty$  and  $(u_j, D_j)$  associated optimal pairs of (1.3), then there exists a subsequence (that we still call  $\sigma_j$ ) and an optimal pair  $(u, D)$  of the problem (1.6) such that  $u_j \rightharpoonup u$  in  $W^{1,p}(\Omega)$ ,  $\chi_{D_j} \xrightarrow{*} \chi_D$  in  $L^\infty(\partial\Omega)$ .*

*Proof.* We consider  $E \subset \partial\Omega$  closed such that  $\mathcal{H}^{n-1}(E) = a$  and  $v \in W^{1,p}(\Omega)$ ,  $\|v\|_{L^p(\partial\Omega)} = 1$  such that  $v = 0$  in  $E$ . Therefore

$$\|u_j\|_{W^{1,p}(\Omega)}^p \leq I(u_j, \chi_{D_j}) = \Lambda(\sigma_j, a) \leq \lambda(\sigma_j, \chi_E) \leq I(v, \chi_E) = \|v\|_{W^{1,p}(\Omega)}^p$$

Hence, the sequence  $u_j$  is bounded in  $W^{1,p}(\Omega)$ . Therefore we can assume that there exists  $u_\infty \in W^{1,p}(\Omega)$  and  $\phi_\infty \in L^\infty(\partial\Omega)$  such that

$$(3.8) \quad u_j \rightharpoonup u_\infty \text{ in } W^{1,p}(\Omega)$$

$$(3.9) \quad u_j \rightarrow u_\infty \text{ in } L^p(\Omega)$$

$$(3.10) \quad u_j \rightarrow u_\infty \text{ in } L^p(\partial\Omega)$$

$$(3.11) \quad \chi_{D_j} \xrightarrow{*} \phi_\infty \text{ in } L^\infty(\partial\Omega)$$

From (3.10) and (3.11) we have that  $\|u_\infty\|_{L^p(\partial\Omega)} = 1$ ,  $\int_{\partial\Omega} \phi_\infty d\mathcal{H}^{n-1} = a$  and  $0 \leq \phi_\infty \leq 1$ . The rest of the proof follows in a completely analogous way, using Lemma 3.3, to [7, Theorem 1.2]  $\square$

#### 4. SYMMETRY

Throughout this section we assume that  $\Omega$  is the unit ball  $B(0, 1)$ . The goal of the section is to show that there exists an optimal pair  $(u, \chi_D)$  of the problem (1.1) with  $D$  a spherical cup in  $S^{n-1} = \partial\Omega$ . A key tool is played by the *spherical symmetrization*.

The spherical symmetrization of a set  $A \subset \mathbb{R}^n$  with respect to an axis given by a unit vector  $e$  is defined as follows: Given  $r > 0$  we consider  $s_r > 0$  such that  $\mathcal{H}^{n-1}(A \cap \partial B(0, r)) = \mathcal{H}^{n-1}(B(re, s_r) \cap \partial B(0, r))$ . We note that the sets  $A \cap \partial B(0, r)$  are  $\mathcal{H}^{n-1}$ -measurable for almost every  $r \geq 0$ . Now we put:

$$A^* = \bigcup_{0 \leq r \leq 1} B(re, s_r) \cap \partial B(0, r)$$

The set  $A^*$  is well defined and measurable whence  $A$  is a measurable set. If  $u \geq 0$  is a measurable function, we define its symmetrized function  $u^*$  so that satisfies the relation  $\{u^* \geq t\} = \{u \geq t\}^*$ . We refer to [13] for an exhaustive study of this symmetrization. In particular, we need the following known results:

**Theorem 4.1.** *Let  $0 \leq u \in W^{1,p}(\Omega)$  and let  $u^*$  be its symmetrized function. Then*

- (1)  $u^* \in W^{1,p}(\Omega)$

- (2)  $u^*$  and  $u$  are equi-measurable, i.e. they have the same distribution function, Hence for every continuous increasing function  $\Phi: \int_{\Omega} \Phi(u^*)dx = \int_{\Omega} \Phi(u)dx$
- (3)  $\int_{\Omega} uvdx \leq \int_{\Omega} u^*v^*dx$ , for every measurable positive function  $v$ .
- (4) In a similar way  $u$  and  $u^*$  are equimeasurable respect to the Hausdorff measure on boundary of balls. Therefore, the two previous items holds with  $\partial\Omega$  and  $d\mathcal{H}^{n-1}$  instead of  $\Omega$  and  $dx$ , respectively.
- (5)  $\int_{\Omega} |\nabla u^*|^p dx \leq \int_{\Omega} |\nabla u|^p dx$ .

With these preliminaries, we can now prove the main result of the section.

**Theorem 4.2.** *Let  $\Omega = B(0, 1)$ . Then there exists an optimal pair  $(u, \chi_E)$  of the problem (1.1) with  $E$  a spherical cup in  $\partial\Omega$ .*

*Proof.* Let  $(u, \chi_D)$  be an optimal pair. We define  $E := ((D^c)^*)^c$ . Since  $(D^c)^*$  is a spherical cup it follows that  $E$  is also a spherical cup.

We note that  $\chi_E = 1 - (\chi_{D^c})^*$ , therefore it is easy to show, from (c) in Theorem 4.1 that

$$\int_{\partial\Omega} \chi_E |u^*|^p d\mathcal{H}^{n-1} \leq \int_{\partial\Omega} \chi_D |u|^p d\mathcal{H}^{n-1}.$$

We note that  $\int_{\partial\Omega} |u^*|^p d\mathcal{H}^{n-1} = 1$ , so  $u^*$  is an admissible function in (1.2). Moreover,

$$\int_{\Omega} |\nabla u^*|^p + |u^*|^p dx + \sigma \int_{\partial\Omega} \chi_E |u^*|^p d\mathcal{H}^{n-1} \leq \int_{\Omega} |\nabla u|^p + |u|^p dx + \sigma \int_{\partial\Omega} \chi_D |u|^p d\mathcal{H}^{n-1}.$$

Consequently,  $(u^*, \chi_E)$  is an optimal pair.  $\square$

## 5. DERIVATIVE OF EIGENVALUES

Henceforth we put  $\Gamma := \partial\Omega$ . In this section we compute derivatives of the eigenvalues  $\lambda(\sigma, \chi_D)$  with respect to perturbations of the set  $D$ . We also assume that the set  $D \subset \Gamma$  is the closure of a regular relatively open set.

For this purpose, we introduce the vector field  $V: \mathbb{R}^n \rightarrow \mathbb{R}^n$  supported on a narrow neighborhood of  $\Gamma$  with  $V \cdot \mathbf{n} = 0$ , where  $\mathbf{n}$  is the outer normal vector. We consider the flow

$$(5.1) \quad \begin{cases} \frac{d}{dt} \Psi_t(x) &= V(\Psi_t(x)) \\ \Psi_0(x) &= x \end{cases}$$

We note that the condition  $V \cdot \mathbf{n} = 0$  implies that  $\Psi_t(\Gamma) = \Gamma$ . From (5.1), it follows the asymptotic expansions

$$(5.2) \quad D\Psi_t = I + tDV + o(t),$$

$$(5.3) \quad (D\Psi_t)^{-1} = I - tDV + o(t),$$

$$(5.4) \quad J\Psi_t = 1 + t\operatorname{div}V + o(t).$$

Here  $D\Psi_t$  and  $J\Psi_t$  denote the differential matrix of  $\Psi_t$  and its jacobian, respectively. See [12].

In order to try with surface integrals, we need the following formulas whose proofs can be founded in [12]. The tangential Jacobian of  $\Psi_t$  is given by

$$J_{\Gamma}\Psi_t(x) = |(D\Psi_t(x))^{-1}\mathbf{n}|J\Psi_t(x) = 1 + t\operatorname{div}_{\Gamma}V + o(t) \quad x \in \Gamma$$

where  $\operatorname{div}_{\Gamma}V$  is the tangential divergence operator defined by

$$\operatorname{div}_{\Gamma}V = \operatorname{div}V - \mathbf{n}^T DV\mathbf{n}.$$

The main result here is the following

**Theorem 5.1.** *Let  $\sigma > 0$  be fixed and  $D \subset \Gamma$  be the closure of a smooth relatively open set. Let  $u \in W^{1,p}(\Omega)$  be the nonnegative normalized eigenfunction for  $\lambda(\sigma, \chi_D)$ .*

*Then, the function  $\lambda(t) := \lambda(\sigma, \chi_{D_t})$  where  $D_t = \Psi_t(D)$  is differentiable at  $t = 0$  and*

$$\lambda'(0) = -\sigma \int_{\partial_\Gamma D} |u_0|^p (\mathbf{n}_\Gamma \cdot V) d\mathcal{H}^{n-2}$$

where  $\mathbf{n}_\Gamma$  denotes the unit normal vector exterior to  $\partial_\Gamma D$  relative to the tangent space of  $\Gamma$ .

**Remark 5.2.** *Observe that the results of Lemma 3.2 immediately imply the continuity of  $\lambda(t)$  at  $t = 0$  and also that the associated eigenfunctions  $u_t$  strongly converge to the associated eigenfunction  $u$  of  $\lambda(0)$  in  $W^{1,p}(\Omega)$ .*

*Proof of Theorem 5.1.* We will follow the same line that [8, Theorem 1.1]. Let  $u \in W^{1,p}(\Omega)$ . We call  $\bar{u} = u \circ \Psi_t$ , then the following asymptotic expansions hold

$$\begin{aligned} (5.5) \quad & \int_{\Omega} |\nabla \bar{u}|^p + |\bar{u}|^p dx = \int_{\Omega} (|D\Psi_t \nabla u|^p + |u|^p) J\Psi_t^{-1} dx \\ & = \int_{\Omega} (|(I + tDV + o(t))\nabla u|^p + |u|^p)(1 - t\operatorname{div}V + o(t)) dx \\ & = \int_{\Omega} |\nabla u|^p + |u|^p dx - t(\operatorname{div}V(|\nabla u|^p + |u|^p) - p|\nabla u|^{p-2}(\nabla u)^t DV \nabla u) dx + o(t) \end{aligned}$$

$$\begin{aligned} (5.6) \quad & \int_{\Gamma} \chi_{D_t} |\bar{u}|^p d\mathcal{H}^{n-1} = \int_{\Gamma} \chi_D |u|^p J_\Gamma \Psi_t^{-1} d\mathcal{H}^{n-1} \\ & = \int_{\Gamma} \chi_D |u|^p (1 - t\operatorname{div}_\Gamma V) d\mathcal{H}^{n-1} + o(t) \end{aligned}$$

$$\begin{aligned} (5.7) \quad & \int_{\Gamma} |\bar{u}|^p d\mathcal{H}^{n-1} = \int_{\Gamma} |u|^p J_\Gamma \Psi_t^{-1} d\mathcal{H}^{n-1} \\ & = \int_{\Gamma} |u|^p (1 - t\operatorname{div}_\Gamma V) d\mathcal{H}^{n-1} + o(t) \end{aligned}$$

From (5.5) and (5.6), we obtain

$$(5.8) \quad I(\bar{u}, \chi_{D_t}) = F(u) - tG(u) + o(t)$$

where

$$(5.9) \quad F(u) = \int_{\Omega} |\nabla u|^p + |u|^p dx + \sigma \int_{\Gamma} \chi_D |u|^p d\mathcal{H}^{n-1}$$

and

$$(5.10) \quad G(u) = \int_{\Omega} \operatorname{div}V(|\nabla u|^p + |u|^p) - p|\nabla u|^{p-2}(\nabla u)^t DV \nabla u dx + \sigma \int_{\Gamma} \chi_D |u|^p \operatorname{div}_\Gamma V d\mathcal{H}^{n-1}$$



Now, take  $u$  to be a normalized eigenfunction associated to  $\lambda(0)$ . Then we have

$$\begin{aligned}\lambda(t) &\leq \frac{I(\bar{u}, \chi_{D_t})}{\int_{\Gamma} |\bar{u}|^p d\mathcal{H}^{n-1}} = \frac{F(u) - tG(u) + o(t)}{\int_{\Gamma} |u|^p d\mathcal{H}^{n-1} - t \int_{\Gamma} |u|^p \operatorname{div} V d\mathcal{H}^{n-1} + o(t)} \\ &= \frac{F(u)}{\int_{\Gamma} |u|^p d\mathcal{H}^{n-1}} + t \left( F(u) \frac{\int_{\Gamma} |u|^p \operatorname{div} V d\mathcal{H}^{n-1}}{\left(\int_{\Gamma} |u|^p d\mathcal{H}^{n-1}\right)^2} - \frac{G(u)}{\int_{\Gamma} |u|^p d\mathcal{H}^{n-1}} \right) + o(t) \\ &= \lambda(0) + t \left( \lambda(0) \int_{\Gamma} |u|^p \operatorname{div} V d\mathcal{H}^{n-1} - G(u) \right) + o(t)\end{aligned}$$

Therefore

$$(5.11) \quad \lambda(t) - \lambda(0) \leq t \left( \lambda(0) \int_{\Gamma} |u|^p \operatorname{div} V d\mathcal{H}^{n-1} - G(u) \right) + o(t)$$

Now, take  $u_t \in W^{1,p}(\Omega)$  a normalized eigenfunction associated to  $\lambda(t)$  and denote by  $\bar{u}_t = u_t \circ \Psi_{-t}$ . So

$$\begin{aligned}\lambda(0) &\leq \frac{I(\bar{u}_t, \chi_D)}{\int_{\Gamma} |\bar{u}_t|^p d\mathcal{H}^{n-1}} = \frac{F(u_t) + tG(u_t) + o(t)}{\int_{\Gamma} |u_t|^p d\mathcal{H}^{n-1} + t \int_{\Gamma} |u_t|^p \operatorname{div}_{\Gamma} V d\mathcal{H}^{n-1} + o(t)} \\ &= \frac{F(u_t)}{\int_{\Gamma} |u_t|^p d\mathcal{H}^{n-1}} - t \left( F(u_t) \frac{\int_{\Gamma} |u_t|^p \operatorname{div} V d\mathcal{H}^{n-1}}{\left(\int_{\Gamma} |u_t|^p d\mathcal{H}^{n-1}\right)^2} - \frac{G(u_t)}{\int_{\Gamma} |u_t|^p d\mathcal{H}^{n-1}} \right) + o(t) \\ &= \lambda(t) - t \left( \lambda(t) \int_{\Gamma} |u_t|^p \operatorname{div} V d\mathcal{H}^{n-1} - G(u_t) \right) + o(t)\end{aligned}$$

This last inequality together with (5.11) give us

$$\begin{aligned}t \left( \lambda(t) \int_{\Gamma} |u_t|^p \operatorname{div} V d\mathcal{H}^{n-1} - G(u_t) \right) + o(t) &\leq \lambda(t) - \lambda(0) \\ &\leq t \left( \lambda(0) \int_{\Gamma} |u|^p \operatorname{div} V d\mathcal{H}^{n-1} - G(u) \right) + o(t)\end{aligned}$$

So, by Remark 5.2 one gets

$$\lambda'(0) = \left( \lambda(0) \int_{\Gamma} |u|^p \operatorname{div} V d\mathcal{H}^{n-1} - G(u) \right).$$

It remains to further simplify the expression for  $\lambda'(0)$ . Let

$$\begin{aligned}G(u) &= \int_{\Omega} \operatorname{div} V (|\nabla u|^p + |u|^p) - p |\nabla u|^{p-2} (\nabla u)^t DV \nabla u \, dx \\ &\quad + \sigma \int_{\Gamma} \chi_D |u|^p \operatorname{div}_{\Gamma} V d\mathcal{H}^{n-1} \\ &= I_1 + I_2\end{aligned}$$

Now using  $V \cdot \nabla u$  as test function in the equation  $-\Delta_p u + |u|^{p-2}u = 0$  and the boundary condition in (1.1) we obtain:

$$\begin{aligned} I_1 &= -p \int_{\Gamma} |\nabla u|^{p-2} \frac{\partial u}{\partial \mathbf{n}} V \cdot \nabla u \, d\mathcal{H}^{n-1} \\ &= -p \int_{\Gamma} \lambda(0) |u|^{p-2} u (V \cdot \nabla u) - \sigma \chi_D |u|^{p-2} u (V \cdot \nabla u) \, d\mathcal{H}^{n-1} \\ &= -\lambda(0) \int_{\Gamma} \nabla(|u|^p) \cdot V \, d\mathcal{H}^{n-1} + \sigma \int_{\Gamma} \chi_D \nabla(|u|^p) \cdot V \, d\mathcal{H}^{n-1} \\ &= \lambda(0) \int_{\Gamma} |u|^p \operatorname{div}_{\Gamma} V \, d\mathcal{H}^{n-1} - \sigma \int_D |u|^p \operatorname{div}_{\Gamma} V \, d\mathcal{H}^{n-1} + \sigma \int_{\partial_r D} |u|^p V \cdot \mathbf{n}_{\Gamma} \, d\mathcal{H}^{n-2} \\ &= \lambda(0) \int_{\Gamma} |u|^p \operatorname{div}_{\Gamma} V \, d\mathcal{H}^{n-1} + \sigma \int_{\partial_r D} |u|^p V \cdot \mathbf{n}_{\Gamma} \, d\mathcal{H}^{n-2} - I_2 \end{aligned}$$

So,

$$G(u) = \lambda(0) \int_{\Gamma} |u|^p \operatorname{div}_{\Gamma} V \, d\mathcal{H}^{n-1} + \sigma \int_{\partial_r D} |u|^p V \cdot \mathbf{n}_{\Gamma} \, d\mathcal{H}^{n-2}$$

and therefore

$$\lambda'(0) = -\sigma \int_{\partial_r D} |u|^p V \cdot \mathbf{n}_{\Gamma} \, d\mathcal{H}^{n-2}$$

This completes the proof of the Theorem.  $\square$

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