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Calculus of Variations

Stability results for the N-dimensional Schiffer conjecture via a perturbation method

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Abstract Given a eigenvalue μ_{0m}^2 of $-\Delta$ in the unit ball B_1, with Neumann boundary conditions, we prove that there exists a class $\mathcal{D}$ of $C^{0,1}$-domains, depending on μ_{0m}, such that if u is a no trivial solution to the following problem $\Delta u + \mu u = 0$ in Ω, $u = 0$ on $\partial\Omega$, and $\int_{\partial\Omega} \partial_{\mathbf{n}} u = 0$, with $\Omega \in \mathcal{D}$, and $\mu = \mu_{0m}^2 + o(1)$, then Ω is a ball. Here μ is a eigenvalue of $-\Delta$ in Ω, with Neumann boundary conditions.

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1 Introduction

The objective of the present paper is to study a overdetermined eigenvalue problem, known in literature as Schiffer conjecture. The latter can be formulated as follows: the only domain Ω such that there exists a no trivial solution φ to the problem

$$\begin{cases} \Delta\varphi + \mu\varphi = 0 & \text{in} \quad \Omega, \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

with

$$\varphi = c \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

is a ball. Here μ and φ are respectively a eigenvalue and a corresponding eigenfunction of $-\Delta$ with Neumann boundary conditions ($\partial_{\mathbf{n}}\varphi$ is the external normal derivative to the boundary $\partial\Omega$), Ω is a sufficiently smooth bounded domain in $\mathbb{R}^N$, with $N \geq 2$, and c is a given constant. By a direct calculation we have that

$$\varphi = I_0(\mu_{0m} r) \quad \text{in} \quad B_1,$$

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solves (1.1), (1.2), when $\Omega = B_1$, and $\mu = \mu_{0m}^2$, for some $m \geq 1$. Here $r = |x|$, $|\cdot|$ denoting the Euclidean norm in $\mathbb{R}^N$, B_1 is the ball of radius 1, centered at zero, and μ_{0m} is the m^{th}-zero of the first derivative of the so-called N-dimensional zero-order Bessel function of first kind I_0, i.e. $I_0'(\mu_{0m}) = 0$, the symbol $'$ denoting the ordinary derivative (see Sect. 2 for more details). Berenstein [1] gives a positive answer to the conjecture by supposing that there exist infinitely many pairs (μ_n, φ_n) satisfying (1.1), (1.2) in $\mathbb{R}^2$. This result has been extended for $N \geq 3$ by Berenstein and Yang [2].

We begin by observing that (see Liu [6]) the following change of variable

$$u = \frac{1}{\mu c}(\varphi - c) \quad \text{in} \quad \Omega, \tag{1.3}$$

implies that φ solves (1.1), (1.2) if and only if u solves

$$\begin{cases} \Delta u + \mu u = -1 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

with

$$\partial_{\mathbf{n}} u = 0 \quad \text{on} \quad \partial\Omega. \tag{1.5}$$

We have that

$$u^{(0)} = \frac{1}{\mu_{0m}^2}\left(\frac{I_0(\mu_{0m} r)}{I_0(\mu_{0m})} - 1\right) \quad \text{in} \quad B_1, \tag{1.6}$$

solves (1.4), (1.5), when $\Omega = B_1$, and $\mu = \mu_{0m}^2$, for some $m \geq 1$. We point out that since problems (1.1), (1.2), and (1.4), (1.5), are invariant up to isometries and up to homotheties of $\mathbb{R}^N$, we have that

$$\varphi = I_0(\mu_{0m}|x - x_0|/1 + R) \quad \text{in } B_{1+R}(x_0),$$

and

$$u^{(0)} = \frac{(1+R)^2}{\mu_{0m}^2}\left(\frac{I_0(\mu_{0m}\,|x - x_0|\,/1 + R)}{I_0(\mu_{0m}/1 + R)} - 1\right) \quad \text{in} \quad B_{1+R}(x_0),$$

solve as well respectively (1.1), (1.2) and (1.4), (1.5), when $\Omega = B_{1+R}(x_0)$, and $\mu = \mu_{0m}^2/(1+R)^2$, where $B_{1+R}(x_0)$ denotes the ball centered at x_0, of radius $1 + R$.

By following [3,4], let us define by E the vector space of $C^{2,\alpha}$-functions defined on the unit sphere ∂B_1, centered at zero, i.e.

$$E = \{k \in C^{2,\alpha}(\partial B_1)\},$$

$\alpha \in (0, 1)$. For $k \in E$, let Ω_k be a domain whose boundary $\partial\Omega_k$ can be written as perturbation of the unit sphere ∂B_1, i.e.

$$\partial\Omega_k = \{x = (1 + k)y,\ y \in \partial B_1\} \tag{1.7}$$

(in particular for $k \equiv 0$ on ∂B_1, $\partial\Omega_0 = \partial B_1$). We begin by proving the following

Theorem 1.1 *Let φ be a no trivial solution to (1.1), (1.2), when $\Omega = \Omega_k$, for some $k \in E$. Then μ is a perturbation of μ_{0m}^2, for some $m \geq 1$, i.e.*

$$\mu = \mu_{0m}^2 + o(1),$$

and φ is a perturbation of $I_0(\mu_{0m}\cdot)$, i.e.

$$\varphi = I_0(\mu_{0m}\cdot) + o(1).$$

Being problem (1.4), (1.5) equivalent, by (1.3), to (1.1), (1.2), in what follows we will study problem (1.4), (1.5).

Let μ be a eigenvalue of $-\Delta$ in Ω_k. Let assume that μ has the form $\mu = \mu_{0m}^2 + o(1)$, for some $m \geq 1$. Let us denote by Φ the operator

$$\Phi : E \to \mathbb{R},$$

defined by

$$\Phi(k) = \int\limits_{\partial\Omega_k} \partial_{\mathbf{n}} u_{\text{par}},$$

where u_{par} is a particular solution to (1.4), when $\Omega = \Omega_k$ ($u_{\text{par}} = u^{(0)}$, when $\Omega = B_1$). Now, since if u solves (1.4), (1.5), when $\Omega = \Omega_k$, it follows that $\Phi(k) = 0$, we will concentrate our attention on studying the *sign* of the operator Φ in a neighborhood of 0 in E. By observing that the sphere of radius $1 + R$, centered at the point $x_0 \in \mathbb{R}^N$, is parameterized by

$$\partial B_{1+R}(x_0) = \{x = (1 + \overline{k}_{R,x_0})y,\ y \in \partial B_1\},$$

where $\overline{k}_{R,x_0}$ is given by

$$\overline{k}_{R,x_0}(y) = x_0 \cdot y - 1 + \sqrt{(1 + R)^2 + |x_0 \cdot y|^2 - |x_0|^2} \tag{1.8}$$

(for R, x_0 such that $(1 + R)^2 + |x_0 \cdot y|^2 - |x_0|^2 \geq 0$ on ∂B_1), we have that Φ vanishes identically on the variety

$$\mathcal{M} = \{k;\ k = \overline{k}_{R,x_0}\}$$

(we observe that $\overline{k}_{R,x_0} \to 0$ in E, as $R, x_0 \to 0$). So the best one can expect is that Φ is different to 0 in $\mathcal{O}\backslash\mathcal{M}$, for some neighborhood $\mathcal{O}$ of 0 in E.

A function $f \in E$ can be written, in Fourier series expansion, as

$$f = f_0 + \sum_{p \geq 1} \sum_{q=1}^{d_p} f_{pq} Y_{pq} \quad \text{on} \quad \partial B_1,$$

where $f_0 = \frac{1}{|\partial B_1|} \int_{\partial B_1} f$, and $f_{pq} = \frac{1}{|\partial B_1|} \int_{\partial B_1} f Y_{pq}$ are respectively the zero-order and the p-order Fourier coefficient of f, and Y_{pq} is the spherical harmonic of degree p. We say that f has the frequency p, if the p-order coefficient of f is different to zero, i.e. $f_{pq} \neq 0$, for some $q \in \{1, \ldots, d_p\}$. On the other hand we say that f doesn't have the frequency p, if the p-order coefficient of f is equal to zero, i.e. $f_{pq} = 0$ for all $q \in \{1, \ldots, d_p\}$.

By studying the behavior of the operator Φ at 0, we prove in a first step that if the eigenvalue μ_{0m}^2 is simple, i.e.

$$L = \{p \in \mathbb{N};\ I'_p(\mu_{0m}) = 0\}$$

is a empty set of positive integers, then Φ is differentiable at 0 in E, and 0 is a critical point of Φ in E (see Theorem 5.2). On the other hand if the eigenvalue μ_{0m}^2 is singular, i.e. L is a no empty (finite) set [whose cardinality depends on the multiplicity of the eigenvalue μ_{0m}^2 (see Sect. 2 for more details)], then Φ is differentiable at 0 in $\bigcup_{p \in L} E_p$, and 0 is a critical point of Φ in $\bigcup_{p \in L} E_p$, where

$$E_p = \{k \in E;\ k_{pq} = 0\} \tag{1.9}$$

is the vector space of functions $k \in E$ which don't have the frequency p. By studying the second derivative of Φ at 0, we can show

Theorem 1.2 *Given a μ_{0m}, for some $m \geq 1$, there exists a neighborhood $\mathcal{O}$ of 0 in E, and two orthogonal spaces V, V' in E, with $\mathcal{O}$, V, V' depending on μ_{0m}, such that Φ is positive in $\mathcal{O}\backslash\{0\} \cap V$, and it is negative in $\mathcal{O}\backslash\{0\} \cap V'$.*

As corollary of Theorem 1.2, we can prove the following

Theorem 1.3 *Given a μ_{0m}, for some $m \geq 1$, there exists a class $\mathcal{D}$ of $C^{2,\alpha}$-domains, depending on μ_{0m}, such that if u is a no trivial solution to (1.4), and*

$$\int\limits_{\partial\Omega} \partial_{\mathbf{n}} u = 0,$$

with $\Omega \in \mathcal{D}$, and $\mu = \mu_{0m}^2 + o(1)$, then $\Omega = B_1$, $\mu = \mu_{0m}^2$, and $u = u^{(0)}$ in B_1.

Since the proof of Theorem 1.3 is short, and it doesn't require particular technical tools, we prove it now.

Proof of Theorem 1.3 Let us denote by $\mathcal{G}$ the class of domains Ω_k, defined by

$$\mathcal{G} = \{\Omega_k; k \in \mathcal{O} \cap (V \cup V')\},$$

with $\mathcal{O}$, V, and V' as in Theorem 1.2. Let Σ be the class of operators ϕ, defined by

$$\sum = \{\phi; \phi = \tau \circ \sigma\},$$

for some homothety τ and isometry σ of $\mathbb{R}^N$. Finally let us denote by $\mathcal{D}$ the class of domains Ω, defined by

$$\mathcal{D} = \{\Omega; \Omega = \phi(\Omega_k)\},$$

for $\Omega_k \in \mathcal{G}$, and $\phi \in \Sigma$. Let assume that u solves (1.4), and $\int_{\partial\Omega} \partial_{\mathbf{n}} u = 0$, with $\Omega \in \mathcal{D}$, and $\mu = \mu_{0m}^2 + o(1)$. Since problem (1.4) is invariant up to isometries and to homotheties, we have that $\int_{\partial\Omega_k} \partial_{\mathbf{n}} u = 0$, for some $k \in \mathcal{O} \cap (V \cup V')$. Now by writing u as

$$u = u_{\text{par}} + u_h \quad \text{in} \quad \Omega_k,$$

where u_h solves the corresponding homogenous problem, and since by Fredholm theorem $-1 \in \ker(\Delta + \mu)^{\perp}$ in Ω_k, by divergence theorem we obtain

$$0 = \int\limits_{\Omega_k} u_h = -\frac{1}{\mu} \int\limits_{\Omega_k} \Delta u_h = -\frac{1}{\mu} \int\limits_{\partial\Omega_k} \partial_{\mathbf{n}} u_h.$$

Then we have

$$0 = \int\limits_{\partial\Omega_k} \partial_{\mathbf{n}} u = \int\limits_{\partial\Omega_k} \partial_{\mathbf{n}} u_{\text{par}},$$

i.e. $\Phi(k) = 0$, which implies that $k = 0$, since, by Theorem 1.2, Φ has a sign in $\mathcal{O}\backslash\{0\} \cap (V \cup V')$. □

Finally by using (1.3), and Theorem 1.3, the following theorem holds true:

Theorem 1.4 *Given a μ_{0m}, for some $m \geq 1$, there exists a class $\mathcal{D}$ of $C^{2,\alpha}$-domains, depending on μ_{0m}, such that if φ is a no trivial solution to (1.1), (1.2), with $\Omega \in \mathcal{D}$, and $\mu = \mu_{0m}^2 + o(1)$, then $\Omega = B_1$, $\mu = \mu_{0m}^2$, and $\varphi = I_0(\mu_{0m} r)$ in B_1.*

We observe that E through the paper is the space of functions of class $C^{2,\alpha}$ on ∂B_1 (this means that we consider only regular perturbations of the unit sphere), but we will prove that the same conclusions hold true in the case where E is the space of functions of class $C^{0,1}$ on ∂B_1, i.e. the boundary $\partial\Omega_k$ is of Lipschitz class.

We recall that in a article appeared in 1976, S. A. Williams [8] proves that Schiffer conjecture is related to the celebrated Pompeiu conjecture. In a series of papers appeared in 1929 the Roumanian mathematician Pompeiu proposes the following problem. We say that a bounded domain Ω has Pompeiu property if and only if the only continuous function f on $\mathbb{R}^N$, $N \geq 2$, such that

$$\int_{\sigma(\Omega)} f = 0, \quad \text{for all} \quad \sigma \in \Sigma,$$

is the function $f \equiv 0$, where Σ denotes the set of isometries of $\mathbb{R}^N$. Pompeiu conjecture says that among bounded domains of $\mathbb{R}^N$, only balls fail to have Pompeiu property. The connection between Schiffer and Pompeiu conjecture asserts that the failure of the Pompeiu property is equivalent to the existence of a no trivial solution to (1.1), (1.2).

A stability result for Pompeiu problem has been proved by F. Segala [7]. By analyzing the asymptotic behavior of the Fourier transform of the characteristic function χ_Ω, Segala proves, for $N = 2$, that if Ω is a domain with Pompeiu property, then all sufficiently small homotheties of Ω have Pompeiu property as well.

It is of interest to recall a application of Pompeiu conjecture. This occurs for example in medical imaging, a technic which consists in determining mass density of a organ in a human body, by measuring the variation of intensity of a X-ray crossing through it. More precisely Pompeiu conjecture says that knowledge of all possible values of variation of intensity of the X-ray (i.e. all isometries $\sigma(\Omega) \in \Sigma$) determines uniquely, up to balls, the mass density of the organ. For further references concerning Pompeiu conjecture, see [9].

The paper is organized as follows: in the next section we give some preliminaries and notations used through the paper. In Sect. 3 we prove Theorem 1.1. In Sects. 4 and 5 we give, via perturbation methods, the first-order approximation, in a neighborhood of 0, respectively of the eigenvalue μ and of the operator Φ. In Sects. 6 and 7 we give the second-order approximation of μ and Φ respectively. In Sect. 8 we prove Theorem 1.2. Finally in Sect. 9 we consider Lipschitz case.

2 Preliminaries and notations

Let us denote by B_1 the unit ball in $\mathbb{R}^N$, centered at zero. By $\overline{B}_1$ we define the Euclidean closure of B_1. Let I_ℓ be the so-called N-dimensional ℓ-order Bessel function of first kind, i.e.

$$I_\ell(r) = r^{-\nu} J_{\nu+\ell}(r), \tag{2.1}$$

where $\nu = \frac{N}{2} - 1$, and $J_{\nu+\ell}$ is the well-known $\nu + \ell$ -order Bessel function of the first kind (we observe that for $N = 2$, I_ℓ coincides with the ℓ-order Bessel function J_ℓ). I_ℓ solves the following Bessel equation

$$I_\ell'' + \frac{N-1}{r} I_\ell' + \left(1 - \frac{\ell(\ell+N-2)}{r^2}\right) I_\ell = 0 \quad \text{in } \mathbb{R}. \tag{2.2}$$

By $\mu_{\ell m}$ we denote the m^{th}-zero of the first derivative of the ℓ-order Bessel function I_ℓ, i.e. $I_\ell'(\mu_{\ell m}) = 0$. We recall in particular that (see Lemma 3.5 in [3])

$$I_0' = -I_1 \quad \text{in} \quad \mathbb{R}.$$

This yields that

$$\mu_{0m} = \lambda_{1m},$$

where λ_{1m} denotes the m^{th}-zero of the one-order Bessel function I_1, i.e. $I_1(\lambda_{1m}) = 0$.

Let $(\mu_n)_{n\geq 1}$ be the sequence, in increasing order, of eigenvalues of $-\Delta$ in B_1 with Neumann boundary conditions. A eigenvalue μ_n, for some $n \in \mathbb{N}$, coincides, for some integer $\ell \geq 0$, and $m \geq 1$, with $\mu_{\ell m}^2$. The corresponding eigenfunctions can be written in polar coordinates (up to a multiplicative constant) as

$$\begin{array}{rcl} \varphi_1 & = & I_\ell(\mu_{\ell m} r) Y_{\ell 1}(\theta), \\ \vdots & \vdots & \vdots \\ \varphi_{d_\ell} & = & I_\ell(\mu_{\ell m} r) Y_{\ell d_\ell}(\theta), \\ \varphi_{p_q} & = & I_p(\mu_{\ell m} r) Y_{pq}(\theta), \end{array}$$

where $p \in L$, and L is a (eventually empty) finite set (by Fredholm theorem) of positive integers such that $I_p'(\mu_{\ell m}) = 0$. We numerate p_q with natural numbers $d_\ell + 1, \ldots, n$. The number of eigenfunctions is called multiplicity of the eigenvalue $\mu_{\ell m}^2$. Here Y_{st} is the spherical harmonic of degree s, with $t = 1, \ldots, d_s$, and

$$d_s = \begin{cases} 1 & \text{if} \quad s = 0, \\ \frac{(2s+N-2)(s+N-3)!}{s!(N-2)!} & \text{if} \quad s \geq 1. \end{cases} \tag{2.3}$$

For example, since $d_1 = N$, the multiplicity of the eigenvalue μ_{1m}^2 is at least equal to N.

Let $\widetilde{k}$ be a $C^{2,\alpha}$-extension of k into $\overline{B}_1$. Let us call A the Jacobian matrix of change of variables

$$x = (1+k)y, \quad y \in \overline{B}_1, \tag{2.4}$$

where we have denoted $\widetilde{k}$ by k. The matrix A is given by

$$A_{ij} = \begin{bmatrix} 1 + k + y_1\partial_1 k & y_1\partial_2 k & \cdots & y_1\partial_n k \\ y_2\partial_1 k & 1 + k + y_2\partial_2 k & \cdots & y_2\partial_n k \\ \vdots & \ddots & \vdots & \vdots \\ y_n\partial_1 k & \cdots & \cdots & 1 + k + y_n\partial_n k \end{bmatrix}.$$

Following [4], the external unit normal vector at the point $x = (1+k)y \in \partial\Omega_k$ is given by

$$\mathbf{n}((1+k)y) = \frac{(A^T)^{-1} y}{\sqrt{G^{-1} y \cdot y}}, \tag{2.5}$$

where G^{-1} is the inverse of the matrix G, and $G = A^T A$. We write the matrix G as

$$G = I_N + G^{(1)} + G^{(2)},$$

where I_N is the N-order identity matrix, and the matrix $G^{(1)}$ and $G^{(2)}$ depend respectively linearly and quadratically on k and ∇k. The matrix $G^{(1)}$ and $G^{(2)}$ (see [4]) are given respectively by

$$G_{ij}^{(1)} = 2kI_N + \begin{bmatrix} 2x_1\partial_1 k & x_1\partial_2 k + x_2\partial_1 k & \cdots & x_1\partial_N k + x_N\partial_1 k \\ x_1\partial_2 k + x_2\partial_1 k & 2x_2\partial_2 k & \cdots & x_2\partial_N k + x_N\partial_2 k \\ \vdots & \ddots & \vdots & \vdots \\ x_1\partial_N k + x_N\partial_1 k & \cdots\cdots & & 2x_N\partial_N k \end{bmatrix}, \tag{2.6}$$

and

$$G_{ij}^{(2)} = k^2 I_N + k \begin{bmatrix} 2x_1\partial_1 k & x_1\partial_2 k + x_2\partial_1 k & \cdots & x_1\partial_N k + x_N\partial_1 k \\ x_1\partial_2 k + x_2\partial_1 k & 2x_2\partial_2 k & \cdots & x_2\partial_N k + x_N\partial_2 k \\ \vdots & \vdots \ddots & & \vdots \\ x_1\partial_N k + x_N\partial_1 k & \cdots\cdots & & 2x_N\partial_N k \end{bmatrix}$$
$$+|x|^2 \begin{bmatrix} (\partial_1 k)^2 & \partial_1 k\partial_2 k & \cdots & \partial_1 k\partial_N k \\ \partial_2 k\partial_1 k & (\partial_2 k)^2 & \cdots & \partial_2 k\partial_N k \\ \vdots & \vdots \ddots & & \vdots \\ \partial_N k\partial_1 k & \cdots\cdots & & (\partial_N k)^2 \end{bmatrix}. \tag{2.7}$$

3 Proof of Theorem 1.1

Let μ be a eigenvalue of $-\Delta$ in Ω_k, and let φ be a corresponding no trivial eigenfunction. We can assume that the eigenvalue μ can be written as

$$\mu = \mu_{\ell m}^2 + o(1),$$

for some $\ell \geq 0$, and $m \geq 1$. By change of variable (2.4), denoting by

$$\widetilde{\varphi}(y) = \varphi((1+k)y) \quad \text{in} \quad \overline{B}_1,$$

and using (2.5), we have that

$$\begin{aligned} \partial_{\mathbf{n}}\varphi((1+k)y) &= (A^T)^{-1}\nabla\widetilde{\varphi}\cdot\mathbf{n}((1+k)y) \\ &= (G^{-1}y\cdot y)^{-1/2}G^{-1}\nabla\widetilde{\varphi}\cdot y \quad \text{on} \quad \partial B_1. \end{aligned}$$

By a direct calculation we obtain that the function $\widetilde{\varphi}$ solves

$$\begin{cases} \operatorname{div}(\sqrt{g}G^{-1}\nabla\widetilde{\varphi}) + \mu\sqrt{g}\widetilde{\varphi} = 0 & \text{in} \quad B_1, \\ G^{-1}\nabla\widetilde{\varphi}\cdot\mathbf{n} = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{3.1}$$

Similarly, let us define by

$$\widetilde{u}(y) = u((1+k)y) \quad \text{in } \overline{B}_1,$$

where u solves (1.4), when $\Omega = \Omega_k$. The function $\widetilde{u}$ solves

$$\begin{cases} \operatorname{div}(\sqrt{g}G^{-1}\nabla\widetilde{u}) + \mu\sqrt{g}\widetilde{u} = -\sqrt{g} & \text{in} \quad B_1, \\ \widetilde{u} = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{3.2}$$

Let us denote $\widetilde{u}$ by u, $\widetilde{\varphi}$ by φ, and y by x. We have that a solution u to (3.2) can be written as

$$u = \frac{1}{\mu_{\ell m}^2}\left(\frac{I_0(\mu_{\ell m}r)}{I_0(\mu_{\ell m})} - 1\right) + u_h^{(0)} + o(1) \quad \text{in} \quad B_1, \tag{3.3}$$

where the function

$$\frac{1}{\mu_{\ell m}^2}\left(\frac{I_0(\mu_{\ell m} r)}{I_0(\mu_{\ell m})}-1\right)$$

is a particular solution to the (unperturbed) problem

$$\begin{cases} \Delta u+\mu_{\ell m}^2 u=-1 & \text{in} \quad B_1, \\ u=0 & \text{on} \quad \partial B_1, \end{cases}$$

and $u_h^{(0)}$ solves the corresponding homogenous unperturbed problem. We observe that $u_h^{(0)}=0$, if the kernel $\ker(\Delta+\mu_{\ell m}^2)=\{0\}$ in B_1 (with Dirichlet boundary conditions). Otherwise, if the kernel $\ker(\Delta+\mu_{\ell m}^2)\neq\{0\}$ in B_1, i.e.

$$\mu_{\ell m}=\lambda_{\ell' m'},$$

for some ℓ, ℓ' (with $\ell \neq \ell'$), then $u_h^{(0)}$ has the form (in polar coordinates)

$$u_h^{(0)}=\sum_{q=1}^{d_{\ell'}} \alpha_{\ell' q} I_{\ell'}(\mu_{\ell m} r) Y_{\ell' q}(\theta)+\sum_{p\in I}\sum_{q=1}^{d_p} \alpha_{pq} I_p(\mu_{\ell m} r) Y_{pq}(\theta),$$

where

$$I=\{p\in\mathbb{N};\, I_p(\mu_{\ell m})=0\}$$

is a (eventually empty) finite set of positive integers, and $\alpha_{\ell' 1},\ldots,\alpha_{\ell' d_{\ell'}},\alpha_{pq}\in\mathbb{R}$. We observe finally that in order that (3.3) makes sense in what follows we will suppose that

$$\mu_{\ell m}\notin\{\lambda_{0n}\}_{n\geq 1}.$$

Proof of Theorem 1.1 We have

$$\mu=\mu_{\ell m}^2+o(1),$$

for some $\ell\geq 0$, and $m\geq 1$. Similarly we have

$$\varphi=\varphi^{(0)}+o(1) \quad \text{in} \quad B_1,$$

where $\varphi^{(0)}$ is a eigenfunction to (1.1), when $\mu=\mu_{\ell m}^2$, and $\Omega=B_1$. By (1.3), we have that u can be written as

$$u=\frac{1}{\mu_{\ell m}^2 c}(\varphi^{(0)}-c)+o(1) \quad \text{in} \quad B_1.$$

By (3.3), the zero-order term of u is given by

$$\frac{1}{\mu_{\ell m}^2}\left(\frac{I_0(\mu_{\ell m}\cdot)}{I_0(\mu_{\ell m})}-1\right)+u_h^{(0)}.$$

Then we obtain

$$\frac{1}{\mu_{\ell m}^2}\left(\frac{I_0(\mu_{\ell m}\cdot)}{I_0(\mu_{\ell m})}-1\right)+u_h^{(0)}=\frac{1}{\mu_{\ell m}^2 c}(\varphi^{(0)}-c),$$

i.e.

$$\varphi^{(0)}=c\frac{I_0(\mu_{\ell m}\cdot)}{I_0(\mu_{\ell m})}+c\mu_{\ell m}^2 u_h^{(0)}.$$

In particular we have

$$\partial_{\mathbf{n}}\varphi^{(0)} = c\mu_{\ell m}\frac{I_0'(\mu_{\ell m})}{I_0(\mu_{\ell m})} + c\mu_{\ell m}^2\partial_{\mathbf{n}}u_h^{(0)} \quad \text{on} \quad \partial B_1. \tag{3.4}$$

Now since $\partial_{\mathbf{n}}\varphi^{(0)} = 0$ on ∂B_1, and $c \neq 0$, by integrating (3.4) over ∂B_1, we obtain $I_0'(\mu_{\ell m}) = 0$, i.e. $\mu_{\ell m} = \mu_{0m}$. So $u_h^{(0)}$ becomes

$$u_h^{(0)} = \sum_{q=1}^{N}\alpha_{1q}I_1(\mu_{0m}r)Y_{1q}(\theta) + \sum_{p\in I}\sum_{q=1}^{d_p}\alpha_{pq}I_p(\mu_{0m}r)Y_{pq}(\theta).$$

Now by multiplying (3.4) by $Y_{ij}(\theta)$, and integrating over ∂B_1, we obtain $\alpha_{ij} = 0$, for $i = 1$, $j = 1, \ldots, N$, and for $i \in I$, $j = 1, \ldots, d_i$. □

4 The first-order approximation of the eigenvalue μ

By writing the matrix $\sqrt{g}G^{-1}$ in (3.1) as

$$\sqrt{g}G^{-1} = I_N + K, \tag{4.1}$$

we have that (3.1) can be written as (we denote $\widetilde{\varphi}$ by φ)

$$\begin{cases} \Delta\varphi + \operatorname{div}(K\nabla\varphi) + \mu\sqrt{g}\varphi = 0 & \text{in} \quad B_1, \\ G^{-1}\nabla\varphi\cdot\mathbf{n} = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{4.2}$$

In particular we obtain

$$\begin{aligned} \sqrt{g}I_N - G &= KG \\ &= (K^{(1)} + K^{(2)})(I_N + G^{(1)} + G^{(2)}) + \cdots, \end{aligned}$$

where $K^{(1)}$ and $K^{(2)}$ denote respectively the one-order and the second-order approximation of the matrix K (the matrix $G^{(1)}$ and $G^{(2)}$ are given respectively by (2.6) and (2.7)). One can verify that

$$\sqrt{g} = (1+k)^N + (1+k)^{N-1}x\cdot\nabla k.$$

This yields that the matrix

$$K^{(1)} = g^{(1)}I_N - G^{(1)}, \tag{4.3}$$

where $g^{(1)}$, the one-order approximation of $\sqrt{g}$, is given by

$$g^{(1)} = Nk + x\cdot\nabla k,$$

and that the matrix

$$K^{(2)} = g^{(2)}I_N - G^{(2)} - K^{(1)}G^{(1)}, \tag{4.4}$$

where $g^{(2)}$, the second-order approximation of $\sqrt{g}$, is given by

$$g^{(2)} = \frac{N(N-1)}{2}k^2 + (N-1)kx\cdot\nabla k.$$

By (4.1), (4.3), and (4.4) we obtain

$$
\begin{aligned}
G^{-1} &= \frac{I_N}{\sqrt{g}} + \frac{1}{\sqrt{g}}(K^{(1)} + K^{(2)}) + \cdots \\
&= I_N - G^{(1)} - g^{(1)2} I_N \\
&\quad + g^{(2)} I_N - G^{(2)} + G^{(1)2} + \cdots .
\end{aligned} \tag{4.5}
$$

Let assume that the eigenvalue μ is a perturbation of μ_{0m}^2, for some $m \geq 1$, i.e. μ has the form

$$\mu = \mu_{0m}^2 + o(1).$$

We say that the eigenvalue μ_{0m}^2 is simple, if

$$L = \{p \in \mathbb{N};\, I'_p(\mu_{0m}) = 0\}$$

is a empty set of positive integers. In this case the eigenspace is generated by the eigenfunction

$$\varphi^{(0)} = I_0(\mu_{0m} r). \tag{4.6}$$

On the other hand we say that the eigenvalue μ_{0m}^2 is singular, if L is a no empty (finite) set of positive integers. In this case the eigenspace is generated by the eigenfunctions

$$
\begin{aligned}
\varphi^{(0)} &= I_0(\mu_{0m} r), \\
\varphi_{p_q}^{(0)} &= I_p(\mu_{0m} r) Y_{pq}(\theta),
\end{aligned}
$$

with $p \in L$. By numerating p_q with natural numbers $2, 3, \ldots, n$, we call multiplicity of μ_{0m}^2 the number n.

Now if μ_{0m}^2 is simple, we can prove that μ can be written as

$$\mu = \mu_{0m}^2 + \mu^{(1)} + o(\|k\|) \quad \text{in} \quad E. \tag{4.7}$$

On the other hand if μ_{0m}^2 is singular, then μ has the same expression as above in $\bigcup_{p \in L} E_p$, where E_p, defined in (1.9), is the space of functions k which don't have the frequency p.

Theorem 4.1 *Let μ_{0m}^2 be simple, then μ can be written as (4.7), where*

$$\mu^{(1)} = -2k_0 \mu_{0m}^2 \quad \text{in} \quad E.$$

If μ_{0m}^2 is singular, the same holds by changing E with the space $\bigcup_{p \in L} E_p$.

Proof of Theorem 4.1 Let us assume that μ can be written as

$$\mu = \mu_{0m}^2 + \mu^{(1)} + o(\|k\|) \quad \text{in} \quad E.$$

Let φ be a corresponding eigenfunction, which, we suppose, can be written as

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + o(\|k\|) \quad \text{in} \quad E,$$

where $\varphi^{(0)} = I_0(\mu_{0m} r)$. By writing the term $\operatorname{div}(K\varphi) + \mu\sqrt{g}\varphi$ in (4.2) as

$$
\begin{aligned}
&\operatorname{div}(K \nabla \varphi) + \sqrt{g}\mu\varphi \\
&= \operatorname{div}((K^{(1)} + K^{(2)})\nabla(\varphi^{(0)} + \varphi^{(1)})) + \mu\varphi + (g^{(1)} + g^{(2)})\mu\varphi + \cdots ,
\end{aligned} \tag{4.8}
$$

one can verify that the one-order terms in (4.8) are

$$\operatorname{div}(K^{(1)} \nabla \varphi^{(0)}) + \mu_{0m}^2 \varphi^{(1)} + \mu^{(1)} \varphi^{(0)} + \mu_{0m}^2 g^{(1)} \varphi^{(0)}.$$

By taking the one-order terms in (3.1), and using (4.5), we obtain that $\varphi^{(1)}$ solves

$$\begin{cases} \Delta\varphi^{(1)} + \mu_{0m}^2\varphi^{(1)} = f^{(1)} & \text{in} \quad B_1, \\ \partial_{\mathbf{n}}\varphi^{(1)} = 0 & \text{on} \quad \partial B_1 \end{cases} \tag{4.9}$$

(since $G^{(1)}\nabla\varphi^{(0)} \cdot \mathbf{n} = \mu_{0m} G^{(1)} x \cdot x I_0'(\mu_{0m}) = 0$ on ∂B_1), where

$$f^{(1)} = -\mu^{(1)}\varphi^{(0)} - \mu_{0m}^2 g^{(1)}\varphi^{(0)} - \operatorname{div}(K^{(1)}\nabla\varphi^{(0)}). \tag{4.10}$$

One can verify that

$$w = x \cdot \nabla\varphi^{(0)} k$$

solves

$$\Delta w + \mu_{0m}^2 w = -\mu_{0m}^2 g^{(1)}\varphi^{(0)} - \operatorname{div}(K^{(1)}\nabla\varphi^{(0)}).$$

So we look for $\varphi^{(1)}$ in the form

$$\varphi^{(1)} = x \cdot \nabla\varphi^{(0)} k + \widetilde{\varphi}_1,$$

where $\widetilde{\varphi}_1$ solves

$$\Delta\widetilde{\varphi}_1 + \mu_{0m}^2\widetilde{\varphi}_1 = -\mu^{(1)}\varphi^{(0)}. \tag{4.11}$$

By writing $\widetilde{\varphi}_1$ as

$$\widetilde{\varphi}_1 = a_0(r) I_0(\mu_{0m} r) + \sum_{p\geq 1}\sum_{q=1}^{d_p} a_{pq} I_p(\mu_{0m} r) Y_{pq}(\theta),$$

where a_0 solves

$$a_0''(r) + q_0(r) a_0'(r) = -\mu^{(1)} \quad \text{in} \quad (0, 1),$$

by a direct calculation we have that

$$a_0'(1) = -\frac{\mu^{(1)}}{I_0^2(\mu_{0m})}\int_0^1 I_0^2(\mu_{0m} r) r^{N-1}.$$

Since the integral (see [4])

$$\int_0^1 I_0^2(\mu_{0m} r) r^{N-1} = \frac{1}{2} I_0^2(\mu_{0m}), \tag{4.12}$$

it follows that

$$a_0'(1) = -\frac{\mu^{(1)}}{2}.$$

Now we have that

$$\partial_{\mathbf{n}}\varphi^{(1)} = 0 \quad \text{on} \quad \partial B_1,$$

if and only if

$$
\begin{aligned}
0 &= -\mu_{0m}^2 I_1'(\mu_{0m})k + \partial_{\mathbf{n}}\widetilde{\varphi}_1 \\
&= -\mu_{0m}^2 I_1'(\mu_{0m})k_0 - \mu_{0m}^2 I_1'(\mu_{0m}) \sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq} Y_{pq}(\theta) \\
&\quad -\frac{\mu^{(1)}}{2} I_0(\mu_{0m}) + \mu_{0m} \sum_{p\geq 1}\sum_{q=1}^{d_p} a_{pq} I_p'(\mu_{0m}) Y_{pq}(\theta).
\end{aligned}
$$

If μ_{0m}^2 is simple, by taking respectively the zero-order and the p-order Fourier coefficient, we obtain

$$\mu^{(1)} = -2k_0\mu_{0m}^2,$$

and

$$a_{pq} = \mu_{0m} k_{pq} I_1'(\mu_{0m})/I_p'(\mu_{0m}).$$

On the other hand if μ_{0m}^2 is singular, it must be $k_{pq} = 0$, for $p \in L$, i.e. $\mu^{(1)}$ has the desired form in the space $\bigcup_{p\in L} E_p$.

Next we prove that

$$\mu - \mu_{0m}^2 - \mu^{(1)} = o(\|k\|) \quad \text{as} \quad k \to 0,$$

and similarly

$$\varphi - \varphi^{(0)} - \varphi^{(1)} = o(\|k\|) \quad \text{as} \quad k \to 0.$$

By defining by

$$\widetilde{\mu} = \mu - \mu_{0m}^2 - \mu^{(1)}, \qquad \text{and} \qquad \widetilde{\varphi} = \varphi - \varphi^{(0)} - \varphi^{(1)},$$

by following [4] one can prove that $\widetilde{\mu} = o(\|k\|)$, and $\|\widetilde{\varphi}\|_{C^{2,\alpha}(\overline{B}_1)} = o(\|k\|)$. □

5 The first-order approximation of the operator Φ

We recall that problem (1.4) cannot have solutions or, if a solution exists, it cannot be unique. This happens all times the kernel

$$\ker(\Delta + \mu) \neq \{0\} \quad \text{in} \quad \Omega,$$

(with Dirichlet boundary conditions). In particular by Fredholm theorem there exists a solution u to (1.4) if and only if $-1 \in \ker(\Delta + \mu)^{\perp}$. In this case u can be written as

$$u = u_{\text{par}} + u_h,$$

where u_{par} is a particular solution to (1.4) such that

$$u_{\text{par}} \in \ker(\Delta + \mu)^{\perp} \quad \text{in} \quad \Omega, \tag{5.1}$$

and u_h solves the corresponding homogenous problem. We observe that u_p is unique and can be written as

$$u_{\text{par}} = \sum_{p\in F^C}\sum_{q=1}^{n_p} \alpha_{pq}\psi_{pq},$$

where $\alpha_{pq} = \frac{\int_\Omega \psi_{pq}}{\mu-\lambda_p}$ is the p-order Fourier coefficient of u, λ_p and ψ_{pq} are respectively the pth-eigenvalue and a corresponding eigenfunction of $-\Delta$ in Ω, and n_p is the dimension of the corresponding eigenspace. F is a finite set of integer (by Fredholm theorem), and F^C is the complementary of F. On the other hand if the kernel $\ker(\Delta + \mu) = \{0\}$ in Ω, then a solution u exists and is unique. Let us denote by Φ the following operator

$$\Phi : E \mapsto \mathbb{R},$$

defined by

$$\Phi(k) := \int_{\partial\Omega_k} \partial_{\mathbf{n}} u_{\text{par}}. \tag{5.2}$$

Here u_{par} is a particular solution to (1.4), verifying (5.1), when $\Omega = \Omega_k$, and μ has the form $\mu = \mu_{0m}^2 + o(1)$. The operator Φ is well-defined, since we suppose that a solution u exists for k lying in some neighborhood $\mathcal{O}$ of 0 in E. By using (2.4), the function $\widetilde{u}$ defined by

$$\widetilde{u}(y) = u((1+k)y) \quad \text{in } \overline{B}_1,$$

solves

$$\begin{cases} \operatorname{div}(\sqrt{g}G^{-1}\nabla\widetilde{u}) + \mu\sqrt{g}\widetilde{u} = -\sqrt{g} & \text{in} \quad B_1, \\ \widetilde{u} = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{5.3}$$

Moreover, since by (2.5) we have that

$$\partial_{\mathbf{n}} u((1+k)y) = (G^{-1}y \cdot y)^{-1/2} G^{-1}\nabla\widetilde{u} \cdot y \quad \text{on } \partial B_1,$$

we obtain

$$\Phi(k) = \int_{\partial B_1} (G^{-1}y \cdot y)^{-1/2} G^{-1}\nabla\widetilde{u}_{\text{par}} \cdot y\sqrt{\widetilde{g}},$$

where $\widetilde{u}_{\text{par}}(y) = u_{\text{par}}((1+k)y)$, and $\sqrt{\widetilde{g}}$ is the surface element of the new variable y. Let us denote $\widetilde{u}_{\text{par}}$ by u_{par}, and y by x. Before proceeding to calculate the first-order derivative of the operator Φ at 0, we need some preliminary lemmas.

Lemma 5.1 *Let $\mu = \mu_{0m}^2 + o(1)$, then*

$$u_{\text{par}} \to u^{(0)} \quad \text{in } E \quad \text{as} \quad k \to 0.$$

Proof of Lemma 5.1 See [4]. □

Theorem 5.2 *Let μ_{0m}^2 be simple. Then Φ is differentiable at* 0 *in E. Moreover* 0 *is a critical point of Φ in E, i.e.*

$$\mathrm{d}\Phi(0) = 0 \quad \text{in} \quad E.$$

If μ_{0m}^2 is singular, the same holds true by changing E with the space $\bigcup_{p\in L} E_p$.

Proof of Theorem 5.2 Let assume that μ_{0m}^2 is simple. Then μ has the form

$$\mu = \mu_{0m}^2 + \mu^{(1)} + o(\|k\|) \quad \text{in} \quad E.$$

Assume that u_{par} can be written as

$$u_{\text{par}} = u^{(0)} + u_{\text{par}}^{(1)} + o(\|k\|) \quad \text{in} \quad E. \tag{5.4}$$

By following step by step (4.8), we have

$$\operatorname{div}(K\nabla u_{\text{par}}) + \sqrt{g}(\mu u_{\text{par}} + 1) = g^{(1)}(\mu_{0m}^2(u^{(0)} + u_{\text{par}}^{(1)}) \\ + \mu^{(1)}(u^{(0)} + u_{\text{par}}^{(1)})) + \operatorname{div}(K^{(1)}\nabla u^{(0)}) + \cdots . \tag{5.5}$$

The one-order terms in (5.5) are

$$g^{(1)}(1 + \mu_{0m}^2 u^{(0)}) + \mu_{0m}^2 u_{\text{par}}^{(1)} + \mu^{(1)} u^{(0)} + \operatorname{div}(K^{(1)}\nabla u^{(0)}).$$

By taking the one-order terms in (5.9), we obtain that $u_{\text{par}}^{(1)}$ solves

$$\begin{cases} \Delta u^{(1)} + \mu_{0m}^2 u^{(1)} = f^{(1)} & \text{in} \quad B_1, \\ u^{(1)} = 0 & \text{on} \quad \partial B_1, \end{cases} \tag{5.6}$$

and $f^{(1)}$ is given by

$$f^{(1)} = -\mu^{(1)} u^{(0)} - g^{(1)}(1 + \mu_{0m}^2 u^{(0)}) - \operatorname{div}(K^{(1)}\nabla u^{(0)}).$$

By Lemma 3.2 in [4], we have that $u_{\text{par}}^{(1)}$ can be written as

$$u_{\text{par}}^{(1)} = -\frac{I_1(\mu_{0m} r)}{\mu_{0m} I_0(\mu_{0m})} rk + v, \tag{5.7}$$

where v is the radial solution to

$$\begin{cases} \Delta v + \mu_{0m}^2 v = -\mu^{(1)} u^{(0)} & \text{in} \quad B_1, \\ v = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{5.8}$$

By (4.5) and (5.4), it follows that

$$\begin{aligned} \Phi(k) &= \int_{\partial B_1} (G^{-1}x \cdot x)^{-1/2} G^{-1} \nabla u^{(0)} \cdot x \sqrt{g} \\ &\quad + \int_{\partial B_1} (G^{-1}x \cdot x)^{-1/2} G^{-1} \nabla u_{\text{par}}^{(1)} \cdot x \sqrt{g} + \cdots \\ &= \int_{\partial B_1} (1 - 2k - 2\partial_{\mathbf{n}} k)^{-1/2} (\partial_{\mathbf{n}} u_{\text{par}}^{(1)} - G^{(1)} \nabla u_{\text{par}}^{(1)} \cdot x) + \cdots . \end{aligned} \tag{5.9}$$

By taking the one-order terms in (5.9), we obtain that the first-order derivative of Φ at 0 is given by

$$\langle \mathrm{d}\Phi(0) \mid k \rangle = \int_{\partial B_1} \partial_{\mathbf{n}} u_{\text{par}}^{(1)}.$$

By writing the radial solution v to (5.8) as

$$v = a_0(r) I_0(\mu_{0m} r),$$

by a direct calculation we have that

$$a_0'(1) = \frac{2k_0}{I_0^2(\mu_{0m})} \int_0^1 \left(\frac{I_0(\mu_{0m} r)}{I_0(\mu_{0m})} - 1 \right) I_0(\mu_{0m} r) r^{N-1}.$$

Since the integral (see [4])

$$\int_0^1 I_0(\mu_{0m} r) r^{N-1} = 0,$$

by (4.12) we obtain

$$\partial_{\mathbf{n}} v = a_0'(1) I_0(\mu_{0m}) = k_0.$$

Finally we have

$$\langle \mathrm{d}\Phi(0) \mid k \rangle = -\frac{I_1'(\mu_{0m})}{I_0(\mu_{0m})} \int_{\partial B_1} k + k_0 \int_{\partial B_1} = 0,$$

where in the last step we use that $\frac{I_1'(\mu_{0m})}{I_0(\mu_{0m})} = 1$. □

In what follows we will assume that the zero-order Fourier coefficient of k is zero, i.e.

$$k_0 = \frac{1}{|\partial B_1|} \int_{\partial B_1} k = 0.$$

6 The second-order approximation of the eigenvalue μ

In this section we calculate the second-order approximation of the eigenvalue μ.

Theorem 6.1 *Let μ_{0m}^2 be simple, then μ can be written as*

$$\mu = \mu_{0m}^2 + \mu^{(2)} + o(\|k\|^2) \quad in\ E,$$

where

$$\mu^{(2)} = \frac{2}{I_0^2(\mu_{0m})} \int_0^1 f_0(r) I_0(\mu_{0m} r) r^{N-1} - \frac{2}{I_0(\mu_{0m})|\partial B_1|} \int_{\partial B_1} G^{(1)} \nabla \varphi^{(1)} \cdot \mathbf{n} \quad in\ E,$$

where f_0 is the zero-order Fourier coefficient of the function

$$\begin{aligned} f = &- \mu_{0m}^2 g^{(1)} \varphi^{(1)} - \operatorname{div}(K^{(1)} \nabla \varphi^{(1)}) \\ &- \mu_{0m}^2 g^{(2)} \varphi^{(0)} - \operatorname{div}(K^{(2)} \nabla \varphi^{(0)}). \end{aligned}$$

If μ_{0m}^2 is singular, the same holds true by changing E with the space $\bigcup_{p \in L \cup L'} E_p$. In particular, for $N = 2$, L' is the following (eventually empty) set of positive integers

$$L' = \{p \in \mathbb{N};\, 2p \in L\}. \tag{6.1}$$

Proof of Theorem 6.1 Let us assume that μ is simple and it can be written as

$$\mu = \mu_{0m}^2 + \mu^{(2)} + o(\|k\|^2) \quad \text{in} \quad E.$$

Let φ be a corresponding eigenfunction, which, we suppose, has the form

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + o(\|k\|^2) \quad \text{in} \quad E.$$

By taking the second-order terms in (3.1), and using (4.5), we have that $\varphi^{(2)}$ solves

$$\begin{cases} \Delta \varphi^{(2)} + \mu_{0m}^2 \varphi^{(2)} = f^{(2)} & \text{in} \quad B_1, \\ \partial_{\mathbf{n}} \varphi^{(2)} - G^{(1)} \nabla \varphi^{(1)} \cdot \mathbf{n} = 0 & \text{on} \quad \partial B_1 \end{cases} \tag{6.2}$$

(since $H^{(2)}\nabla\varphi^{(0)}\cdot\mathbf{n}=0$ on ∂B_1, $H^{(2)}$ being the second-order approximation of the matrix G^{-1}), where $f^{(2)}$ is given by

$$f^{(2)} = -\mu^{(2)}\varphi^{(0)} - \mu_{0m}^2 g^{(1)}\varphi^{(1)} - \operatorname{div}(K^{(1)}\nabla\varphi^{(1)}) \\ -\mu_{0m}^2 g^{(2)}\varphi^{(0)} - \operatorname{div}(K^{(2)}\nabla\varphi^{(0)}). \tag{6.3}$$

We look for $\varphi^{(2)}$ in the form

$$\varphi^{(2)} = w + \widetilde{\varphi}^{(2)},$$

where w solves

$$\Delta w + \mu_{0m}^2 w = f,$$

with f given by

$$f = -\mu_{0m}^2 g^{(1)}\varphi^{(1)} - \operatorname{div}(K^{(1)}\nabla\varphi^{(1)}) \\ -\mu_{0m}^2 g^{(2)}\varphi^{(0)} - \operatorname{div}(K^{(2)}\nabla\varphi^{(0)}), \tag{6.4}$$

and where $\widetilde{\varphi}^{(2)}$ solves

$$\Delta\widetilde{\varphi}^{(2)} + \mu_{0m}^2\widetilde{\varphi}^{(2)} = -\mu^{(2)}\varphi^{(0)}.$$

By following the proof of Theorem 4.1, we obtain

$$\partial_{\mathbf{n}}\widetilde{\varphi}^{(2)} = -\frac{\mu^{(2)}}{2} I_0(\mu_{0m}) + \mu_{0m}\sum_{p\geq 1}\sum_{q=1}^{d_p} a_{pq} I_p'(\mu_{0m}) Y_{pq}(\theta).$$

By passing in polar coordinates, we write w as

$$w = w_0(r) + \sum_{p\geq 1}\sum_{q=1}^{d_p} w_{pq}(r) Y_{pq}(\theta),$$

where w_0 solves

$$w_0''(r) + \frac{N-1}{r} w_0'(r) + \mu_{0m}^2 w_0(r) = f_0(r) \quad \text{in } (0,1),$$

and where f_0 is the zero-order Fourier coefficient of f. By writing w_0 as

$$w_0(r) = b_0(r) I_0(\mu_{0m} r),$$

by a direct calculation we obtain

$$b_0'(1) = \frac{1}{I_0^2(\mu_{0m})}\int_0^1 f_0(r) I_0(\mu_{0m} r) r^{N-1}.$$

Now we have that

$$\partial_{\mathbf{n}}\varphi^{(2)} - G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial B_1,$$

if and only if

$$0 = b_0'(1)I_0(\mu_{0m}) + \sum_{p\geq 1}\sum_{q=1}^{d_p} w_{pq}'(1)Y_{pq}(\theta) \tag{6.5}$$
$$-\frac{\mu^{(2)}}{2}I_0(\mu_{0m}) + \mu_{0m}\sum_{p\geq 1}\sum_{q=1}^{d_p} a_{pq}I_p'(\mu_{0m})Y_{pq}(\theta) - G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n}.$$

By taking the zero-order Fourier coefficient we obtain

$$\mu^{(2)} = \frac{2}{I_0^2(\mu_{0m})}\int_0^1 f_0(r)I_0(\mu_{0m}r)r^{N-1} - \frac{2}{I_0(\mu_{0m})|\partial B_1|}\int_{\partial B_1} G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n}.$$

On the other hand if μ_{0m}^2 is singular, (6.5) holds true if and only if

$$\int_{\partial B_1}(\partial_{\mathbf{n}}w - G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n})Y_{pq} = 0 \quad \text{for} \quad p\in L. \tag{6.6}$$

Before proceeding with the proof of the theorem, we need the following □

Lemma 6.2 *Let μ_{0m}^2 be singular, then for $p\in L$ we have that*

$$\int_{\partial B_1} G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n}Y_{pq} = \mu_{0m}I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s} k_{st}^2\frac{I_s(\mu_{0m})}{I_s'(\mu_{0m})}s(s+N-2)\int_{\partial B_1} Y_{st}^2Y_{pq} \quad in \bigcup_{s\in L} E_s. \tag{6.7}$$

Proof of Lemma 6.2 Since for $y\in\mathbb{R}^N$ we have

$$G^{(1)}y = 2ky + x\cdot y\nabla k + y\cdot\nabla kx,$$

it follows that

$$G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n} = \nabla\varphi^{(1)}\cdot\nabla k \quad \text{on} \quad \partial B_1.$$

By passing in polar coordinates it follows that

$$\nabla\varphi^{(1)}\cdot\nabla k = \partial_{\mathbf{n}}\varphi^{(1)}\partial_{\mathbf{n}}k + \sum_{i=1}^{N-1}G_{ii}^{-1}\partial_{\theta_i}\varphi^{(1)}\partial_{\theta_i}k,$$

where G^{-1} is the inverse matrix of the $N-1$ diagonal matrix G, G being the Euclidean metric tensor induced on the sphere ∂B_1. We obtain

$$\partial_{\theta_i}\varphi^{(1)}\partial_{\theta_i}k = \mu_{0m}I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s}k_{st}^2 I_s(\mu_{0m})/I_s'(\mu_{0m})(\partial_{\theta_i}Y_{st})^2$$
$$+2\mu_{0m}I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t\neq n=1}^{d_s}k_{st}k_{sn}I_s(\mu_{0m})/I_s'(\mu_{0m})\partial_{\theta_i}Y_{st}\partial_{\theta_i}Y_{sn}.$$

By orthogonality of spherical harmonics, we obtain

$$\int_{\partial B_1} G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n}$$
$$= \mu_{0m}I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s}k_{st}^2 I_s(\mu_{0m})/I_s'(\mu_{0m})\int_{\partial B_1} G^{-1}\nabla Y_{st}\cdot\nabla Y_{st}. \tag{6.8}$$

By recalling that spherical harmonics Y_{st} solve

$$\frac{1}{\sqrt{g}}\mathrm{div}(\sqrt{g}G^{-1}\nabla Y_{st}) = -s(s+N-2)Y_{st},$$

where $g=|\det G|$, by multiplying by $Y_{st}Y_{pq}$, and integrating over ∂B_1, we obtain

$$\int_{\partial B_1}\frac{1}{\sqrt{g}}\mathrm{div}(\sqrt{g}G^{-1}\nabla Y_{st})Y_{st}Y_{pq} = -s(s+N-2)\int_{\partial B_1}Y_{st}^2Y_{pq}.$$

Now, by divergence theorem and orthogonality of spherical harmonics (by recalling that $p\in L$, and $s\notin L$), it follows that the surface integral

$$\int_{\partial B_1}\frac{1}{\sqrt{g}}\mathrm{div}(\sqrt{g}G^{-1}\nabla Y_{st})Y_{st}Y_{pq} = \int_K \mathrm{div}(\sqrt{g}G^{-1}\nabla Y_{st})Y_{st}Y_{pq}$$
$$= -\int_{\partial B_1}G^{-1}\nabla Y_{st}\cdot\nabla Y_{st}Y_{pq}.$$

Then we have

$$\int_{\partial B_1}G^{-1}\nabla Y_{st}\cdot\nabla Y_{st}Y_{pq} = s(s+N-2)\int_{\partial B_1}Y_{st}^2Y_{pq},$$

which, by (6.8), yields (6.7). □

Next we conclude the proof of Theorem 6.1. Since $\varphi^{(1)} = x\cdot\nabla\varphi^{(0)}k + \widetilde{\varphi}^{(1)}$, where

$$\widetilde{\varphi}^{(1)} = \mu_{0m}I_1'(\mu_{0m})\sum_{p\geq 1}\sum_{q=1}^{d_p}k_{pq}I_p(\mu_{0m}r)/I_p'(\mu_{0m})Y_{pq}(\theta),$$

and since $x\cdot\nabla\widetilde{\varphi}^{(1)}k$ solves

$$\Delta(x\cdot\nabla\widetilde{\varphi}^{(1)}k)+\mu_{0m}^2 x\cdot\nabla\widetilde{\varphi}^{(1)}k=-\mu_{0m}^2 g^{(1)}\widetilde{\varphi}^{(1)}-\operatorname{div}(K^{(1)}\nabla\widetilde{\varphi}^{(1)}),$$

we can write w as

$$w=x\cdot\nabla\widetilde{\varphi}^{(1)}k+\widetilde{w},$$

where $\widetilde{w}$ solves

$$\begin{aligned}\Delta\widetilde{w}+\mu_{0m}^2\widetilde{w}=&-\mu_{0m}^2 g^{(1)}x\cdot\nabla\varphi^{(0)}k-\operatorname{div}(K^{(1)}\nabla(x\cdot\nabla\varphi^{(0)}k))\\&-\mu_{0m}^2 g^{(2)}\varphi^{(0)}-\operatorname{div}(K^{(2)}\nabla\varphi^{(0)}).\end{aligned}$$

Then we obtain

$$\begin{aligned}\partial_{\mathbf{n}}w&=\partial_{\mathbf{n}}\widetilde{\varphi}^{(1)}k+\partial_r^2\widetilde{\varphi}^{(1)}k+\partial_{\mathbf{n}}\widetilde{\varphi}^{(1)}\partial_{\mathbf{n}}k+\partial_{\mathbf{n}}\widetilde{w}\\&=\mu_{0m}^2 I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s}k_{st}Y_{st}k+\mu_{0m}^3 I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s}k_{st}I_s''(\mu_{0m})/I_s'(\mu_{0m})Y_{st}k\\&\quad+\partial_{\mathbf{n}}\widetilde{\varphi}^{(1)}\partial_{\mathbf{n}}k+\partial_{\mathbf{n}}\widetilde{w}\quad\text{on }\partial B_1.\end{aligned}$$

Since $\partial_{\mathbf{n}}w$ doesn't depend on the extension of k into $\overline{B}_1$, it follows that the term $\partial_{\mathbf{n}}\widetilde{\varphi}^{(1)}\partial_{\mathbf{n}}k$ must simplify with some terms of $\partial_{\mathbf{n}}\widetilde{w}$. For sake of simplicity we continue to define by $\partial_{\mathbf{n}}\widetilde{w}$ the new term $\partial_{\mathbf{n}}\widetilde{w}$. By (2.2) it follows that

$$\begin{aligned}\partial_{\mathbf{n}}w=&-(N-2)\mu_{0m}^2 I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s}k_{st}Y_{st}k\\&-\mu_{0m}^3 I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s}k_{st}I_s(\mu_{0m})/I_s'(\mu_{0m})Y_{st}k\\&+\mu_{0m}I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s}s(s+N-2)k_{st}I_s(\mu_{0m})/I_s'(\mu_{0m})Y_{st}k+\partial_{\mathbf{n}}\widetilde{w}\quad\text{on }\partial B_1.\end{aligned}$$

By orthogonality of spherical harmonics, we obtain

$$\begin{aligned}\int_{\partial B_1}\partial_{\mathbf{n}}wY_{pq}=&-(N-2)\mu_{0m}^2 I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s}k_{st}^2\int_{\partial B_1}Y_{st}^2Y_{pq}\\&-\mu_{0m}^3 I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s}k_{st}^2 I_s(\mu_{0m})/I_s'(\mu_{0m})\int_{\partial B_1}Y_{st}^2Y_{pq}\\&+\mu_{0m}I_1'(\mu_{0m})\sum_{s\notin L}\sum_{t=1}^{d_s}s(s+N-2)k_{st}^2 I_s(\mu_{0m})/I_s'(\mu_{0m})\int_{\partial B_1}Y_{st}^2Y_{pq}+\int_{\partial B_1}\partial_{\mathbf{n}}\widetilde{w}Y_{pq}.\end{aligned}$$

Comparing with (6.7), we obtain

$$\int_{\partial B_1} (\partial_{\mathbf{n}} w - G^{(1)} \nabla \varphi^{(1)} \cdot \mathbf{n}) Y_{pq}$$

$$= -(N-2)\mu_{0m}^2 I_1'(\mu_{0m}) \sum_{s \notin L} \sum_{t=1}^{d_s} k_{st}^2 \int_{\partial B_1} Y_{st}^2 Y_{pq}$$

$$- \mu_{0m}^3 I_1'(\mu_{0m}) \sum_{s \notin L} \sum_{t=1}^{d_s} k_{st}^2 I_s(\mu_{0m})/I_s'(\mu_{0m}) \int_{\partial B_1} Y_{st}^2 Y_{pq} + \int_{\partial B_1} \partial_{\mathbf{n}} \widetilde{w} Y_{pq}.$$

Since Y_{st}^2, for $N = 2$, written in Fourier series expansion, has only even terms with frequency $2s$, it follows that (6.6) holds true for all integers $q \in \{1, \ldots, d_p\}$ if $2s \notin L$. On the other hand if $2s \in L$, then (6.6) holds true for all integers $q \in \{1, \ldots, d_p\}$ such that Y_{pq} is odd, while for q such that Y_{pq} is not odd, it must be $k_{st} = 0$. □

7 The second-order approximation of the operator Φ

In order to calculate the second-order derivative of the operator Φ at 0, we need the second-order approximation of the particular solution u_{par}. Let us write u_{par} in polar coordinates as

$$u_{\text{par}} = u_{\text{par}_0}(r) + \sum_{p \geq 1} \sum_{q=1}^{d_p} u_{\text{par}_{pq}}(r) Y_{pq}(\theta),$$

where, as usual, $u_{\text{par}_0}(r) = \frac{1}{|\partial B_1|} \int_{\partial B_1} u_{\text{par}}(r, \theta)$ and $u_{\text{par}_{pq}} = \frac{1}{|\partial B_1|} \int_{\partial B_1} u_{\text{par}} Y_{pq}$ are respectively the zero-order and the p-order Fourier coefficient of u_{par}. Let us define by

$$v_{\text{par}} = \sum_{p \geq 1} \sum_{q=1}^{d_p} u_{\text{par}_{pq}}(r) Y_{pq}(\theta)$$

the non-radial part of u_{par}.

Theorem 7.1 *Let μ_{0m}^2 be simple. Then the operator Φ is two-times differentiable at 0 in E. Moreover we have*

$$\langle \mathrm{d}^2 \Phi(0) k \mid k \rangle = \mu_{0m} \sum_{p \geq 1} \sum_{q=1}^{d_p} k_{pq}^2 I_p(\mu_{0m})/I_p'(\mu_{0m}) \quad \textit{in } E. \tag{7.1}$$

If μ_{0m}^2 is singular, then

$$\langle \mathrm{d}^2 \Phi(0) k \mid k \rangle = \mu_{0m} \sum_{p \notin L} \sum_{q=1}^{d_p} k_{pq}^2 I_p(\mu_{0m})/I_p'(\mu_{0m}) \quad \textit{in } \bigcup_{p \in L \cup L'} E_p.$$

Proof of Theorem 7.1 Let μ_{0m}^2 be simple. Then μ has the form

$$\mu = \mu_{0m}^2 + \mu^{(2)} + o(\|k\|^2) \quad \text{in } E.$$

Let assume that u_{par_0} can be written as

$$u_{\mathrm{par}_0} = u^{(0)} + u^{(1)}_{\mathrm{par}_0} + u^{(2)}_{\mathrm{par}_0} + o(\|k\|^2) \quad \text{in } E.$$

By taking the second-order terms of the zero-order coefficient in (5.3), we obtain that $u^{(2)}_{\mathrm{par}_0}$ is the radial solution to

$$\begin{cases} \Delta u^{(2)} + \mu_{0m}^2 u^{(2)} = f_0^{(2)} & \text{in} \quad B_1, \\ u^{(2)} = 0 & \text{on} \quad \partial B_1, \end{cases} \tag{7.2}$$

where $f_0^{(2)}$ is the zero-order Fourier coefficient of the function

$$\begin{aligned} f^{(2)} = &-\mu^{(2)} u^{(0)} - \mu_{0m}^2 g^{(1)} u^{(1)}_{\mathrm{par}} - \mathrm{div}(K^{(1)} \nabla u^{(1)}_{\mathrm{par}}) \\ &- g^{(2)} - \mu_{0m}^2 g^{(2)} u^{(0)} - \mathrm{div}(K^{(2)} \nabla u^{(0)}). \end{aligned}$$

Now we prove that Φ is two-times differentiable at 0 in E. We have

$$\begin{aligned} \Phi(k) &= \int_{\partial B_1} (G^{-1} x \cdot x)^{-1/2} G^{-1} (\nabla u_{\mathrm{par}_0} + \nabla v_{\mathrm{par}}) \cdot x \sqrt{\tilde{g}} + \cdots \\ &= \int_{\partial B_1} (1 + k + \partial_{\mathbf{n}} k)(\partial_{\mathbf{n}} u_{\mathrm{par}_0} + \partial_{\mathbf{n}} v_{\mathrm{par}} - G^{(1)} (\nabla u_{\mathrm{par}_0} + \nabla v_{\mathrm{par}}) \cdot x) \sqrt{\tilde{g}} + \cdots, \end{aligned} \tag{7.3}$$

where in the last step we use that the surface element $\sqrt{\tilde{g}}$ is given by

$$\sqrt{\tilde{g}} = 1 + (N-1)k + o(\|k\|) \quad \text{on } \partial B_1.$$

By taking the second-order terms in (7.3), we obtain that the second-order derivative of Φ at 0 is given by

$$\begin{aligned} \langle \mathrm{d}^2 \Phi(0) k \mid k \rangle = &\int_{\partial B_1} \partial_{\mathbf{n}} u^{(2)}_{\mathrm{par}_0} + \int_{\partial B_1} (k + \partial_{\mathbf{n}} k) \partial_{\mathbf{n}} u^{(1)}_{\mathrm{par}} \\ &+ (N-1) \int_{\partial B_1} k \partial_{\mathbf{n}} u^{(1)}_{\mathrm{par}} - \int_{\partial B_1} G^{(1)} \nabla u^{(1)}_{\mathrm{par}} \cdot x. \end{aligned} \tag{7.4}$$

Since

$$G^{(1)} \nabla u^{(1)}_{\mathrm{par}} \cdot x = 2(k + \partial_{\mathbf{n}} k) \partial_{\mathbf{n}} u^{(1)}_{\mathrm{par}} \quad \text{on } \partial B_1,$$

substituting in (7.4), it follows that

$$\langle \mathrm{d}^2 \Phi(0) k \mid k \rangle = \int_{\partial B_1} \partial_{\mathbf{n}} u^{(2)}_{\mathrm{par}_0} - (N-2) \int_{\partial B_1} k^2 + \int_{\partial B_1} \partial_{\mathbf{n}} k, \tag{7.5}$$

(we use that $\partial_{\mathbf{n}} u^{(1)}_{\mathrm{par}} = -k$ on ∂B_1). By writing the function $u^{(2)}_{\mathrm{par}_0}$ as

$$u^{(2)}_{\mathrm{par}_0} = a_0(r) I_0(\mu_{0m} r),$$

by a direct calculation we have that

$$a_0'(1) = \frac{1}{I_0^2(\mu_{0m})}\int_0^1 f_0^{(2)} I_0(\mu_{0m}r)r^{N-1}$$
$$= -\frac{\mu^{(2)}}{2I_0(\mu_{0m})\mu_{0m}^2} + \frac{1}{I_0^2(\mu_{0m})}\int_0^1 g_0(r)I_0(\mu_{0m}r)r^{N-1},$$

where g_0 is the zero-order Fourier coefficient of

$$g = -\mu_{0m}^2 g^{(1)} u_{\text{par}}^{(1)} - \operatorname{div}(K^{(1)}\nabla u_{\text{par}}^{(1)})$$
$$-\mu_{0m}^2 g^{(2)} u^{(0)} - g^{(2)} - \operatorname{div}(K^{(2)}\nabla u^{(0)}).$$

By recalling that

$$\mu^{(2)} = \frac{2}{I_0^2(\mu_{0m})}\int_0^1 f_0(r)I_0(\mu_{0m}r)r^{N-1} - \frac{2}{I_0(\mu_{0m})|\partial B_1|}\int_{\partial B_1} G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n},$$

where f_0 is zero-order Fourier coefficient of

$$f = -\mu_{0m}^2 g^{(1)}\varphi^{(1)} - \operatorname{div}(K^{(1)}\nabla\varphi^{(1)})$$
$$-\mu_{0m}^2 g^{(2)}\varphi^{(0)} - \operatorname{div}(K^{(2)}\nabla\varphi^{(0)}),$$

we obtain

$$a_0'(1) = -\frac{1}{I_0^3(\mu_{0m})\mu_{0m}^2}\int_0^1 f_0(r)I_0(\mu_{0m}r)r^{N-1} + \frac{1}{I_0^2(\mu_{0m})}\int_0^1 g_0(r)I_0(\mu_{0m}r)r^{N-1}$$
$$+\frac{1}{I_0^2(\mu_{0m})\mu_{0m}^2|\partial B_1|}\int_{\partial B_1} G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n}. \tag{7.6}$$

We have that

$$-f + g = \mu_{0m}^2 g^{(1)}\varphi^{(1)} + \operatorname{div}(K^{(1)}\nabla\varphi^{(1)})$$
$$+\mu_{0m}^2 g^{(2)}\varphi^{(0)} + \operatorname{div}(K^{(2)}\nabla\varphi^{(0)})$$
$$-\mu_{0m}^2 g^{(1)} u_{\text{par}}^{(1)} - \operatorname{div}(K^{(1)}\nabla u_{\text{par}}^{(1)})$$
$$-\mu_{0m}^2 g^{(2)} u^{(0)} - g^{(2)} - \operatorname{div}(K^{(2)}\nabla u^{(0)}). \tag{7.7}$$

By writing $u^{(0)}$, $\nabla u^{(0)}$, $\varphi^{(1)}$ respectively as

$$u^{(0)} = \frac{1}{\mu_{0m}^2}\left(\frac{\varphi^{(0)}}{I_0(\mu_{0m})} - 1\right),$$
$$\nabla u^{(0)} = \frac{1}{\mu_{0m}^2 I_0(\mu_{0m})}\nabla\varphi^{(0)},$$
$$\varphi^{(1)} = \mu_{0m}^2 I_0(\mu_{0m})u_{\text{par}}^{(1)} + \widetilde{\varphi}^{(1)},$$

and by substituting in (7.7), we obtain

$$-\frac{1}{I_0^3(\mu_{0m})\mu_{0m}^2} f + \frac{1}{I_0^2(\mu_{0m})} g = \frac{1}{I_0^3(\mu_{0m})}(g^{(1)}\widetilde{\varphi}^{(1)} + \frac{1}{\mu_{0m}^2}\operatorname{div}(K^{(1)}\nabla\widetilde{\varphi}^{(1)})).$$

Then (7.6) becomes

$$a_0'(1) = \frac{1}{I_0^3(\mu_{0m})} \int_0^1 (g^{(1)}\widetilde{\varphi}^{(1)} + \frac{1}{\mu_{0m}^2} \mathrm{div}(K^{(1)}\nabla\widetilde{\varphi}^{(1)}))_0 I_0(\mu_{0m}r)r^{N-1}$$
$$+ \frac{1}{I_0^2(\mu_{0m})\mu_{0m}^2|\partial B_1|} \int_{\partial B_1} G^{(1)}\nabla\varphi^{(1)} \cdot \mathbf{n},$$

where $(g^{(1)}\widetilde{\varphi}^{(1)} + \frac{1}{\mu_{0m}^2}\mathrm{div}(K^{(1)}\nabla\widetilde{\varphi}^{(1)}))_0$ is the zero-order Fourier coefficient of $g^{(1)}\widetilde{\varphi}^{(1)} + \frac{1}{\mu_{0m}^2}\mathrm{div}(K^{(1)}\nabla\widetilde{\varphi}^{(1)})$.

Next we compute the integral

$$\frac{1}{I_0^3(\mu_{0m})} \int_0^1 (g^{(1)}\widetilde{\varphi}^{(1)} + \frac{1}{\mu_{0m}^2} \mathrm{div}(K^{(1)}\nabla\widetilde{\varphi}^{(1)}))_0 I_0(\mu_{0m}r)r^{N-1}. \tag{7.8}$$

Let us consider the problem

$$\begin{cases} \Delta w + \mu_{0m}^2 w = -\mu_{0m}^2 g^{(1)}\widetilde{\varphi}^{(1)} - \mathrm{div}(K^{(1)}\nabla\widetilde{\varphi}^{(1)}) & \text{in} \quad B_1, \\ w = 0 & \text{on} \quad \partial B_1. \end{cases} \tag{7.9}$$

By writing w_0, the radial part of w, as

$$w_0 = b_0(r)I_0(\mu_{0m}r),$$

by a direct computation we obtain

$$\partial_{\mathbf{n}} w_0 = -\frac{\mu_{0m}^2}{I_0(\mu_{0m})} \int_0^1 (g^{(1)}\widetilde{\varphi}^{(1)})_0 I_0(\mu_{0m}r)r^{N-1} \tag{7.10}$$
$$-\frac{1}{I_0(\mu_{0m})} \int_0^1 (\mathrm{div}(K^{(1)}\nabla\widetilde{\varphi}^{(1)}))_0 I_0(\mu_{0m}r)r^{N-1}.$$

Comparing (7.8) with (7.10), we obtain

$$\frac{1}{I_0^3(\mu_{0m})} \int_0^1 (g^{(1)}\widetilde{\varphi}^{(1)} + \frac{1}{\mu_{0m}^2} \mathrm{div}(K^{(1)}\nabla\widetilde{\varphi}^{(1)}))_0 I_0(\mu_{0m}r)r^{N-1} dr$$
$$= -\frac{1}{\mu_{0m}^2 I_0^2(\mu_{0m})} \partial_{\mathbf{n}} w_0.$$

On the other hand, since a particular solution to (7.9) can be written as

$$w = x \cdot \nabla\widetilde{\varphi}^{(1)} k + \widetilde{w},$$

where $\widetilde{w}$ solves

$$\begin{cases} \Delta\widetilde{w} + \mu_{0m}^2\widetilde{w} = 0 & \text{in} \quad B_1, \\ \widetilde{w} = -\mu_{0m}^2 I_1'(\mu_{0m})k^2 & \text{on} \quad \partial B_1 \end{cases}$$

(since $x \cdot \nabla\widetilde{\varphi}^{(1)} k = \mu_{0m}^2 I_1'(\mu_{0m})k^2$ on ∂B_1), we obtain that w_0 has the form

$$w_0 = \frac{r}{|\partial B_1|} \int_{\partial B_1} \partial_r \widetilde{\varphi}^{(1)} k - \mu_{0m}^2 I_1'(\mu_{0m}) I_0(\mu_{0m}r) \frac{1}{|\partial B_1|} \int_{\partial B_1} k^2.$$

We have that

$$\partial_{\mathbf{n}} w_0 = \frac{1}{|\partial B_1|}\int\limits_{\partial B_1} \partial_r \widetilde{\varphi}^{(1)}(1,\theta)k(1,\theta) + \frac{1}{|\partial B_1|}\int\limits_{\partial B_1} \partial_{rr} \widetilde{\varphi}^{(1)}(1,\theta)k(1,\theta)$$
$$+\frac{1}{|\partial B_1|}\int\limits_{\partial B_1} \partial_r \widetilde{\varphi}^{(1)}(1,\theta)\partial_r k(1,\theta).$$

Since

$$\partial_r \widetilde{\varphi}^{(1)}(1,\theta) = \mu_{0m}^2 I_1'(\mu_{0m})k,$$

and

$$\partial_{rr} \widetilde{\varphi}^{(1)}(1,\theta) = \mu_{0m}^3 I_1'(\mu_{0m}) \sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq} I_p''(\mu_{0m})/I_p'(\mu_{0m}) Y_{pq}(\theta),$$

we obtain

$$\partial_{\mathbf{n}} w_0 = \mu_{0m}^2 I_1'(\mu_{0m})\frac{1}{|\partial B_1|}\int\limits_{\partial B_1} k^2 + \mu_{0m}^3 I_1'(\mu_{0m}) \sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq}^2 I_p''(\mu_{0m})/I_p'(\mu_{0m})$$
$$+\mu_{0m}^2 \frac{I_1'(\mu_{0m})}{|\partial B_1|}\int\limits_{\partial B_1} k\partial_{\mathbf{n}} k.$$

Then, by (2.2), it follows that

$$\partial_{\mathbf{n}} w_0 = -(N-2)\mu_{0m}^2 I_1'(\mu_{0m})\frac{1}{|\partial B_1|}\int\limits_{\partial B_1} k^2$$
$$-\mu_{0m}^3 I_1'(\mu_{0m}) \sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq}^2 I_p(\mu_{0m})/I_p'(\mu_{0m})$$
$$+\mu_{0m} I_1'(\mu_{0m}) \sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq}^2 p(p+N-2) I_p(\mu_{0m})/I_p'(\mu_{0m})$$
$$+\mu_{0m}^2 \frac{I_1'(\mu_{0m})}{|\partial B_1|}\int\limits_{\partial B_1} k\partial_{\mathbf{n}} k.$$

Finally we have

$$\partial_{\mathbf{n}} u_{\mathrm{par}_0}^{(2)} = \frac{N-2}{|\partial B_1|}\int\limits_{\partial B_1} k^2 + \mu_{0m} \sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq}^2 I_p(\mu_{0m})/I_p'(\mu_{0m})$$
$$-\frac{1}{|\partial B_1|}\int\limits_{\partial B_1} k\partial_{\mathbf{n}} k - \frac{1}{\mu_{0m}} \sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq}^2 p(p+N-2) I_p(\mu_{0m})/I_p'(\mu_{0m})$$
$$+\frac{1}{I_0(\mu_{0m})\mu_{0m}^2}\frac{1}{|\partial B_1|}\int\limits_{\partial B_1} G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n}.$$

Since by (6.7), for $Y_{1q} = 1$, we have

$$\frac{1}{|\partial B_1|}\int\limits_{\partial B_1} G^{(1)}\nabla\varphi^{(1)}\cdot\mathbf{n} = \mu_{0m}I_1'(\mu_{0m})\sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq}^2\, p(p+N-2)I_p(\mu_{0m})/I_p'(\mu_{0m}),$$

it follows that

$$\partial_{\mathbf{n}}u^{(2)}_{\text{par}_0} = \frac{N-2}{|\partial B_1|}\int\limits_{\partial B_1} k^2 + \frac{\mu_{0m}}{|\partial B_1|}\sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq}^2 I_p(\mu_{0m})/I_p'(\mu_{0m})$$
$$-\frac{1}{|\partial B_1|}\int\limits_{\partial B_1} k\partial_{\mathbf{n}}k.$$

Finally we have

$$\langle \mathrm{d}^2\Phi(0)k \mid k\rangle = \int\limits_{\partial B_1}\partial_{\mathbf{n}}u^{(2)}_{\text{par}_0} - (N-2)\int\limits_{\partial B_1} k^2 + \int\limits_{\partial B_1}\partial_{\mathbf{n}}kk$$
$$= \mu_{0m}\sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq}^2 I_p(\mu_{0m})/I_p'(\mu_{0m}). \tag{7.11}$$

Let us suppose now that $k_0 \neq 0$. Then we have

$$\langle \mathrm{d}^2\Phi(0)k \mid k\rangle = \alpha k_0^2 + \mu_{0m}\sum_{p\geq 1}\sum_{q=1}^{d_p} k_{pq}^2 I_p(\mu_{0m})/I_p'(\mu_{0m}),$$

for some constant α. Now since $\Phi(k_0) = 0$, it follows that $\langle \mathrm{d}^2\Phi(0)k_0 \mid k_0\rangle = 0$, and then $\alpha = 0$. □

8 Proof of Theorem 1.2

We begin our analysis by assuming that μ_{0m}^2 is simple. Two cases can happen: either μ_{0m}^2, as eigenvalue with Dirichlet boundary conditions, has multiplicity equal to N, i.e. the set

$$I = \{p \geq 2;\ I_p(\mu_{0m}) = 0\} \tag{8.1}$$

is a empty set of positive integers, or μ_{0m}^2 has multiplicity bigger than N, i.e. I is a no empty (finite) set of positive integers. If μ_{0m}^2 has multiplicity equal to N, then (7.11) is equal to zero for $k \in < 1, Y_{11}, \ldots, Y_{1N} >$ (the symbol $<, f_1, \ldots, f_N >$ denoting the vector space generated by the vectors $f_1, \ldots, f_N$), i.e. for k having the form

$$k = k_0 + \sum_{q=1}^{N} k_{1q}Y_{1q}.$$

We observe that the vector space $< 1, Y_{11}, \ldots, Y_{1N} >$ coincides with the tangent space to the variety

$$\mathcal{M} = \{k;\ k = \overline{k}_{R,x_0}\},$$

at 0, where $\overline{k}_{R,x_0}$, defined in (1.8), parametrizes the sphere $\partial B_{1+R}(x_0)$ of radius $1+R$, centered at x_0. So the best that one can expect is that Φ has a sign in the space

$$H = \bigcup_{p\in\{0,1\}} E_p, \tag{8.2}$$

of functions k which don't have neither the frequency zero, nor the frequency 1. We observe that H is orthogonal to the space $< 1, Y_{11}, \ldots, Y_{1N} >$. In what follows we prove the following

Lemma 8.1 *There exists a neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^N$ such that the function $\overline{k}_{R,x_0}$ has the frequency 1 for $x_0 \in \mathcal{O}$.*

Proof of Lemma 8.1 Let x_0 be such that $x_{0q} \neq 0$, for some $q \in \{1, \ldots, N\}$. We have that

$$\begin{aligned}\frac{1}{|\partial B_1|}\int_{\partial B_1} \overline{k}_{R,x_0} Y_{1q} &= \sum_{n=1}^{N} x_{0n} \frac{1}{|\partial B_1|}\int_{\partial B_1} Y_{1n}Y_{1q} + \frac{1}{|\partial B_1|}\int_{\partial B_1} hY_{1q} \\ &= x_{0q} + \frac{1}{|\partial B_1|}\int_{\partial B_1} hY_{1q}.\end{aligned}$$

Since the function

$$h(x_0, y) = \sqrt{(1+R)^2 + |x_0 \cdot y|^2 - |x_0|^2}$$

is even on ∂B_1, it follows that $\int_{\partial B_1} hY_{1q} = 0$ for all such that Y_{1q} is odd. Let $q \in \{1, \ldots, N\}$ be such that $\int_{\partial B_1} hY_{1q} \neq 0$. Since

$$h(x_0, y) = 1 + R + o(|x_0|), \quad \text{as} \quad x_0 \to 0,$$

the thesis follows. □

Now if μ_{0m}^2 has multiplicity bigger than N, as eigenvalue with Dirichlet boundary conditions (i.e. I is a no empty set), then (7.11) is equal to zero for $k \in < 1, Y_{11}, \ldots, Y_{1N}, Y_{p1}, \ldots Y_{pd_p} >$, i.e. for k having the form

$$k = k_0 + \sum_{q=1}^{N} k_{1q}Y_{1q} + \sum_{p\in I}\sum_{q=1}^{d_p} k_{pq}Y_{pq}.$$

Finally, if μ_{0m}^2 is singular, the same conclusions hold true, by changing E with the space $\bigcup_{p\in L\cup L'} E_p$.

Before proceeding with the proof of Theorem 1.2, we need some preliminary lemmas. We begin by studying the sign of the term $I_p(\mu_{0m})/I_p'(\mu_{0m})$ in (7.11). We can prove the following

Lemma 8.2 *There exists a positive integer p_0, depending on μ_{0m}, such that, for all $p \geq p_0$,*

$$I_p(\mu_{0m})/I_p'(\mu_{0m}) > 0. \tag{8.3}$$

Proof of Lemma 8.2 Since the $\lim_{p\to+\infty} \mu_{p1} = +\infty$, we have that there exists a p_0 such that $\mu_{p1} \geq \mu_{0m}$, for all $p \geq p_0$. Now since the function $I_p/I_p'$is positive on the interval $(0, \mu_{p1})$, (8.3) follows. □

Lemma 8.3 *There exists a neighborhood $\mathcal{O}$ of the origin in E, such that if $k \in \mathcal{O} \cap E_1^C$, then the mass center $\overline{x}$ of Ω_k is different to zero.*

Here

$$E_1^C = \{k \in E; k_{1q} \neq 0 \quad \text{for some} \quad q = 1, \dots, N\},$$

the complementary of E_1, is the set of functions k which have the frequency 1. We recall that the mass center of a domain Ω is the point $\overline{x}$ of coordinates

$$\overline{x}_i = \frac{1}{|\Omega|} \int_{\Omega} x_i, \quad i = 1, \dots, N.$$

This lemma implies that if the mass center of Ω_k, for $k \in \mathcal{O}$, is at the point zero, then k doesn't have the frequency 1 , i.e. $k \in E_1$. In particular we have that a domain Ω_k, with $k \in \mathcal{O} \cap E_1$ is either a domain with mass center at 0, or $\Omega_k = \tau(\Omega_{\widetilde{k}})$, for some translation τ of $\mathbb{R}^N$, and some domain $\Omega_{\widetilde{k}}$, where $\Omega_{\widetilde{k}}$ has mass center at zero.

Proof of Lemma 8.3 See [4]. □

Lemma 8.4 *There exists a neighborhood $\mathcal{U}$ of the origin in E, with $\mathcal{U}$ contained in $\mathcal{O}$, such that given a domain Ω_k, with $k \in \mathcal{U}$, one can find a $\widetilde{k} \in \mathcal{O} \cap H$ such that*

$$\tau \circ \sigma(\Omega_{\widetilde{k}}) = \Omega_k,$$

for some translation τ, and some homothety σ of $\mathbb{R}^N$.

As consequence of this lemma, since the operator Φ is invariant up to isometries and up to homotheties, we obtain that Φ has a sign in $\mathcal{U}$, if it has a sign in $\mathcal{O} \cap H$.

Proof of Lemma 8.4 Let us consider the set

$$F = \{k \in \mathcal{O}; \overline{x} = 0\},$$

where the point $\overline{x}$ is the mass center of the domain Ω_k. Let $\mathcal{U}$ be a neighborhood of 0 in E, $\mathcal{U}$ contained in $\mathcal{O}$. If $k \in \mathcal{U} \cap H$, it is right. On the other hand if $k \notin H$, then either

$$k \in E_1,$$

or

$$k \notin E_1.$$

If $k \in E_1$, then $k_0 \neq 0$, then $\widetilde{k} = k - k_0$ lies in $\mathcal{U} \cap H$, and $\sigma(\Omega_{\widetilde{k}}) = \Omega_k$, for some homothety σ of $\mathbb{R}^N$. Now if $k \notin E_1$, let $\overline{x}$ be the mass center of Ω_k (we have that $\overline{x} \neq 0$, otherwise $k \in F$, and then $k \in E_1$). We have that k can be written as (see [4])

$$k(y) = k'((1 + \overline{k}_{1,\overline{x}})y - \overline{x}) + \overline{k}_{1,\overline{x}}(y)(1 + k'((1 + \overline{k}_{1,\overline{x}})y - \overline{x})),$$

with k' such that $\Omega_{k'}$ has mass center at 0. Then

$$\|k'\| \leq \|k' - k\| + \|k\|$$

Now since

$$k(y) - k'((1 + \overline{k}_{1,\overline{x}})y - \overline{x}) \to 0, \quad \text{as} \quad \overline{x} \to 0,$$

we obtain that $k' \in F$, and the result follows. □

Now we can prove Theorem 1.2.

Proof of Theorem 1.2 *Case (i):* μ_{0m}^2 is *simple*. Let assume that μ_{0m}^2 has multiplicity equal to N, as eigenvalue with Dirichlet boundary conditions.

Step 1. Let V be the space

$$V = \{k \in H; k_{pq} = 0, p \in K\},$$

where

$$K = \{p \in \mathbb{N}; I_p(\mu_{0m})/I'_p(\mu_{0m}) < 0\}$$

is a (eventually empty) finite set of positive integers (by Lemma 8.2). Let V' be the space

$$V' = \{k; k \in < k_{p1}, \dots, k_{pd_p} > p \in K\}.$$

We observe that V' is orthogonal to V, and

$$H = V \oplus V'.$$

Step 2. First we study the sign of (7.11) in V'. Let us denote by

$$M = \max_{p \in K} I_p(\mu_{0m})/I'_p(\mu_{0m}).$$

We have that $M < 0$. We obtain

$$\begin{aligned} \langle \mathrm{d}^2\Phi(0)k \mid k\rangle &= \mu_{0m} \sum_{p\in K} \sum_{q=1}^{d_p} k_{pq}^2 I_p(\mu_{0m})/I'_p(\mu_{0m}) \\ &\le M\mu_{0m}, \end{aligned}$$

for all $k \in V'$, with $\|k\|_{V'} = 1$. So there exists a neighborhood $\mathcal{O}$ of the origin in E such that Φ is negative in $\mathcal{O}\backslash\{0\} \cap V'$.

Step 3. Let us study the sign of (7.11) in V. Since

$$\frac{I_p(r)}{I'_p(r)} = \frac{r}{p\left(1 - r\frac{I_{p+1}(r)}{I_p(r)}\right)},$$

for $I'_p(r) \neq 0$ (see [5, pp. 486]), and since

$$\frac{I_{p+1}(r)}{I_p(r)} \sim \frac{r}{2p} \quad \text{as } p \to +\infty$$

(see [3, pp. 23]), we obtain

$$\frac{1}{(1 - \mu_{0m} I_{p+1}(\mu_{0m})/I_p(\mu_{0m}))} \ge 1.$$

Then the general term in series (7.11) becomes

$$\begin{aligned} k_{pq}^2 I_p(\mu_{0m})/I'_p(\mu_{0m}) &= \frac{k_{pq}^2}{p} \frac{\mu_{0m}}{(1 - \mu_{0m} I_{p+1}(\mu_{0m})/I_p(\mu_{0m}))} \\ &\ge \frac{k_{pq}^2}{p}\mu_{0m}, \end{aligned}$$

which yields that

$$\langle d^2\Phi(0)k \mid k\rangle = \mu_{0m} \sum_{p\in K^C}\sum_{q=1}^{d_p} k_{pq}^2 I_p(\mu_{0m})/I'_p(\mu_{0m})$$
$$\geq \mu_{0m}^2 \sum_{p\in K^C}\sum_{q=1}^{d_p} \frac{k_{pq}^2}{p}$$
$$\geq \mu_{0m}^2,$$

for all $k \in H$, with $\|k\|_V = 1$ (we have normed V with the weighted $L^2(\partial B_1)$-norm $\|k\|^2 = \sum_{p=1}^{+\infty}\sum_{q=1}^{d_p} k_{pq}^2/p$). So there exists a neighborhood $\mathcal{O}$ of the origin in E such that Φ is positive in $\mathcal{O}\backslash\{0\} \cap V$.

We point out that if $I_p(\mu_{0m})/I'_p(\mu_{0m}) > 0$, for all $p \geq 2$, then the set $K = \emptyset$. In this case Φ is positive in $\mathcal{O}\backslash\{0\} \cap H$, and, by Lemma 8.4, it follows that Φ is positive in $\mathcal{U}\backslash\mathcal{M}$, i.e. the result is optimal. On the other hand if $K \neq \emptyset$, then Φ must *change* sign in H.

Let assume that μ_{0m}^2 has multiplicity bigger than N (as eigenvalue with Dirichlet boundary conditions). In this case V becomes

$$V = \{k \in H; k_{pq} = 0, p \in K \cup I\},$$

being the set I defined in (8.1), and V' becomes

$$V' = \{k; k \in < k_{p1}, \ldots, k_{pd_p} >, p \in K \cup I\}.$$

Case (ii): μ_{0m}^2 is *singular*. Let assume that μ_{0m}^2 has multiplicity equal to N. Let $\widetilde{V}$ be the space

$$\widetilde{V} = \{k \in H; k_{pq} = 0, p \in K \cup L \cup L'\}.$$

Let V' be the space

$$V' = \{k; k \in < k_{p1}, \ldots, k_{pd_p} > p \in K\}.$$

By using the same arguments as in previous case (i), we obtain that Φ is negative in $\mathcal{O}\backslash\{0\}\cap V'$, and it is positive in $\mathcal{O}\backslash\{0\} \cap \widetilde{V}$. Now since Φ is continuous in E, and the space $\bigcup_{p\in L\cup L'} E_p$ has zero Lebesgue measure in E, it follows that Φ is positive in $\mathcal{O}\backslash\{0\} \cap V$, with $V = \{k \in H; k_{pq} = 0, p \in K\}$. Finally if μ_{0m}^2 has multiplicity bigger than N the same conclusion holds true, with $V = \{k \in H; k_{pq} = 0, p \in K \cup I\}$. □

9 Lipschitz case

In this section we examine briefly Lipschitz case, i.e. the case where

$$E = \{k \in C^{0,1}(\partial B_1)\}.$$

By classical regularity results we know that $u \in C^{\omega}_{\text{loc}}(\Omega_k) \cap C^{0,1}(\overline{\Omega}_k)$ solves (1.4) in a weak sense, when $\Omega = \Omega_k$, i.e.

$$\int_{\Omega_k} \nabla u \cdot \nabla \phi - \mu \int_{\Omega_k} u\phi = \int_{\Omega_k} \phi,$$

for all $\phi \in C_c^{\infty}(\Omega_k)$. By repeating the same arguments as in the regular case, we can prove the following

Theorem 9.1 *Given a μ_{0m}, for some $m \geq 1$, there exists a class $\mathcal{D}$ of $C^{0,1}$-domains (depending on μ_{0m}), such that if u is a weak no trivial solution to* (1.4)*, and*

$$\int_{\partial\Omega} \partial_{\mathbf{n}} u = 0,$$

with $\Omega \in \mathcal{D}$, and $\mu = \mu_{0m}^2 + o(1)$, then $\Omega = B_1$, $\mu = \mu_{0m}^2$, and $u = u^{(0)}$ in B_1.

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