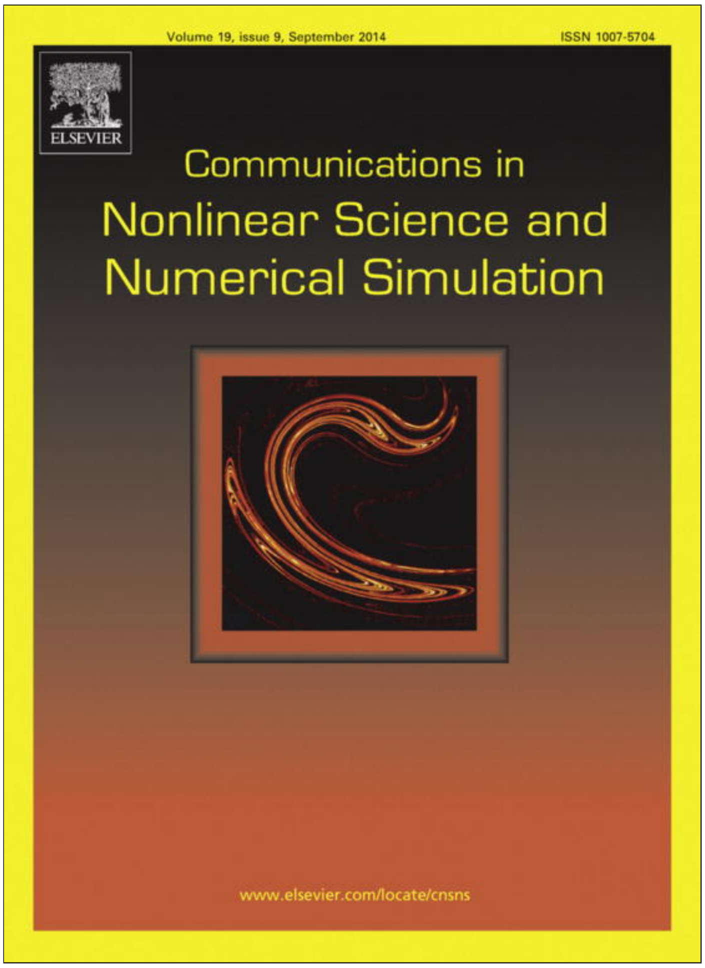
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Existence theorems for some abstract nonlinear non-autonomous systems with delays ☆,☆☆

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A B S T R A C T

For some abstract classes of nonlinear non-autonomous systems with variable and state-dependent delays existence, non-existence and multiplicity of periodic solutions are discussed. To illustrate the efficiency of the method, we obtain some well-known results for applied systems as corollaries of our existence theorems.

1. Introduction

Lotka–Volterra delayed systems are extensively used to model prey-predator population dynamics. For example, the system

$$x_i'(t) = x_i(t)\left[c_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t-\tau_i(t))\right] \tag{1.1}$$

was under study in [9,19,21,26,28,34]; Gilpin-Ayala model

$$x_i'(t) = x_i(t)\left[c_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j^{\theta_j}(t-\tau_{ij}(t))\right] \tag{1.2}$$

in [8,9]; logarithmic Lotka–Volterra

$$x_i'(t) = x_i(t)\left[a_i(t) - \sum_{j}^{n} b_{ij}(t)\ln x_j(t) - \overset{n}{\underset{j}{m}} c_{ij}(t)\ln x_j(t-\tau_{ij}(t))\right] \tag{1.3}$$

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in [43]; Hopfield neuron network models

$$x_i'(t) = -a_i(t)x_i(t) + \sum_j^n a_{ij}(t)f_{ij}(x(t)) + \sum_j^n b_{ij}(t)f_{ij}(x(t - \tau_{ij}(t))) \quad (1.4)$$

were studied in [7,13,16,25,29,44].

In [15,30,36,39] the following general models for an n-dimensional vector $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ were under investigation

$$x'(t) = A(t)x(t) + \lambda f(t, x(t - \tau(t))), \quad (1.5)$$

$$x'(t) = A(t, x(t))x(t) + f(t, x(t - \tau)) \quad (1.6)$$

and

$$x'(t) = \nabla g(x(t)) + f(t, x(t - \tau)). \quad (1.7)$$

Here, $A(t,x)$ is a continuous $n \times n$ matrix, $f(t,x)$ is a continuous n-dimensional vector function and $g(x)$ is a C^1 scalar function.

There have been various approaches developed to examine the existence of periodic solutions for delay differential equations since the first study published by Browder in 1962, such as fixed point theorems, Hopf bifurcation theorems, Poincaré–Bendixson theorems, Lyapunov functions, the spectral theory of matrices, Morse theory, Galerkin methods and coincidence degree theory (see, for example, [1–4,10,11,27]). Some interesting results were recently obtained in [17,22,24,32,33,39,40,42]. Multiple systems of population dynamics were recently studied in [3,5,6,18,20,23,35,37,38,41].

Motivated by these models, we introduce and study the most general system and discuss its applications. Some new and interesting sufficient conditions are obtained to guarantee the existence, non-existence and multiplicity of periodic solutions.

2. Continuation theorem for the abstract model and applications

Let

$$X := \{x \in C(\mathbb{R}, \mathbb{R}^n) : x(t + \omega) = x(t) \quad \text{for all } t\}$$

and consider the functional differential equation

$$x'(t) = \Phi(x)(t), \quad (2.1)$$

where $\Phi : X \to X$ is continuous and maps bounded sets into bounded sets. For $x \in X$, its absolute maximum and minimum values and its average $\frac{1}{\omega}\int_0^\omega x(t)\,dt$ are denoted by x_{max}, x_{min} and $\bar{x}$, respectively. The euclidian norm of a vector $y \in \mathbb{R}^n$ shall be denoted by $|y|$. Let U be an open and bounded subset of X and denote its closure by $cl(U)$. If $\mathscr{K} : cl(U) \to X$ is compact with $\mathscr{K}u \neq u$ for $u \in \partial U$, then the Leray–Schauder degree of the Fredholm operator $\mathscr{F} = Id - \mathscr{K}$ at 0 shall be denoted by $deg_{LS}(\mathscr{F}, U, 0)$ (for a detailed definition and properties of the degree see for example [27]). Finally, we identify the subset of constant functions of X with $\mathbb{R}^n$; thus, a vector $\gamma \in \mathbb{R}^n$ may be interpreted as an element of X so the function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\phi(\gamma) := \overline{\Phi(\gamma)} = \frac{1}{\omega}\int_0^\omega \Phi(\gamma)(t)\,dt$$

is well defined. The following continuation theorem will be the key for further studies. The proof follows essentially the same outline of analogous results (see e. g. [10]) so it is omitted.

Theorem 2.1. *Assume there exists a bounded open subset $U \subset X$ such that*

1. *If $x'(t) = \lambda\Phi(x)(t)$ for some $x \in cl(U)$ and $0 < \lambda < 1$, then $x \in U$.*
2. *$\phi(x) \neq 0$ for $x \in \partial U \cap \mathbb{R}^n$.*
3. *$deg_B(\phi, U \cap \mathbb{R}^n, 0) \neq 0$ (deg_B stands for the Brouwer degree).*

Then (2.1) has at least one solution $x \in cl(U)$.

Consider the system

$$x'(t) = F(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_m)) \quad (2.2)$$

with $F : \mathbb{R} \times \mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ continuous and ω-periodic in t and $\tau_j = \tau_j(t, x(t))$, with $\tau_j : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ continuous, positive and ω-periodic in t. For convenience, given an arbitrary bounded open set $\Omega \subset \mathbb{R}^n$, we define

$$X_\Omega := \{x \in X : x(t) \in \Omega \quad \text{for all } t\}.$$

Assume, for simplicity, that Ω is smooth, but a more general version of the result can be also obtained. The Euler characteristic of $\overline{\Omega}$ shall be denoted by $\chi(\overline{\Omega})$ and the outer normal vector of Ω at x shall be denoted by ν_x.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$ be smooth, open and bounded with $\chi(\overline{\Omega}) \neq 0$ and assume that*

$$\nu_x \cdot F(t, x, y_1, \ldots, y_m) < 0 \tag{2.3}$$

or

$$\nu_x \cdot F(t, x, y_1, \ldots, y_m) > 0 \tag{2.4}$$

for all t, $x \in \partial\Omega$ and $y_j \in \overline{\Omega}$ such that

$$|y_j - x| \leqslant \tau_{max} \sup_{t\in[0,\omega], y_k, z \in \overline{\Omega}} |F(t, z, y_1, \ldots, y_m)|.$$

Then (2.2) admits at least one ω-periodic solution in X_Ω.

Proof. Let $x \in cl(X_\Omega)$ satisfy $x'(t) = \lambda F(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_m))$ for some $\lambda \in (0, 1)$. If $x(t_0) \in \partial\Omega$ for some $t_0 \in [0, \omega]$, then $x'(t_0)$ is tangent to $\partial\Omega$ at the point $x(t_0)$, that is

$$\nu_{x(t_0)} \cdot \lambda F(t_0, x(t_0), x(t_0 - \tau_1), \ldots, x(t_0 - \tau_m)) = \nu_{x(t_0)} \cdot x'(t_0) = 0.$$

On the other hand, for all j it is seen that

$$|x(t_0) - x(t_0 - \tau_j)| = \left| \int_{t_0-\tau_j}^{t_0} x'(t)\,dt \right| \leqslant \lambda \int_{t_0-\tau_j}^{t_0} |F(t, x(t), x(t-\tau_1), \ldots, x(t-\tau_m))|\,dt$$

$$< \tau_{max} \sup_{t\in[0,\omega], x, x_k \in \overline{\Omega}} |F(t, x, y_1, \ldots, y_m)|;$$

so using (2.3) or (2.4) a contradiction is obtained, thus, $x \in X_\Omega$. Finally, for $x \in \mathbb{R}^n$ observe that

$$\phi(x) = \frac{1}{\omega} \int_0^\omega F(t, x, \ldots, x)\,dt$$

and hence $\nu_x \cdot \phi(x) \neq 0$ for all $x \in \partial\Omega$. This implies that ϕ or $-\phi$ is homotopic to the outer normal vector field, and the conclusion follows from a theorem by Hopf [12], which establishes that the degree of the outer normal is equal to $\chi(\overline{\Omega})$. □

Remark 2.3. The preceding theorem is rather general but it can be applied to specific situations in order to obtain more precise results. In particular, if Ω is homeomorphic to a ball then the condition on the Euler characteristic is automatically fulfilled since $\chi(\Omega) = 1$; this is the case in [11], where it was assumed that Ω is convex. However, the general version allows Ω to have holes, and thus can be also adapted to deal with singular problems, as shown in the following example.

Example 2.4. Consider the system

$$w'(t) = a(t, x(t-\tau)) \frac{x(t)}{|x(t)|^n} + b(t, x(t-\tau)) \frac{x(t) - v}{|x(t) - v|^m} := F(t, x(t), x(t-\tau)) \tag{2.5}$$

where $v \in \mathbb{R}^n \setminus \{0\}$ is a fixed vector, $a, b : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are continuous, bounded and ω-periodic in the first coordinate, $\tau > 0$ and $n, m > 1$. Assume:

1. There exists $R > |v|$ such that $a(t, x) > 0$ and $b(t, x) > 0$ for all t and all x such that $|x| = R$.
2. $a(t, 0) < 0$ and $b(t, v) < 0$ for all t.

Then (2.5) admits at least one ω-periodic solution, provided that τ is small enough.
Indeed, we may consider $\Omega := B_R(0) \setminus (cl(B_r(0)) \cup cl(B_r(v)))$ for some $r < \frac{|v|}{2}$ such that

$$\frac{a(t, x)}{r^{n-2}} + b(t, x) \frac{(x - v) \cdot x}{|x - v|^m} < 0 \quad \text{for all } t \quad \text{and} \quad |x| = r,$$

$$a(t, x) \frac{(x - v) \cdot x}{|x|^n} + \frac{b(t, x)}{r^{m-2}} < 0 \quad \text{for all } t \quad \text{and} \quad |x - v| = r.$$

Observe that

$$|F(t, x, y)| \leqslant \|a\|_\infty r^{1-n} + \|b\|_\infty r^{1-m} := M(r)$$

and $F(t,x,x)\cdot x > 0$ for $|x| = R$ and all t, so taking τ small enough we deduce:

$$F(t,x,y)\cdot x > 0 \quad \text{for all } t, |x| = R \quad \text{and} \quad R - \tau M(r) \leqslant |y| \leqslant R.$$

Moreover, letting τ to be smaller if necessary, it is verified that

$$\frac{a(t,y)}{r^{n-2}} + b(t,y)\frac{(x-v)\cdot x}{|x-v|^m} < 0 \quad \text{for all } t, |x| = r \quad \text{and} \quad |y-x| < \tau M(r),$$

$$a(t,y)\frac{(x-v)\cdot x}{|x|^n} + \frac{b(t,y)}{r^{m-2}} < 0 \quad \text{for all } t, |x-v| = r \quad \text{and} \quad |y-x| < \tau M(r).$$

Finally, note that $\chi(\Omega) = 1 - 2(-1)^n \neq 0$. Thus, all the conditions of Theorem 2.2 are satisfied. It is worth noticing that, when n is odd, the same approach can be also applied to a system with an odd number of singularities.

Concerning the nonlinearity F, a special case of remarkable interest is

$$F(t,x,y_1,\ldots,y_m) = A(t,x)x - \lambda H(t,x,y_1,\ldots,y_m) \tag{2.6}$$

with $A : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^{n\times n}$ continuous and ω-periodic and $H : \mathbb{R}\times\mathbb{R}^{n(m+1)} \to \mathbb{R}^n$ continuous and ω-periodic in its first coordinate and $\lambda > 0$. Consequently, existence results for (1.4), (1.5) and (1.6) are obtained.

In [31] a planar system was firstly introduced and studied

$$\begin{aligned} x_1'(t) &= -a_1(t)x_1(t) + b_1(t)f_1(x_1(t-\tau_{11}(t)), x_2(t-\tau_{12}(t))) \\ x_2'(t) &= -a_2(t)x_2(t) + b_2(t)f_2(x_1(t-\tau_{21}(t)), x_2(t-\tau_{22}(t))). \end{aligned} \tag{2.7}$$

The following result was obtained in [14], as a corollary of a more general existence theorem: if $a_i, b_i, \tau_{ij} > 0$ are continuous and ω-periodic and $f_i \in C(\mathbb{R}^2,\mathbb{R})$ is bounded, then system (2.7) has at least one periodic solution [14, Corollary 2.1]. This fact follows trivially from Theorem 2.2. Indeed, we may set

$$A(t) := \begin{pmatrix} -a_1(t) & 0 \\ 0 & -a_2(t) \end{pmatrix}, \quad H(t,y_{11},y_{12},y_{21},y_{22}) := \begin{pmatrix} -b_1(t)f_1(y_{11},y_{12}) \\ -b_2(t)f_2(y_{21},y_{22}) \end{pmatrix}.$$

If we take $\Omega = B_R(0) \subset \mathbb{R}^n$, then $v_x = \frac{x}{|x|}$ for all $x \in \partial\Omega$ and hence

$$v_x\cdot F(t,x,y_{11},\ldots,y_{22}) = -\frac{1}{|x|}\left(a_1(t)x_1^2 + a_2(t)x_2^2 - b_1(t)x_1f_1(y_{11},y_{12}) - b_2(t)x_2f_2(y_{21},y_{22})\right).$$

As $a_1(t), a_2(t) > 0$ for all t and f_1, f_2 are bounded, it is seen that (2.3) is fulfilled when R is large enough. We remark that, in this particular case, a simpler proof can be given using Schauder Theorem which, compared with the degree method, is usually more restrictive, since it is required to find a convex bounded and closed subset C that is invariant for the fixed point operator, i.e., $T(C) \subset C$. Whereas using degree, the homotopy invariance is a very strong tool, since in many situations the problem is reduced to a degree computation of a finite-dimensional mapping. More generally, using the fact that $x\cdot A(t)x = x\cdot\frac{A(t,x)+A(t,x)^T}{2}x$ for all t and all x, we obtain:

Corollary 2.5. *Let F be defined by (2.6), $\Omega = B_R(0) \subset \mathbb{R}^n$, and denote by $\mu_1(t,x) \leqslant \ldots \leqslant \mu_n(t,x)$ the eigenvalues of the symmetric matrix $\frac{A(t,x)+A(t,x)^T}{2}$. If*

$$\lambda x\cdot H(t,x,y_1,\ldots,y_m) < \mu_1(t,x)R^2$$

or

$$\lambda x\cdot H(t,x,y_1,\ldots,y_m) > \mu_n(t,x)R^2$$

for all $t\in\mathbb{R}, x\in\partial B_R(0)$ and $y_1,\ldots,y_m$ as in Theorem 2.2, then Eq. (2.2) admits at least one ω-periodic solution in X_Ω.

Corollary 2.6. *Let F be defined by (2.6) and assume there exists a constant $c > 0$ such that $\mu_1(t,x) \geqslant c$ or $\mu_n(t,x) \leqslant -c$ for all t and x. Then there exists $\lambda_* > 0$ such that the problem*

$$x'(t) = A(t,x(t))x(t) - \lambda H(t,x(t),x(t-\tau_1),\ldots,x(t-\tau_m))$$

has at least one ω-periodic solution for $|\lambda| < \lambda_*$.

Note that if F is C^1 with respect to $x, y_1,\ldots,y_m$, then the previous corollary may be deduced from the implicit function theorem.

The same function F provides very elementary non-existence results, for example:

Proposition 2.1. *Let F be defined by (2.6) and assume that*

$$\lambda x\cdot H(t,x,y_1,\ldots,y_m) < \mu_1(t,x)|x|^2$$

or

$$\lambda x \cdot H(t, x, y_1, \ldots, y_m) > \mu_n(t, x)|x|^2$$

for all $x \neq 0$ *and* y_j *such that* $|y_j| \leqslant |x|$ *for* $j = 1 \ldots, m$, *then the problem has no nontrivial* ω*-periodic solutions.*

Proof. Suppose that x is a nontrivial ω-periodic solution and the first condition holds, then

$$x(t) \cdot x'(t) = -x(t) \cdot A(t, x(t))x(t) + \lambda x(t) \cdot H(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_m)).$$

Now consider $\theta(t) = \frac{|x(t)|^2}{2}$, and let t_{max} be the point where the absolute maximum of θ is achieved. Then $|x(t_{max} - \tau_j)| \leqslant |x(t_{max})|$ for all j and

$$0 = \theta'(t_{max}) = x(t_{max}) \cdot x'(t_{max})$$

$$< -\mu_1(t_{max}, x(t_{max}))|x(t_{max})|^2 + \mu_1(t_{max}, x(t_{max}))|x(t_{max})|^2 = 0,$$

a contradiction. □

Corollary 2.7. *Let F be defined by* (2.6) *and assume that*

$$\liminf_{|x|\to\infty, |y_j|\geqslant|x|} \frac{x \cdot H(t, x, y_1, \ldots, y_m)}{|x|^2} \geqslant c$$

uniformly on t for some positive constant c. Then there exists $\lambda^* > 0$ *such that the problem*

$$x'(t) = A(t, x(t))x(t) - \lambda H(t, x(t), x(t - \tau_1), \ldots, x(t - \tau_m))$$

has no nontrivial ω*-periodic solutions for* $\lambda > \lambda^*$.

Example 2.8. In [29], the following system was studied:

$$x_i'(t) = a_i(t)g_i(x(t)) - \lambda b_i(t)f_i(x(t - \tau(t))). \tag{2.8}$$

Here, we shall assume that $a_i, b_i, \tau : \mathbb{R} \to \mathbb{R}$ are continuous, positive and ω-periodic and $\lambda > 0$. Further, assume that

1. $f_i(x) > 0$ for all $x \neq 0$ and all i.
2. g is bounded, $g(x) \neq 0$ for $x \neq 0$ and $deg_B(g, B_1(0), 0) \neq 0$.
3.

$$\lim_{x\to 0} \frac{f_i(x)}{|x|} = 0 < \liminf_{x\to 0, g_i(x)>0} \frac{g_i(x)}{|x|}$$

and

$$\lim_{|x|\to\infty} f_i(x) = 0 < \liminf_{|x|\to\infty, g_i(x)>0} g_i(x)$$

for all i.

Then there exists $\lambda^* > 0$ such that the problem has at least two nontrivial solutions for $\lambda > \lambda^*$.

In this case the proof does not follow from Theorem 2.2, but we may apply a direct argument using Theorem 2.1. In first place, fix an arbitrary s such that $s > (a_i)_{max}(g_i)_{max} + 1$ for all i. If $x \in X$ is such that $x_i'(t) = \sigma[a_i(t)g_i(x(t)) - \lambda b_i(t)f_i(x(t - \tau(t)))]$ for some $\sigma \in (0, 1)$, then $x_i'(t) < s - 1$ and hence $(x_i)_{max} < (x_i)_{min} + s - 1$. In particular, if $x_i(\xi) = \|x_i\|_\infty = s$, then $1 \leqslant |x_i(\xi - \tau(\xi))| \leqslant s$ and thus, if λ is large enough, then it is seen that $x_i'(\xi) \neq 0$, a contradiction. Moreover, in this case $\phi_i(x) = \bar{a}_i g_i(x) - \lambda \bar{b}_i f_i(x)$ so it is clear that $deg_B(\phi, B_s(0), 0) = 0$. On the other hand, if $\|x_i\|_\infty = r < s$, then

$$a_i(\xi)\frac{g_i(x(\xi))}{|x(\xi)|} = \lambda b_i(\xi)\frac{f_i(x(\xi - \tau(\xi)))}{|x(\xi - \tau(\xi))|}\frac{|x(\xi - \tau(\xi)|}{|x(\xi)|} \to 0$$

as $r \to 0$, a contradiction. By the same token, if $\|x_i\|_\infty = R > s$, then

$$a_i(\xi)g_i(x(\xi)) = \lambda b_i(\xi)f_i(x(\xi - \tau(\xi))) \to 0$$

as $R \to \infty$, a contradiction. Furthermore, observe that

$$deg_B(\phi, B_r(0), 0) = deg_B(\phi, B_R(0), 0) = deg(g, B_1(0), 0) \neq 0,$$

and we conclude that the problem has at least one solution in each of the following sets:

$$U_r^s := \{x \in X : r < \|x\|_\infty < s\}$$

$$U_s^R := \{x \in X : s < \|x\|_\infty < R\}.$$

Next, consider the following system:

$$x'(t) = -B(t) + H(t, x(t), x(t-\tau_1), \dots, x(t-\tau_m)), \tag{2.9}$$

with $B : \mathbb{R} \to \mathbb{R}^n$ continuous and ω-periodic, $B_i(t) > 0$ for all t and $H : \mathbb{R} \times \mathbb{R}^{n(m+1)} \to (0, +\infty)^n$ continuous and ω-periodic in t.

Theorem 2.9. *Let $\rho, \sigma \in \mathbb{R}^n$ satisfy $\rho_i < \sigma_i$ for $i = 1, \dots, n$ and $\Omega := \Pi_{i=1}^n (\rho_i, \sigma_i)$. Assume that*

$$[H_i(t, r, r^1, \dots, r^m) - B_i(t)].[H_i(t, s, s^1, \dots, s^m) - B_i(t)] < 0 \tag{2.10}$$

for all $t \in \mathbb{R}$, $i = 1, \dots, n$ and for any $r, r^j, s, s^j \in \overline{\Omega}$ with $r_i = \rho_i$, $s_i = \sigma_i$, $r_i^j \leqslant \rho_i + \omega\overline{B}_i$, $s_i^j \geqslant \sigma_i - \omega\overline{B}_i$ for $j = 1, \dots, m$. Then (2.9) admits at least one ω-periodic solution in X_Ω.

Proof. It suffices to apply the continuation theorem, with

$$\Phi(x)(t) := -B(t) + H(t, x(t), \quad x(t-\tau_1), \dots, x(t-\tau_m)).$$

If $x'(t) = \lambda\Phi(x)(t)$ for some $\lambda \in (0,1)$, then $x_i'(t) > B_i(t)$ for all t and all i. Thus, for $0 \leqslant t_1 \leqslant t_2 \leqslant \omega$ we deduce that $x_i(t_2) - x_i(t_1) > -\omega\overline{B_i}$ and, by periodicity, that $(x_i)_{max} < (x_i)_{min} + \omega\overline{B}_i$. If $x(t) \in \overline{\Omega}$ for all t and, for example, $x_i(\eta) = (x_i)_{max} = \sigma_i$, then $x_i'(\eta) = 0$ and thus

$$H_i(\eta, x(\eta), x(\eta - \tau_1), \dots, x(\eta - \tau_m)) - B_i(\eta) = 0.$$

This contradicts inequality (2.10) with $s = x(\eta), s^j = x(\eta - \tau_j)$. A similar conclusion holds if we suppose that $(x_i)_{min} = \rho_i$ for some i. Furthermore, as

$$\phi_i(x) = \frac{1}{\omega}\int_0^\omega H_i(t, x, \dots, x)\, dt - \overline{B}_i$$

it follows that

$$\phi_i(x)\phi_i(y) < 0$$

for all $x, y \in \overline{\Omega}$ such that $x_i = r_i$ and $y_i = s_i$. This implies that $deg(\phi, \Omega, 0) \neq 0$ and the proof is complete. □

It is worthy to observe that, in particular, condition (2.10) implies that, for each t, over two opposite faces of the domain Ω, the field $-B(t) + H(t, x, \dots, x)$ is both inwardly or outwardly pointing.

Remark 2.10.

1. The same result holds for the "mirror" equation

$$x'(t) = B(t) - H(t, x(t), x(t-\tau_1), \dots, x(t-\tau_m)).$$

In particular, the theorem can be applied to models (1.1), (1.2) and (1.3). For example, take (1.1) and substitute $y_i(t) = \ln(x_i(t))$ to obtain the system

$$y_i'(t) = c_i(t) - \sum_{j=1}^n a_{ij}(t)e^{y_j(t-\tau_i)}.$$

Fix σ_i such that

$$c_i(t) < a_{ii}(t)e^{\sigma_i - [c_i]_\omega}$$

for all t, then (1.1) has at least one ω-periodic positive solution, provided that

$$c_i(t) > \sum_{j \neq i} a_{ij}(t)e^{\sigma_j}$$

for all t. However, in this case a direct analysis allows to obtain a more precise sufficient condition. Indeed, it was proven in [34] that the system has a positive ω-periodic solution, provided that the linear system

$$\sum_{j=1}^n \overline{a_{ij}}x_j = \overline{c_i} \quad i = 1, \dots, n$$

has a positive solution.

Also, we may consider system (1.7) where $g : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and ω-periodic in its first coordinate. Here, we shall assume that τ is a constant. For convenience, we define the functions $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\psi(t,x) := \nabla g(x) + f(t,x)$$

and

$$\Psi(x) := \overline{\psi(\cdot,x)} = \nabla g(x) + \frac{1}{\omega}\int_0^\omega f(t,x)\,dt.$$

Theorem 2.11. *Assume that $|f(t,x)| \leqslant c|x| + d$ for some constants c, d with $0 < c < \frac{2\pi}{\omega(1+\pi)}$. Furthermore, assume there exists an open $\Omega \subset B_R(0) \subset \mathbb{R}^n$ for some $R > 0$ large enough such that*

1. *$0 \notin co(\psi([0,\omega] \times B_\rho(x)))$ for all $x \in \partial\Omega$, where $co(\mathscr{A})$ denotes the convex hull of a set $\mathscr{A} \subset \mathbb{R}^n$ and $\rho := \frac{\pi\omega R}{2\pi - c\omega}$.*
2. *$deg(\Psi, \Omega, 0) \neq 0$.*

Then system (1.7) has at least one ω-periodic solution.

Proof. We shall apply the continuation theorem over the set

$$U := \{x \in X : \|x - \overline{x}\|_\infty < \rho, \quad \overline{x} \in \Omega\}.$$

Thus, it suffices to prove that the ω-periodic solutions of the equation

$$x'(t) = \lambda(\nabla g(x(t)) + f(t, x(t-\tau))$$

with $\lambda \in (0,1)$ and $\overline{x} \in cl(\Omega)$ satisfy:

1. $\|x - \overline{x}\|_\infty < \rho$.
2. $\overline{x} \notin \partial\Omega$.

Multiply by $x'(t)$ the previous equation and integrate to obtain

$$\int_0^\omega |x'(t)|^2\,dt < \int_0^\omega (c|x(t-\tau)| + d)|x'(t)|\,dt \leqslant c\int_0^\omega |x(t-\tau) - \overline{x}|.|x'(t)|\,dt + (d + c|\overline{x}|)\omega^{1/2}\|x'\|_{L^2}$$
$$\leqslant \frac{c\omega}{2\pi}\|x'\|_{L^2}^2 + (d + c|\overline{x}|)\omega^{1/2}\|x'\|_{L^2}.$$

Hence,

$$\|x'\|_{L^2} < \frac{2\pi}{2\pi - c\omega}(d + c|\overline{x}|)\omega^{1/2}.$$

Moreover, as

$$\|x - \overline{x}\|_\infty \leqslant \frac{\omega^{1/2}}{2}\|x'\|_{L^2},$$

we deduce that

$$\|x - \overline{x}\|_\infty < \frac{\omega\pi}{2\pi - c\omega}(d + c|\overline{x}|).$$

In other words, if $|\overline{x}| \leqslant R$ with R large enough, then it follows that $\|x - \overline{x}\|_\infty < \rho$. Next, suppose that $\overline{x} \in \partial\Omega$ and integrate the equation to obtain:

$$\int_0^\omega [\nabla g(x(t)) + f(t, x(t-\tau))]dt = 0.$$

By periodicity,

$$\int_0^\omega [\nabla g(x(t-\tau)) + f(t, x(t-\tau))]dt = 0,$$

that is: $\int_0^\omega \psi(t, x(t-\tau))\,dt = 0$. As $x(t-\tau) \in B_\rho(\overline{x})$ for all t, from the first condition we conclude that $0 \notin co(Im\Gamma)$, where $\Gamma(t) = \psi(t, x(t-\tau))$. This contradicts the mean value theorem for vector-valued integrals. $\square$

Example 2.12. In [15, 1999], the following result was proven. Suppose there are constants a, b, c, d such that $a > c > 0, c\omega < 1$ and, for all $(t, x) \in [0, \omega] \times \mathbb{R}^n$,

1. $x \cdot \nabla g(x) \geqslant a|x|^2 - b$.
2. $|f(t,x)| \leqslant c|x| + d$.

Then (1.7) has at least one ω-periodic solution. This result can be easily obtained as a corollary of our methods. Indeed, in this case

$$x \cdot \psi(t,x) > (a-c)|x|^2 - d|x| - b > \frac{a-c}{2}|x|^2 > 0$$

for $|x| \gg 0$. This implies that Ψ has degree equal to 1 over large balls centered at 0.

Note that our results yields in fact a generalization of the result in [15], since the smallness condition on c can be relaxed. Furthermore Theorem 2.11 also covers some cases not contained in [15]. For example, assume that ∇g is sublinear and that $f(t,x) = c(t)x + h(t,x)$ where $c(t)$ is continuous and ω-periodic such that $0 < |c(t)| < \frac{2\pi}{\omega(1+\pi)}$ for all t and h is continuous and ω-periodic in t with $\frac{h(t,x)}{|x|} \to 0$ uniformly on t as $|x| \to \infty$. Then the problem has at least one ω-periodic solution.

Moreover, Theorem 2.11 provides examples of multiple solutions. For simplicity, assume that $|f(t,x)| \leqslant d$ for all t and all x, then the previous theorem is valid for arbitrary $R > 0$ and $\rho = d\omega^{1/2}$. Then, we may consider the radial case $\nabla g(x) = \eta(|x|)x$, where $\eta : [0, +\infty) \to \mathbb{R}$ is continuous, which corresponds to $g(x) = \theta(|x|)$, with $\theta(u) := \int_0^u s\eta(s)\,ds$. From the theorem, existence of ω-periodic solutions is guaranteed if there exist intervals $I = (a_1 - \rho, a_1 + \rho)$ and $J = (a_2 - \rho, a_2 + \rho)$ with $a_2 - \rho > a_1 + \rho > 2\rho$ such that $\eta(r)\eta(s) < 0$ for all $r \in I$ and $s \in J$, provided that $\eta(a_j)a_j \gg d$ and that n is odd. Indeed, in this case we may take $\Omega = B_{a_2}(0) \setminus cl(B_{a_1}(0))$: the first condition is fulfilled since the angle between $\nabla g(z)$ and $\nabla g(x)$ is less than π for all $z \in B_\rho(x)$ and $|x| = a_j$; the second condition follows from the fact that $deg(\nabla g, B_{a_j}(0), 0) = [sgn(\eta(a_j))]^n = sgn(\eta(a_j))$ (since n is odd), and hence $deg(\nabla g, \Omega, 0) = sgn(\eta(a_2)) - sgn(\eta(a_1)) \neq 0$. In particular, if η oscillates, and oscillations are 'large enough', then the problem has infinitely many solutions. This is the case, for example, of the system

$$x'(t) = cos(|x|)x + f(t, x(t-\tau))$$

with f continuous and ω-periodic in t such that $|f(t,x)| \leqslant 1$ for all $t \in \mathbb{R}, x \in \mathbb{R}^n$. If $\omega < \frac{\pi^2}{4}$ and n is odd, then the system has infinitely many ω-periodic solutions. Indeed, in the previous situation it suffices to take $a_1 = k\pi,\ a_2 = (k+1)\pi$: if $k \in \mathbb{N}$ is sufficiently large, then there exists at least one ω-periodic solution x such that $k\pi < |\bar{x}| < (k+1)\pi$.

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