FAMILIES AND UNFOLDINGS OF SINGULAR HOLOMORPHIC LIE ALGEBROIDS

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Abstract. In this paper, we investigate families of singular holomorphic Lie algebroids on complex analytic spaces. We introduce and study a special type of deformation called by unfoldings of Lie algebroids which generalizes the theory due to Suwa for singular holomorphic foliations. We show that there is a one to one correspondence between transversal unfoldings and holomorphic flat connections on a natural Lie algebroid on the bases.

1. Introduction

Definition 1.1. Let $\mathscr{A}$ be a reflexive sheaf of $\mathcal{O}_X$-modules over a complex manifold X, equipped with a $\mathcal{O}_X$-morphism $a\colon \mathscr{A} \to T_X$. We say that $\mathscr{A}$ is a *Lie algebroid* of *anchor* a if there is a $\mathbb{C}$-bilinear map $\{\cdot\,,\cdot\}\colon \mathscr{A} \otimes_{\mathcal{O}_X} \mathscr{A} \to \mathscr{A}$ such that

(a) $\{v,u\} = -\{u,v\}$;
(b) $\{u,\{v,w\}\} + \{v,\{w,u\}\} + \{w,\{u,v\}\} = 0$;
(c) $\{g\cdot u, v\} = g\cdot\{u,v\} - a(v)(g)\cdot u$ for all $g \in \mathcal{O}_X$ and $u,\, v \in \mathscr{A}$.

The singular set of $\mathscr{A}$ is defined by

$$\mathrm{Sing}(\mathscr{A}) = \mathrm{Sing}(\mathrm{Coker}(a)).$$

The Lie algebroid $a\colon \mathscr{A} \to T_X$ induces a holomorphic foliation $Im(a) \subset T_X$.

Definition 1.2. (Pullback of a Lie alebroid) Given a Lie algebroid $\mathscr{A}$ over a variety X and a morphism $f : Y \to X$ we define an algebroid $f^{\bullet}\mathscr{A}$ over Y. The underlying sheaf of $f^{\bullet}\mathscr{A}$ is defined as the fibered product of the diagram

$$\begin{array}{ccc} f^{\bullet}\mathscr{A} := f^{*}\mathscr{A} \oplus_{f^{*}T_X} T_Y & \longrightarrow & T_Y \\ \big\downarrow & & \big\downarrow{\scriptstyle Df} \\ f^{*}\mathscr{A} & \xrightarrow{\quad a \quad} & f^{*}T_X \end{array}$$

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The anchor map is the top horizontal map in the above diagram. The Lie algebra structure is induced by restriction of the direct sum bracket in $f^*\mathscr{A} \oplus T_Y$ to the subsheaf $f^\bullet\mathscr{A}$.

Let $f : X \to S$ be a smooth morphism of analytic spaces and consider $T_{X|S}$ the relative tangent sheaf, which is naturally a subsheaf of T_X.

Definition 1.3. A family of singular holomorphic Lie algebroids over X is a reflexive sheaf $\mathscr{A}$ of modules over X which is flat over S, equipped with a $\mathcal{O}_X$-morphism $a_S \colon \mathscr{A} \to T_{X|S}$ and an $f^{-1}\mathcal{O}_S$-linear map $\{\cdot,\cdot\}_S \colon \mathscr{A} \otimes_{f^{-1}\mathcal{O}_S} \mathscr{A} \to \mathscr{A}$ such that

(a) $\{\alpha,\beta\} = -\{u,v\}$;
(b) $\{\alpha,\{\beta,\gamma\}\} + \{\beta,\{\gamma,\alpha\}\} + \{\gamma,\{\alpha,\beta\}\} = 0$;
(c) $\{f\cdot\alpha,\beta\} = f\{\alpha,\beta\} - a(\beta)(f)\cdot\alpha$ for all $f \in \mathcal{O}_X$ and $\alpha,\, \beta \in \mathscr{A}$.

Remark 1.4. Given a family $\mathscr{A}_S$ over a smooth morphism $p : X \to S$ and a morphism $f : R \to S$ we consider the fibered product

$$\begin{array}{ccc} X_R & \xrightarrow{f} & X \\ {\scriptstyle p_R}\downarrow & & \downarrow{\scriptstyle p} \\ R & \xrightarrow{f} & S \end{array}$$

If we take de Lie algebdroid pull-back $f^\bullet\mathscr{A}$ we obtain a family over the smooth morphism $p_R : X_R \to R$. Observe that, using a covering of X by open sets of the form $Y \times U$ with $U \subseteq S$ an open set, we can give local generators $\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_d}$ of $T_{X|S}$ where $y_1,\ldots,y_d$ are local coordinates of Y. Using such a covering we obtain a covering for X_R of the form $Y \times V$ where $V = f^{-1}(U) \subseteq R$ which give local generators of $T_{X_R|R}$ of the form $\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_d}$.

In other words we have $f^*T_{X|S} \simeq T_{X_R|R}$, so the underlying sheaf of the algebroid $f^\bullet\mathscr{A}$ in this case is just $f^*\mathscr{A}$.

Example 1.5. An important caveat to have in mind here is that, contrary to what one may suppose, the algebroid pull-back is not an associative operations. In other words, in general $g^\bullet f^\bullet\mathscr{A} \not\cong (f\circ g)^\bullet\mathscr{A}$. There is however a canonical morphism $c_{gf} : g^\bullet f^\bullet\mathscr{A} \to (f\circ g)^\bullet\mathscr{A}$ given by the pull-back property of $(f\circ g)^\bullet\mathscr{A}$, but is not an isomorphism in general.

To see an example of this lets take $X = \mathbb{A}^2$, and $a : \mathscr{A} \to T^2_{\mathbb{A}}$ to be the inclusion of the sheaf generated by the vector field $v = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Let $f : \mathbb{A}^1 \to \mathbb{A}^2$ be the inclusion of the axis $(y = 0)$ and $g : (0,0) \to \mathbb{A}^1$ the inclusion of the origin in the axis. It follows from the definition that $f^\bullet\mathscr{A}$ is the pull-back in the diagram

$$\begin{array}{ccc} (0) & \longrightarrow & T_{\mathbb{A}^1} \\ \downarrow & & \downarrow \\ f^*\mathscr{A} & \xrightarrow[{[v]\mapsto[x\frac{\partial}{\partial y}]}]{} & f^*T_{\mathbb{A}^2} \end{array}$$

so $f^\bullet\mathscr{A} = (0)$, and therefore $g^\bullet f^\bullet\mathscr{A} = (0)$. But the pull-back of the inclusion of the origin in the plane gives $(fg)^\bullet\mathscr{A}$ is

$$\begin{array}{ccc} (fg)^\bullet\mathscr{A} & \longrightarrow & (0) \\ \simeq\big\downarrow & & \big\downarrow \\ (fg)^*\mathscr{A} & \xrightarrow[{[v]\mapsto 0}]{} & (fg)^*T_{\mathbb{A}^2} \end{array}$$

so $(fg)^\bullet\mathscr{A}$ is an algebdroid over the point consisting of a vector space of dimension 1 with trivial bracket and trivial anchor, i.e.: an abelian one dimensional Lie algebra.

2. Unfolding of Lie algebroids

Definition 2.1. (Unfolding of Lie algebroid) Let $a_S : (\mathscr{A}_S, \{\cdot,\cdot\}_S) \to T_{X|S}$ be a family of holomorphic Lie algebroids over a smooth morphism $\pi : X \to S$. An unfolding of $\mathscr{A}_S$ is a Lie algebroid $\mathscr{A}$ on X with anchor $a : \mathscr{A} \to T_X$ such that

(a) The family of Lie algebroids $\mathscr{A}_S$, is recovered as $\mathscr{A}_S = a^{-1}(Im(a_S))$.
(b) $\mathrm{rank}(\mathscr{A}) = \mathrm{rank}(\mathscr{A}_S) + \dim(S)$.

An unfolding is defined to be *isotrivial* if the induced flat family of algebroids is trivial.

Definition 2.2. (Transversal unfolding) Let $\mathscr{A}$ be an unfolding of a family $\mathscr{A}_S$. Let $N\mathscr{A}$ be the cokernel of the map $a : \mathscr{A} \to T_X$ and $N_S\mathscr{A}$ be the cokernel of the map $a_S : \mathscr{A}_S \to T_{X|S}$. Notice that the maps $\mathscr{A}_S \to \mathscr{A}$ and $T_{X|S} \to T_X$ induce a map $N_S\mathscr{A} \to N\mathscr{A}$. The unfolding $\mathscr{A}$ is said to be *transversal* if the map $N_S\mathscr{A} \to N\mathscr{A}$ is an isomorphism.

Following T. Suwa [6] the third named author has showed the in [4] following result:

Theorem 2.3. *Let X be a non-singular variety and $\mathscr{F}_0$ a foliation on X. There is, for each scheme S, a 1 to 1 correspondence:*

$$\left\{\begin{array}{r} \textit{isotrivial transversal unfoldings} \\ \textit{of } \mathscr{F} \textit{ parametrized by } S \end{array}\right\} \longleftrightarrow \left\{\begin{array}{r} \textit{sections } \upsilon \in H^0(S, \Omega^1_S) \otimes \Upsilon(\mathscr{F}_0) \\ \textit{s.t.: } d\upsilon + \dfrac{1}{2}[\upsilon, \upsilon] = 0 \end{array}\right\}.$$

There is a similar result in the real case in [5, Theorem 4.6].

We observe that a section $\upsilon \in H^0(S, \Omega^1_S) \otimes \Upsilon(\mathscr{F}_0)$, satisfying $d\upsilon + \frac{1}{2}[\upsilon, \upsilon] = 0$, is such that the induced $\mathcal{O}_S$-morphism

$$\pi_*\upsilon : \pi_*\Upsilon(\mathscr{F}_0) \to T_S$$

is the anchor map of the Lie algebroid associated to the foliation $\pi_*\upsilon(\pi_*\Upsilon(\mathscr{F}_0)) \subset T_S$.

In this work we generalize this result for holomorphic Lie algebroids.

Definition 2.4. A differential operator ψ of order ≤ 1 on a quasi-coherent sheaf $\mathscr{A}$ over X is a morphism of $\mathbb{C}$-modules $\psi : \mathscr{A} \to \mathscr{A}$ such that for each local section $f \in \mathcal{O}_X(U)$ there is a local section $\psi f \in \mathcal{O}_X(U)$ such that, if $m_f : \mathscr{A}|_U \to \mathscr{A}|_U$ is multiplication by f, then

$$\psi|_U \circ m_f - m_f \circ \psi|_U = m_{\psi f}.$$

We denote by $D_X^{\leq 1}(\mathscr{A})$ the $\mathcal{O}_X$-module of differential operators of order ≤ 1 of $\mathscr{A}$.

Definition 2.5. Given a differential operator ψ of order ≤ 1 the map $f \mapsto \psi f$ determines a derivation of $\mathcal{O}_X$. This derivation is called *the symbol* of the operator ψ.

Remark 2.6. For any torsion free sheaf $\mathscr{F}$, the sheaf $D_X^{\leq 1}(\mathscr{F})$ has a natural structure of Lie algebroid, as the commutator of differential operators define a bracket whose anchor is the symbol.

Remark 2.7. The above definition of the sheaf of differential operator differs from that of [3, 16.8]. In loc. cit. a differential operator between two sheaves $\mathscr{F}$ and $\mathscr{G}$ is an element of the sheaf $\mathcal{H}om_X(\mathcal{P}_X^1(\mathscr{F}), \mathscr{G})$, where $\mathcal{P}_X^1(\mathscr{F})$ is the *sheaf of principal parts of order* 1 of $\mathscr{F}$ (see [3, 16.7]). in the case $\mathscr{F} = \mathscr{G} = \mathcal{O}_X$ we have a short exact sequence

$$0 \to \Omega_X^1 \to \mathcal{P}_X^1 \to \mathcal{O}_X \to 0,$$

tensoring with a sheaf $\mathscr{F}$ gives the sequence $0 \to \Omega_X^1 \otimes \mathscr{F} \to \mathcal{P}_X^1(\mathscr{F}) \to \mathscr{F} \to 0$. Then applying $\mathcal{H}om_X(-, \mathscr{F})$ gives the exact sequence

$$0 \to \mathcal{E}nd(\mathscr{F}) \to \mathcal{H}om_X(\mathcal{P}_X^1(\mathscr{F}), \mathscr{F}) \to T_X \otimes \mathcal{E}nd(\mathscr{F}) \to \mathcal{E}xt_X^1(\mathscr{F}, \mathscr{F}).$$

Using the natural map $\mathcal{O}_X \to \mathcal{E}nd(\mathscr{F})$ we have the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}nd(\mathscr{F}) & \longrightarrow & D_X^{\leq 1}(\mathscr{F}) & \longrightarrow & T_X \\ & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{E}nd(\mathscr{F}) & \longrightarrow & \mathcal{H}om_X(\mathcal{P}_X^1(\mathscr{F}), \mathscr{F}) & \longrightarrow & T_X \otimes \mathcal{E}nd(\mathscr{F}) \end{array}$$

in which the right square is a pull-back diagram.

Lemma 2.8. *Let $\mathscr{F}$ be a sheaf over X, and let $f : R \to S$ be a morphism. Denote by Y the pull-back of X by f and by $f^*\mathscr{F}$ the corresponding pull-back of $\mathscr{F}$ as a sheaf over Y. There is a canonical morphism*

$$f^\bullet D_X^{\leq 1}(\mathscr{F}) \to D_Y^{\leq 1}(f^*\mathscr{F}).$$

Proof. To a local section of $f^\bullet D_X^{\leq 1}(\mathscr{F})$ we can explicitly assign a differential operator on $f^*\mathscr{F}$ by the following formula. A local section α of $f^\bullet D_X^{\leq 1}(\mathscr{F})$ is by definition

$$\alpha = \left(\sum_i \psi_i \otimes r_i, v \right),$$

where $\psi_i \otimes r_i$ are local sections of $D_X^{\leq 1}(\mathscr{F}) \otimes \mathcal{O}_{X_R}$ and v is a local section of T_Y such that $\sigma(\psi_i \otimes r_i) = Df(v)$. Now, given a local section $a = \sum_i a_i \otimes t_i$ of $\mathscr{F} \otimes \mathcal{O}_Y$ we define

$$\alpha(a) := \sum_{ij} (\psi_i(a_j) \otimes r_i t_j + a_j \otimes v(t_j) s_i \in f^*\mathscr{F}.$$

□

Lemma 2.9. *Let $\mathscr{F}$ be such that both $\mathscr{F}$ and $f^*\mathscr{F}$ are* reflexive *sheaves over X and Y respectively, then there is a morphism*

$$\mathcal{H}om_Y(\mathcal{P}_Y^1(f^*\mathscr{F}), f^*\mathscr{F}) \to f^*\left(\mathcal{H}om_X(\mathcal{P}_X^1(\mathscr{F}), \mathscr{F})\right).$$

Proof. By [3] there is always a morphism

$$\mathcal{H}om_Y(\mathcal{P}_Y^1, \mathcal{O}_Y) \to f^*\left(\mathcal{H}om_X(\mathcal{P}_X^1, \mathcal{O}_X)\right)$$

In the case where $\mathscr{F}$ is locally free we have canonical isomorphisms

$$\mathcal{H}om_Y(\mathcal{P}_Y^1(f^*\mathscr{F}), f^*\mathscr{F}) \cong \mathcal{H}om_Y(\mathcal{P}_Y^1(f^*\mathscr{F}), \mathcal{O}_Y) \otimes f^*\mathscr{F}$$

and

$$f^*\left(\mathcal{H}om_X(\mathcal{P}_X^1, \mathcal{O}_X)\right) \otimes f^*\mathscr{F} \cong f^*\left(\mathcal{H}om_X(\mathcal{P}_X^1(\mathscr{F}), \mathscr{F})\right).$$

Then from the above morphism between the sheaves of principal parts we get

$$\mathcal{H}om_Y(\mathcal{P}_Y^1(f^*\mathscr{F}), \mathcal{O}_Y) \otimes f^*\mathscr{F} \to f^*\left(\mathcal{H}om_X(\mathcal{P}_X^1, \mathcal{O}_X)\right) \otimes f^*\mathscr{F},$$

which gives us a morphism $\mathcal{H}om_Y(\mathcal{P}_Y^1(f^*\mathscr{F}), f^*\mathscr{F}) \to f^*\left(\mathcal{H}om_X(\mathcal{P}_X^1(\mathscr{F}), \mathscr{F})\right)$ if $\mathscr{F}$ is locally free. If more generally $\mathscr{F}$ is reflexive then there is an open set U such that $\mathscr{F}|_U$ is locally free and such that every section of $\mathscr{F}(U)$ extends to a global section, in other words, if $j : U \to X$ is the inclusion then we have an isomorphism $j_*(\mathscr{F}|_U) \simeq \mathscr{F}$. If also $f^*\mathscr{F}$ is reflexive then $f^*\mathscr{F}|_{f^{-1}U}$ is locally free and $j_*(f^*\mathscr{F}|_{f^{-1}U}) \simeq f^*\mathscr{F}$.

Local sections of $\mathcal{H}om_X(\mathcal{P}_X^1(\mathscr{F}), \mathscr{F})$ over an open set V are in natural correspondence with $\mathbb{C}$-linear morphisms $\mathscr{F}(V) \to \mathscr{F}(V)$ that are differential maps in the sense of [3]. In particular every section of $\mathcal{H}om_X(\mathcal{P}_X^1(\mathscr{F}), \mathscr{F})(U)$ extends to a global section. And the same can be said about sections of $\mathcal{H}om_Y(\mathcal{P}_Y^1(f^*\mathscr{F}), f^*\mathscr{F})$ over $(f^{-1}U)$ extending to global sections. Now over $f^{-1}U$ we have a morphism

$$\mathcal{H}om_Y(\mathcal{P}_Y^1(f^*\mathscr{F}), f^*\mathscr{F})|_{f^{-1}U} \to f^*\left(\mathcal{H}om_X(\mathcal{P}_X^1(\mathscr{F}), \mathscr{F})|_U\right).$$

Which extends to a morphism $\mathcal{H}om_Y(\mathcal{P}_Y^1(f^*\mathscr{F}), f^*\mathscr{F}) \to f^*\left(\mathcal{H}om_X(\mathcal{P}_X^1(\mathscr{F}), \mathscr{F})\right)$ as wanted. □

Lemma 2.10. *Let $\mathscr{F}$ be a reflexive sheaf over X and $U \subseteq X$ be an open set such that $\mathscr{F}|_U$ is locally free and such that every section of $\mathcal{O}_X(U)$ extends to a global section. Then there is a section $\mathcal{E}nd(\mathscr{F}) \to \mathcal{O}_X$ to the canonical inclusion $\mathcal{O}_X \to \mathcal{E}nd(\mathscr{F})$.*

Proof. Let $j : U \hookrightarrow X$ be the inclusion, then over U we have $\mathcal{E}nd(\mathscr{F})|_U \simeq \mathscr{F}|_U \otimes \mathscr{F}^\vee|_U$, so there is the evaluation map $\mathscr{F}|_U \otimes \mathscr{F}^\vee|_U \to \mathcal{O}_X|_U$, which divided by the generic rank of $\mathscr{F}$ is a section of the inclusion $\mathcal{O}_X \to \mathcal{E}nd(\mathscr{F})$. As both $\mathscr{F} \simeq j_*(\mathscr{F}|_U)$ and $\mathcal{O}_X \simeq j_*(\mathcal{O}_X|_U)$ by hypothesis, we have $\mathcal{E}nd(\mathscr{F}) \simeq j_*(\mathscr{F}|_U \otimes \mathscr{F}^\vee|_U)$ from which we get the morphism $\mathcal{E}nd(\mathscr{F}) \to \mathcal{O}_X$. □

Definition 2.11. A closed subset $Z \subseteq X$ is of *relative codimension* c over S iff for every point $s \in S$ we have $\dim X_s - \dim Z_s = c$

Theorem 2.12. *Let $\mathscr{F}$ be a reflexive sheaf over X, flat over S, with X smooth over S and such that there is an open set $U \subseteq X$ with $\mathscr{F}|_U$ locally free and such that $Z := X \setminus U$ is of relative codimension ≥ 2. Let $f : R \to S$ be a morphism. Denote by X_R the pull-back of X by f and by $f^*\mathscr{F}$ the corresponding pull-back of $\mathscr{F}$ as a sheaf over X_R flat over R. Then the canonical morphism $f^\bullet D_X^{\leq 1}(\mathscr{F}) \to D_{X_R}^{\leq 1}(f^*\mathscr{F})$ is an isomorphism of Lie algebroids.*

Proof. Lemma 2.8 gives the existence of a morphism $f^\bullet D_X^{\leq 1}(\mathscr{F}) \to D_{X_R}^{\leq 1}(\mathscr{F})$. Here we will show the existence of an inverse to this morphism.

By Lemma 2.9 there is a morphism

$$\mathcal{H}om(\mathcal{P}^1_{X_R}(f^*\mathscr{F}), f^*\mathscr{F}) \to f^*\left(\mathcal{H}om(\mathcal{P}^1_X(\mathscr{F}), \mathscr{F})\right).$$

Now, as $X \setminus U$ is of relative codimension 2 over S and $X \to S$ is smooth, then every section of $\mathcal{O}_X$ over U extends to a global section, then we are in condition to apply Lemma 2.10 and get a splitting $\mathcal{E}nd(\mathscr{F}) \to \mathcal{O}_X$. The fact that $D_X^{\leq 1}(\mathscr{F})$ is a pull-back of $\mathcal{H}om(\mathcal{P}^1_X(\mathscr{F}), \mathscr{F})$ by T_X over $T_X \otimes \mathcal{E}nd(\mathscr{F})$ gives in turn a splitting $\mathcal{H}om(\mathcal{P}^1_X(\mathscr{F}), \mathscr{F}) \to D_X^{\leq 1}(\mathscr{F})$. So we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{H}om(\mathcal{P}^1_{X_R}(f^*\mathscr{F}), f^*\mathscr{F}) & \longrightarrow & f^*\left(\mathcal{H}om(\mathcal{P}^1_X(\mathscr{F}), \mathscr{F})\right) \\
\uparrow & & \downarrow \\
D_{X_R}^{\leq 1}(f^*\mathscr{F}) & \longrightarrow & f^* D_X^{\leq 1}(\mathscr{F}).
\end{array}$$

From the bottom arrow of this diagram and the pull-back property of $f^\bullet D_X^{\leq 1}(\mathscr{F})$ we get a morphism

$$D_{X_R}^{\leq 1}(f^*\mathscr{F}) \to f^\bullet D_X^{\leq 1}(\mathscr{F}).$$

Because $f^\bullet D_X^{\leq 1}(\mathscr{F})$ is defined as a pull-back, the composition

$$f^\bullet D_X^{\leq 1}(\mathscr{F}) \to D_{X_R}^{\leq 1}(f^*\mathscr{F}) \to f^\bullet D_X^{\leq 1}(\mathscr{F})$$

is the identity, so $f^\bullet D_X^{\leq 1}(\mathscr{F}) \to D_{X_R}^{\leq 1}(f^*\mathscr{F})$ is a monomorphism. The other composition, that is $D_{X_R}^{\leq 1}(f^*\mathscr{F}) \to f^\bullet D_X^{\leq 1}(\mathscr{F}) \to D_{X_R}^{\leq 1}(f^*\mathscr{F})$ gives the central vertical morphism in the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{E}nd(f^*\mathscr{F}) & \longrightarrow & D^{\leq 1}_{X_R}(f^*\mathscr{F}) & \longrightarrow & T_{X_R} \\ & & \| & & \downarrow & & \| \\ 0 & \longrightarrow & \mathcal{E}nd(f^*\mathscr{F}) & \longrightarrow & D^{\leq 1}_{X_R}(f^*\mathscr{F}) & \longrightarrow & T_{X_R}, \end{array}$$

so $f^\bullet D^{\leq 1}_X(\mathscr{F}) \to D^{\leq 1}_{X_R}(f^*\mathscr{F})$ is also an epimorphism. □

Remark 2.13. If $a : \mathscr{A} \to T_X$ is a Lie algebroid and $\alpha \in \mathscr{A}(U)$ is a local section, then the map

$$\psi|_U : \mathscr{A}|_U \to \mathscr{A}|_U$$

defined by $\psi|_U(\alpha) := \{\alpha, -\}$ is both a derivation of the Lie algebra structure of $\mathscr{A}|_U$ and a differential operator of order ≤ 1. Indeed, we have that for each local section $f \in \mathcal{O}_X(U)$ and $\alpha \in \mathscr{A}|_U$ we obtain a section $\psi f := -a(\{\alpha, -\})(f) \in \mathcal{O}_X(U)$. Thus by the identity

$$\{f \cdot \alpha, -\} = f \cdot \{\alpha, -\} - a(\{\alpha, -\})(f) \cdot \alpha.$$

we conclude that

$$(\psi|_U \circ m_f)(\alpha) - (m_f \circ \psi|_U)(\alpha) = m_{\psi f}(\alpha)$$

for all $f \in \mathcal{O}_X(U)$ and $\alpha \in \mathscr{A}|_U$.

We will denote by $\mathrm{Der}_{\mathrm{Lie}}(\mathscr{A})$ the sheaf of derivations of the Lie algebra structure of $\mathscr{A}$. It is a sub-sheaf of the sheaf $\mathrm{End}_{\mathcal{O}_X}(\mathscr{A})$ of $\mathcal{O}_X$-linear endomorphisms.

Definition 2.14. Given a family of algebroids $(\mathscr{A}_S, \{\cdot, \cdot\}_S)$, we denote by $\sigma : D^{\leq 1}_X(\mathscr{A}) \to T_X$ the symbol map of differential operators, then we have the diagram

$$\begin{array}{ccccc} \mathscr{A}_S & \longrightarrow & \mathrm{Der}_{\mathrm{Lie}}(\mathscr{A}_S) \cap D^{\leq 1}_X & \longrightarrow & \Big(\mathrm{Der}_{\mathrm{Lie}}(\mathscr{A}_S) \cap D^{\leq 1}_X(\mathscr{A}_S)\Big) / \mathscr{A}_S \\ \downarrow{\scriptstyle a_S} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ T_{X|S} & \longrightarrow & T_X & \longrightarrow & \pi^* T_S. \end{array}$$

Where the intersection is taken as subsheaves of the sheaf $\mathcal{H}om_{f^{-1}\mathcal{O}_S}(\mathscr{A}_S, \mathscr{A}_S)$ of endomorphisms of $f^{-1}\mathcal{O}_S$-modules of $\mathscr{A}_S$. We define the sheaf $\mathfrak{u}(\mathscr{A}_S)$ as

$$\mathfrak{u}(\mathscr{A}_S) := \sigma^{-1}(\pi^{-1}T_S) \subseteq \Big(\mathrm{Der}_{\mathrm{Lie}}(\mathscr{A}_S) \cap D^{\leq 1}_X(\mathscr{A}_S)\Big) / \mathscr{A}_S.$$

Note that $\mathfrak{u}(\mathscr{A}_S)$ is *not* a coherent sheaf over X but only a sheaf of $f^{-1}\mathcal{O}_S$-modules.

Remark 2.15. The sheaf $\mathfrak{u}(\mathscr{A}_S)$ inherits from $\mathrm{Der}_{\mathrm{Lie}}(\mathscr{A}_S)$ the structure of a sheaf of Lie algebras. Indeed, as the inclusion $\mathscr{A}_S \subseteq \mathrm{Der}_{\mathrm{Lie}}(\mathscr{A}_S)$ is an ideal of Lie algebras, the Lie bracket $[\psi, \phi] = \psi \circ \phi - \phi \circ \psi$ passes to the quotient to a bracket in $\mathfrak{u}(\mathscr{A}_S)$. In particular, we have that $\Upsilon(\mathscr{A}_S)$ is a Lie algebra over $\Gamma(S, \mathcal{O}_S)$.

Let us begin by considering an unfolding $(\mathscr{A}, \{\cdot\,,\cdot\})$ of a family of algebroids $(\mathscr{A}_S, \{\cdot\,,\cdot\}_S)$ parametrized by S. Note that, when the unfolding is transversal, we have that the anchor $a : \mathscr{A} \to T_X$ determines an isomorphism $[a] : \mathscr{A}/\mathscr{A}_S \xrightarrow{\cong} \pi^* T_S$, so we can consider the morphism $\upsilon_{\mathscr{A}} : \pi^* T_S \to \mathscr{A}/\mathscr{A}_S$ defined as the inverse of the isomorphism determined by the anchor.

Proposition 2.16. *If $(\mathscr{A}, \{\cdot\,,\cdot\})$ is transversal to S then $\upsilon_{\mathscr{A}}(\pi^{-1}T_S)$ is a subsheaf of $\mathfrak{u}(\mathscr{A})$.*

Proof. Note that the statement is making reference to $\pi^{-1}T_S \subset \pi^* T_S$, that is the sheaf of vector fields that are constant along the fibers of π, also known as *basic vector fields*. Then, given a local section $s \in \upsilon_{\mathscr{A}}(\pi^{-1}T_S) \subseteq \mathscr{A}/\mathscr{A}_S$ we need to show that, for any lifting $\tilde{s}$ of s in $\mathscr{A}$ we have that $\{\mathscr{A}_S, \tilde{s}\} \subseteq \mathscr{A}_S$. In other words we need to show that if α is a local section of $\mathscr{A}_S$, then $a(\{\tilde{s}, \alpha\}) \in T_{X|S}$. Locally in X we can take $a(\tilde{s})$ of the form $Y + Z$ with $Y \in T_{X|S}$ and $Z \in \pi^{-1}T_S$.

Noting $W = a(\alpha) \in T_{X|S}$ we compute

$$a(\{\tilde{s}, \alpha\}) = [W, a(\tilde{s})] = [W, Y + Z] = [W, Y] - Z(W),$$

since $W(Z) = 0$, being Z in $\pi^{-1}T_S$. Then $a(\{\tilde{s}, \alpha\})$ is in $T_{X|S}$, and also in $a(\mathscr{A})$, so it is in $a(\mathscr{A}_S)$. □

Theorem 2.17. *Let X be a non-singular variety, and $(\mathscr{A}_S, \{\cdot\,,\cdot\}_S)$ a family of algebroids on X parametrized by a scheme of finite type S. There is, for each scheme S, a 1 to 1 correspondence:*

$$\left\{ \begin{array}{c} \textit{transversal unfoldings} \\ \textit{of } \mathscr{A}_S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{morphisms } \pi^{-1}(T_S) \to \mathfrak{u}(\mathscr{A}_S) \\ \textit{respecting brackets} \end{array} \right\}.$$

Proof. Morphism associated to an unfolding: given a transversal unfolding $\mathscr{A}$ of $\mathscr{A}_S$ we have $\mathscr{A}/\mathscr{A}_S \cong \pi^*(T_S)$. Then, it follows from Proposition 2.16 that we have a map

$$\upsilon_{\mathscr{A}} : \pi^{-1}(T_S) \to \mathfrak{u}(\mathscr{A}_S).$$

Now, in order to establish the first part of the correspondence we need to prove that this is a map of sheaves of $\pi^{-1}\mathcal{O}_S$-Lie algebras. Considere $\upsilon_{\mathscr{A}}(\pi^{-1}(T_S)) \subseteq \mathfrak{u}(\mathscr{A}_S)$ and we take $A \subseteq \mathscr{A}$ its pre-image under the morphism $\mathscr{A} \to \mathscr{A}/\mathscr{A}_S$. Since $A/\mathscr{A}_S$ is a subsheaf of $\mathfrak{u}(\mathscr{A}_S)$, then $\mathscr{A}_S \subseteq A$ is a Lie ideal, so we have a diagram

$$\begin{array}{ccccc} \mathscr{A}_S & \longrightarrow & A & \longrightarrow & \upsilon_{\mathscr{A}}(\pi^{-1}(T_S)) \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & & \downarrow \\ a(\mathscr{A}_S) & \longrightarrow & a(A) & \longrightarrow & \pi^{-1}(T_S), \end{array}$$

as $\mathscr{A}_S$ and $a(\mathscr{A}_S)$ are Lie ideals of A and $a(A)$ respectively, and the anchor a is a morphism of Lie algebras. Then the morphism induced in the quotient $\upsilon_{\mathscr{A}}(\pi^{-1}(T_S)) \to \pi^{-1}(T_S)$ is also a Lie algebra morphism, so also its inverse $\upsilon_{\mathscr{A}}$ is a morphism of sheaves of Lie algebras.

Unfolding associated with a morphism: given a morphism of sheaves of Lie algebras

$$\upsilon : \pi^{-1}(T_S) \to \mathfrak{u}(\mathscr{A}_S),$$

we get an extension of sheaves of Lie algebras over $\pi^{-1}(\mathcal{O}_S)$

$$0 \to \mathscr{A}_S \to A \to \pi^{-1}(T_S) \to 0.$$

Indeed, the Lie algebra A is defined as the pull-back of the diagram

$$\begin{array}{ccc} A & \longrightarrow & \pi^{-1}(T_S) \\ \downarrow & & \downarrow \\ \mathrm{Der}_{\mathrm{Lie}}(\mathscr{A}_S) \cap D_X^{\leq 1}(\mathscr{A}_S) & \longrightarrow & \mathfrak{u}(\mathscr{A}_S). \end{array}$$

Moreover, we have a morphism $\tilde{a} : A \to T_X$ defined by the composition

$$A \to D_X^{\leq 1} \to T_X.$$

However, A is not a quasi-coherent module over X, to get a module over $\mathcal{O}_X$ we need to modify this sheaf a little. For this we define the sheaf of sub-modules $B \subseteq A \otimes_{\pi^{-1}\mathcal{O}_S} \mathcal{O}_X$ as the quasi-coherent subsheaf generated by the stalks of the form $\alpha \otimes f - f\alpha \otimes 1$, where α is a stalk of $\mathscr{A}_S$ and f a stalk of $\mathcal{O}_X$. Then we define

$$\mathscr{A} :=_{\mathrm{def}} (A \otimes_{\pi^{-1}\mathcal{O}_S} \mathcal{O}_X)/B.$$

The map $\tilde{a}$ can be extended to an $\mathcal{O}_X$-linear map $a' : A \otimes \mathcal{O}_X \to T_X$. As $a'|_B = 0$, we get an $\mathcal{O}_X$-linear map $\mathscr{A} \to T_X$ extending the map $A \to T_X$. Also notice that $\mathscr{A}_S$ is a subsheaf of $\mathscr{A}$ and that

$$\mathscr{A}/\mathscr{A}_S = A/\mathscr{A}_S \otimes \mathcal{O}_X = \pi^{-1}T_S \otimes \mathcal{O}_X = \pi^* T_S.$$

We can extend the Lie bracket of A to a Lie bracket in $A \otimes \mathcal{O}_X$ by the formula

$$\{\alpha \otimes f, \beta \otimes g\} = \beta \otimes (f \cdot \tilde{a}(\alpha)(g)) + \alpha \otimes (g \cdot \tilde{a}(\beta)(f)) + \{\alpha, \beta\} \otimes f \cdot g.$$

With this bracket the subsheaf B is a sheaf of Lie ideals. Therefore we get that $\mathscr{A}$ has a Lie algebroid structure and it is an unfolding of $\mathscr{A}_S$.

The construction of the morphism $\pi^{-1}(T_S) \to \mathfrak{u}(\mathscr{A}_S)$ associated with an unfolding and of the unfolding associated with the morphism are inverse to each other. □

Proposition 2.18. *$\pi_* \mathfrak{u}(\mathscr{A}_S)$ has the structure of a Lie algebroid over S.*

Proof. Since $\mathfrak{u}(\mathscr{A}_S)$ is a $\pi^{-1}\mathcal{O}_S$-module then $\pi_* \mathfrak{u}(\mathscr{A}_S)$ is an $\mathcal{O}_S$-module. It is endowed with a Lie algebra bracket which is the push-forward of the bracket of $\mathfrak{u}(\mathscr{A}_S)$. Its anchor map can be defined as follows: Taking the natural map $D_X^{\leq 1} \to$

T_X one gets by considering the derivation defined by a differential operator we get a diagram

$$\begin{array}{ccccc} \mathscr{A}_S & \longrightarrow & \mathrm{Der}_{\mathrm{Lie}}(\mathscr{A}_S) \cap D^{\leq 1}_X & \longrightarrow & \mathfrak{u}(\mathscr{A}_S) \\ \downarrow{\scriptstyle a_S} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle a_{\mathfrak{u}}} \\ T_{X|S} & \longrightarrow & T_X & \longrightarrow & \pi^{-1}T_S. \end{array}$$

When the morphism π is proper we have a natural isomorphism $\pi_*\pi^*T_S \cong T_S$. The anchor map of $\pi_*\mathfrak{u}(\mathscr{A}_S)$ is then

$$\pi_* a_{\mathfrak{u}} : \pi_*\mathfrak{u}(\mathscr{A}_S) \to \pi_*\pi^{-1}T_S \cong T_S.$$

□

Recall that a flat connection on an algebroid $\mathscr{A}$ over a space X is a section of the anchor map $s : T_X \to \mathscr{A}$ respecting Lie brackets. Then we get the following.

Corollary 2.19. *There is a 1 to 1 correspondence*

$$\left\{ \begin{array}{c} \textit{transversal unfoldings} \\ \textit{of } \mathscr{A}_S \end{array} \right\} \longleftrightarrow \left\{ \textit{flat connections on the algebroid } \pi_*\mathfrak{u}(\mathscr{A}_S) \right\}.$$

In particular to have an unfolding of a family $\mathscr{A}_S$ of algebroids we must have an epimorphic anchor map on the algebroid $\pi_*\mathfrak{u}(\mathscr{A}_S)$. So for any family $\mathscr{A}_S$ we have a foliation in the base space S induced by the algebroid $\pi_*\mathfrak{u}(\mathscr{A}_S)$. Any unfolding of a restriction of the family $\mathscr{A}_S$ must be over a leaf of said foliation (compare with [2]).

Proposition 2.20. *Given a pull-back diagram of holomorphic spaces with smooth vertical arrows*

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & X \\ {\scriptstyle \pi_R}\downarrow & & \downarrow{\scriptstyle \pi} \\ R & \xrightarrow{f} & S. \end{array}$$

And a family $\mathscr{A}_S$ of algebdroids over $\pi : X \to S$ such that $\mathrm{sing}(\mathscr{A})$ has relative codimension greater than 2 as a subscheme of X/S. We have a canonical morphism

$$\pi_{R*}\mathfrak{u}(\phi^\bullet\mathscr{A}) \to f^\bullet\pi_*\mathfrak{u}(\mathscr{A}),$$

as algebdroids on R.

Proof. By remark 1.4 we have that the sheaf underlying the algebroid $\phi^\bullet\mathscr{A}$ is $\phi^*\mathscr{A}$.

By the hypothesis on $\mathrm{sing}\mathscr{A}$ we can apply Proposition 2.12 which says that we have an isomorphism $\phi^\bullet D^{\leq 1}_X(\mathscr{A}) \simeq D^{\leq 1}_Y(\phi^*\mathscr{A})$. Also, as the Lie algebra structure of $\mathscr{A}$ is $\mathcal{O}_S$-linear, and $\mathcal{O}_Y = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_R$, then the Lie algebra structure of $\phi^*\mathscr{A}$ is $\mathcal{O}_R$-linear. If a local section $\varphi \in D^{\leq 1}_Y(\phi^*\mathscr{A})(U)$ acts on $\phi^*\mathscr{A}(U)$ as a derivation of the Lie algebra structure, then it acts as a derivation on sections of $\phi^*\mathscr{A}$ of the form $s \otimes 1$, so the image of φ by the composition

$$D_Y^{\leq 1}(\phi^*\mathscr{A}) \simeq \phi^\bullet D_X^{\leq 1}(\mathscr{A}) \to \phi^* D_X^{\leq 1}(\mathscr{A})$$

is in the subsheaf $\phi^*(D_X^{\leq 1}(\mathscr{A}) \cap \mathrm{Der}_{\mathrm{Lie}}(\mathscr{A}))$. The fact that the diagram

$$\begin{array}{ccc} \phi^\bullet D_X^{\leq 1}(\mathscr{A}) & \longrightarrow & \phi^* D_X^{\leq 1}(\mathscr{A}) \\ \downarrow & & \downarrow \\ T_Y & \longrightarrow & \phi^* T_X \end{array}$$

commutes implies with the above that we have a morphism $\mathfrak{u}(\phi^\bullet\mathscr{A}) \to \phi^*\mathfrak{u}(\mathscr{A})$ which in turn gives a morphism $\mathfrak{u}(\phi^\bullet\mathscr{A}) \to \phi^\bullet\mathfrak{u}(\mathscr{A})$. The Proposition follows from the fact that $\pi_{R*}\phi^\bullet\mathfrak{u}(\mathscr{A}) \simeq f^\bullet\pi_*\mathfrak{u}(\mathscr{A})$ □

3. Examples

3.1. **Holomorphic foliations.** Let $a_S \colon \mathscr{F}_S \to T_{X|S}$ a family of singular holomorphic foliation over X . An unfolding of $\mathscr{F}_S$ is a foliation $\mathscr{F}$ on X with anchor $a : \mathscr{F} \to T_X$ such that that

(a) $\mathscr{F}_S = a^{-1}(a_S(\mathscr{F}_S))$.
(b) $\dim(\mathscr{F}) = \dim(\mathscr{F}_S) + \dim(S)$.

Now suppose we have $\mathscr{F} = \mathcal{L}$ is a line bundle, in other words we have a foliation by curves. Let v be a local generating section of $\mathcal{L}$ and ψ a local $\mathbb{C}$-linear endomorphism of $\mathcal{L}$, so $\psi(v) = f_\psi \cdot v$ for some local section f_ψ of $\mathcal{O}_X$. If ψ is a derivation for the Lie algebra structure of $\mathcal{L}$ (which is induced by the inclusion $\mathcal{L} \subseteq T_X$) Then for any local section g of $\mathcal{O}_X$ we get

$$\psi([v, g \cdot v]) = [\psi(v), g \cdot v] + [v, \psi(g \cdot v)],$$
$$\psi(v(g) \cdot v) = [f_\psi \cdot v, g \cdot v] + [v, \psi(g \cdot v)]$$

If ψ is also a differential operator we have $\psi(v(g) \cdot v) - v(g)\psi(v) = \sigma(\psi)(v(g)) \cdot v$, where σ denotes the symbol of ψ. Then we have

$$\psi(v(g) \cdot v) = [f_\psi \cdot v, g \cdot v] + [v, \psi(g \cdot v)] =$$
$$v(g) f_\psi \cdot v - \sigma(\psi)(v(g)) \cdot v = f_\psi v(g) \cdot v + v(g) f_\psi \cdot v - v(\sigma(\psi)(g)) \cdot v.$$

In other words, $[\sigma(\psi), v](g) = f_\psi v(g)$, as this happens for every local section g of $\mathcal{O}_X$ then

$$[\sigma(\psi), v] = f_\psi \cdot v = \psi(v). \qquad (1)$$

Denoting $p : T_X \to \pi^* T_S$ the projection, lets call for an open set $V \subseteq X$

$$U(\mathcal{L})(V) := \left\{\psi \in \left(\mathrm{Der}_{\mathrm{Lie}}(\mathcal{L}) \cap D_X^{\leq 1}(\mathcal{L})\right)(V), \ \text{ s.t: } p \circ \sigma(\psi) \in \pi^{-1} T_S\right\}.$$

We have then $\mathfrak{u}(\mathcal{L}) = U(\mathcal{L})/\mathcal{L}$ and an inclusion of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & U(\mathcal{L}) & \longrightarrow & a(U(\mathcal{L})) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & D_X^{\leq 1}(\mathcal{L}) & \xrightarrow{\sigma} & T_X & \longrightarrow & 0.
\end{array}
\tag{2}
$$

Notice that equation (1) implies that sections of $a(U(\mathcal{L}))$ act on $\mathcal{L}$ as differential operator, so the top short exact sequence in diagram (2) splits, so every section ψ of $U(\mathcal{L})$ can be written as $\psi = m_a + Y$ where m_a is multiplication by a local section a of $\mathcal{O}_X$ and Y is a local vector field on X. Moreover, equation (1) implies that if $m_a + Y$ is a section of $U(\mathcal{L})$ then for a section X of $\mathcal{L}$ we have $[Y, X] = [Y, X] + a \cdot X$, so $a = 0$ then the sheaf K of diagram (2) is null. In conclusion we can characterize $\mathfrak{u}(\mathcal{L})$ as

$$\mathfrak{u}(\mathcal{L}) = \left(Y \in T_X : p(Y) \in \pi^{-1} T_S,\ [Y, \mathcal{L}] \subseteq \mathcal{L}\right)/\mathcal{L}.$$

In this case the kernel of the algebroid $\pi_* \mathfrak{u}$ is the $\mathcal{O}_S$-linear Lie algebra

$$\mathfrak{g}(\mathcal{L}) = \left(Y \in T_{X|S} : [Y, \mathcal{L}] \subseteq \mathcal{L}\right)/\mathcal{L},$$

which is the algebra of infinitesimal symmetries of the foliation.

3.2. **Sheaf of Lie algebra.** Let $\mathscr{A}_S$ be a family of sheaf of Lie algebra. In this case the anchor map $a_S = 0$. An unfolding of $\mathscr{A}_S$ is a Lie algebroid $\mathscr{A}$ on X with anchor $a : \mathscr{A} \to T_X$ such that

(a) The family of Lie algebroids $\mathscr{A}_S$, is recovered as $\mathscr{A}_S = a^{-1}(0) = Ker(a)$.
(b) $\operatorname{rank}(\mathscr{A}) = \operatorname{rank}(\mathscr{A}_S) + \dim(S)$.

That is, $\mathscr{A}_S$ is an isotropy sub-Lie algebroid of a Lie algebroid $a : \mathscr{A} \to T_X$ and the dimension of the foliation associated this Lie algebroid has dimension equal to $\dim(S)$ by the condition b). We have

$$0 \to \mathscr{A}_S \to \mathscr{A} \to Im(a) \to 0.$$

Therefore, if the unfolding is transversal, we have the isomorphism $[a] : \mathscr{A}/\mathscr{A}_S \xrightarrow{\cong} \pi^* T_S$. That is $Im(a) \cong \pi^* T_S$, this in turn define a splitting of the short exact sequence $0 \to T_{X|S} \to T_X \to \pi^* T_S \to 0$.

In particular we can take any sheaf $\mathscr{F}$ flat over S and take the algebroid $\mathscr{A}_S$ to be $\mathscr{F}$ with the structure of an abelian Lie algebra and the zero anchor map. In this case we get the extension of Lie algebras

$$0 \to \mathscr{F} \to \mathscr{A} \to \pi^* T_S \to 0.$$

This extension defines an action of $\pi^* T_S$ on $\mathscr{F}$, in particular we have a flat connection on $\pi_* \mathscr{F}$. The extension also defines an homology clas c on the Chevalley-Eilenberg cohomology $c \in H^2(\pi^* T_S, \mathscr{F})$. Reciprocally, given a splitting of $0 \to T_{X|S} \to T_X \to \pi^* T_S \to 0$, and a flat connection ∇ on the quasi-coherent sheaf $\pi_* \mathscr{F}$, we get a Lie algebra action of $\pi^* T_S$ on $\mathscr{F}$. Indeed, let $p \in X$, v be a local section of $\pi^*T_{S_p}$ and $x \in \mathscr{F}_p$. As $\mathscr{F}_p$ is a localization of $\mathscr{F}_{\pi^{-1}(\pi(p))}$ we can write x as $\sum_i f_i y_i$ with $y_i \in \mathscr{F}_{\pi^{-1}(\pi(p))}$ and $f_i \in \mathcal{O}_{X,p}$, we can also assume $v = g \cdot w$ with $w \in T_{S,\pi(p)}$ and $g \in \mathcal{O}_{X,p}$. Now, denoting by $\iota : \pi^*T_S \to T_X$ the splitting, we can define the action of π^*T_S in $\mathscr{F}$ as

$$\nabla_v(x) = \sum_i g \cdot \iota(w)(f_i)y_i + f_i \nabla_w(y_i).$$

Now, given an element of the Chevalley-Eilenberg cohomology $c \in H^2(\pi^*T_S, \mathscr{F})$, where $\mathscr{F}$ is taken as a π^*T_S-module with the action just defined, we get an abelian extension of Lie algebras

$$0 \to \mathscr{F} \to \mathscr{A} \to \pi^*T_S \to 0.$$

Which is an unfolding of $\mathscr{F}$ as abelian Lie algebroid with trivial anchor.

3.3. **Poisson structures.** Let $a_S : (\Omega^1_{X|S}, \{\cdot,\cdot\}_S) \to T_{X|S}$ be a family of holomorphic Poisson structure over a smooth morphism $\pi : X \to S$. A Poisson structure $a : (\Omega^1_X, \{\cdot,\cdot\}) \to T_X$ on X is an unfolding of $a_S : (\Omega^1_{X|S}, \{\cdot,\cdot\}_S) \to T_{X|S}$ if $\Omega^1_{X|S}$ is the pre-image by a of the associated symplectic foliation of $(\Omega^1_{X|S}, \{\cdot,\cdot\}_S)$, since $\operatorname{rank}(\Omega^1_X) = \operatorname{rank}(\Omega^1_{X|S}) + \dim(S)$.

We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\Omega^1_S & \xrightarrow{i} & \Omega^1_X & \longrightarrow & \Omega^1_{X|S} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a_S} & & \\ & & \pi^*T_S & \xleftarrow[i^*]{} & T_X & \longleftarrow & T_{X|S} & \longleftarrow & 0, \end{array} \tag{3}$$

where $\rho := i^* \circ a \circ i$. This implies that the map $\rho : \pi^*\Omega^1_S \to \pi^*T_S$ induces a Poisson structure on S by $\pi_*\rho : \Omega^1_S \to T_S$.

If the unfolding is transversal, we have that the isomorphism $[a] : \Omega^1_X/\Omega^1_{X|S} \to \pi^*T_S$ provides a spliting for the sequence

$$0 \longrightarrow \pi^*\Omega^1_S \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X|S} \longrightarrow 0 \tag{4}$$

which implies that $\rho : \pi^*\Omega^1_S \to \pi^*T_S$ is an isomorphism, i.e, $\pi_*\rho : \Omega^1_S \to T_S$ is a symplectic structure on S.

It would be interesting to study how unfoldings of holomorphic Poisson structures behave under Morita equivalence [1].

3.4. **Sheaf of logarithmic forms.** Let X be a smooth projective variety and D an effective normal crossing divisor on X. Denote $\imath : T_X(-\log D) \to T_X$ the inclusion anchor map. A deformation of the pair $(X, D) \to S$ can be interpreted as a family of Lie algebroids $\imath_S : T_{X|S}(-\log D) \to T_{X|S}$, where $T_{X|S}(-\log D) = \imath(T_X(-\log D)) \cap T_{X|S}$ and $\imath_S := \imath|_S$. Since the rank of $T_{X|S}(-\log D)$ is $\dim(X) - \dim(S)$ and

$\imath^{-1}(T_{X|S}(-\log D)) = T_X(-\log D)$, then $\imath : T_X(-\log D) \to T_X$ is an unfolding of $T_{X|S}(-\log D)$.

If the unfolding is transversal, we have the isomorphism

$$[\imath] : T_X(-\log D)/T_{X|S}(-\log D) \xrightarrow{\cong} \pi^* T_S.$$

Now, we have the holomorphic Bott's partial connection on $T_X(-\log D)/T_{X|S}(-\log D)$

$$\nabla : T_X(-\log D)/T_{X|S}(-\log D) \to \Omega^1_{X|S}(\log D) \otimes [T_X(-\log D)/T_{X|S}(-\log D)]$$

by setting

$$\nabla_u(q) = \phi([i_S(u), \tilde{q}]),$$

where $\phi : T_X(-\log D) \to T_X(-\log D)/T_{X|S}(-\log D)$ denotes the projection, $\tilde{q} \in T_X(-\log D)$ such that $\phi(\tilde{q}) = q$ and $u \in T_{X|S}(-\log D)$. Since ∇ is flat along $T_{X|S}(-\log D)$ and the unfolding is transversal we conclude that it induces a holomorphic connection on T_S given by $\tilde{\nabla} := \pi_*(\nabla \circ [\imath])$.

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