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# Characterization of classical graph classes by weighted clique graphs

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ABSTRACT

Given integers $m_1, \ldots, m_\ell$, the *weighted clique graph* of $G$ is the clique graph $K(G)$, in which there is a weight assigned to each complete set $S$ of size $m_i$ of $K(G)$, for each $i = 1, \ldots, \ell$. This weight equals the cardinality of the intersection of the cliques of $G$ corresponding to $S$. We characterize weighted clique graphs in similar terms as Roberts and Spencer's characterization of clique graphs. Further we characterize several classical graph classes in terms of their weighted clique graphs, providing a common framework for describing some different well-known classes of graphs, as hereditary clique-Helly graphs, split graphs, chordal graphs, interval graphs, proper interval graphs, line graphs, among others.


## 1. Introduction

The clique graph of a graph $G$ is the intersection graph of the maximal cliques of $G$. Clique graphs have been studied extensively, for over forty years. The subject has attracted the attention of several researchers and many problems still remain unsolved in the area.

In this paper, we consider the generalization where weights are assigned to complete subsets of vertices of the clique graph, having certain prescribed sizes. The weights equal the cardinalities of the intersections of the cliques of $G$ which correspond to the complete sets under consideration. This concept has been considered before for weights assigned only to the edges of the clique graph, in [29,30], in [16,19–21,31,33,38] in the context of chordal graphs, and recently in [13] in the context of complex networks analysis. The more general concept where weights might be assigned to any complete subset of vertices is introduced in this work. An extended abstract containing partial results of the paper has appeared as [6]. We recently knew about a work by McKee [32] that considers weights assigned to edges and triangles with the aim of characterizing two graph classes, in the same spirit of the present paper.

There are some distinct motivations for this study. First, within clique graph theory itself. We describe a characterization of weighted clique graphs, in similar terms as Roberts and Spencer's classical characterization of clique graphs. The problem of recognizing weighted clique graphs then arises naturally. Observe that weighted clique graphs carry information about sizes of intersecting maximal cliques, not found in the usual clique graphs. The first question is whether weighted clique graphs are easier to recognize than clique graphs. In this direction, we show that it is NP-complete to recognize weighted clique graphs, if only intersections of size one are given. That is, when all vertices of $K(G)$ are weighted with the cardinality of the corresponding clique in $G$.

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We remark that in many instances there is interest to obtain a pre-image of a clique graph that has some structural properties, as diamond-free, claw-free, chordal, etc. Another possibility is to ask for a pre-image where sizes of the corresponding cliques, or their intersections, are prescribed. This leads to the notion of weighted clique graphs, considered in this paper. We also investigate connections between properties of graphs and their weighted clique graphs. In many instances, some properties of a graph carry over to its clique graph, which makes it possible to characterize clique graphs of specific graph classes. So, it is natural to ask the question in the context of weighted clique graphs, in other words, what properties of weighted clique graphs, of some class of graphs, characterize the class itself. That is, the statement we look for is of the form: $G$ has property $X$ if and only its weighted clique graph has property $Y$. It turns out that many standard graph classes admit such a characterization. These are presented in the second part of the paper.

The above consists of a further motivation for this work: describing classical graph classes under a unified approach, where a general description of each class differs from the other by some parameter in this description. We show that this can be achieved by employing weighted clique graphs. In fact, we describe new characterizations for some classical well-known classes of graphs, as chordal graphs, interval graphs, diamond-free graphs, among others, all of them in terms of weighted clique graphs. They differ by the way the weights of the graph are related. The aim is to state characterizations in terms of weights corresponding to complete sets of vertices of sizes as small as possible. In fact, we show that some of the characterizations are best possible, in the sense that there is no characterization employing weights corresponding to complete subsets of smaller size.

Finally, we refer to an application of weighted clique graphs, in complex networks, where the weights are restricted to complete sets of size two. The purpose of [13] was to employ clique graphs to study overlapping communities. In fact, the concept of clique graphs fits naturally into the idea of overlapping communities. Furthermore, by relaxing the constraints that the weights ought to correspond to complete sets of size two, and considering general weighted clique graphs instead, we would have a tool to analyze larger number of overlapping communities.

We now describe the terminology employed in the paper.

Given a graph, a *complete set* is a set of pairwise adjacent vertices. A *clique* is an inclusion-wise maximal complete set. We will denote by $\mathcal{M}(G)$ the set of cliques of a graph $G$, and by $\mathcal{M}_G(v)$ the set of cliques containing the vertex $v$ in $G$. We will denote by $\omega(G)$ the maximum size of a clique of $G$.

Consider a finite family of non-empty sets. The *intersection graph* of this family is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect.

The *clique graph* $K(G)$ of $G$ is the intersection graph of the cliques of $G$, that is, $V(K(G)) = \mathcal{M}(G)$ and $M, M'$ are adjacent in $K(G)$ if and only if $M \cap M' \neq \emptyset$. A graph $G$ is a *clique graph* if there exists a graph $H$ such that $K(H) = G$.

Let $\mathcal{A}$ be a class of graphs. We write $K(\mathcal{A})$ to denote the class of clique graphs of the graphs in $\mathcal{A}$, that is, $\mathcal{B} = K(\mathcal{A})$ if and only if for each $G$ in $\mathcal{A}$, $K(G)$ belongs to $\mathcal{B}$ and for each $H$ in $\mathcal{B}$, there exists $G$ in $\mathcal{A}$ such that $K(G) = H$.

Given a graph $G$, the set of its cliques can be computed in $O(mnp)$ time [42], where $n$, $m$ and $p$ are the number of vertices, edges and cliques of $G$, respectively. So, employing a straightforward algorithm that would just apply the definition, the clique graph of $G$ can be computed in $O(mnp + np^2)$ time. Note that the number of cliques of a graph with $n$ vertices can grow exponentially in $n$, so this time complexity is not necessarily polynomial in the size of $G$. In fact, even deciding if the clique graph of a given graph $G$ is a complete graph is already a co-NP-complete problem [28].

The converse problem is also apparently not easy to solve. Clique graphs have been characterized by Roberts and Spencer in [37] (see Theorem 4), but the problem of deciding if a graph is a clique graph is NP-complete [1].

A family $\mathcal{F}$ of subsets of a set $S$ is *separating* if for every pair of different elements $x, y$ in $S$, there is a subset in $\mathcal{F}$ that contains $x$ and does not contain $y$ or, equivalently, if for each $x$ in $S$, the intersection of all the subsets in $\mathcal{F}$ containing $x$ is $\{x\}$.

A family of subsets of a set satisfies the *Helly property* if every subfamily of it consisting of pairwise intersecting subsets has a common element. A graph is *clique-Helly* if its cliques satisfy the Helly property.

Clique-Helly graphs are clique graphs [22]. In that case, given a graph $H$, the problem of building a graph $G$ such that $K(G) = H$ can be solved with the same time complexity as building $K(H)$. Nevertheless, the problem of deciding whether the clique graph of a given graph $G$ is clique-Helly is NP-hard [10].

Given a graph $G$ and integers $m_1, \ldots, m_\ell$, an $\{m_1, \ldots, m_\ell\}$-*weighting* of $G$ is a function $w$ that assigns a non-negative weight to each complete set $S$ of $G$ of size $|S| \in \{m_1, \ldots, m_\ell\}$. A *full-weighting* of $G$ is a function $w$ that assigns a non-negative weight to each complete set of $G$.

An $\{m_1, \ldots, m_\ell\}$-*weighted graph* is a graph $G$ together with an $\{m_1, \ldots, m_\ell\}$-weighting $w$ of $G$. Analogously, a *fully-weighted graph* is a graph $G$ together with a full-weighting $w$ of $G$.

If $\ell = 1$, we will write simply $m_1$-weighting and $m_1$-weighted graph, instead of $\{m_1\}$-weighting and $\{m_1\}$-weighted graph. Also, we will omit the brackets and write $w(M_1, \ldots, M_s)$ instead of $w(\{M_1, \ldots, M_s\})$ for the weight of the set $\{M_1, \ldots, M_s\}$.

The $\{m_1, \ldots, m_\ell\}$-*weighted clique graph* of $G$ is the clique graph $K(G)$ together with an $\{m_1, \ldots, m_\ell\}$-weighting $w$ of $K(G)$ such that for each $s \in \{m_1, \ldots, m_\ell\}$ and each complete set $\{M_1, \ldots, M_s\}$ of $K(G)$ satisfies $w(M_1, \ldots, M_s) = |M_1 \cap \cdots \cap M_s|$. Analogously, the *fully-weighted clique graph* of $G$ is the clique graph $K(G)$ together with a full-weighting $w$ of $K(G)$ such that for each complete set $\{M_1, \ldots, M_s\}$ of $K(G)$ satisfies $w(M_1, \ldots, M_s) = |M_1 \cap \cdots \cap M_s|$. We write $K^w_{m_1,\ldots,m_\ell}(G)$ to denote the $\{m_1, \ldots, m_\ell\}$-weighted clique graph of $G$ and $K^w_{\text{full}}(G)$ to denote the fully-weighted clique graph of $G$. Observe that $w$ is

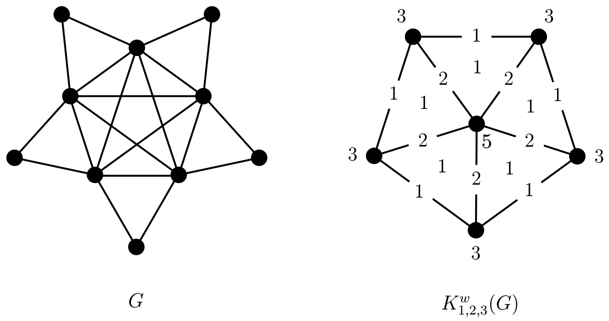


**Fig. 1.** A graph $G$ and its $\{1, 2, 3\}$-weighted clique graph.

non-decreasing with respect to set inclusion, i.e., $w(S) \geq w(S')$ whenever $S \subseteq S'$. In Fig. 1, a graph $G$ and its weighted clique graph $K^w_{1,2,3}(G)$ are shown.

Note that the graphs $K^w_1(G)$, $K^w_{1,2}(G)$, $K^w_2(G)$ are graphs with special vertex and/or edge weights, where each vertex gets the size of the corresponding clique in $G$ as weight and each edge gets the size of the intersection of the two corresponding cliques in $G$ as weight. Also, by definition of $K(G)$, in $K^w_{1,2}(G)$ and $K^w_2(G)$, we have that $w(M, M') > 0$ for every edge $MM'$ of $K(G)$.

The organization of this paper is as follows. In Section 2, we introduce some definitions and results related to clique graphs. In Section 3, we prove the characterization of weighted clique graphs. As mentioned, one of the contributions of this work is to characterize several classical and well-known graph classes by means of their weighted clique graph, and this is given in Section 4. We prove a characterization of hereditary clique-Helly graphs in terms of $K^w_3$ and show that $K^w_{1,2}$ is not sufficient to characterize neither hereditary clique-Helly graphs nor clique-Helly graphs. For chordal graphs and their subclass *UV* graphs, we obtain a characterization by means of $K^w_{2,3}$. We show furthermore that $K^w_{1,2}$ is not sufficient to characterize *UV* graphs. We describe also a characterization of interval graphs in terms of $K^w_{2,3}$ and of proper interval graphs in terms of $K^w_{1,2}$. In addition, we prove that $\{K^w_1, K^w_2\}$ is not sufficient to characterize proper interval graphs. For split graphs and line graphs, we give a characterization by means of $K^w_{1,2}$, and also prove that $\{K^w_1, K^w_2\}$ is neither sufficient to characterize split graphs nor line graphs. Finally, we characterize trees in terms of $K^w_1$ and block graphs in terms of $K^w_2$, and show that the latter cannot be characterized by means of their 1-weighted clique graphs.

## 2. Preliminaries

We shall consider finite, simple, loopless, undirected graphs. Let $G$ be a graph. Denote by $V(G)$ its vertex set and by $E(G)$ its edge set. Given a vertex $v$ of $G$, denote by $N_G(v)$ the set of neighbors of $v$ in $G$ and by $N_G[v]$ the set $N_G(v) \cup \{v\}$. A vertex $v$ of $G$ is called *universal* if $N_G[v] = V(G)$. *Diamond* is the unique graph on 4 vertices with 5 edges. *Claw* consists of a vertex adjacent to three pairwise non-adjacent vertices. If $H$ is a graph, a graph $G$ is *H-free* if $G$ does not contain $H$ as an induced subgraph.

A *stable set* in a graph is a set of pairwise non-adjacent vertices.

A graph is a *split graph* if its vertices can be partitioned into a clique and a stable set. A graph is a (not necessarily induced) *star* if it has a universal vertex. In that case, any universal vertex is called a *center* of the star.

A graph $G$ is an *interval graph* if $G$ is the intersection graph of a finite family of intervals of the real line, and it is a *proper interval graph* if it admits an intersection model in which no interval properly contains another. A *unit interval graph* is the intersection graph of a finite family of intervals of the real line, all of the same length. Proper interval graphs and unit interval graphs coincide, and they are exactly the claw-free interval graphs [36].

**Theorem 1** (*Fulkerson and Gross, 1965 [14]*)**.** *A graph $G$ is an interval graph if and only if its cliques can be linearly ordered such that, for each vertex $v$ of $G$, the cliques containing $v$ are consecutive.*

Such an ordering is called a *canonical ordering* of the cliques.

**Theorem 2** (*Roberts, 1969 [36]*)**.** *A graph $G$ is a proper interval graph if and only if its vertices can be linearly ordered such that, for each clique $M$ of $G$, the vertices contained in $M$ are consecutive.*

Such an ordering is called a *canonical ordering* of the vertices.

The *line graph* $L(G)$ of $G$ is the intersection graph of the edges of $G$. A graph $G$ is a *line graph* if there exists some simple graph $H$ such that $G = L(H)$. Line graphs were extensively studied and there are several characterizations of them. We will use in this work a characterization by Krausz [26], that is also related with our characterization of weighted clique graphs.

**Theorem 3** (*Krausz, 1943 [26]*)**.** *A graph $G$ is a line graph if and only if there is a collection $\mathcal{F}$ of complete sets of $G$ such that every edge of $G$ is contained in exactly one complete set of $\mathcal{F}$, and each vertex of $G$ is contained in at most two complete sets of $\mathcal{F}$.*

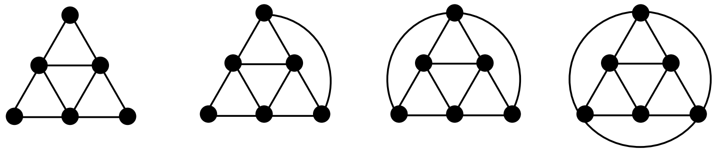

**Fig. 2.** Pyramids.

**Table 1**
Clique graphs of some graph classes.

| Class $\mathcal{A}$ | $K(\mathcal{A})$ | Reference |
|---|---|---|
| Block | Block | [23] |
| Trees | Block | [23] |
| Chordal | Dually chordal | [7,18,41] |
| *UV* | Dually chordal | [41] |
| Dually chordal | Chordal ∩ clique-Helly | [7,18] |
| Clique-Helly | Clique-Helly | [11] |
| Hereditary clique-Helly | Hereditary clique-Helly | [35] |
| Interval | Proper interval | [24] |
| Proper interval | Proper interval | [24] |
| Diamond-free | Diamond-free | [9] |
| Split | Stars | |
| Triangle-free | Linear domino | [34] |
| Linear domino | Triangle-free | [9] |
| *UV* | Dually chordal | [41] |
| Trivially perfect | Component complete | |

A graph $G$ is a *tree* if it is connected and contains no cycle. A graph is *chordal* if it contains no chordless cycle of length at least 4. Equivalently, a graph is chordal if it is the intersection graph of subtrees of a tree [8,15,45]. A graph is a *UV graph* if it is the intersection graph of paths of a tree. *UV* graphs are also called *path graphs* in the literature. A graph is *strongly chordal* when it is chordal and each of its cycles of even length at least 6 has an odd chord.

A graph $G$ is *trivially perfect* if for all induced subgraphs $H$ of $G$, the cardinality of the maximum stable set of $H$ is equal to the number of cliques of $H$. Equivalently, a graph is trivially perfect if it contains no chordless cycle of length 4 or chordless path of length 3 [17].

A graph $G$ is a *block graph* if each 2-connected subgraph of $G$ is a complete subgraph. Equivalently, a graph is a block graph if it is chordal and diamond-free.

A graph is a *domino* if each of its vertices belongs to at most two cliques. If in addition, each of its edges belongs to at most one clique, then $G$ is a *linear domino graph*. Linear domino graphs coincide with {claw, diamond}-free graphs [25]. Note that sometimes a cycle of length 6 with a unique chord joining two vertices at distance three in the cycle is also called a domino. Notice that this graph does not belong to the class of dominoes according to the notation in [25].

A graph $G$ is *dually chordal* if it admits a spanning tree $T$ such that, for every edge $vz$ of $G$, the vertices of the $v$–$z$ path in $T$ induce a complete subgraph in $G$ [7,41]. In that case, $T$ is called a *compatible tree* of $G$.

A graph $G$ is *hereditary clique-Helly* if $H$ is clique-Helly for every induced subgraph $H$ of $G$. Equivalently, a graph $G$ is hereditary clique-Helly if none of the pyramids (Fig. 2) is an induced subgraph of $G$ [35].

Clique graphs of many graph classes have been characterized. The known results involving the graph classes that will be considered in this paper are summarized in Table 1.

## 3. Characterization of weighted clique graphs

The characterization of clique graphs is as follows.

**Theorem 4** (*Roberts and Spencer, 1971 [37]*)**.** *A graph $H$ is a clique graph if and only if there is a collection $\mathcal{F}$ of complete sets of $H$ such that every edge of $H$ is contained in some complete set of $\mathcal{F}$, and $\mathcal{F}$ satisfies the Helly property.*

A characterization for 2-weighted clique graphs, formulated in similar terms, was presented independently in [29,37]. The result was presented in terms of multigraphs, but restated with our notation it reads as follows.

**Theorem 5** (*Roberts and Spencer, 1971 [37], McKee, 1991 [29]*)**.** *Let $H$ be a graph and let $w$ be a 2-weighting of $H$. Then there exists a graph $G$ such that $H = K_2^w(G)$ if and only if there is a collection $\mathcal{F}$ of complete sets of $H$, not necessarily distinct, such that every edge $e$ of $H$ is contained in $w(e)$ complete sets of $\mathcal{F}$ and $\mathcal{F}$ satisfies the Helly property.*

We extend this characterization to weighted clique graphs with arbitrary weights.

**Theorem 6.** *Let $H$ be a graph and let $w$ be an $\{m_1, \ldots, m_\ell\}$-weighting of $H$. Then there exists a graph $G$ such that $H = K^w_{m_1,\ldots,m_\ell}(G)$ if and only if there is a collection $\mathcal{F}$ of complete sets of $H$, not necessarily distinct, such that:*

(1) *every edge of $H$ is contained in some complete set of $\mathcal{F}$,*
(2) *$\mathcal{F}$ satisfies the Helly property,*
(3) *$\mathcal{F}$ is separating,*
(4) *for every $1 \le j \le \ell$, each complete set $S$ of size $m_j$ in $H$ satisfies $S \subseteq F$ for exactly $w(S)$ sets $F \in \mathcal{F}$.*

**Proof.** If $|V(H)| = 1$, the theorem trivially holds. So, from now on, we will assume $|V(H)| \ge 2$.

($\Rightarrow$) Let $G$ be a graph such that $H = K^w_{m_1,\ldots,m_\ell}(G)$ and let $\mathcal{F} = \{\mathcal{M}_G(v)\}_{v \in V(G)}$. It is clear that the elements of the family $\mathcal{F}$ are complete sets of $H$ since for each $v$ in $V(G)$, all cliques in $\mathcal{M}_G(v)$ contain $v$, so they are pairwise adjacent as vertices of $H$. Let $MM'$ be an edge of $H$. Then the cliques $M$ and $M'$ share a vertex $v$ in $G$, thus both belong to $\mathcal{M}_G(v)$, which is a complete set of $\mathcal{F}$. This proves (1). Now, let $\mathcal{F}'$ be a pairwise intersecting subfamily of $\mathcal{F}$. That is, $\mathcal{F}' = \{\mathcal{M}_G(v)\}_{v \in V' \subseteq V(G)}$. If $|V'| = 1$, there is nothing to prove. Otherwise, let $v, w$ be two vertices of $V'$. Since $\mathcal{M}_G(v)$ has non-empty intersection with $\mathcal{M}_G(w)$, there is a clique of $G$ containing both $v$ and $w$ and, in particular, $v$ and $w$ are adjacent. So $V'$ is a complete set of $G$, and it is contained in some clique $M$ of $G$, which in turn belongs to every element in $\mathcal{F}'$. This proves (2). Let $M, M'$ be two distinct vertices of $H$. Since they are two different cliques of $G$, by maximality, neither of them is contained in the other one. So there is a vertex $v$ in $M \setminus M'$, and some vertex $v'$ in $M' \setminus M$. In $\mathcal{F}$, $M$ belongs to $\mathcal{M}_G(v) \setminus \mathcal{M}_G(v')$ and $M'$ belongs to $\mathcal{M}_G(v') \setminus \mathcal{M}_G(v)$. This proves (3). Finally, let $1 \le j \le \ell$ and let $\{M_1, \ldots, M_{m_j}\}$ be a complete set of size $m_j$ in $H$. Let $V' = M_1 \cap \cdots \cap M_{m_j} \subseteq V(G)$. By definition of $\mathcal{F}$, the subfamily of all sets $F \in \mathcal{F}$ such that $\{M_1, \ldots, M_{m_j}\} \subseteq F$ is $\mathcal{F}' = \{\mathcal{M}_G(v)\}_{v \in V'}$. Since $H = K^w_{m_1,\ldots,m_\ell}(G)$, by definition of $w$, we conclude $w(M_1, \ldots, M_{m_j}) = |V'|$. This proves (4).

($\Leftarrow$) Let $\mathcal{F}$ be a collection of complete sets of $H$ satisfying (1)–(4). Let $G$ be the intersection graph of $\mathcal{F}$. We will prove that $H = K^w_{m_1,\ldots,m_\ell}(G)$. For each vertex $M$ of $H$, let $\mathcal{F}_M$ be the set of elements of $\mathcal{F}$ containing $M$. We will prove that the cliques of $G$ are exactly $\{\mathcal{F}_M\}_{M \in V(H)}$, that they are pairwise different, and that $\mathcal{F}_M$ intersects $\mathcal{F}_{M'}$ if and only if $MM'$ is an edge of $H$. For each $M \in V(H)$, the elements in $\mathcal{F}_M$ are mutually intersecting, so they form a complete set of $G$, which in turn is contained in some clique of $G$. Suppose that $Q$ is a clique of $G$, that is, a maximal set of pairwise intersecting elements of $\mathcal{F}$. Since $\mathcal{F}$ satisfies the Helly property, $Q$ has a common element $M'$, so $F \subseteq \mathcal{F}_{M'}$ and, by maximality, $Q = \mathcal{F}_{M'}$. Now, let $M, M'$ be two different vertices of $H$. Since $\mathcal{F}$ is separating, the intersection of all the members of $\mathcal{F}_M$ is $\{M\}$ and the intersection of all the members of $\mathcal{F}_{M'}$ is $\{M'\}$, so $\mathcal{F}_M$ and $\mathcal{F}_{M'}$ are different and neither is contained in the other one. In addition, $\mathcal{F}_M$ and $\mathcal{F}_{M'}$ have non-empty intersection if and only if there is some set of $\mathcal{F}$ containing both $M$ and $M'$. Since the members of $\mathcal{F}$ are complete sets of $H$ and each edge of $H$ is contained in some member of $\mathcal{F}$, that happens if and only if $MM'$ is an edge of $H$. So $K(G)$ is isomorphic to $H$, with bijection $\mathcal{F}_M \mapsto M$. Finally, let $1 \le j \le \ell$ and let $\{M_1, \ldots, M_{m_j}\}$ be a complete set of size $m_j$ in $H$. Then $\mathcal{F}_{M_1} \cap \cdots \cap \mathcal{F}_{M_{m_j}}$ is the family of all members $F \in \mathcal{F}$ with $\{M_1, \ldots, M_{m_j}\} \subseteq F$. By (4), $|\mathcal{F}_{M_1} \cap \cdots \cap \mathcal{F}_{M_{m_j}}| = w(M_1, \ldots, M_{m_j})$. Therefore $H = K^w_{m_1,\ldots,m_\ell}(G)$. $\square$

It seems interesting to analyze the computational complexity of deciding whether a weighted graph is a weighted clique graph. For 1-weightings, the problem is apparently already difficult.

**Theorem 7.** *The problem of deciding whether a graph with vertex weights is a* 1*-weighted clique graph is NP-complete.*

**Proof.** We will reduce the problem of deciding if a given graph is a clique graph to the problem of deciding if a 1-weighted graph is a 1-weighted clique graph. Since the decision problem for clique graphs is NP-complete [1], the theorem will follow. Let $H$ be an instance of the first problem, and define $w$ as $w(M) = |E(H)| + 1$ for each vertex $M$ of $H$. It is clear that if the 1-weighted graph $H$ is a 1-weighted clique graph, then $H$ is a clique graph. So let us prove the converse, and suppose that $H$ is a clique graph. By Theorem 4, there is a family $\mathcal{F}$ of complete sets of $H$ that covers all the edges of $H$ and satisfies the Helly property. Note that the Helly property is hereditary, so every subfamily of $\mathcal{F}$ satisfies it as well, and so we can consider a subfamily of $\mathcal{F}$ with at most $|E(H)|$ elements, one covering each edge of $H$. This observation was previously done in [12]. We can add to the family one-vertex sets in such a way that each vertex of $H$ is covered $|E(H)| + 1$ times. Note that for each vertex $M$ of $H$, the set $\{M\}$ will be added at least once, so the new family is separating. Also, adding one-vertex sets does not alter the Helly property. Then, by Theorem 6, $H$ with the 1-weighting $w$ is a 1-weighted clique graph. It remains to prove that the problem belongs to NP. If some graph $H$ provided with a 1-weighting $w$ is a 1-weighted clique graph, by Theorem 6, there is a family of complete sets of $H$ that is separating, satisfies the Helly property, covers all the edges of $H$ and covers each vertex $M$ of $H$ exactly $w(M)$ times. We can always choose a subfamily of it of size at most $|E(H)|$ that covers all the edges, and then split every other complete set in the family into one-vertex sets. This operation does neither alter the separability nor the Helly property, and the resulting family still covers all the edges of $H$ and covers each vertex $M$ of $H$ exactly $w(M)$ times. So, as a certificate of polynomial size, we can give a family $\mathcal{F}'$ of complete sets of $H$ such that $|\mathcal{F}'| \le |E(H)|$, $\mathcal{F}'$ covers all the edges of $H$, each vertex $M$ of $H$ is covered at most $w(M)$ times and, if it is covered exactly $w(M)$ times, then $\mathcal{F}'$ separates $M$, that is, the intersection of all the complete sets containing $M$ is $\{M\}$. All these properties can be verified in polynomial time (for the Helly property, see [5]). $\square$

The clique graph operator is far from being one-to-one. Indeed, the clique graph of any graph with a universal vertex is a complete graph, and moreover a graph $G$ such that $K(G)$ is complete does not necessarily have a universal vertex [28]. If we have a full-weighting of a graph $H$, we will show that we can actually either build the unique graph $G$ such that $H = K_{\text{full}}(G)$, or decide that such a graph does not exist. But the size of a full-weighting of $H$ can be exponential in the size of $H$. The *compact fully-weighted clique graph* of $G$ is the clique graph $K(G)$ together with a full-weighting $\bar{w}$ of $K(G)$ such that for each complete set $S$ of $K(G)$ satisfies $\bar{w}(S) = |\{v \in V(G) : \mathcal{M}_G(v) = S\}|$. We write $K^{\bar{w}}_{\text{full}}(G)$ to denote the compact fully-weighted clique graph of $G$. It is "compact" in the sense that $\bar{w}(S) > 0$ for $O(|V(G)|)$ sets $S$. We can compute $K^{w}_{\text{full}}(G)$ from $K^{\bar{w}}_{\text{full}}(G)$ by defining $w(S) = \sum_{S' \supseteq S} \bar{w}(S')$ for each complete set $S$ of $K(G)$. Analogously, we can compute $K^{\bar{w}}_{\text{full}}(G)$ from $K^{w}_{\text{full}}(G)$ by defining $\bar{w}(S) = w(S)$ if $S$ is a maximum size clique of $K(G)$ and $\bar{w}(S) = w(S) - \sum_{S' \supsetneq S} \bar{w}(S')$ otherwise.

**Theorem 8.** *Let $H$ be a graph and let $\bar{w}$ be a full-weighting of $H$. Then there exists a graph $G$ such that $H = K^{\bar{w}}_{\text{full}}(G)$ if and only if $\{S : \bar{w}(S) > 0\}$ covers the edges of $H$, is separating and satisfies the Helly property. In that case, $G$ is unique and obtained as the intersection graph of $\bar{w}(S)$ copies of the set $S$ for each complete set $S$ of $H$.*

**Proof.** ($\Rightarrow$) Suppose that there exists a graph $G$ such that $H = K^{\bar{w}}_{\text{full}}(G)$. As it was argued in the proof of Theorem 6, the family $\mathcal{F} = \{\mathcal{M}_G(v)\}_{v \in V(G)}$ covers the edges, is separating and satisfies the Helly property. But, by definition of the compact fully-weighted clique graph, if $S$ is a complete set of $K^{\bar{w}}_{\text{full}}(G)$ then $\bar{w}(S) > 0$ if and only if $S = \mathcal{M}_G(v)$ for $\bar{w}(S)$ vertices $v$ of $G$. So the sets $\{\mathcal{M}_G(v)\}_{v \in V(G)}$ and $\{S : \bar{w}(S) > 0\}$ are equal. The last statement is a consequence of the known fact that every graph $G$ is the intersection graph of the (multi)family $\{\mathcal{M}_G(v)\}_{v \in V(G)}$.

($\Leftarrow$) Suppose that $\{S : \bar{w}(S) > 0\}$ covers the edges of $H$, is separating and satisfies the Helly property. Let $G$ be the intersection graph of a family $\mathcal{F}$ composed by $\bar{w}(S)$ copies of the set $S$ for each complete set $S$ of $H$. We will prove that $H = K^{\bar{w}}_{\text{full}}(G)$. For each vertex $M$ of $H$, let $\mathcal{F}_M$ be the set of elements of $\mathcal{F}$ containing $M$. We will prove that the cliques of $G$ are exactly $\{\mathcal{F}_M\}_{M \in V(H)}$, that they are pairwise different, and that $\mathcal{F}_M$ intersects $\mathcal{F}_{M'}$ if and only if $MM'$ is an edge of $H$. For each $M \in V(H)$, the elements in $\mathcal{F}_M$ are mutually intersecting, so they form a complete set of $G$, which in turn is contained in some clique of $G$. Suppose that $Q$ is a clique of $G$, that is, a maximal set of pairwise intersecting elements of $\mathcal{F}$. Since $\mathcal{F}$ satisfies the Helly property, $Q$ has a common element $M'$, so $Q \subseteq \mathcal{F}_{M'}$ and, by maximality, $Q = \mathcal{F}_{M'}$. Now, let $M, M'$ be two different vertices of $H$. Since $\mathcal{F}$ is separating, the intersection of all the members of $\mathcal{F}_M$ is $\{M\}$ and the intersection of all the members of $\mathcal{F}_{M'}$ is $\{M'\}$, so $\mathcal{F}_M$ and $\mathcal{F}_{M'}$ are different and neither is contained in the other one. In addition, $\mathcal{F}_M$ and $\mathcal{F}_{M'}$ have non-empty intersection if and only if there is some set of $\mathcal{F}$ containing both $M$ and $M'$. Since the members of $\mathcal{F}$ are complete sets of $H$ and each edge of $H$ is contained in some member of $\mathcal{F}$, that happens if and only if $MM'$ is an edge of $H$. So $K(G)$ is isomorphic to $H$, with bijection $\mathcal{F}_M \mapsto M$. For $F \in \mathcal{F} = V(G)$ and $M \in V(H)$, $F \in \mathcal{F}_M$ if and only if $M \in F$, thus $\mathcal{M}_G(F)$ maps to $F$ in $H$. As $\mathcal{F}$ contains $\bar{w}(S)$ copies of the set $S$ for each complete set $S$ of $H$, it follows that $H = K^{\bar{w}}_{\text{full}}(G)$. $\square$

**Corollary 9.** *Given a graph $H$ and a full-weighting $\bar{w}$ of it. Then, the question "is there a graph $G$ such that $H = K^{\bar{w}}_{\text{full}}(G)$?" can be answered in $O(|V(H)|^3|\{S : \bar{w}(S) > 0\}|)$ time. In particular, if the answer is positive, this it is $O(|V(H)|^3|V(G)|)$, and the unique graph $G$ such that $H = K^{\bar{w}}_{\text{full}}(G)$ can be build in $O(|V(H)|^3|V(G)| + |E(G)|)$ time.*

**Proof.** The Helly property of the sets $\{S : \bar{w}(S) > 0\}$ can be tested in $O(|V(H)|^3|\{S : \bar{w}(S) > 0\}|)$ time [5]. It is not difficult to see that the remaining conditions can be tested within that computational complexity as well. Finally, the fact that if the answer is positive then $|V(G)| = \sum_S \bar{w}(S)$ concludes the proof of this corollary. $\square$

**Corollary 10.** *Let $H$ be a graph and let $w$ be a full-weighting of $H$. Then there exists a graph $G$ such that $H = K^{w}_{\text{full}}(G)$ if and only if the function $\bar{w}$ defined on the complete sets of $H$ as $\bar{w}(S) = w(S)$, if $S$ is a maximum size clique of $H$, and $\bar{w}(S) = w(S) - \sum_{S' \supsetneq S} \bar{w}(S')$, otherwise, is non-negative and such that $\{S : \bar{w}(S) > 0\}$ covers the edges of $H$, is separating and satisfies the Helly property. In that case, $G$ is unique and obtained as the intersection graph of $\bar{w}(S)$ copies of the set $S$ for each complete set $S$ of $H$.*

**Proof.** ($\Rightarrow$) Is a direct consequence of Theorem 8 and the fact that if there exists a graph $G$ such that $H = K^{w}_{\text{full}}(G)$, then the weight $\bar{w}$ associated with the compact fully-weighted clique graph of $G$ can be computed as $\bar{w}(S) = w(S)$, if $S$ is a maximum size clique of $K(G)$, and $\bar{w}(S) = w(S) - \sum_{S' \supsetneq S} \bar{w}(S')$, otherwise.

($\Leftarrow$) If the function $\bar{w}$ is non-negative, then it is a full-weighting of $H$. By Theorem 8, there is a unique graph $G$ such that $H$ provided with the full-weighting $\bar{w}$ is the compact fully-weighted clique graph of $G$, and $G$ is obtained as the intersection graph of $\bar{w}(S)$ copies of the set $S$ for each complete set $S$ of $H$. Finally, observe that by the definition of $\bar{w}$ in terms of $w$, it holds $w(S) = \sum_{S' \supseteq S} \bar{w}(S')$ for each complete set $S$ of $H$. Thus $H$ provided with the full-weighting $w$ is the fully-weighted clique graph of $G$. $\square$

## 4. Characterizations by means of the weighted clique operator

Some graph classes can be naturally defined in terms of their weighted clique graphs. This is the case for clique-Helly graphs and their generalizations. A family of subsets of a set satisfies the $(p, q, r)$-*Helly property* if every subfamily with the property that any $p$ of its members have $q$ elements in common, has a total intersection of at least $r$ elements. A graph is $(p, q, r)$-*clique-Helly* if its cliques satisfy the $(p, q, r)$-Helly property [10].

The following claims follow directly from the respective definitions.

**Proposition 11.** *Let $G$ be a graph. Then $G$ is clique-Helly if and only if $K^w_{3,\dots,\omega(K(G))}(G)$ satisfies $w(S) > 0$ for every complete set $S$ of $K(G)$ of size at least 3.*

**Proposition 12.** *Let $G$ be a graph. Then $G$ is $(p, q, r)$-clique-Helly if and only if $K^w_{p,\dots,\omega(K(G))}(G)$ satisfies that every complete set of size at least $p$ in which all its subsets of size $p$ have weight at least $q$, has weight at least $r$.*

By the results in [11] shown in Table 1, we have the following corollary.

**Corollary 13.** *Let $H$ be a graph and let $w$ be a $\{3, \dots, \omega(H)\}$-weighting of $H$ that is strictly positive over every complete set of $H$ of size at least 3. If there is a graph $G$ such that $H = K^w_{3,\dots,\omega(H)}(G)$, then $H$ is clique-Helly.*

Diamond-free graphs have also a natural characterization in terms of their weighted clique graphs. It is well-known (e.g. [9]) that a graph is diamond-free if and only each edge belongs to exactly one clique. This property can be restated as follows.

**Proposition 14.** *Let $G$ be a graph. Then $G$ is diamond-free if and only if $K^w_2(G)$ satisfies $w(M, M') = 1$ for every edge $MM'$ of $K(G)$.*

In particular, by the results in [9] shown in Table 1, we have the following corollary, that was also pointed out in [29].

**Corollary 15.** *Let $H$ be a graph and let $w$ be a 2-weighting of $H$. If $w(M, M') = 1$ for every $MM'$ in $E(H)$, then there exists some graph $G$ such that $H = K^w_2(G)$ if and only if $H$ is diamond-free.*

Moreover, since diamond-free graphs are clique-Helly, we have that in a fully-weighted clique graph of a diamond-free graph, the weight of each complete set of size at least two is exactly one. In [2], the authors establish whether a 1-weighted graph $H$ is $K^w_1(G)$ for some diamond-free graph $G$, thus completing the characterization of weighted clique graphs of diamond-free graphs.

**Theorem 16** (*Barrionuevo and Calvo, 2004 [2]*). *Let $H$ be a graph and let $w$ be a 1-weighting of $H$. Then there exists some diamond-free graph $G$ such that $H = K^w_1(G)$ if and only if $H$ is diamond-free and $w(M) \geq \max\{2, |\mathcal{M}_H(M)|\}$ for each $M$ in $V(H)$.*

The result above can be obtained also as a corollary of Theorem 6. Joining it with Proposition 14, we have shown the following corollary.

**Corollary 17.** *Let $H$ be a graph and let $w$ be a $\{1, 2\}$-weighting of $H$, such that $w(M, M') = 1$ for each edge $MM'$ of $H$. Then there exists a graph $G$ such that $H = K^w_{1,2}(G)$ if and only if $H$ is diamond-free and $w(M) \geq \max\{2, |\mathcal{M}_H(M)|\}$ for each $M$ in $V(H)$.*

It is clear that diamond-free graphs cannot be characterized solely by their 1-weighted clique graph, since the diamond and two triangles sharing a vertex have the same 1-weighted clique graph.

A connected graph $G$ with at least two vertices is triangle-free if and only if $w(M) = 2$ for each vertex $M$ of $K^w_1(G)$. Indeed, the results in [34] showed in Table 1 imply the following proposition.

**Proposition 18.** *Let $H$ be a graph and let $w$ be a 1-weighting of $H$ such that $w(M) = 2$ for each vertex $M$ of $H$. Then there exists a graph $G$ such that $H = K^w_1(G)$ if and only if $H$ is linear domino.*

Also linear domino graphs can be naturally defined in terms of their weighted clique graph.

**Proposition 19.** *Let $G$ be a graph. Then $G$ is linear domino if and only if $K^w_2(G)$ is triangle-free and satisfies $w(M, M') = 1$ for every edge $MM'$ of $K(G)$.*

In the remainder of this section, we will present characterizations of some classical and extensively studied graph classes in terms of their weighted clique graphs. Many of them are subclasses of chordal and/or clique-Helly graphs.

### *4.1. Hereditary clique-Helly graphs*

First, we characterize hereditary clique-Helly graphs in terms of their weighted clique graphs as presented in the following theorem.

**Theorem 20.** *Let $G$ be a graph. Then $G$ is hereditary clique-Helly if and only if $K^w_{2,3}(G)$ satisfies $w(M_1, M_2, M_3) = \min\{w(M_1, M_2), w(M_2, M_3), w(M_1, M_3)\}$, for every triangle $M_1, M_2, M_3$ of $K(G)$.*

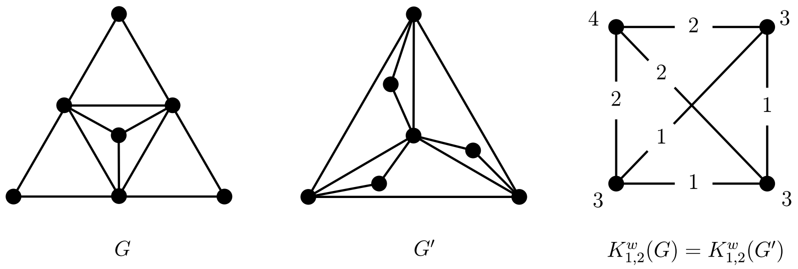


**Fig. 3.** Two graphs $G$, $G'$ such that $K^w_{1,2}(G) = K^w_{1,2}(G')$. The graph $G'$ is hereditary clique-Helly, the graph $G$ is not even clique-Helly. On the other hand, $G$ is a *UV* graph, while the graph $G'$ is not.

**Proof.** ($\Rightarrow$) Suppose that, for some triangle $M_1, M_2, M_3$ of $K(G)$, we have $w(M_1, M_2, M_3) < \min\{w(M_1, M_2), w(M_2, M_3), w(M_1, M_3)\}$. Let $v_1$ be a vertex in $(M_2 \cap M_3) \setminus M_1$, $v_2$ be a vertex in $(M_1 \cap M_3) \setminus M_2$, and $v_3$ be a vertex in $(M_1 \cap M_2) \setminus M_3$. Then $v_1, v_2, v_3$ are pairwise adjacent, since each pair of them is contained in a clique. Since $v_1$ does not belong to $M_1$, there is a vertex $v'_1$ in $M_1$ that is not adjacent to $v_1$. Since, $v_2$ and $v_3$ are in $M_1$, $v'_1$ is adjacent to both. Analogously, define $v'_2$ and $v'_3$. Then $v_1, v_2, v_3, v'_1, v'_2, v'_3$ induce a pyramid in $G$, a contradiction.

($\Leftarrow$) Suppose that $K^w_{2,3}(G)$ satisfies $w(M_1, M_2, M_3) = \min\{w(M_1, M_2), w(M_2, M_3), w(M_1, M_3)\}$, for every triangle $M_1, M_2, M_3$ of $K(G)$. If $G$ is not hereditary clique-Helly, then $G$ contains one of the pyramids as an induced subgraph. That is, there are six vertices $x, y, z, x', y', z'$ in $G$ such that $x$ is adjacent to all but $x'$, $y$ is adjacent to all but $y'$, and $z$ is adjacent to all but $z'$. Let $M_x$, $M_y$ and $M_z$ be cliques of $G$ containing $\{x', y, z\}$, $\{x, y', z\}$ and $\{x, y, z'\}$, respectively. Then $x$ belongs to $M_y \cap M_z$ but not to $M_x$, $y$ belongs to $M_x \cap M_z$ but not to $M_y$, and $z$ belongs to $M_x \cap M_y$ but not to $M_z$. Therefore, $M_x, M_y, M_z$ form a triangle in $K(G)$ but $w(M_x, M_y, M_z) < \min\{w(M_x, M_y), w(M_x, M_z), w(M_y, M_z)\}$, a contradiction. $\square$

Moreover, this property holds also for $\{2, m\}$-weightings, with $m \geq 3$. We can re-state some of the results of [35] and [44] as follows.

**Theorem 21** (*Prisner, 1993 [35], Wallis and Zhang, 1990 [44]*). *Let $G$ be a hereditary clique-Helly graph, and let $m \geq 3$. Then $K^w_{2,m}(G)$ satisfies $w(S) = \min\{w(M, M') : M, M' \in S\}$, for every complete set $S$ of size $m$ in $K(G)$.*

The examples in Fig. 3 show that $K^w_{1,2}$ is not sufficient to characterize neither hereditary clique-Helly graphs nor clique-Helly graphs. But we can obtain a characterization of hereditary clique-Helly graphs in terms of $K^w_3$.

**Theorem 22.** *Let $G$ be a graph. Then $G$ is hereditary clique-Helly if and only if $K^w_3(G)$ satisfies $w(M_1, M_2, M_3) \geq \min\{w(M_1, M_2, M_4), w(M_2, M_3, M_4), w(M_1, M_3, M_4)\}$, for every complete set $\{M_1, M_2, M_3, M_4\}$ of size four in $K(G)$.*

**Proof.** ($\Rightarrow$) Suppose that for some complete set $M_1, M_2, M_3, M_4$ of size four in $K^w_3(G)$, we have $w(M_1, M_2, M_3) < \min\{w(M_1, M_2, M_4), w(M_2, M_3, M_4), w(M_1, M_3, M_4)\}$. Then, in $K^w_{2,3}(G)$, $w(M_1, M_2, M_3) < \min\{w(M_1, M_2, M_4), w(M_2, M_3, M_4), w(M_1, M_3, M_4)\} \leq \min\{w(M_1, M_2), w(M_2, M_3), w(M_1, M_3)\}$. By Theorem 20, $G$ is not hereditary clique-Helly.

($\Leftarrow$) If $G$ is not hereditary clique-Helly, then $G$ contains one of the pyramids as an induced subgraph. That is, there are six vertices $x, y, z, x', y', z'$ in $G$ such that $x$ is adjacent to all but $x'$, $y$ is adjacent to all but $y'$, and $z$ is adjacent to all but $z'$. Let $M_x$, $M_y$ and $M_z$ be cliques of $G$ containing $\{x', y, z\}$, $\{x, y', z\}$ and $\{x, y, z'\}$, respectively. Let $S = M_x \cap M_y \cap M_z$. None of $x, y, z$, belong to $S$, but $S \cup \{x, y, z\}$ is a complete set of $G$. Let $M$ be a clique of $G$ containing it, thus different from $M_x$, $M_y$ and $M_z$. Then $S \cup \{x\} \subseteq (M_y \cap M_z \cap M)$, $S \cup \{y\} \subseteq (M_x \cap M_z \cap M)$, and $S \cup \{z\} \subseteq (M_x \cap M_y \cap M)$. Therefore, $w(M_x, M_y, M_z) < \min\{w(M_x, M_y, M), w(M_x, M_z, M), w(M_y, M_z, M)\}$ in $K^w_3(G)$. $\square$

### *4.2. Trees and block graphs*

The characterizations of trees and block graphs are as follows.

**Theorem 23.** *Let $G$ be a graph, $|V(G)| > 1$. Then $G$ is a tree if and only if $K^w_1(G)$ is a connected block graph such that $w(M) = 2$ for all $M \in V(K(G))$.*

**Proof.** ($\Rightarrow$) Let $G$ be a tree. Then $K(G)$ is a block graph (see Table 1). Moreover, $K(G)$ is connected and all cliques of $G$ have size 2.

($\Leftarrow$) Let $G$ be a graph such that $K^w_1(G)$ satisfies the hypotheses of the theorem. Then $G$ is triangle-free since all of its cliques have size 2, and connected since $K^w_1(G)$ is connected. Suppose $G$ is not a tree, and let $C$ be the shortest cycle of $G$. Since $G$ is triangle-free, $|C| \geq 4$ and every edge of $C$ is a clique. Then these cliques induce a chordless cycle of length at least four in $K^w_1(G)$, a contradiction with the fact that it is a block graph. $\square$

**Theorem 24.** *Let $G$ be a connected graph. Then $G$ is a block graph if and only if $K^w_2(G)$ is a connected block graph such that $w(M, M') = 1$, for every edge $MM'$ of $K(G)$.*

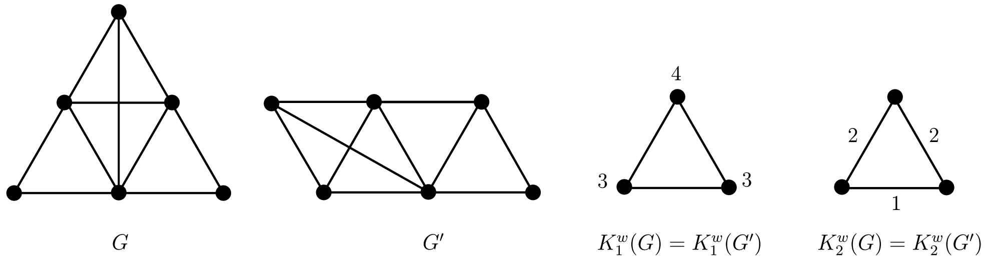


**Fig. 4.** Two graphs $G$, $G'$ such that $K_1^w(G) = K_1^w(G')$ and $K_2^w(G) = K_2^w(G')$. The graph $G$ is a split graph, the graph $G'$ is not. On the other hand, the graph $G'$ is a proper interval graph, while the graph $G$ is not. Furthermore, the graph $G'$ is a line graph, while the graph $G$ is not.

**Proof.** ($\Rightarrow$) Let $G$ be a connected block graph. Then $G$ is diamond-free, so by Proposition 14, $K_2^w(G)$ satisfies $w(M, M') = 1$, for every edge $MM'$ of $K(G)$. Moreover, $K(G)$ is connected since $G$ is connected and it is a block graph (see Table 1).

($\Leftarrow$) Let $G$ be a graph such that $K_2^w(G)$ satisfies the hypotheses of the theorem. Then $G$ is diamond-free, by Proposition 14, and connected since $K(G)$ is connected. Suppose $G$ is not a block graph. So $G$ is not chordal. Let $C = v_1v_2 \cdots v_kv_1$ be the shortest chordless cycle of $G$ with $k \geq 4$. Since $C$ is chordless, each edge $v_iv_j$ of $C$ is contained in a clique of $G$ whose intersection with $C$ is exactly the set $\{v_i, v_j\}$. These cliques induce a 2-connected subgraph in $K(G)$ and, since $K(G)$ is a block graph, they are pairwise adjacent. Let $M$ be the clique containing $v_1v_2$, $M'$ be the clique containing $v_3v_4$, and $z$ be their common vertex. Since $C$ is chordless, $z$ is none of $v_1$, $v_2$, $v_3$ or $v_4$, and it is adjacent to all of them, so $\{v_1, v_2, v_3, z\}$ induce a diamond in $G$, a contradiction. □

The same example used in the case of diamond-free graphs shows that block graphs cannot be characterized by their 1-weighted clique graph.

### 4.3. Split graphs

A characterization of split graphs in terms of $K_{1,2}^w$ is the following.

**Theorem 25.** *Let $G$ be a graph. Then $G$ is a connected split graph if and only if $K_{1,2}^w(G)$ is a star with a center $M$ such that $w(M, M') = w(M') - 1$ for every vertex $M' \in V(K(G))$, $M' \neq M$.*

**Proof.** ($\Rightarrow$) Let $G$ be a connected split graph with split partition consisting of a maximal clique $M$ and a stable set $S$. Since $G$ is connected, each $v \in S$ is adjacent to some vertex of $M$. Since $S$ is a stable set, each $v \in S$ belongs to exactly one clique $M' \neq M$, and $M' \cap M \neq \emptyset$. Then, $K(G)$ is a star with center $M$. Moreover, each clique $M' \neq M$ contains exactly one vertex not in $M$, so $w(M', M) = w(M') - 1$, for each clique $M' \neq M$.

($\Leftarrow$) Let $G$ be a graph and suppose that $K_{1,2}^w(G)$ is a star with center $M$ and $w(M', M) = w(M') - 1$, for each $M' \in V(K(G))$, $M' \neq M$. Let $S = V(G) \setminus M$. Since for each clique $M'$ of $G$ different from $M$ we have $w(M', M) = w(M') - 1$, there is no clique of $G$ containing two vertices from $S$. So $S$ is a stable set, and $G$ is a split graph. □

The examples in Fig. 4 show that $\{K_1^w, K_2^w\}$ is not sufficient to characterize split graphs. Notice that having $\{K_1^w(G), K_2^w(G)\}$ is not the same as having $K_{1,2}^w(G)$, since when $K(G)$ has a non-trivial automorphism, one does not necessarily know the correspondence between vertices of $K_1^w(G)$ and vertices of $K_2^w(G)$.

### 4.4. Interval graphs

For interval and proper interval graphs, we have the following characterizations.

**Theorem 26.** *Let $G$ be a graph. Then $G$ is an interval graph if and only if $K_{2,3}^w(G)$ admits a linear ordering $M_1, \ldots, M_p$ of its vertices such that for every $1 \leq i < j < k \leq p$, we have $w(M_i, M_j, M_k) = w(M_i, M_k)$.*

**Proof.** ($\Rightarrow$) Let $G$ be an interval graph and let $M_1, \ldots, M_p$ be a canonical ordering of its cliques. Let $v \in M_i \cap M_k$, $i < j < k$, then $v \in M_j$. Therefore, $w(M_i, M_j, M_k) = w(M_i, M_k)$.

($\Leftarrow$) Let $\sigma = M_1, \ldots, M_p$ be a linear ordering of the vertices of $K_{2,3}^w(G)$ satisfying $w(M_i, M_j, M_k) = w(M_i, M_k)$ for every $1 \leq i < j < k \leq p$. Then, $\sigma$ corresponds to a canonical ordering of the cliques of $G$. Otherwise, there are three cliques $M_i, M_j, M_k$, $i < j < k$ in $\sigma$, and $v \in V(G)$ such that $v \in M_i, M_k$ and $v \notin M_j$. In that case, $w(M_i, M_k) \geq w(M_i, M_j, M_k) + 1$, thus $w(M_i, M_k) > w(M_i, M_j, M_k)$, a contradiction. □

**Theorem 27.** *Let $G$ be a graph. Then $G$ is a proper interval graph if and only if $K_{1,2}^w(G)$ admits a linear ordering $M_1, \ldots, M_p$ of its vertices such that for every triangle $M_i, M_j, M_k$, $1 \leq i < j < k \leq p$, we have $w(M_j) = w(M_i, M_j) + w(M_j, M_k) - w(M_i, M_k)$.*

**Proof.** ($\Rightarrow$) Let $G$ be a proper interval graph and $\sigma$ a canonical ordering of its vertices. Define an ordering $\prec$ over the cliques such that if $M, M'$ are two different cliques, $M \prec M'$ if the smallest vertex of $M$, with respect to $\sigma$, is smaller than all the vertices of $M'$. Since $\sigma$ is a canonical ordering, $\prec$ defines a total ordering $M_1, \ldots, M_p$ of the cliques of $G$. Let $M_i, M_j, M_k$ be three pairwise intersecting cliques such that $i < j < k$. Clearly, $|M_i \cap M_j \cap M_k| = |M_i \cap M_k|$, because $M_i, M_j, M_k$ are distinct cliques. Moreover, $|M_j| = |M_i \cap M_j| + |M_j \cap M_k| - |M_i \cap M_k|$, otherwise the vertices of some of the cliques would not be consecutive in $\sigma$. Then $M_1, \ldots, M_p$ is an ordering of the vertices of $K^w_{1,2}(G)$ satisfying $w(M_j) = w(M_i, M_j) + w(M_j, M_k) - w(M_i, M_k)$ for every triangle $M_i, M_j, M_k$, with $1 \le i < j < k \le p$.

($\Leftarrow$) For every three pairwise intersecting cliques $M_i, M_j, M_k$, it is clear that

$$w(M_j) \ge w(M_i, M_j) + w(M_j, M_k) - w(M_i, M_j, M_k). \tag{1}$$

By hypothesis, $K^w_{1,2}(G)$ admits a linear ordering of its cliques $\sigma = M_1, \ldots, M_p$ such that for every triangle $M_i, M_j, M_k$, $1 \le i < j < k \le p$,

$$w(M_j) = w(M_i, M_j) + w(M_j, M_k) - w(M_i, M_k). \tag{2}$$

From (1) and (2), we have that $w(M_i, M_j, M_k) \ge w(M_i, M_k)$, thus $w(M_i, M_j, M_k) = w(M_i, M_k)$. By Theorem 1, $G$ is an interval graph. If $G$ is claw-free, then it is a proper interval graph. So suppose that $G$ contains a claw induced by $\{v, x, y, z\}$, where $\{x, y, z\}$ is a stable set and $v$ is adjacent to $x$, $y$ and $z$. Let $M_i, M_j, M_k$ be cliques of $G$ containing $vx$, $vy$, $vz$, respectively. By symmetry, without loss of generality, we may assume $i < j < k$. By elementary finite set theory, $|M_j| = |M_i \cap M_j| + |M_j \cap M_k| - |M_i \cap M_j \cap M_k| + |M_j \setminus (M_i \cup M_k)|$. By (2) and since $w(M_i, M_j, M_k) = w(M_i, M_k)$, we conclude that $|M_j \setminus (M_i \cup M_k)| = 0$, that is, $M_j \subseteq M_i \cup M_k$. This is a contradiction, because $y$ does neither belong to $M_i$ nor to $M_k$, because it is not adjacent to $x$ and $z$, respectively. Therefore, $G$ is claw-free and, consequently, a proper interval graph. $\square$

The examples in Fig. 4 show that $\{K^w_1, K^w_2\}$ is not sufficient to characterize proper interval graphs.

### 4.5. Chordal and UV graphs

It is a known result that clique graphs of chordal graphs are dually chordal graphs [7,18,41]. Moreover, it holds that, for a chordal graph $G$, there is some compatible tree $T$ of $K(G)$ such that, for every vertex $v$ of $G$, the subgraph of $T$ induced by $\mathcal{M}_G(v)$ is a subtree. Such a tree is called a *clique tree* of $G$. Gavril [16] and independently Shibata [38] proved that those trees are exactly the maximum weight spanning trees of $K^w_2(G)$. Also in the context of chordal graphs, 2-weighted clique graphs where considered in [19–21,27,31]. Very recently, strongly chordal graphs and trivially perfect graphs were characterized by means of their $\{2, 3\}$-weighted clique graphs.

**Theorem 28** (*McKee, 2012 [32]*)**.** *Let $G$ be a graph. Then $G$ is strongly chordal if and only if, for every $k \ge 1$, every cycle of $K^w_{2,3}(G)$ whose edges have weight at least $k$ either has a chord of weight at least $k$ or is a triangle of weight at least $k$.*

**Theorem 29** (*McKee, 2012 [32]*)**.** *Let $G$ be a graph. Then $G$ is trivially perfect if and only if, for every $k \ge 1$, every two adjacent edges of $K^w_{2,3}(G)$ of weight at least $k$ lie in a triangle of weight at least $k$.*

In this work, we characterize chordal and *UV* graphs by means of their $\{2, 3\}$-weighted clique graphs.

**Theorem 30.** *Let $G$ be a connected graph. Then $G$ is chordal if and only if $K^w_{2,3}(G)$ admits a spanning tree $T$ such that for every three different vertices $M_1, M_2, M_3$ of $T$, if $M_2$ belongs to the path $M_1$–$M_3$ in $T$, then $w(M_1, M_2, M_3) = w(M_1, M_3)$.*

**Proof.** ($\Rightarrow$) Let $G$ be a connected chordal graph and $T$ be a clique tree of $G$. Then $T$ is a spanning tree of $K(G)$. If $G$ is a complete graph, the property trivially holds. Otherwise, let $M_1$ and $M_3$ be two cliques of $G$. The property is clearly true when $M_1 \cap M_3 = \emptyset$. So, suppose that $M_1$ and $M_3$ contain a common vertex $v$. The subtree $T_v$ of $T$ induced by $\mathcal{M}_G(v)$ contains vertices $M_1$ and $M_3$, and, being connected, it also contains all the vertices $M_2$ in the path $M_1$–$M_3$ in $T$. Then, those vertices are also in $\mathcal{M}_G(v)$, i.e., $M_2$ contains the vertex $v$. Therefore, $w(M_1, M_2, M_3) = w(M_1, M_3)$.

($\Leftarrow$) Let $T$ be a spanning tree of $K^w_{2,3}(G)$ satisfying the property above. Suppose that $G$ is not a chordal graph. Then, there is some vertex $v$ such that the graph $T_v$ induced by $\mathcal{M}_G(v)$ in $T$ is not connected. Since $T$ is connected, there is a path $M_1$–$M_3$ in $T$ joining two different connected components $C_1, C_2$ of $T_v$, with $M_1 \in C_1$ and $M_3 \in C_2$. That path contains at least one vertex $M_2$ corresponding to a clique not in $\mathcal{M}_G(v)$. In that case, $w(M_1, M_3) \ge w(M_1, M_2, M_3) + 1$, thus $w(M_1, M_3) > w(M_1, M_2, M_3)$, a contradiction. $\square$

Let $G$ be a connected *UV* graph, and let $(T, \mathcal{F})$ be a representation of $G$ as the intersection graph of a family of paths of a tree $T$, where $\mathcal{F}$ is the family of paths. By taking a smallest such tree $T$, we obtain that $V(T) = \mathcal{M}(G)$ and each path in $\mathcal{F}$ representing vertex $v$ corresponds to $\mathcal{M}_G(v)$ [19]. This is called a *clique tree* of the *UV* graph $G$.

**Theorem 31.** *Let $G$ be a connected graph. Then $G$ is UV if and only if $K^w_{2,3}(G)$ admits a spanning tree $T$ such that for every three different vertices $M_1, M_2, M_3$ of $T$, if $M_2$ belongs to the path $M_1$–$M_3$ in $T$, then $w(M_1, M_2, M_3) = w(M_1, M_3)$, and for every $M$ in $T$ and $M_1, M_2, M_3$ distinct neighbors of $M$ in $T$, we have $w(M_1, M_2, M_3) = 0$.*

**Proof.** ($\Rightarrow$) Let $G$ be a connected *UV* graph and $T$ be a clique tree of $G$. Since $T$ is also a clique tree in the sense of chordal graphs, following the proof of Theorem 30, $T$ satisfies that for every three different vertices $M_1, M_2, M_3$ of $T$, if $M_2$ belongs to the path $M_1$–$M_3$ in $T$, then $w(M_1, M_2, M_3) = w(M_1, M_3)$. In addition, for each vertex $v$ of $G$, the subgraph of $T$ induced by $\mathcal{M}_G(v)$ is a path. So, for every $M$ in $T$ and $M_1, M_2, M_3$ distinct neighbors of $M$ in $T$, we conclude $w(M_1, M_2, M_3) = 0$.

($\Leftarrow$) Let $T$ be a spanning tree of $K^w_{2,3}(G)$ satisfying the property above. In particular, by Theorem 30, $G$ is a chordal graph and, following the proof of that theorem, $T$ is a clique tree of $G$ in the sense of chordal graphs. So, for every vertex $v$ of $G$, the graph $T_v$ induced by $\mathcal{M}_G(v)$ in $T$ is connected. If for some vertex $v$ the tree $T_v$ it is not a path, then it should have a vertex $M$ of degree at least three. Let $M_1, M_2, M_3$ three distinct neighbors of $M$ in $T_v$. All of them contain vertex $v$, so $w(M_1, M_2, M_3) > 0$, a contradiction. □

The examples in Fig. 3 show that $K^w_{1,2}$ is not sufficient to characterize *UV* graphs.

### 4.6. Line graphs

Line graphs have been characterized in several ways, for instance by minimal forbidden induced subgraphs [3,4,39], coverings [26] and structural properties [40,43]. We present here a characterization in terms of $K^w_{1,2}$.

**Theorem 32.** *Let $G$ be a graph. Then $G$ is the line graph of a simple graph if and only if $K^w_{1,2}(G)$ admits a partition of its vertices into two (possibly empty) subsets $V_1, V_2$, such that:*

(a) *The weight of each vertex in $V_2$ is* 3*.*
(b) *The weight of each edge whose endpoints are in the same set of the partition is* 1*, and the weight of each edge whose endpoints are in different sets of the partition is* 2*.*
(c) *Each vertex in $V_2$ has at most three neighbors in $V_1$, and they are pairwise adjacent.*
(d) *Each pair of adjacent vertices in $V_2$ has exactly two common neighbors in $V_1$.*
(e) *Each triangle in $V_1$ has a common neighbor in $V_2$.*

**Proof.** ($\Rightarrow$) It is a known result that if $G$ is the line graph of a simple graph $H$, then a clique of $G$ is formed either by the edges of a triangle of $H$ or by the edges incident to a common vertex $v$ of $H$, when they are not properly contained in a triangle of $H$ (recall that this structure is called *star*, and $v$ is called the *center* of the star). Let $V_1$ be the set of cliques of $G$ corresponding to stars of $H$ and let $V_2$ be the set of cliques of $G$ corresponding to triangles of $H$. Point (a) is trivial by definition of $V_2$. If a triangle and a star of $H$ share an edge, then the star is centered at some vertex of the triangle, and in fact, they share two edges. On the other hand, two triangles can share at most one edge, and two stars centered at $v$ and $z$ respectively, share at most the edge $vz$, when it does exist, since $H$ is simple. This proves the points (b) and (c). If two triangles of $H$ share an edge $vz$, then vertices $v$ and $z$ have degree at least three in $H$, and the stars centered at them are maximal cliques of $G$, thus corresponding to vertices of $V_1$. These are the two neighbors of the vertices of $V_2$ corresponding to the triangles. Note that they do not share the (possible) third neighbor. This proves item (d). For the last point, if the stars of $H$ centered at vertices $u$, $v$ and $z$ pairwise share an edge, then $u$, $v$ and $z$ are pairwise adjacent, and there is a vertex in $V_2$ corresponding to the clique of $G$ formed by the edges of the triangle $uvz$, that intersects the three cliques corresponding to the stars.

($\Leftarrow$) Let $G$ be a graph such that $K^w_{1,2}(G)$ admits a partition of its vertices into two (possibly empty) subsets $V_1, V_2$, satisfying (a)–(e). We will use Krausz's characterization by building a collection $\mathcal{F}$ of complete sets of $G$ such that every edge of $G$ is contained in exactly one complete set of $\mathcal{F}$, and each vertex of $G$ is contained in at most two complete sets of $\mathcal{F}$.

Let us first observe some properties of the cliques of $G$. Let $M$ be a vertex of $V_2$ having three neighbors in $V_1$, say $M_1, M_2, M_3$ (by (c), $M$ cannot have more than three neighbors in $V_1$ and $M_1, M_2, M_3$ are pairwise adjacent). Note that, by (a) and (b), each of $M_1, M_2, M_3$ shares two vertices with $M$, $M$ has only three vertices, and each pair of cliques in $\{M_1, M_2, M_3\}$ shares one vertex. Then, $M_1 \cap M_2 \cap M_3 = \emptyset$ and each of $M_1, M_2, M_3$ covers a different edge of $M$. Thus, it follows by (e) that no vertex of $G$ belongs to three cliques corresponding to vertices in $V_1$.

Let $M$ be a vertex of $V_2$ having no neighbors in $V_1$. By (d), $M$ is an isolated vertex. By the same fact, if $M$ has only one neighbor in $V_1$, then it has no neighbors in $V_2$. In that case, if $M_1$ is the neighbor of $M$, by (b), then $M_1$ covers one of the three edges of $M$. No other clique of $G$ in $(V_1 \cup V_2) \setminus \{M, M_1\}$ contains a vertex of $M$.

Let $M$ be a vertex of $V_2$ having two neighbors $M_1, M_2$ in $V_1$. Each of $M_1, M_2$ shares two vertices with $M$ and they have only one vertex in common, so $M_1$ and $M_2$ cover two of the three edges of $M$. Namely, we can denote the vertices of $M$ as $v_1, v_2, v_3$ in such a way that $v_3 \in M_1 \cap M_2$, $v_1 \in M_1$ and $v_2 \in M_2$. Let us suppose that there is some other clique $M'$ in $V_2$, adjacent to $M$ and having less than three neighbors in $V_1$. By (d), $M'$ has exactly two neighbors in $V_1$ and they are $M_1$ and $M_2$, otherwise $M$ would have 3 neighbors in $V_1$. Reasoning as above, $M_1$ and $M_2$ share a common vertex with $M'$ and, as $|M_1 \cap M_2| = 1$ and $|M \cap M'| = 1$, it follows that $\{v_3\} = M \cap M'$. As a consequence, the endpoints of the edge of $M$ that is not covered by the cliques in $V_1$ do not belong to any other clique in $V_2$ having less than three neighbors in $V_1$.

The collection $\mathcal{F}$ will be composed by the cliques of $G$ that are vertices of $V_1$, plus the cliques of $G$ that are vertices of $V_2$ with no neighbors in $V_1$, plus two cliques of size 2 for each vertex of $V_2$ with one neighbor in $V_1$ (namely, the two edges that are not covered by the cliques corresponding to vertices in $V_1$, as discussed above), plus one clique of size 2 for each vertex of $V_2$ with two neighbors in $V_1$ (namely, the edge that is not covered by the cliques corresponding to vertices in $V_1$).

Each edge of $G$ belongs to some clique corresponding to a vertex in $(V_1 \cup V_2)$, and, by the analysis above, all the edges of $G$ are covered by $\mathcal{F}$. Moreover, by the way of selecting edges from cliques of $G$ corresponding to vertices in $V_2$ and since the intersection of two cliques corresponding to vertices in $V_1$ is at most one vertex, each edge of $G$ is covered by exactly one complete set of $\mathcal{F}$, as required.

In order to complete the proof, we point out that no vertex of $G$ belongs to three cliques corresponding to vertices in $V_1$. Also, by the analysis above, each vertex that is covered by some complete set of $\mathcal{F}$ that does not correspond to a vertex in $V_1$, is either covered by the clique corresponding to an isolated vertex in $V_2$, or by two edges of the clique corresponding to a vertex in $V_2$, or by one edge of the clique corresponding to a vertex in $V_2$ and one clique corresponding to a vertex in $V_1$, and does not belong to any other complete set in $\mathcal{F}$. □

The examples in Fig. 4 show that $\{K_1^w, K_2^w\}$ is not sufficient to characterize line graphs.

## 5. Conclusions

We have defined the concept of weighted clique graph $K^w_{m_1,\ldots,m_\ell}(G)$, which consists of adding a weight function $w$ to the clique graph $K(G)$. For each $i = 1, \ldots, \ell$ and each complete set formed by $m_i$ vertices of $K^w_{m_1,\ldots,m_\ell}(G)$, the weighting $w$ assigns to the set the cardinality of the intersection of the corresponding cliques of $G$. We have described a general characterization of weighted clique graphs, in terms similar to the characterization of Roberts and Spencer for clique graphs. We have then proved that the problem of recognizing 1-weighted clique graphs is NP-complete. Further, we have then formulated characterizations of several known classes of graphs, in terms of their weighted clique graphs, specifying the sets $\{m_1, \ldots, m_\ell\}$ needed in each case. These classes include chordal graphs, interval graphs, line graphs, proper interval graphs, trees, among others.

Some questions related to the above results remain.

(1) The first one concerns the recognition of clique graphs. The question is whether the additional information carried by weighted clique graphs with respect to clique graphs can be used to recognize them, in polynomial time in the size of $K(G)$. More precisely, can we find some set of integers $\{m_1, \ldots, m_\ell\}$ and some polynomial-time algorithm to recognize $\{m_1, \ldots, m_\ell\}$-weighted clique graphs?
(2) In case of an affirmative answer of the above question, what would be the minimum $m_\ell$ and minimum size of $|\{m_1, \ldots, m_\ell\}|$ to obtain such an algorithm?
(3) What is the complexity of recognizing $\{m_1, \ldots, m_\ell\}$-weighted clique graphs for different sets $\{m_1, \ldots, m_\ell\}$?
(4) Describe a characterization of circular-arc graphs, in terms of their weighted clique graphs.

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