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On the affine group of a normal homogeneous manifold

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Abstract A very important class of homogeneous Riemannian manifolds are the so-called normal homogeneous spaces, which have associated a canonical connection. In this study, we obtain geometrically the (connected component of the) group of affine transformations with respect to the canonical connection for a normal homogeneous space. The naturally reductive case is also treated. This completes the geometric calculation of the isometry group of naturally reductive spaces. In addition, we prove that for normal homogeneous spaces the set of fixed points of the full isotropy is a torus. As an application of our results it follows that the holonomy group of a homogeneous fibration is contained in the group of (canonically) affine transformations of the fibers; in particular, this holonomy group is a Lie group (this is a result of Guijarro and Walschap).

Keywords Naturally reductive · Normal homogeneous · Canonical connection · Transvection group · Affine group · Isometry group

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1 Introduction

Compact normal homogeneous Riemannian manifolds, or, more generally, naturally reductive spaces, are a very important family of homogeneous spaces. These spaces appear in a natural way as a generalization of symmetric spaces. Associated with a naturally reductive space there is a canonical connection. For symmetric spaces, the Levi-Civita connection is a canonical connection. For general naturally reductive spaces, this is not anymore true, but they have the following important property that characterizes them: there exists a canonical connection which has the same geodesics as the Levi-Civita connection.

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Examples of naturally reductive and normal homogeneous spaces are the so-called isotropy irreducible spaces, strongly or not, which were classified by Wolf [14], in the strong case, and Wang and Ziller [13] in the general case.

If $M = G/H$ is a compact naturally reductive space with associated canonical connection ∇^c, then the group $\mathrm{Aff}(\nabla^c)$, of ∇^c-affine transformations, is a compact subgroup of $\mathrm{Iso}(M)$ (this is done in Sect. 3). Moreover, if M is a normal homogeneous space, then we shall prove that the flow of any G-invariant field is ∇^c-affine and therefore, an isometry (see Theorem 3.2 and Corollary 3.5). The aim of this article is to compute, in a geometric way, the (connected component of the identity of the) full ∇^c-affine group. Namely, our main result is the following theorem.

Theorem 1.1 *Let $M = G/H$ be a compact normal homogeneous space, $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. Let ∇^c be the canonical connection associated with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. Then, the affine group of the canonical connection is given by*

$$\mathrm{Aff}_0(\nabla^c) = G_1 \times K \quad (\textit{almost direct product}),$$

where K denotes the connected subgroup of ∇^c-affine transformations whose Lie algebra consists of the G-invariant fields and G_1 is the semisimple part of G.

Note that given $p \in M$, K is naturally identified with F_p, the connected component by p of the set of fixed points of the isotropy group $H = G_p$. There is another natural identification for F_p. Let $\bar{K}(p) = \{k|_{F_p} : k \in G,\ k \cdot F_p = F_p\}$, then $\bar{K}(p)$ acts simply transitively on F_p. In this way, F_p is naturally identified with a Lie group. Both groups K and $\bar{K}(p)$ are isomorphic, and their actions on F_p can be regarded as right and left multiplication, respectively.

Theorem 1.1 also holds for M naturally reductive if one replaces $\mathrm{Lie}(K)$ by the $\mathrm{Tr}(\nabla^c)$-invariant fields whose associated flow are ∇^c-affine (see Remark 4.3). Recall that $\mathrm{Tr}(\nabla^c)$ is the group of transvections of the canonical connection, which is a normal subgroup of $\mathrm{Aff}(\nabla^c)$.

It is a non-trivial problem to decide, for naturally reductive metrics, when the group $\mathrm{Aff}_0(\nabla^c)$ coincides with the full (connected) isometry group $\mathrm{Iso}_0(M)$. In [10], it is proved that this is always true, except for spheres. Such a result depends strongly on a Berger-type result, the so-called Skew-torsion Holonomy Theorem, see also [1] and [9]. Namely,

Theorem 1.2 ([10]) *Let $M = G/H$ be a compact naturally reductive space, which is locally irreducible, and let ∇^c be the associated canonical connection. Assume that M is neither (globally) isometric to a sphere, nor to a real projective space. Then,*

(i) $\mathrm{Iso}_0(M) = \mathrm{Aff}_0(\nabla^c)$.
(ii) *If* $\mathrm{Iso}(M) \not\subset \mathrm{Aff}(\nabla^c)$ *then M is isometric to a simple Lie group, endowed with a bi-invariant metric.*

We have the following corollary which determines, in a geometric way, the full isometry group.

Corollary 1.3 *Let $M = G/G_p$, $p \in M$, be a compact normal homogeneous space. Assume that M is locally irreducible and that $M \neq S^n$, $M \neq \mathbb{R}P^n$. Write $G = G_0 \times G_1$ as a almost direct product, where G_0 is abelian and G_1 is a semisimple Lie group of the compact type. Then,*

$$\mathrm{Iso}_0(M) = G_1 \times K,$$

where K denotes the connected component by p of the set of fixed points of G_p (regarded as a Lie group). In particular, $\mathrm{Iso}(M)$ *is semisimple if and only if K is semisimple.*

We also study the set of fixed points of the isotropy of the full ∇^c-affine group. By making use of the above corollary, we obtain the following result.

Theorem 1.4 *Let M be a compact normal homogeneous space and let S_p the connected component by p of the set of fixed points of* $\mathrm{Iso}_0(M)_p$*. Then, S_p is a torus (eventually trivial).*

As an application of our main results it follows that the holonomy group of a homogeneous fibration is contained in the group of (canonically) affine transformations of the fibers; in particular, this holonomy group must be a Lie group (this is a result of Guijarro and Walschap [5]).

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2 Preliminaries

Let $M = G/H$ be a compact homogeneous Riemannian manifold, where G is a Lie subgroup of $\mathrm{Iso}(M)$. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

be a reductive decomposition of the Lie algebra of G (where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathfrak{g}$). Associated with this decomposition, there is a canonical connection ∇^c in M, which is G-invariant. The ∇^c-geodesics through $p = eH$ are given by

$$\mathrm{Exp}(tX) \cdot p, \quad X \in \mathfrak{m}.$$

Moreover, the ∇^c-parallel transports along these geodesics are given by $\mathrm{Exp}(tX)_*$. The canonical connection has parallel curvature and torsion. In general, any G-invariant tensor is ∇^c-parallel. Therefore, in particular, any geometric tensor, such as the metric tensor and the Riemannian curvature tensor, is ∇^c-parallel.

An important class of this spaces are the so-called naturally reductive spaces. Geometrically, M is naturally reductive if the Riemannian geodesics coincide with the ∇^c-geodesics. This definition is equivalent to the following algebraic condition:

$$\langle [X, Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X, Z]_{\mathfrak{m}}\rangle = 0$$

for all $X, Y, Z \in \mathfrak{m} \simeq T_{pM}$. In fact, this follow from the Koszul formula and the Killing equation.

For a naturally reductive space, we can compute explicitly the Levi-Civita connection and the canonical connection. In fact,

$$(\nabla_{\widetilde{X}} \widetilde{Y})_p = \frac{1}{2}[\widetilde{X}, \widetilde{Y}](p) = -\frac{1}{2}[X, Y]_{\mathfrak{m}},$$
$$(\nabla^c_{\widetilde{X}} \widetilde{Y})_p = [\widetilde{X}, \widetilde{Y}](p) = -[X, Y]_{\mathfrak{m}},$$

where $\widetilde{W}$ is the Killing field on M induced by $W \in \mathfrak{m}$. In fact, since $\nabla_{\widetilde{X}} \widetilde{X} = 0$, one has that $\nabla_{\widetilde{X}} \widetilde{Y} = -\nabla_{\widetilde{Y}} \widetilde{X}$, and therefore

$$[\widetilde{X}, \widetilde{Y}] = \nabla_{\widetilde{X}} \widetilde{Y} - \nabla_{\widetilde{Y}} \widetilde{X} = 2\nabla_{\widetilde{X}} \widetilde{Y}.$$

The second identity is a direct consequence of the formula of the Lie derivative in terms of the flow.

A distinguishes class of naturally reductive spaces are the normal homogeneous spaces. In this case, $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{h}$ with respect to a bi-invariant metric in G, i.e.,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$$

is the associated reductive decomposition. Then, the quotient projection $G \to M$ is a Riemannian submersion and therefore, maps horizontal geodesics of G into geodesics of M.

3 G-invariant fields and the canonical connection

Let $M = G/G_{\mathrm{p}}$ $(p \in M)$ be a Riemannian homogeneous space with a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. If $g \in G$, $\mathfrak{g} = \mathrm{Ad}(g)\mathfrak{h} \oplus \mathrm{Ad}(g)\mathfrak{m}$ is also a reductive decomposition, which has associated with the same canonical connection, since ∇^c is G-invariant. Thus, we obtain the geodesics through $q = g \cdot p$, which are given by $\mathrm{Exp}(tY)g \cdot p$, $Y \in \mathrm{Ad}(g)\mathfrak{m}$, and the parallel transports along these geodesics are given by $\mathrm{Exp}(tY)_*$.

Next, we prove that the flow of any G-invariant fields is ∇^c-affine, for a normal homogeneous space.

Let G be a Lie group acting on M by ∇^c-affine diffeomorphisms and let X be a G-invariant field on M with local flow φ_{t} (i.e., $g_*(X) = X$ for all $g \in G$, which is equivalent to the fact that any element of G commutes with φ_{t}). Let

$$F_p = \{q \in M : G_p \cdot q = q\}$$

be the set of fixed points of the isotropy at p. Then, as it is well-known, F_p is a closed and totally geodesic submanifold of M. Moreover,

$$\varphi_{\mathrm{t}}(F_p^o) = F_p^o, \tag{3.1}$$

where F_p^o is the connected component of F_p that contains p. In fact, let $q \in F_p$, then $\varphi_{\mathrm{t}}(q) = \varphi_{\mathrm{t}}(G_p \cdot q) = G_p \cdot \varphi_{\mathrm{t}}(q)$, hence $\varphi_{\mathrm{t}}(F_p) \subset F_p$. Applying this to φ_{-t}, we obtain $\varphi_{\mathrm{t}}(F_p^o) = F_p^o$.

The next step is to observe that if the canonical connection is nice (the closure of G is compact in this case, since M is compact) and G acts transitively on M, then the isotropy group does not change along the fixed points, i.e.,

$$G_p = G_q \quad \text{for all } q \in F_p. \tag{3.2}$$

This is easy to prove. In fact, if $q = g \cdot p \in F_p$ then $G_q = gG_pg^{-1}$, but $G_p = gG_pg^{-1}$.

Remark 3.1 Let G be a Lie group and H a Lie subgroup of G, then the Lie algebra of H is $\mathrm{Ad}(N(H))$-invariant, where $N(H)$ is the normalizer of H in G. Moreover, if G admits a bi-invariant metric then, the orthogonal subspace to the Lie algebra of H is also $\mathrm{Ad}(N(H))$-invariant.

Theorem 3.2 *Let $M = G/G_p$ be a compact normal homogeneous space and let ∇^c be the canonical connection associated with the reductive decomposition $\mathfrak{g} = \mathfrak{g}_p \oplus \mathfrak{g}_p^{\perp}$. Then, the flow of any G-invariant field is ∇^c-affine.*

Proof Let X be a G-invariant field with associated flow φ_t. We will prove that φ_t maps geodesics into geodesics and ∇^c-parallel fields along these geodesics into ∇^c-parallel fields along the image geodesics. Note that from this fact, we obtain that φ_t is ∇^c-affine.

By the homogeneity of M, it suffices to prove this for geodesics starting at p. These geodesics are given by $\mathrm{Exp}(sY) \cdot p$, $Y \in \mathfrak{g}_p^{\perp}$. Since X is G-invariant, φ_t commutes with G, and hence

$$\varphi_t(\mathrm{Exp}(sY) \cdot p) = \mathrm{Exp}(sY) \cdot \varphi_t(p).$$

Now, by equalities 3.1 and 3.2, $\varphi_t(p) \in F_p$ and $G_{\varphi_t(p)} = G_p$. Let $g \in G$ be such that $\varphi_t(p) = g \cdot p$. Observe that g must lie in the normalizer of G_p in G. In fact,

$$(g^{-1}G_p g) \cdot p = (g^{-1}G_p) \cdot (g \cdot p) = (g^{-1} \cdot (G_{g \cdot p} \cdot (g \cdot p))) = p.$$

Hence $g^{-1}G_p g \subset G_p$, and thus $g \in N(G_p)$.

However, since $\mathrm{Ad}(g)\mathfrak{g}_p^{\perp} = \mathfrak{g}_p^{\perp}$ (by Remark 3.1), $\varphi_t(\mathrm{Exp}(sY) \cdot p)$ is a geodesic through $g \cdot p$. Finally, the ∇^c-parallel transport along $\mathrm{Exp}(sY) \cdot p$ are given by $\mathrm{Exp}(sY)_*$, which is mapped to $\mathrm{Exp}(sY)_* \circ d\varphi_t$. Hence, ∇^c-parallel fields along geodesics are mapped into ∇^c-parallel fields. □

Remark 3.3 Note that, with the same proof of Theorem 3.2, if φ is a diffeomorphism that commutes with G, then φ is ∇^c-affine.

Corollary 3.4 *Let $M = G/H$ be a compact normal homogeneous space and let X be a G-invariant field on M, then X is a Killing field.*

The proof Corollary 3.4 is consequence from Theorem 3.2 and the following proposition.

Proposition 3.5 *Let M be a compact naturally reductive space with associated canonical connection ∇^c. Then,* $\mathrm{Aff}_0(\nabla^c) \subset \mathrm{Iso}(M)$.

Proof Recall that ∇^c and ∇ have both the same geodesics. Now, if φ is ∇^c-affine, φ maps (Riemannian) geodesics into geodesics, hence φ is ∇-affine (since ∇ is torsion free, see [11, p. 107]). In order to complete the proof, we need the following well-known lemma, for which we include a conceptual proof. □

Lemma 3.6 *Let M be a compact Riemannian manifold and let X be an affine Killing field on M (i.e., the flow associated to X preserves the Levi-Civita connection). Then, X is a Killing field on M.*

Proof Since X is an affine Killing field, then X is a Jacobi field along any geodesic (since the flow of X map geodesics into geodesics, due to the fact that the flow of X is given by affine transformations). Take a unit speed geodesic $\gamma(t)$ and let us write $X(\gamma(t)) = J^T(t) + J^{\perp}(t)$ as the sum of two perpendicular Jacobi fields along $\gamma(t)$, where $J^T(t) = (at+b)\gamma'(t)$. Since M is compact, any field is bounded, thus $a = 0$. Hence $J(t) = b\gamma'(t) + J^{\perp}(t)$. Differentiating with respect to t both sides of the equation $b = \langle J(t), \gamma'(t)\rangle$, we obtain that

$$\langle \nabla_{\gamma'} X, \gamma' \rangle = b \langle \nabla_{\gamma'} \gamma', \gamma' \rangle = 0.$$

Since $\gamma(t)$ is arbitrary, X satisfies the Killing equation. □

Remark 3.7 Let M be a compact naturally reductive space. By Proposition 3.5, the connected component of the group of affine transformations of ∇^c is contained in $\mathrm{Iso}(M)$. Moreover, as it is not difficult to see, $\mathrm{Aff}_0(\nabla^c)$ is a closed subgroup of $\mathrm{Iso}(M)$, and thus $\mathrm{Aff}_0(\nabla^c)$ is compact. For an arbitrary linear connection in a compact manifold, the connected component of the affine group needs not to be compact (see, for instance, [16]).

4 The transvections and the affine group

Let $M = G/H$ be a compact normal homogeneous space and consider $\mathfrak{k}$ the Lie algebra of G-invariant fields on M. In the previous section, we see that $\mathfrak{k} \subset \mathfrak{aff}(\nabla^c)$, the Lie algebra of $\mathrm{Aff}_0(\nabla^c)$. Therefore, $\mathfrak{k}$ determines a connected Lie subgroup $K \subset \mathrm{Aff}_0(\nabla^c)$. If X is a G-invariant field, then the flow associated to X commutes with G. Hence G and K commute, and we have

$$G \subset G \cdot K \subset \mathrm{Aff}_0(\nabla^c).$$

The purpose of this section is to prove that $G \cdot K = \mathrm{Aff}_0(\nabla^c)$.

The key of the proof is to show that G is a normal subgroup of $\mathrm{Aff}_0(\nabla^c)$ and choose a complementary ideal of $\mathfrak{g} = \mathrm{Lie}(G)$, which is contained in $\mathfrak{k}$. Note that this depends strongly on the fact that $\mathrm{Aff}_0(\nabla^c)$ is a compact Lie group (Remark 3.7). In fact, we will prove that G coincides with the group of transvections of the canonical connection.

The canonical transvection group consists of all ∇^c-affine transformations that preserve the ∇^c-holonomy sub-bundles of the orthogonal frame bundle.

Remark 4.1 Let $M = G/H$ be a homogeneous space, where H is the isotropy at $p \in M$. It is well-known that the ∇^c-parallel transport along any curve is realized by elements of G. Then, if φ is a transvection, $\varphi \in G$. Hence,

$$\mathrm{Tr}(\nabla^c) \subset G.$$

Moreover, $\mathrm{Tr}(\nabla^c)$ is a connected and normal subgroup of $\mathrm{Aff}_0(\nabla^c)$ (and therefore of G), which is transitive on M. The Lie algebra of $\mathrm{Tr}(\nabla^c)$, as it is well-known, is given by

$$\mathfrak{tr}(\nabla^c) = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$$

(not a direct sum, in general). The transvection group needs not to be a closed subgroup of G. If we present $M = \mathrm{Tr}(\nabla^c)/\mathrm{Tr}(\nabla^c)_{\mathrm{p}}$, then the original canonical connection coincides with the canonical connection associated with the reductive decomposition $\mathfrak{tr}(\nabla^c) = [\mathfrak{m}, \mathfrak{m}]_{\mathfrak{h}} + \mathfrak{m}$.

Proposition 4.2 *Let $M = G/H$ be a compact normal homogeneous space with associated canonical connection ∇^c. Then, $G = \mathrm{Tr}(\nabla^c)$. In particular, G is a normal subgroup of* $\mathrm{Aff}_0(\nabla^c)$.

Proof Recall that $\mathrm{Tr}(\nabla^c)$ is a normal subgroup of G. Let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{tr}(\nabla^c)$, i.e., $\mathfrak{g} = \mathfrak{tr}(\nabla^c) \oplus \mathfrak{a}$ (orthogonal sum). Since $\mathfrak{tr}(\nabla^c) = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$, then $\mathfrak{a}$ is orthogonal to $\mathfrak{m}$, and hence $\mathfrak{a} \subset \mathfrak{h}$. Therefore, $\mathfrak{a}$ is invariant by any isotropy group, since any two isotropy groups are conjugates. Since G acts effectively on M, we conclude that $\mathfrak{a} = 0$. □

Proof of Theorem 1.1 We will make use of G-invariant fields (whose associated flow is ∇^c-affine, by Theorem 3.2). Now, since G is a normal subgroup, then the Lie algebra of G, $\mathfrak{g}$ is an ideal of $\mathfrak{aff}(\nabla^c)$. Let $\mathfrak{g}'$ be a complementary ideal, i.e.

$$\mathfrak{aff}(\nabla^c) = \mathfrak{g} \oplus \mathfrak{g}'$$

(for example, choosing $\mathfrak{g}'$ to be the orthogonal complement with respect to a bi-invariant metric in $\mathrm{Aff}_0(\nabla^c)$, which is compact). If $X \in \mathfrak{g}'$, then $[X, \mathfrak{g}] = 0$. Therefore, X belongs to $\mathfrak{k}$, the Lie algebra of G-invariant fields. Then $\mathrm{Aff}_0(\nabla^c) \subset G \cdot K$, which implies the equality. In order to complete the proof of Theorem 1.1 see the proof of Corollary 1.3 as follows. □

Proof of Corollary 1.3 From Theorems 1.1 and 1.2, one has that

$$\mathrm{Iso}_0(M) = \mathrm{Aff}_0(\nabla^c) = G \cdot K.$$

Now, since G_0 is abelian and G commutes with K, $G_0 \subset K$. Moreover, if X is a Killing field induced by G_1 which coincides with a Killing field induced by K, one has $[X, \mathfrak{g}_1] = 0$, where $\mathfrak{g}_1 = \mathrm{Lie}(G_1)$, since G_1 commutes with K. Hence $X = 0$, since G_1 is semisimple. □

Remark 4.3 *The naturally reductive case*: Let M be a compact naturally reductive space with associated canonical connection ∇^c, and let $p \in M$. One has that $\mathrm{Aff}_0(\nabla^c)$ is a compact subgroup of $\mathrm{Iso}(M)$ (by Remark 3.7). Take the presentation $M = \mathrm{Tr}(\nabla^c)/\mathrm{Tr}(\nabla^c)_p$. In this case, one cannot prove that the flows of $\mathrm{Tr}(\nabla^c)$-invariant fields are ∇^c-affine, since the reductive complement is not necessarily $\mathrm{Ad}(N(\mathrm{Tr}(\nabla^c)_p))$-invariant (see Remark 3.1). However, with the same arguments of this section, to calculate the affine group one can supplement $\mathfrak{tr}(\nabla^c)$ with an ideal of $\mathfrak{aff}(\nabla^c)$. In fact, if $\tilde{\mathfrak{k}}$ is the Lie algebra generated by the $\mathrm{Tr}(\nabla^c)$-invariant fields whose flows are ∇^c-affine, and if $\widetilde{K}$ is the connected Lie subgroup of $\mathrm{Aff}(\nabla^c)$ associated with $\tilde{\mathfrak{k}}$, then

$$\mathrm{Aff}_0(\nabla^c) = \mathrm{Tr}(\nabla^c) \cdot \widetilde{K},$$

and $\mathrm{Tr}(\nabla^c)$ commutes with $\widetilde{K}$.

Remark 4.4 Let $M = G/H$ be (compact) naturally reductive with an associated canonical connection ∇^c. It is a well-known fact that the affine group $\mathrm{Aff}(\nabla^c)$ is given by the diffeomorphisms of M which map ∇^c-geodesics into ∇^c-geodesics and preserve the torsion tensor. If M is simply connected, there is a standard way of enlarging the group G. More precisely, any linear isometry $\ell : T_pM \to T_qM$, with $\ell(R^c_p) = R^c_q$ and $\ell(T^c_p) = T^c_q$ extends to an isometry of M (since the canonical connection has ∇^c-parallel curvature R^c and torsion T^c). In certain cases, the standard extension is trivial. For example, for $S^7 = \mathrm{Spin}(7)/G_2$ (strongly isotropy irreducible presentation, see [14]).

4.1 The fixed points of the isotropy

We close this section with a comment on the fixed points of the full isotropy. Let $M = G/G_p$ be a (compact) normal homogeneous space and let ∇^c be the associated canonical connection. Let F_p be the connected component by p of the set of fixed points of G_p (recall that the isotropy does not change along the set of fixed points, by 3.2). This induces a foliation $\mathcal{F}$ of M with totally geodesics leaves F_p, since ∇^c-affine transformations are isometries of M. Note that F_p is a Lie group with a bi-invariant metric. In fact, consider the subgroup H of G such that leaves F_p invariant. Since G is transitive and preserves the foliation $\mathcal{F}$, one must have that H is simply transitive on F_p. Therefore, F_p is naturally identified with H endowed with a bi-invariant metric. In fact, the right invariant fields on $H \simeq F_p$ are the restrictions to F_p of the Killing fields induced by H. Moreover, the left invariant fields on H correspond to the restrictions to F_p of the G-invariant fields on M, which are Killing fields on M, since the metric is normal homogeneous (see Theorem 3.2).

Proof of Theorem 1.4 In order to apply Corollary 1.3, we can assume M to be irreducible (eventually, by passing to the universal cover of M; see [10, Remark 6.5]). We keep the notation of the above paragraph. Let S_p be the connected component by p of the set of fixed points of $\mathrm{Iso}_0(M)_p$. Then, S_p is a totally geodesic submanifold of M which is contained in F_p. Moreover, it is not hard to see that S_p coincides (locally) with the Euclidean de Rham factor of the symmetric space F_p (see the above paragraph). This implies that S_p is a flat torus. □

5 Application to homogeneous fibrations

In this section, we will show that the holonomy group of a homogeneous fibration (i.e., the fibers are given by the orbits of a compact Lie group) may be regarded as a subgroup of the affine transformations of the fibers, with respect to some canonical connection. In particular, this implies that this holonomy group must be a Lie group [5].

General results on submersions $\pi : M \to B$ are due to Ehresmann and Hermann. Namely, *if any fiber is connected and compact, then π is a fiber bundle* [3,12]. Moreover, *if the submersion is Riemannian, M is complete, and M, B are both connected, then π is a fiber bundle* [6,12].

Let $\pi : M \to B$ be a G-homogeneous (metric) fibration, where π is a Riemannian submersion, M is complete and G is a compact subgroup of the isometries of M (and all the G-orbits are principal). One can decompose orthogonally $TM = \mathcal{V} \oplus \mathcal{H}$, where $\mathcal{V} = \ker d\pi$ is the vertical distribution and $\mathcal{H}$ is the so-called horizontal distribution. One can lift horizontally curves in B, and this lift is unique for any arbitrary initial condition in the fiber. Denote by $M_b := \pi^{-1}(\{b\}) = G \cdot b$ the fiber at b. The holonomy group Φ_b of π at $b \in B$ is the group of holonomy diffeomorphisms of M_b induced by horizontal lifts of loops at b. Observe that, in general, Φ_b is not necessarily a Lie group, a counterexample can be found in [2, Remark 9.57].

Let us show that the holonomy group Φ_b is a subgroup of affine transformations with respect to a canonical connection in the fiber M_b (which is a normal homogeneous space, since G is compact). We fix some notation before continuing. Let $\alpha : [0, 1] \to B$, $\alpha(0) = \alpha(1) = b$, be a loop at b. For $p \in M_b$, let $\tilde{\alpha}_p$ be the unique horizontal lift of α with $\tilde{\alpha}_p(0) = p$. The holonomy diffeomorphism $\varphi \in \Phi_b$, induced by α, is given by $\varphi(p) = \tilde{\alpha}_p(1)$. Since the horizontal distribution is G-invariant, φ commutes with G, i.e., $\varphi(g \cdot p) = g \cdot \varphi(p)$.

Each fiber M_{b} can be regarded as a normal homogeneous space. In fact, let H be the isotropy subgroup at p and let ∇^c be the canonical connection associated with the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. Let $\varphi \in \Phi_b$. Since φ commutes with G, we can apply Remark 3.3 to obtain that φ is ∇^c-affine. Hence $\Phi_b \subset \mathrm{Aff}(\nabla^c)$. Moreover, any element of $\mathrm{Lie}(\Phi_b)$ is a G-invariant field.

Corollary 5.1 [[5]] *Let $\pi : M \to B$ be a G-homogeneous fibration, where G is a compact Lie group, and let $b \in B$. Then, Φ_b is a Lie group.*

Proof From the previous comments (see Corollary 5.1), we have that $(\Phi_b)_0$ is an abstract subgroup of $\mathrm{Aff}_0(\nabla^c)$. Hence, from a result of Goto [4] (see also [15]), Φ_b is a Lie group, since $\mathrm{Aff}(\nabla^c)$ is a Lie group (see [7,8]). □

Remark 5.2 By Remark 3.7, $\mathrm{Aff}(\nabla^c)$ is a compact Lie group. Hence, the closure of the holonomy group of a homogeneous fibration Φ_b is compact in the affine group $\mathrm{Aff}(\nabla^c)$. Moreover, by Proposition 3.5, Φ_b is a Lie subgroup of $\mathrm{Iso}(M_b)$, where $M_b = G \cdot b$ is endowed with a normal homogeneous metric.

Remark 5.3 Let $\pi : M \to B$ be a G-homogeneous fibration, and let $b \in B$. The fiber M_b at b is a (normal) homogeneous space, namely $M_b = G/H$, where H is the isotropy group at a point $p \in M_b$. Assume that p is the only fixed point of H in M_b. Then, the holonomy group Φ_b at b is trivial. In fact, let α be a loop at b and let $\tilde{\alpha}$ be the horizontal lift of α with $\tilde{\alpha}(0) = p$. Since $\tilde{\alpha}(0)$ is fixed by H, and H maps horizontal curves into horizontal curves, we have that $\tilde{\alpha}(t)$ is fixed by the isotropy group, for all t. In particular, $\tilde{\alpha}(1) = p$. Observe that p is arbitrary, since the isotropy groups at different points are conjugated.

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