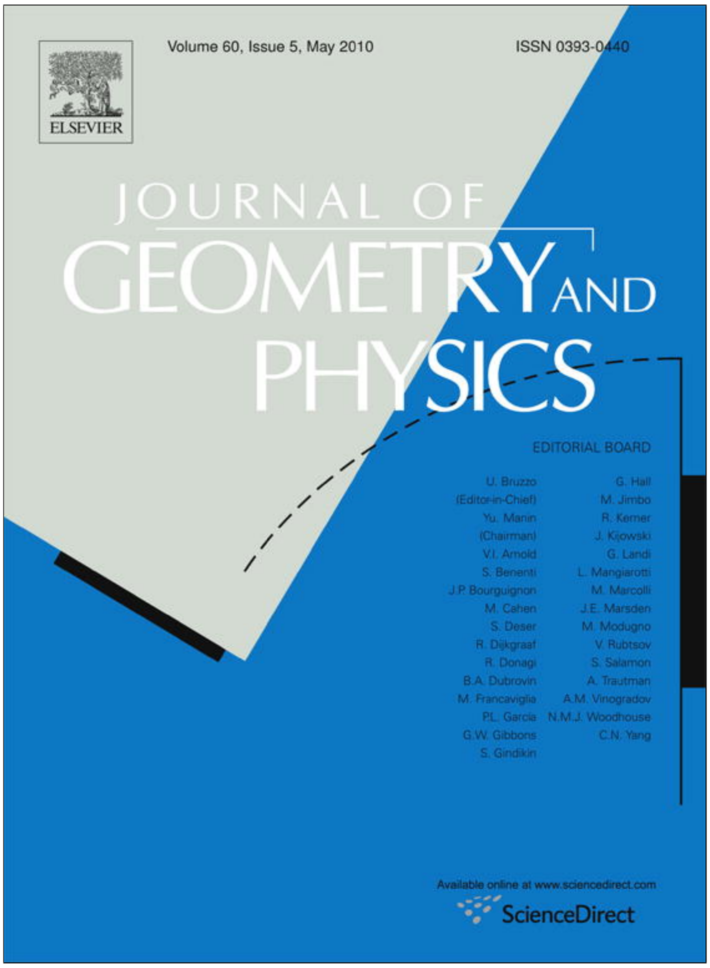
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$\mathbb{Z}_2$-cohomology and spectral properties of flat manifolds of diagonal type

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ABSTRACT

We study the cohomology groups with $\mathbb{Z}_2$-coefficients for compact flat Riemannian manifolds of diagonal type $M_\Gamma = \Gamma \setminus \mathbb{R}^n$ by explicit computation of the differentials in the Lyndon–Hochschild–Serre spectral sequence. We obtain expressions for $H^j(M_\Gamma, \mathbb{Z}_2)$, $j = 1, 2$ and give an effective criterion for the non-vanishing of the second Stiefel–Whitney class $\mathrm{w}_2(M_\Gamma)$. We apply the results to exhibit isospectral pairs with special cohomological properties; for instance, we give isospectral 5-manifolds with different $H^2(M_\Gamma, \mathbb{Z}_2)$, and isospectral 4-manifolds M, M' having the same $\mathbb{Z}_2$-cohomology where $\mathrm{w}_2(M) = 0$ and $\mathrm{w}_2(M') \neq 0$. We compute the $\mathbb{Z}_2$-cohomology of all generalized Hantzsche–Wendt n-manifolds for $n = 3, 4, 5$ and we study H^2 and w_2 for a large n-dimensional family, $\mathcal{K}_n$, with explicit computation for a subfamily of examples due to Lee and Szczarba.

0. Introduction

Let Γ be a Bieberbach group, M_Γ the associated compact flat Riemannian manifold and let F be the holonomy group of Γ. The main goal of this paper is to make a step in the computation of the cohomology groups $H^*(M_\Gamma, \mathbb{Z}_2)$ and to find connections with spectral geometry of M_Γ under the assumption $F \cong \mathbb{Z}_2^k$.

Since M_Γ is an Eilenberg–Mac Lane space, its cohomology coincides with the group cohomology of Γ; that is, $H^*(M_\Gamma, R) \cong H^*(\Gamma, R)$ for any ring of coefficients R. If m is the characteristic of R and $(|F|, m) = 1$, then it is well known that $H^*(\Gamma, R) \cong \bigwedge^*(\mathbb{Z}^n \otimes R)^F$, the F-invariants in the holonomy representation (see [1]). However, if $(|F|, m) \neq 1$ the Lyndon–Hochschild–Serre (LHS) spectral sequence does not degenerate at E_2 and hence, in order to compute the cohomology, it is necessary to investigate the maps in the spectral sequence more deeply. A typical case occurs when $m = 2$ and the holonomy group $F \cong \mathbb{Z}_2^k$.

We shall follow the approach in [2] by studying the differentials that occur in the LHS spectral sequence. For a large class of flat manifolds, we will obtain explicit formulas for $H^2(\Gamma, \mathbb{Z}_2)$, which in many cases can be efficiently computed.

Since the $E_2^{p,q}$-term of the LHS spectral sequence involves $H^p(\mathbb{Z}_2^k, \mathbb{Z}_2)$ (see (2.4)) and since the cohomology $H^*(\mathbb{Z}_2^k, \mathbb{Z}_2)$ is a polynomial algebra in k indeterminates, it will turn out that the first and second cohomology of M_Γ can be expressed in terms of the ranks of the differentials which, in turn, can be obtained from the images of what we call $\mathbb{Z}_2$-*class polynomials*, which are polynomials in the variables $x_1, \ldots, x_k$ obtained by a simple rule (see Section 1.4, Remark 1.4). The main results in Section 2 are stated in Theorems 2.5 and 2.7.

In Section 3, we will use again the $\mathbb{Z}_2$-class polynomials to give a useful criterion to decide on the non-vanishing of the second Stiefel–Whitney class w_2 of M_Γ (see Theorem 3.3).

In Section 4, we apply the results in the previous sections to study the $\mathbb{Z}_2$-cohomology of families and pairs having similar spectral properties.

We concentrate primarily on the generalized Hantzsche–Wendt manifolds, GHW manifolds for short (see [3]). These are the n-dimensional flat manifolds with holonomy group $\mathbb{Z}_2^{n-1}$; the orientable ones (a minority) are called HW manifolds and

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are generalizations of the classical Hantzsche–Wendt 3-manifold (or *didicosm*) in dimension 3 (see [4]). For HW manifolds, the cohomology groups $H^j(M_\Gamma, R)$ vanish for $R = \mathbb{Q}$, $\mathbb{Z}_p$, p odd, $0 < j < n$. In particular, these manifolds are rational cohomology spheres.

Some related results have been obtained by Vasquez (see [5]) showing examples of the non-vanishing of w_2, Putrycz [6] computing the abelianization of HW manifolds for $n > 3$, and Dekimpe and Petrosyan [7], who give an algorithm that allows one to compute the integral homology of HW manifolds, and effectively compute it for all of them in dimension 7.

We now summarize the new examples concerning non-homeomorphic manifolds having similar spectral properties that are given in this paper.

(1) We determine the $\mathbb{Z}_2$-cohomology of all GHW manifolds in dimensions 3, 4 and 5, listing all isospectral classes (see Example 2.8, Appendix A and Table 2).
(2) For $n = 4$, we exhibit p-isospectral pairs for all p, M, M', that have the same cohomology but such that $w_2(M) \neq 0$ and $w_2(M') = 0$ (see manifolds labelled $(1, 1, 0)$ and $(1, 0, 1)$ in Theorem 4.4; see also [8]).
(3) We consider all four-dimensional flat manifolds of diagonal type with $F \equiv \mathbb{Z}_2^2$ or $F \equiv \mathbb{Z}_2^3$ and we show several isospectral or p-isospectral pairs, with $1 \leq p \leq 3$, having different $\mathbb{Z}_2$-cohomology groups and where some of them have different lengths of closed geodesics (Appendix A).
(4) We find, for $n = 5$, many isospectral pairs with $F \equiv \mathbb{Z}_2^4$ having different $H^2(M_\Gamma, \mathbb{Z}_2)$ and having the same $H^1(M_\Gamma, \mathbb{Z}_2)$ (Section 4.1). Such examples are not possible to obtain in dimension 4.
(5) We obtain expressions for $H^1(M_\Gamma, \mathbb{Z}_2)$ and $H^2(M_\Gamma, \mathbb{Z}_2)$ for an infinite family, $\mathcal{K}_n$, studied in [9], and we compute it explicitly for the subfamily introduced by Lee and Szczarba in [10] (Theorems 4.4 and 4.5).

Finally, we include several problems that are of interest to us (see Problems 3.5, 4.1 and 4.3).

1. Preliminaries

1.1. Compact flat manifolds and Bieberbach groups

A *crystallographic group* is a discrete, cocompact subgroup Γ of the isometry group of $\mathbb{R}^n$, $\mathrm{I}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$. If Γ is also torsion free, then Γ is a *Bieberbach group*. Such Γ acts properly discontinuously and freely on $\mathbb{R}^n$; thus $M_\Gamma = \Gamma \setminus \mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group Γ. Any compact flat manifold arises in this way.

The translations in Γ form a normal maximal abelian subgroup of finite index, L_Λ, Λ a lattice in $\mathbb{R}^n$. The quotient $F := \Lambda \setminus \Gamma$ gives the linear holonomy group of the Riemannian manifold M_Γ.

Any $\gamma \in I(\mathbb{R}^n)$ can be written uniquely as $\gamma = BL_b$, where $B \in O(n)$ and L_b is a translation by $b \in \mathbb{R}^n$. The restriction to Γ of the canonical projection $r : I(\mathbb{R}^n) \to O(n)$, $r(BL_b) = B$, is a homomorphism with kernel $\Lambda \simeq L_\Lambda$, and $r(\Gamma) \cong F$ is a finite subgroup of $O(n)$ called the *point group* of Γ.

In algebraic terms, Γ is an extension of F by Λ, i.e., there is an exact sequence

$$0 \to \Lambda \to \Gamma \xrightarrow{r} F \to 1. \tag{1.1}$$

Furthermore, since L_Λ is a normal subgroup of Γ, and $(BL_b)L_\lambda(BL_b)^{-1} = L_{B\lambda}$, then Λ is B-stable for any $\gamma = BL_b \in \Gamma$. This action defines an integral representation of F called the *holonomy representation*.

1.2. Diagonal Bieberbach groups

In this article we will mainly work with Bieberbach groups such that the holonomy representation is of *diagonal type*; that is, there is an orthogonal basis $\mathcal{B}$ of the lattice Λ such that the elements in the point group F diagonalize in $\mathcal{B}$ with eigenvalues ± 1. We shall usually take $\Lambda = \mathbb{Z}^n$ and $\mathcal{B}$ as the canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. For such Γ we have $F \cong \mathbb{Z}_2^k$ and we may also assume that $b \in \frac{1}{2}\mathbb{Z}^n$ for any element $\gamma = BL_b \in \Gamma$ (see [11]). Furthermore, if $\Gamma = \langle B_i L_{b_i},\ i = 1 \ldots k;\ L_{\mathbb{Z}^n} \rangle$, with $B_i^2 = Id$, then we can assume that $b_j \cdot e_i = 0$ or $\frac{1}{2}$ for every $1 \leq i \leq n$, $1 \leq j \leq k$. Although this definition clearly imposes a restriction on Γ, this is still a very rich class of Bieberbach groups (see for instance [9]).

A standard way to represent Bieberbach groups of diagonal type is by using the *column notation*, i.e., writing the diagonal matrices in columns, and as a subscript, the corresponding translation vector modulo the lattice.

Example 1.1 (*Hantzsche–Wendt 3-manifold or Didicosm*). Let M_Γ be the flat manifold of dimension $n = 3$, with holonomy group $\mathbb{Z}_2^2$, with generators of the corresponding group described as follows: let $B_1 = \mathrm{diag}(1, -1, -1)$, $B_2 = \mathrm{diag}(-1, 1, -1)$, $b_1 = \frac{e_1+e_3}{2}$, $b_2 = \frac{e_1+e_2}{2}$; i.e., $\Gamma = \langle B_1 L_{b_1},\ B_2 L_{b_2};\ L_{\mathbb{Z}^n} \rangle$. It is clear that M_Γ is orientable since $\det B = 1$ for every $BL_b \in \Gamma$. In column notation,

B_1	B_2
$1_{\frac{1}{2}}$	$-1_{\frac{1}{2}}$
-1	$1_{\frac{1}{2}}$
$-1_{\frac{1}{2}}$	-1

or

B_1	B_2	B_1B_2
$1_{\frac{1}{2}}$	$-1_{\frac{1}{2}}$	-1
-1	$1_{\frac{1}{2}}$	$-1_{\frac{1}{2}}$
$-1_{\frac{1}{2}}$	-1	$1_{\frac{1}{2}}$

.

It is easy to see that M_Γ is a rational homology sphere; that is, the first and second Betti numbers are equal to zero. In the next section we will compute the $\mathbb{Z}_2$-cohomology of M_Γ and of other low-dimensional flat manifolds (see Example 2.8).

1.3. Cohomology of (Bieberbach) groups

We recall that the cohomology of a group G with coefficients in a G-module A, denoted by $H^*(G, A)$, can be calculated by means of the so-called *reduced bar resolution* (see [12, Ch. 1]) that gives rise to a cochain complex defined as follows. Let $C^m(G, A)$ be the G-module of functions

$$f : \underbrace{G \times G \times \cdots \times G}_{m \text{ times}} \to A$$

with coboundary given by

$$(df)(g_1, \ldots, g_{m+1}) = g_1 \cdot f(g_2, g_3, \ldots, g_{m+1}) + \sum_{i=1}^{m} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{m+1}) + (-1)^{m+1} f(g_1, \ldots, g_m).$$

The cohomology of this complex gives $H^*(G, A)$, which has the structure of a graded ring, with product given by the standard cup product. One has that $H^0(G, A) \cong A^G$, the G-invariants in A, and furthermore the second cohomology group $H^2(G, A)$ classifies the isomorphism classes of all extensions of G by A in which the induced action of G on A agrees with the given action.

Going back to our setting, we consider a compact flat manifold M_Γ with fundamental group Γ. Thus M_Γ is an Eilenberg–Mac Lane space of type $K(\Gamma, 1)$; hence, for any coefficient ring R, $H^*(M_\Gamma, R) \cong H^*(\Gamma, R)$, where, in the right-hand side, R is regarded as a trivial Γ-module.

Given Γ, we recall that a group extension of the form (1.1) is determined by a cohomology class $\beta \in H^2(F, \Lambda)$, which is defined as follows. Consider the map $\alpha : F \to \Lambda \setminus \mathbb{R}^n$, such that $\alpha(B) = [Bb]$ for any $BL_b \in \Gamma$, where $[Bb]$ denotes the class of Bb in $\Lambda \setminus \mathbb{R}^n$. Note that α is well defined since $B(\Lambda) = \Lambda$. In particular, given $BL_b, CL_c \in \Gamma$ one has $\alpha(BC) = [(BC)(C^{-1}b + c)]$. We observe that in our context – diagonal type Bieberbach groups – the action of B does not change the class $[b]$, i.e. $\alpha(B) = [Bb] = [b]$, and also $\alpha(BC) = [(BC)(C^{-1}b + c)] = [b + c]$. We fix $\tilde{\alpha} : F \to \mathbb{R}^n$, a 'lift' of α, so that $\tilde{\alpha}(B) \equiv Bb \mod \Lambda$, for any $B \in F$, and we furthermore assume that $\tilde{\alpha}(I) = 0$.

The map α represents a cohomology class in $H^1(F, \Lambda \setminus \mathbb{R}^n)$ and, by the long exact sequence in cohomology, we have the isomorphism $H^1(F, \Lambda \setminus \mathbb{R}^n) \underset{d}{\cong} H^2(F, \Lambda)$, where d is the Bockstein homomorphism. The group Γ in the extension (1.1) is determined uniquely, up to equivalence, by the class $\beta = d\alpha \in H^2(F, \Lambda)$ represented by the map $F \times F \to \Lambda$:

$$(B, C) \mapsto B \cdot \tilde{\alpha}(C) - \tilde{\alpha}(BC) + \tilde{\alpha}(B), \tag{1.2}$$

which takes values in Λ, since

$$B \cdot \tilde{\alpha}(C) - \tilde{\alpha}(BC) + \tilde{\alpha}(B) \equiv B \cdot Cc - BC(C^{-1}b + c) + Bb \equiv 0 \mod \Lambda.$$

We will abuse notation by writing the map in (1.2) by $\beta(B, C)$ or $d\alpha(B, C)$.

In the computation of group cohomology, we will make use of the LHS spectral sequence, which has terms $E_2^{p,q} \cong H^p(\Lambda \setminus \Gamma, H^q(\Lambda, \mathbb{Z}_2))$, which converges to $H^{p+q}(\Gamma, \mathbb{Z}_2)$. Therefore, the following fact will be systematically used in our computations of differentials.

$$H^*(\mathbb{Z}^n, A) \cong \bigwedge^* (\mathbb{Z}^n)^* \otimes A, \quad \text{for any trivial } \mathbb{Z}^n\text{-module } A. \tag{1.3}$$

Furthermore, we will need a result due to Borel on the group cohomology ring of $G = \mathbb{Z}_2^k$ with $\mathbb{Z}_2$-coefficients.

Theorem 1.2 *(see [13, Thm. 7.1] or [14, Thm. 4.4, p. 69]). As a graded algebra*

$$H^*(\mathbb{Z}_2^k, \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \ldots, x_k],$$

with generators $x_1, \ldots, x_k$ in dimension 1. In particular,

$$H^1(\mathbb{Z}_2^k, \mathbb{Z}_2) \cong \mathrm{Hom}\,(\mathbb{Z}_2^k, \mathbb{Z}_2),$$

generated by the homomorphisms x_i $(i = 1, \ldots k)$ mapping the i-th generator of $\mathbb{Z}_2^k$ to 1 and all other generators to 0.

Thus, $H^2(\mathbb{Z}_2^k, \mathbb{Z}_2)$ can be identified with the space of degree-two homogeneous polynomials in k variables, with $\mathbb{Z}_2$-coefficients.

1.4. Computation of the class $\bar{\beta}$

Let now Γ be a Bieberbach group with holonomy group $F \cong \mathbb{Z}_2^k$ and let β be the associated class as defined in (1.2). This class determines a $\mathbb{Z}_2$-cohomology class

$$\bar{\beta} \in H^2(\mathbb{Z}_2^k, \Lambda^* \otimes \mathbb{Z}_2) \cong \left(H^2(\mathbb{Z}_2^k, \mathbb{Z}_2)\right)^n. \tag{1.4}$$

The components $\bar{\beta}_i$ of $\bar{\beta}$ are homogeneous polynomials of degree two that will be very important for us. They will be called the $\mathbb{Z}_2$-*class polynomials* of Γ, or 2-*class polynomials*, for short.

In the next proposition we show that the $\bar{\beta}_i$ can be computed by using the ring structure of $H^2(\mathbb{Z}_2^k, \mathbb{Z}_2)$, given in Theorem 1.2. Indeed, we have the following.

Proposition 1.3. *Let Γ be a Bieberbach group of diagonal type and holonomy group $\mathbb{Z}_2^k = \langle B_1, \dots, B_k\rangle$. The ℓ-th coordinate of $\bar{\beta}$ is given by*

$$\bar{\beta}_\ell = \sum_{\substack{i:B_ie_\ell=e_\ell\\ b_{i\ell}=\frac12}} x_i^2 + \sum_{i:b_{i\ell}=\frac12} \sum_{\substack{j\neq i\\ B_je_\ell=-e_\ell}} x_ix_j, \tag{1.5}$$

where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$.

Proof. As a first observation, we note that x_i^2 is represented by a map $\mathbb{Z}_2^k \times \mathbb{Z}_2^k \to \mathbb{Z}_2$ with value 1 on (B_i, B_i) and value zero on the other generators. Hence the coefficient of x_i^2 in $\bar{\beta}_\ell$ is $\bar{\beta}_\ell(B_i, B_i)$. Using the above formula (1.2) for the differential of α, we have

$$\bar{\beta}(B_i, B_i) = d\alpha(B_i, B_i) = B_i \cdot \alpha(B_i) + \alpha(B_i) = \sum_\ell b_{i\ell}(B_ie_\ell + e_\ell).$$

Thus the coefficient equals 1 if and only if $b_{i\ell} = \frac12$ and $B_ie_\ell = e_\ell$. This verifies the coefficients of terms of the form x_i^2 in formula (1.5).

In order to verify the coefficient of the mixed terms x_ix_j $(j \neq i)$ in (1.5), we observe that, for $i \neq j$,

$$\begin{aligned}\bar{\beta}(B_i, B_j) &= d\alpha(B_i, B_j) = B_i \cdot \alpha(B_j) - \alpha(B_iB_j) + \alpha(B_i) = B_ib_j - B_jb_i - b_j + b_i\\ &= \sum_\ell \big(b_{j\ell}(B_ie_\ell - e_\ell) - b_{i\ell}(B_je_\ell - e_\ell)\big).\end{aligned}$$

The coefficient of x_ix_j in $\bar{\beta}_\ell$ is $\bar{\beta}_\ell(B_i, B_j)$. The above formula shows that this equals $1 \mod 2$, for i, j such that $b_{i\ell} = \frac12$ and $B_je_\ell = -e_\ell$, or $b_{j\ell} = \frac12$ and $B_ie_\ell = -e_\ell$, but not both simultaneously. This completes the proof. □

Remark 1.4. The result in the proposition can be described in more graphic terms. In column notation, the ℓ-component $\bar{\beta}_\ell$ is a polynomial obtained from the ℓ-th row of the diagram, by including the term x_i^2 if the entry (ℓ, i) is $1_{\frac12}$, and including the term x_ix_j, for $i \neq j$, if the entries (ℓ, i), (ℓ, j) in any order, are either $(1_{\frac12}, -1_u)$, or $(-1_u, -1_{\frac12-u})$, where u is either 0 or $\frac12$. Visually, we have

B_i	2-class polynomial
$1_{\frac12}$	x_i^2

B_i	B_j	2-class polynomial
$1_{\frac12}$	-1_u	$x_i^2 + x_ix_j$
-1_u	$-1_{\frac12-u}$	x_ix_j

We will apply this rule in the case of the didicosm (see Example 1.1). We have that α is given on generators (modulo the lattice $\mathbb{Z}^n$) by $\alpha(B_1) = (0, \frac12, \frac12)$, $\alpha(B_2) = (\frac12, \frac12, 0)$. Thus, we have

B_1	B_2	2-class polynomial
$1_{\frac12}$	$-1_{\frac12}$	$\bar{\beta}_1 = x_1^2 + x_1x_2$
-1_0	$1_{\frac12}$	$\bar{\beta}_2 = x_2^2 + x_1x_2$
$-1_{\frac12}$	-1_0	$\bar{\beta}_3 = x_1x_2$

Thus, by using formula (1.5) we obtain in this case

$$\bar{\beta} = (x_1^2 + x_1x_2,\; x_2^2 + x_1x_2,\; x_1x_2).$$

2. The Lyndon–Hochschild–Serre spectral sequence

As mentioned in the Introduction, the Lyndon–Hochschild–Serre spectral sequence will be the main tool in our study of the $\mathbb{Z}_2$-cohomology of a Bieberbach group Γ. The E_2-term of this sequence is

$$E_2^{p,q} \cong H^p(F, H^q(\Lambda, R)),$$

which converges to $H^{p+q}(\Gamma, R)$, where the coefficient ring R is regarded as a trivial Γ-module and $p, q \geq 0$.

It is a standard fact that, for any F-module A, the order of F annihilates $H^p(F, A)$, i.e., $|F|\,\eta = 0$ for any $\eta \in H^p(F, A)$, if $p > 0$. Furthermore, if multiplication by $|F|$ in A is an isomorphism, then $H^p(F, A) = 0$ for $p > 0$ (see for instance [14, Cor. 5.4]).

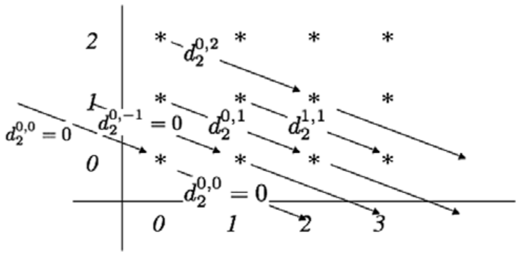


Fig. 1. Low dimensional differential in E_2.

Now, by (1.3), $H^q(\Lambda, R) \cong R^{\binom{n}{q}}$ for any coefficient ring R. Hence, if multiplication by $|F|$ on R is an isomorphism (in particular if R is a field of characteristic zero or odd), it follows that $E_2^{p,q} = H^p(F, H^q(\Lambda, R)) = 0$, for $p > 0$. Hence only the first column in E_2 does not vanish; thus the spectral sequence already degenerates at the first column of E_2. Therefore (see Section 1.3)

$$H^q(\Gamma, R) = \left(\bigwedge^q (\Lambda \otimes_{\mathbb{Z}} R)^*\right)^F. \tag{2.1}$$

In particular, in this case, $H^*(\Gamma, R)$ is determined by the holonomy representation only.

However, when multiplication by $|F|$ is not surjective, the determination of the cohomology is much harder and one needs to investigate deeper into the maps in the LHS spectral sequence. In particular, this is the case if the Bieberbach group Γ has holonomy group F with $|F| = 2^k$ and $R = \mathbb{Z}_2$. This is the main object of study in this paper.

2.1. The Charlap–Vasquez method

As mentioned, if the spectral sequence does not degenerate in E_2 to compute cohomology we need to look at the E_3 term, obtained by computing the cohomology of the complexes in E_2, as the picture in Fig. 1 shows for low dimensions.

To illustrate the main difficulties to be met in our computations, we observe that $E_2^{1,0} = E_3^{1,0} = E_4^{1,0} = \cdots = E_\infty^{1,0}$ and $E_3^{0,1} = \ker d_2^{0,1} = E_4^{0,1} = \cdots = E_\infty^{0,1}$. Thus

$$H^1(\Gamma, R) = \ker d_2^{0,1} \oplus E_2^{1,0} = \ker d_2^{0,1} \oplus H^1(F, \mathbb{Z}_2).$$

The second cohomology group, $H^2(\Gamma, R)$, comes from the sum of the terms in positions $(2, 0)$, $(1, 1)$, $(0, 2)$ in E_∞. For example, $E_3^{2,0} = E_2^{2,0}/\mathrm{Im}\, d_2^{0,1} = E_\infty^{2,0}$, since the differentials d_3 in the E_3 complex, issuing from, and arriving at, entry $(2, 0)$ vanish. Similarly, $E_3^{1,1} = \ker d_2^{1,1} = E_4^{1,1} = \cdots = E_\infty^{1,1}$. However, the differential $d_3^{0,2}$ in $E_3^{0,2} = \ker d_2^{0,2}$ need not vanish in general, so in order to determine this summand of H^2 one often needs to compute the third differential. However, as we shall see, $E_3^{0,2}$ vanishes for a large class of $\mathbb{Z}_2^k$-manifolds and then one can get an explicit expression for $H^2(\Gamma, R)$ (see Lemma 2.6 and Theorem 2.7).

Remark 2.1. Of course, the first cohomology group can also be computed by using the fact that $H_1(M, \mathbb{Z}) = [\Gamma, \Gamma] \setminus \Gamma$, together with the universal coefficient theorem.

Relative to the computation of $H^2(M_\Gamma, R)$, Charlap and Vasquez [2] obtained an expression for the second differential d_2 for any group extension of a group F by a finitely generated abelian group. We now introduce and explain the terms in their formula in the case when $R = \mathbb{Z}_2$. This is the case of interest in this paper.

Take $P : \Lambda \cong H_1(\Lambda, \mathbb{Z}) \to \mathrm{Hom}(H_{q-1}(\Lambda, \mathbb{Z}_2), H_q(\Lambda, \mathbb{Z}_2))$ to be the map induced by Pontrjagin multiplication on $H_*(\Lambda, \mathbb{Z}_2)$. Given that $H^{q-1}(\Lambda, H_q(\Lambda, \mathbb{Z}_2)) \cong \mathrm{Hom}(H_{q-1}(\Lambda, \mathbb{Z}_2), H_q(\Lambda, \mathbb{Z}_2))$, then P maps Λ to $H^{q-1}(\Lambda, H_q(\Lambda, \mathbb{Z}_2))$ and induces (by functoriality on coefficients) a homomorphism

$$P_* : H^2(F, \Lambda) \to H^2(F, H^{q-1}(\Lambda, H_q(\Lambda, \mathbb{Z}_2))).$$

The composition map from $\mathrm{Hom}(H_q(\Lambda, \mathbb{Z}_2), \mathbb{Z}_2) \otimes \mathrm{Hom}(H_{q-1}(\Lambda, \mathbb{Z}_2), H_q(\Lambda, \mathbb{Z}_2)) \to \mathrm{Hom}(H_{q-1}(\Lambda, \mathbb{Z}_2), \mathbb{Z}_2)$ induces a pairing

$$H^q(\Lambda, \mathbb{Z}_2) \otimes H^{q-1}(\Lambda, H_q(\Lambda, \mathbb{Z}_2)) \to H^{q-1}(\Lambda, \mathbb{Z}_2).$$

We shall denote by

$$\cup : H^p(F, H^q(\Lambda, \mathbb{Z}_2)) \otimes H^2(F, H^{q-1}(\Lambda, H_q(\Lambda, \mathbb{Z}_2))) \to H^{p+2}(F, H^{q-1}(\Lambda, \mathbb{Z}_2))$$

the cup product induced by this pairing.

Theorem 2.2 ([2]). *Let Γ be a Bieberbach group with holonomy group F. If $\xi \in E_2^{p,q} \cong H^p(F, H^q(\Lambda, \mathbb{Z}_2))$ and $\beta \in H^2(F, \Lambda)$ is the class corresponding to the extension (1.1), we have*

$$d_2^{p,q}(\xi) = (-1)^p \xi \cup (-P_*(\beta) + v^q), \tag{2.2}$$

where $v^q \in H^2(F, H^{q-1}(\Lambda, H_q(\Lambda, \mathbb{Z}_2)))$ is the q-th characteristic class of the F-module Λ.

One can actually write the Charlap–Vasquez formula in terms of the Pontrjagin product on $\Lambda \cong \mathbb{Z}^n$. Indeed, as shown in [15, pp. 4-09ff], for $\Lambda \cong \mathbb{Z}^n$ the Pontrjagin product P on $H_*(\Lambda, \mathbb{Z}) \cong \bigwedge_*(\mathbb{Z}^n)$ agrees with the wedge product.

Thus $P : \Lambda \cong H_1(\Lambda, \mathbb{Z}) \to \mathrm{Hom}(H_{q-1}(\Lambda, \mathbb{Z}_2), H_q(\Lambda, \mathbb{Z}_2))$ is such that

$$z \mapsto \{w_1 \wedge w_2 \wedge \cdots \wedge w_{q-1} \mapsto \bar{z} \wedge w_1 \wedge w_2 \wedge \cdots \wedge w_{q-1}\},$$

where $\bar{z} \in H_1(\Lambda, \mathbb{Z}_2)$ is the $\mathbb{Z}_2$-cohomology class of z. We have the following explicit description of $P_*(\beta) \in H^2(F, H^{q-1}(\Lambda, H_q(\Lambda, \mathbb{Z}_2)))$:

$$\begin{aligned} P_*(\beta): \quad & F \times F \to H^{q-1}(\Lambda, H_q(\Lambda, \mathbb{Z}_2)) \cong \mathrm{Hom}(H_{q-1}(\Lambda, \mathbb{Z}_2), H_q(\Lambda, \mathbb{Z}_2)) \\ & (B_1, B_2) \mapsto \{w_1 \wedge \cdots \wedge w_{q-1} \mapsto \bar{\beta}(B_1, B_2) \wedge w_1 \wedge \cdots \wedge w_{q-1}\}, \end{aligned}$$

where $\bar{\beta}$ is the class of β in $\mathbb{Z}_2$-cohomology (see (1.4)).

Thus (2.2) can be restated in terms of the wedge product, by making explicit also the cup product multiplication.

Theorem 2.3. *Let $\varepsilon^1, \ldots, \varepsilon^n$ be a basis of $\Lambda^* \otimes \mathbb{Z}_2 \cong (\mathbb{Z}_2^n)^*$, and let $\bar{\beta}$ be as in (1.4). If $\xi : F^p \to \mathrm{Hom}(H_q(\Lambda, \mathbb{Z}_2), \mathbb{Z}_2)$, then $d_2\xi : F^{p+2} \longrightarrow \mathrm{Hom}(H_{q-1}(\Lambda, \mathbb{Z}_2), \mathbb{Z}_2)$ is given by*

$$d_2\xi(v, B_1, B_2)(u) = \xi(v)(\bar{\beta}(B_1, B_2) \wedge u) + v^q \tag{2.3}$$

for any $v \in F^p$ and $u \in H_{q-1}(\Lambda, \mathbb{Z}_2)$.

We shall assume from now on that the Bieberbach group Γ is an extension (1.1) of $F \cong \mathbb{Z}_2^k$ by a lattice $\Lambda \cong \mathbb{Z}^n$, and furthermore that F acts diagonally on Λ. Here we note that the action of $F \cong \mathbb{Z}_2^k$ on $H^*(\Lambda, \mathbb{Z}_2) \cong \bigwedge^*(\mathbb{Z}_2^n)$ is trivial.

We have the following.

Lemma 2.4. *If F acts diagonally on Λ then $v^q = 0$ for any q.*

Proof. By [2, Prop. 7.1, p. 37], $v^1 = 0$ for any F-module. For rank-one modules, $v^2 = 0$ by dimension reasons and, since by [16, Thm. 7] v^2 is additive on direct sums, then $v^2 = 0$ for a diagonal holonomy action. Now, by [16, Cor. of Thm. 6], if $v^2 = 0$ then $v^q = 0$ for any q. □

Under the assumption that $F \cong \mathbb{Z}_2^k$, the terms in the spectral sequence take a special form. Indeed,

$$E_2^{p,q} = H^p\left(\mathbb{Z}_2^k, \bigwedge^q (\mathbb{Z}_2^n)^*\right) \cong H^p(\mathbb{Z}_2^k, \mathbb{Z}_2) \otimes \bigwedge^q (\mathbb{Z}_2^n)^*. \tag{2.4}$$

Theorem 2.5. *Let $\mathbb{Z}_2^k$ act diagonally on $\Lambda = \mathbb{Z}^n$. Let $\varepsilon^1, \ldots, \varepsilon^n$ be a basis of $\Lambda^* \otimes \mathbb{Z}_2 \cong (\mathbb{Z}_2^n)^*$, and let $\bar{\beta}_i$ be the i-th component of $\bar{\beta}$ (see (1.5)).*

(i) *If $\xi : (\mathbb{Z}_2^k)^p \to \mathrm{Hom}(H_q(\Lambda, \mathbb{Z}_2), \mathbb{Z}_2)$, then $d_2\xi : (\mathbb{Z}_2^k)^{p+2} \longrightarrow \mathrm{Hom}(H_{q-1}(\Lambda, \mathbb{Z}_2), \mathbb{Z}_2)$ is given by*

$$d_2\xi(v, B_1, B_2)(u) = \xi(v)(\bar{\beta}(B_1, B_2) \wedge u) \tag{2.5}$$

for any $v \in (\mathbb{Z}_2^k)^p$ and $u \in H_{q-1}(\Lambda, \mathbb{Z}_2)$.

(ii) *The differentials $d_2^{p,q}$ are given as follows:*

For $d_2^{0,1} : E_2^{0,1} \cong H^1(\mathbb{Z}^n, \mathbb{Z}_2) \cong (\mathbb{Z}_2^n)^* \longrightarrow E_2^{2,0} \cong H^2(\mathbb{Z}_2^k, \mathbb{Z}_2)$

$$\text{we have } d_2^{0,1}\varepsilon^i = \bar{\beta}_i \quad i = 1, \ldots, n. \tag{2.6}$$

For $d_2^{1,1} : E_2^{1,1} \cong H^1(\mathbb{Z}_2^k, \mathbb{Z}_2) \otimes \mathbb{Z}_2^n \longrightarrow E_2^{3,0} \cong H^3(\mathbb{Z}_2^k, \mathbb{Z}_2)$

$$\text{we have } d_2^{1,1}(x_i \otimes \varepsilon_j) = x_i \cup \bar{\beta}_j \quad i = 1, \ldots, k, \ j = 1, \ldots, n, \tag{2.7}$$

where $x_1, \ldots, x_k$ is a basis of $H^1(\mathbb{Z}_2^k, \mathbb{Z}_2)$.

For $d_2^{0,2} : E_2^{0,2} \cong \bigwedge^2 (\mathbb{Z}_2^n)^* \longrightarrow E_2^{2,1} \cong H^2(\mathbb{Z}_2^k, (\mathbb{Z}_2^n)^*)$

$$\text{we have} \quad d_2^{0,2}(\varepsilon^i \wedge \varepsilon^j)(\varepsilon^k) = \delta_{ik}\bar{\beta}_j + \delta_{jk}\bar{\beta}_i = \begin{cases} \bar{\beta}_j & \text{if } k = i, \\ \bar{\beta}_i & \text{if } k = j, \\ 0 & \text{otherwise}, \end{cases} \tag{2.8}$$

where $\{\varepsilon^i \wedge \varepsilon^j\}$ $(i < j)$ is a basis of $\bigwedge^2 \mathbb{Z}_2^n$.

Proof. Part (i) is a direct consequence of the formula for the Pontrjagin product.

Relative to (ii), we will check (2.7) and (2.8), leaving the verification of (2.6) to the reader.

To check (2.7), we note that $x_i \otimes \varepsilon_j \in H^1(\mathbb{Z}_2^k, \mathbb{Z}_2) \otimes \mathbb{Z}_2^n$ is identified with $\xi \in H^1(\mathbb{Z}_2^k, H^1(\Lambda, \mathbb{Z}_2)) \cong H^1(\mathbb{Z}_2^k, \mathbb{Z}_2^n)$, where

$$\begin{aligned} \xi :\quad & \mathbb{Z}_2^k \longrightarrow, \mathbb{Z}_2^n \\ & f \longrightarrow (0, \dots, \underbrace{x_i(f)}_{\text{at place j}}, \dots\dots, 0) \ . \end{aligned}$$

By expression (2.3) for d_2, we have

$$\begin{aligned} & d_2\xi(v, B_1, B_2) = \xi(v)(\bar\beta(B_1, B_2)), \\ & \text{hence } \ d_2(x_i \otimes \varepsilon_j)(v, B_1, B_2) \ = \ x_i(v)(\bar\beta_j(B_1, B_2)) \\ & \qquad\qquad\qquad\qquad\qquad\quad = \ x_i \cup \bar\beta_j(v, B_1, B_2). \end{aligned}$$

Thus (2.7) follows.

We now verify (2.8). A basis of $\bigwedge^2(\mathbb{Z}_2^n)^*$ is $\{\varepsilon^i \wedge \varepsilon^j : i < j\}$, where $\{\varepsilon_i\}$ is a basis of $\mathbb{Z}_2^n$. Hence, by (2.3) again

$$\begin{aligned} d_2(\varepsilon^i \wedge \varepsilon^j)(B_1, B_2)(\varepsilon_k) &= \varepsilon^i \wedge \varepsilon^j(\bar\beta(B_1, B_2) \wedge \varepsilon_k) \\ &= \varepsilon^i \wedge \varepsilon^j \left(\sum_\ell \bar\beta_\ell(B_1, B_2)\varepsilon_\ell \wedge \varepsilon_k \right) \\ &= (\delta_{ik}\bar\beta_j + \delta_{jk}\bar\beta_i)(B_1, B_2) \end{aligned}$$

This completes the proof. □

For the computation of $E_\infty^{0,2}$ it is important to study the differential $d_2^{0,2}$. To this end, the following lemma – despite its somewhat restrictive assumptions – is not hard to prove and it will be sufficient for the applications to be given in later sections.

Lemma 2.6. *Let $n - 1 \le k(k+1)$. If there are $n - t$ linearly independent $\mathbb{Z}_2$-class polynomials, then*

$$\dim \ker d_2^{0,2} = \binom{t}{2}. \tag{2.9}$$

In particular, if there is only one linearly dependent $\mathbb{Z}_2$-class polynomial, then $\ker d_2^{0,2} = \{0\}$; thus $E_3^{0,2} = E_\infty^{0,2} = \{0\}$.

Proof. *Step* 1. We first show that if there are $m - 1$ (out of m) linearly independent $\mathbb{Z}_2$-class polynomials, then rank $d_2^{0,2} \ge \binom{m}{2}$. Indeed, look at the (transpose) matrix of $d_2^{0,2}$ in block form:

	$\varepsilon^1 \wedge \varepsilon^2$	$\varepsilon^1 \wedge \varepsilon^3$	$\dots$	$\dots$	$\varepsilon^1 \wedge \varepsilon^m$	$\varepsilon^2 \wedge \varepsilon^3$	$\dots$	$\varepsilon^2 \wedge \varepsilon^m$	$\dots$	$\varepsilon^{m-1} \wedge \varepsilon^m$
ε^1	$\bar\beta_2$	$\bar\beta_3$	$\dots$	$\dots$	$\bar\beta_m$	0	$\dots$	0	0	0
ε^2	$\bar\beta_1$					$\bar\beta_3$	$\dots$	$\bar\beta_m$	0	0
ε^3		$\bar\beta_1$				$\bar\beta_2$			$\vdots$	0
ε^4			$\ddots$							0
$\vdots$				$\ddots$						
ε^m					$\bar\beta_1$			$\bar\beta_2$	$\dots$	$\bar\beta_{m-1}$

where $\bar\beta_j$ stands for the column matrix of the coefficients of the $\mathbb{Z}_2$-class polynomial $\bar\beta_j$. By a standard elimination process, it is not difficult to see that the rank of this matrix is maximal, namely $\binom{m}{2}$. Indeed, since $\bar\beta_2, \dots, \bar\beta^m$ are linearly independent, we see that the first $m - 1$ columns have maximal rank. Continuing with the elimination process, one sees that the full rank is maximal.

Step 2. Suppose now that the first t $\mathbb{Z}_2$-class polynomials $\bar\beta_1, \dots, \bar\beta_t$ vanish and the remaining $n-t$ are linearly independent. Then all columns corresponding to $\varepsilon^i \wedge \varepsilon^j$, $1 \le i < j \le t$ vanish, so the rank reduces by $\binom{t}{2}$. Now, by applying a similar argument as before, one obtains that the matrix has maximal rank.

Step 3. If instead of having t vanishing $\mathbb{Z}_2$-class polynomials, we have that exactly $n - t$ of them are linearly independent, then one can carry out the proof in a similar way, though it is more involved. □

2.2. Use of the LHS spectral sequence

Let Γ be a Bieberbach group with diagonal holonomy $F \cong \mathbb{Z}_2^k$ and lattice $\Lambda = \mathbb{Z}^n$.

Our next goal is to obtain explicit formulas for the dimensions of $H^1(M, \mathbb{Z}_2)$ and $H^2(M, \mathbb{Z}_2)$. To this end, we will compute $E_2^{p,q} = H^p(\mathbb{Z}_2^k, \mathbb{Z}_2) \otimes \bigwedge^q (\mathbb{Z}_2^n)^*$ together with the differentials.

Recall that $H^p(\mathbb{Z}_2^k, \mathbb{Z}_2)$ can be identified with the space of homogeneous polynomials of degree p in k variables, which has dimension $\binom{k+p-1}{p}$. Moreover, $\bigwedge^q (\mathbb{Z}_2^n)^*$ has dimension $\binom{n}{q}$. Thus, the dimension of $E_2^{p,0}$ equals $\binom{k+p-1}{p}$ $\left(\text{in particular } \dim E_2^{2,0} = \binom{k+1}{2}\right)$ and the dimension of $E_2^{0,q}$ is $\binom{n}{q}$.

q				
2	$\binom{n}{2}$	$k\binom{n}{2}$	$\binom{k+1}{2}\binom{n}{2}$	$\dots$
1	n	kn	$\binom{k+1}{2}n$	$\dots$
0	1	k	$\binom{k+1}{2}$	$\dots$
	0	1	2	3 $\quad p$

$E_2^{p,q}$

Observe also that the q-th line in the spectral sequence is obtained from the first one by multiplying by $\binom{n}{q}$. The figure shows the dimensions of $E_2^{p,q}$ for the terms that are relevant for computing the first and second cohomology groups.

Theorem 2.7. *Let M be an n-dimensional compact flat manifold with diagonal holonomy $\mathbb{Z}_2^k$. Then*

$$\dim H^1(M, \mathbb{Z}_2) = n - \operatorname{rank} d_2^{0,1} + k. \tag{2.10}$$

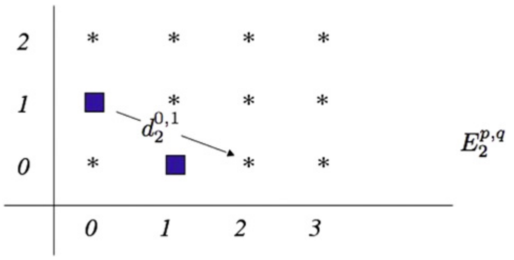
$$\dim H^2(M, \mathbb{Z}_2) = \binom{k+1}{2} - \operatorname{rank} d_2^{0,1} + kn - \operatorname{rank} d_2^{1,1} + \dim E_\infty^{0,2}. \tag{2.11}$$

Furthermore, if there are $n-1$ linearly independent $\mathbb{Z}_2$-class polynomials, then

$$\dim H^2(M, \mathbb{Z}_2) = \binom{k+1}{2} - \operatorname{rank} d_2^{0,1} + kn - \operatorname{rank} d_2^{1,1}. \tag{2.12}$$

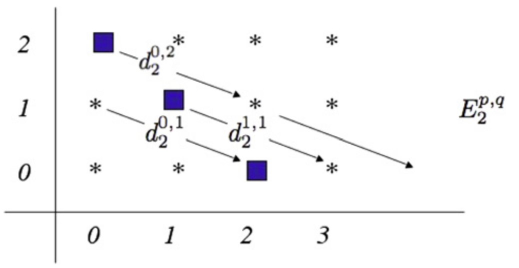
Note that $\operatorname{rank} d_2^{0,1}$ *coincides with the number of linearly independent $\mathbb{Z}_2$-class polynomials $\bar{\beta}_\ell$, $\ell = 1, \dots, n$ (see (2.6)).*

Proof (*Computation of $H^1(M, \mathbb{Z}_2)$*). The relevant terms in the spectral sequence are $E_\infty^{1,0} = E_2^{1,0}$, of dimension k, and $E_\infty^{0,1} = E_3^{0,1} = \ker d_2^{0,1}$, of dimension $n - \operatorname{rank} d_2^{0,1}$. Hence, since $H^1(M, \mathbb{Z}_2) = E_\infty^{0,1} \oplus E_\infty^{1,0}$, our assertion follows.



Computation of $H^2(M, \mathbb{Z}_2)$.

The relevant terms in E_2 are shown in black boxes in the picture.



Observe that $(2, 0)$ and $(1, 1)$ already stabilize at level 3, but this is not in general the case for the $(0, 2)$ term. So we have

$$H^2(M, \mathbb{Z}_2) = E_2^{2,0}/\operatorname{Im} d_2^{0,1} \oplus \ker d_2^{1,1} \oplus E_\infty^{0,2}.$$

As remarked at the beginning of this section, $\dim E_2^{2,0} = \binom{k+1}{2}$; thus assertion (2.11) follows.

Finally, the last assertion follows from Lemma 2.6 and (2.11). $\square$

Example 2.8. As a warm up, we begin by computing the $\mathbb{Z}_2$-cohomology of all three-dimensional flat manifolds with diagonal holonomy $\mathbb{Z}_2^k$, $k = 1, 2$.

• $k = 1$: there are two groups of this type: Γ_{c2} and Γ_{+a1}, corresponding to manifolds named *dicosm* and *first amphicosm*. In column notation:

$$\Gamma_{c2}: \begin{array}{|c|}\hline B_1 \\ \hline -1 \\ -1 \\ 1_{\frac{1}{2}} \\ \hline \end{array}, \qquad \Gamma_{+a1}: \begin{array}{|c|}\hline B_1 \\ \hline -1 \\ 1 \\ 1_{\frac{1}{2}} \\ \hline \end{array}.$$

The $\mathbb{Z}_2$-class polynomials are

$$\bar{\beta}_{\Gamma_{c2}} = (0, 0, x_1^2), \qquad \bar{\beta}_{\Gamma_{+a1}} = (0, x_1^2, 0).$$

So $\operatorname{rank} d_2^{0,1} = 1$ in both cases, and applying formula (2.10) in Theorem 2.7 we have

$$\dim H^1(c2, \mathbb{Z}_2) = \dim H^1(+a1, \mathbb{Z}_2) = 3 - 1 + 1 = 3$$

hence, by Poincaré duality also

$$\dim H^2(c2, \mathbb{Z}_2) = \dim H^2(+a1, \mathbb{Z}_2) = 3.$$

As a way of illustration, we will also carry out the computation of $H^2(c2, \mathbb{Z}_2)$ and $H^2(+a1, \mathbb{Z}_2)$ by the method in the previous section, i.e. by using (2.11). Since the set of $\mathbb{Z}_2$-class polynomials is the same for both manifolds, $H^2(c2, \mathbb{Z}_2) = H^2(+a1, \mathbb{Z}_2)$.

We have already seen that $\operatorname{rank} d_2^{0,1} = 1$ and, on the other hand, (2.7) implies that $\operatorname{rank} d_2^{1,1} = 1$. It remains to determine $\dim E_\infty^{0,2}$. Since $\operatorname{rank} d_2^{1,1} = 1$, it follows that the spectral sequence stabilizes at level 3, that is $E_3^{0,2} = E_\infty^{0,2}$; hence $\dim E_\infty^{0,2} = \dim \ker d_2^{0,2} = \dim E_2^{0,2} - \operatorname{rank} d_2^{0,2}$. Using (2.8), we compute that $\operatorname{rank} d_2^{0,2} = 2$ (the computations are similar to those in Example 2.9, supra). Thus, $\dim E_\infty^{0,2} = 3 - 2 = 1$. Hence, $\dim H^2(M_{\Gamma_{c2}}, \mathbb{Z}_2) = 0 + 2 + 1 = 3$.

• $k = 2$. There are three groups of this type, $\Gamma_{c22}, \Gamma_{+a2}, \Gamma_{-a2}$, corresponding, respectively, to the manifolds called *didicosm*, and *first* and *second amphidicosms*.

$$\Gamma_{c22}: \begin{array}{|cc|}\hline B_1 & B_2 \\ \hline 1_{\frac{1}{2}} & -1_{\frac{1}{2}} \\ -1 & 1_{\frac{1}{2}} \\ -1_{\frac{1}{2}} & -1 \\ \hline \end{array}, \qquad \Gamma_{+a2}: \begin{array}{|cc|}\hline B_1 & B_2 \\ \hline 1 & 1_{\frac{1}{2}} \\ 1_{\frac{1}{2}} & -1 \\ -1 & -1 \\ \hline \end{array}, \qquad \Gamma_{-a2}: \begin{array}{|cc|}\hline B_1 & B_2 \\ \hline 1 & 1_{\frac{1}{2}} \\ 1_{\frac{1}{2}} & -1 \\ -1_{\frac{1}{2}} & -1 \\ \hline \end{array}.$$

We will carry out the computations only for the didicosm or the three-dimensional HW manifold, since the other cases are similar. The $\mathbb{Z}_2$-class polynomials are

$$\bar{\beta}_{\Gamma_{c22}} = (x_1x_2 + x_1^2, x_1x_2 + x_2^2, x_1x_2).$$

So $\operatorname{rank} d_2^{0,1} = 3$ and applying formula (2.10) in Theorem 2.7 we get that

$$\dim H^1(c22, \mathbb{Z}_2) = 2,$$

and by Poincaré duality also $\dim H^2(c22, \mathbb{Z}_2) = 2$.

Similarly, for the first and second amphidicosms one computes

$$\dim H^1(+a2, \mathbb{Z}_2) = \dim H^2(+a2, \mathbb{Z}_2) = 3,$$
$$\dim H^1(-a2, \mathbb{Z}_2) = \dim H^2(-a2, \mathbb{Z}_2) = 2.$$

Example 2.9. We now give an example where we apply Theorem 2.7 to compute the full $\mathbb{Z}_2$-cohomology ring of a four-dimensional compact flat manifold with $F \cong \mathbb{Z}_2^2$. Later (in Appendix A), we give a complete table of the $\mathbb{Z}_2$-cohomology of all four-dimensional flat manifolds with diagonal holonomy $\mathbb{Z}_2^k$, $k = 2, 3$, that is obtained by the same method.

We remark that, in this example, stabilization of the LHS spectral sequence for the relevant terms to compute the first and second cohomology groups, only occurs at level 3.

Consider the four-dimensional manifold with Bieberbach group Γ_{18} given in column notation by

$$\Gamma_{18}: \begin{array}{|cc|}\hline B_1 & B_2 \\ \hline 1 & -1 \\ 1 & -1 \\ -1 & -1_{\frac{1}{2}} \\ 1_{\frac{1}{2}} & 1_{\frac{1}{2}} \\ \hline \end{array}.$$

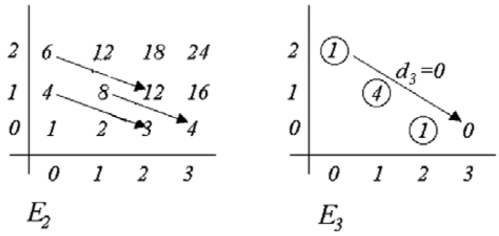


Fig. 2. E_2 and E_3 in Example 2.9.

Then, using formula (1.5), we have

$$\bar{\beta}_{\Gamma_{18}} = (0, 0, x_1x_2, x_1^2 + x_2^2).$$

Since two $\mathbb{Z}_2$-class polynomials are linearly independent, one sees immediately that rank $(d_2^{0,1} : \mathbb{Z}_2^4 \to \mathbb{Z}_2^3)$ is 2. Therefore, by (2.10) in Theorem 2.7,

$$\dim H^1(M_{\Gamma_{18}}, \mathbb{Z}_2) = 4 - 2 + 2 = 4.$$

We now compute the differential $d_2^{1,1} : \mathbb{Z}_2^8 \to \mathbb{Z}_2^4$, which sends $x_i \otimes \varepsilon_j \mapsto x_i \cup \bar{\beta}_j = x_i\bar{\beta}_j$ (under the standard identification) for every i, j:

	$d_2^{1,1}$
$x_1 \otimes \varepsilon_1$	0
$x_1 \otimes \varepsilon_2$	0
$x_1 \otimes \varepsilon_3$	$x_1^2x_2$
$x_1 \otimes \varepsilon_4$	$x_1^3 + x_1x_2^2$

	$d_2^{1,1}$
$x_2 \otimes \varepsilon_1$	0
$x_2 \otimes \varepsilon_2$	0
$x_2 \otimes \varepsilon_3$	$x_1x_2^2$
$x_2 \otimes \varepsilon_4$	$x_1^2x_2 + x_2^3$

We see that the image of $d_2^{1,1}$ is generated by all possible products of $\mathbb{Z}_2$-class polynomials by the monomials x_j. Thus, in this case rank $d_2^{1,1} = 4$.

Note that we cannot apply (2.12), since two $\mathbb{Z}_2$-class polynomials vanish. However, since $E_3^{3,0} \cong \mathbb{Z}_2^4/\mathrm{Im}\, d_2^{1,1} = \{0\}$, we have $d_3^{0,2} = 0$, so $E_3^{0,2} = E_\infty^{0,2} = \ker d_2^{0,2}$. Let us compute this map. Although this may follow from Lemma 2.6, we will carry out the details to illustrate the use of the method. We have (see (2.8))

$$d_2^{0,2} : \bigwedge^2 (\mathbb{Z}_2^4)^* \cong \mathbb{Z}_2^6 \longrightarrow \mathbb{Z}_2^{12}$$
$$\varepsilon^i \wedge \varepsilon^j \mapsto \left(\varepsilon^k \mapsto \begin{cases} \bar{\beta}_j & \text{if } k = i, \\ \bar{\beta}_i & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases} \right)$$

For Γ_{18} we have

		$\varepsilon^1 \wedge \varepsilon^2$	$\varepsilon^1 \wedge \varepsilon^3$	$\varepsilon^1 \wedge \varepsilon^4$	$\varepsilon^2 \wedge \varepsilon^3$	$\varepsilon^2 \wedge \varepsilon^4$	$\varepsilon^3 \wedge \varepsilon^4$
	x_1^2	0	0	1	0	0	0
ε^1	x_1x_2	0	1	0	0	0	0
	x_2^2	0	0	1	0	0	0
	x_1^2	0	0	0	0	1	0
ε^2	x_1x_2	0	0	0	1	0	0
	x_2^2	0	0	0	0	1	0
	x_1^2	0	0	0	0	0	1
ε^3	x_1x_2	0	0	0	0	0	0
	x_2^2	0	0	0	0	0	1
	x_1^2	0	0	0	0	0	0
ε^4	x_1x_2	0	0	0	0	0	1
	x_2^2	0	0	0	0	0	0

Its rank is 5, so $\dim \ker d_2^{0,2} = 6 - 5 = 1$. Thus, by (2.11) in Theorem 2.7 (see Fig. 2), we get

$$\dim H^2(M_{\Gamma_{18}}, \mathbb{Z}_2) = 3 - 2 + 8 - 4 + 1 = 6.$$

Note that in this example the first two rows of the Bieberbach group (written in column notation) do not contain any $\frac{1}{2}$. In this situation the flat manifold M_Γ is a flat toral extension of a flat manifold $\bar{M}$ of smaller dimension (see [17]). In the present case $\bar{M}$ is a Klein bottle and there is a fibration $\mathbb{T}^2 \to M_\Gamma \to \bar{M}$, where the fiber is a torus. Thus the cohomology of M_Γ can be computed using the Leray–Serre spectral sequence of the above fibration that converges to $H^*(M_\Gamma)$ and it has $E_2 = H^*(\bar{M}, H^*(\mathbb{T}^2))$. This procedure yields the same result obtained above for the $\mathbb{Z}_2$-cohomology of $M_{\Gamma_{18}}$.

3. Second Stiefel–Whitney class

The characteristic classes of a compact Riemannian manifold are closely related to its geometry. We note that Pontrjagin classes for any compact flat manifold vanish, since they can be expressed in terms of the curvature tensor (Chern–Weyl theorem); however, the Stiefel–Whitney classes need not vanish, a somewhat unexpected fact first noticed by Auslander and Szczarba [18]. We recall furthermore that an oriented Riemannian manifold M admits a spin structure if and only if $\mathrm{w}_2(M) = 0$ [13].

The goal of this section will be to describe an explicit method to decide whether $\mathrm{w}_2(M) = 0$ for a compact flat manifold M with diagonal holonomy $\mathbb{Z}_2^k$, generalizing results in [10,5] for some families of $\mathbb{Z}_2^k$-manifolds. Indeed, in Theorem 3.3 we give a simple criterion for $\mathrm{w}_2(M)$ to vanish or not, in terms of the $\mathbb{Z}_2$-class polynomials $\bar{\beta}_\ell$.

We first recall that there is a one to one correspondence between real vector bundles ξ over a paracompact n-manifold M and homotopy classes of maps $M \to BO(n)$, where $BO(n)$ is the classifying space of $O(n)$. The inclusion of the subgroup $D(n) \cong \mathbb{Z}_2^n$ of diagonal matrices in $O(n)$ induces a monomorphism

$$\phi^* : H^*(BO(n), \mathbb{Z}_2) \to H^*(\mathbb{Z}_2^n, \mathbb{Z}_2) = \mathbb{Z}_2[x_1, \ldots, x_n],$$

with image the subring $\mathbb{Z}_2[\sigma_1, \ldots, \sigma_n]$, generated by the elementary symmetric functions σ_j of degree j in $x_1, \ldots, x_n$. The preimage $\mathrm{w}_j = (\phi^*)^{-1}\sigma_j$ generates $H^j(BO(n), \mathbb{Z}_2)$ and is called the universal j-th Stiefel–Whitney class. If ξ is a real vector bundle over M with classifying map $f_\xi : M \to BO(n)$, the *Stiefel–Whitney classes of* ξ are defined by

$$\mathrm{w}_j(\xi) = f_\xi^*(\mathrm{w}_j) \in H^j(M, \mathbb{Z}_2), \quad \text{for } 1 \le j \le n.$$

When ξ is the tangent bundle of M, $\mathrm{w}_j(TM) =: \mathrm{w}_j(M)$ is called the j-th *Stiefel–Whitney class* of M.

Let now Γ be the Bieberbach group of a $\mathbb{Z}_2^k$-manifold of diagonal type. We observe that, in this setting, we have a trivial action of $F \cong \mathbb{Z}_2^k$ on $H^*(\Lambda, \mathbb{Z}_2) \cong \bigwedge^*(\mathbb{Z}_2^n)$.

Our next goal is to show how to compute the second Stiefel–Whitney class w_2 of a $\mathbb{Z}_2^k$-manifold with diagonal holonomy by using the LHS spectral sequence.

Observe first that the map $\Gamma \xrightarrow{r} F$ induces a map $\mu(r) : M_\Gamma \to B(F)$, where $B(F)$ is the classifying space for F and M_Γ identifies with the classifying space for Γ.

Remark 3.1. Since $B(F)$ (resp. M_Γ) is the Eilenberg–Mac Lane space of F (resp. Γ), the induced homomorphism in singular cohomology $\mu(r)^* : H^*(B(F), \mathbb{Z}_2) \to H^*(M_\Gamma, \mathbb{Z}_2)$ can be identified with the homomorphism $r^* : H^*(F, \mathbb{Z}_2) \to H^*(\Gamma, \mathbb{Z}_2)$, where $\mathbb{Z}_2$ is regarded as a trivial module in both cases. Note also that r^* agrees with the inflation map, since the $\mathbb{Z}_2$-action is trivial (see e.g. [19, Section 6.7] for the definition of inflation).

The composition of morphisms

$$\Gamma \xrightarrow{r} F \cong \mathbb{Z}_2^k \xhookrightarrow{i} D(n) \cong \mathbb{Z}_2^n \hookrightarrow O(n)$$

induces a map of M into $BO(n)$ which is the classifying map for the tangent bundle of M. Let $x_1, \ldots, x_k$ be a basis of $H^1(\mathbb{Z}_2^k, \mathbb{Z}_2)$ and let $x'_1, \ldots, x'_n$ be the standard basis of $H^1(D(n), \mathbb{Z}_2)$. The classes

$$\omega_\ell = i^*(x'_\ell) = \sum_m a_{m\ell} x_m,$$

are called the *2-weights* of the map i.

Let $\sigma_j(\omega_1, \ldots, \omega_n)$ be the j-th elementary symmetric function in $\omega_1, \ldots, \omega_n$. The following proposition is a direct consequence of the results in [20, Section 11] (cf. also [10, (2.1)]).

Proposition 3.2. *Let $\Gamma \xrightarrow{r} F = \mathbb{Z}_2^k$ be the map in (1.1). Then*

$$\mathrm{w}_j(M) = r^*\sigma_j(\omega_1, \ldots, \omega_n). \tag{3.1}$$

We now focus on the case of $\mathrm{w}_2(M)$. For this purpose, we consider the exact sequence associated to the LHS spectral sequence (cf. [21, Cor. 7.2.3]),

$$\cdots \to H^1(\Lambda, \mathbb{Z}_2) \xrightarrow{d_2^{0,1}} H^2(F, \mathbb{Z}_2) \xrightarrow{r^*} H^2(\Gamma, \mathbb{Z}_2)$$

and recall that inflation agrees with r^* in this case. We thus have

$$\operatorname{Im} d_2^{0,1} = \ker r^*. \tag{3.2}$$

Hence, from (2.6) and (3.2), we obtain the following.

Theorem 3.3. *Let M_Γ be a n-dimensional compact flat manifold with diagonal holonomy $\mathbb{Z}_2^k$. Then $\mathrm{w}_2 \neq 0$ if and only if the symmetric function $\sigma_2(\omega_1, \ldots, \omega_n)$ is not a sum of $\mathbb{Z}_2$-class polynomials.*

Example 3.4. We apply the criterion in the theorem to determine w_2 in the case of the three-dimensional manifolds with holonomy group $\mathbb{Z}_2^2$, in the notation of Example 2.8.

We first compute the 2-weights in the case of Γ_{c22}. We have

$$\omega_1 = x_2, \qquad \omega_2 = x_1, \qquad \omega_3 = x_1 + x_2.$$

Thus $\sigma_2(\omega_1, \omega_2, \omega_3) = x_1^2 + x_2^2 + x_1x_2$. Recall that

$$\bar{\beta} = (x_1x_2 + x_1^2, x_1x_2 + x_2^2, x_1x_2),$$

so σ_2 is the sum of the three $\mathbb{Z}_2$-class polynomials. Hence $\mathrm{w}_2 = 0$.

For Γ_{+a2}, the 2-weights are

$$\omega_1 = 0, \qquad \omega_2 = x_2, \qquad \omega_3 = x_1 + x_2.$$

Thus $\sigma_2(\omega_1, \omega_2, \omega_3) = x_1x_2 + x_2^2$. Furthermore,

$$\bar{\beta} = (x_2^2, x_1^2 + x_1x_2, 0),$$

so σ_2 is not the sum of $\mathbb{Z}_2$-class polynomials. Hence $\mathrm{w}_2(M_{\Gamma_{+a2}}) \neq 0$.

In the case of Γ_{-a2} the 2-weights are the same as for Γ_{+a2}. Recall that

$$\bar{\beta} = (x_2^2, x_1^2 + x_1x_2, x_1x_2).$$

Thus, $\sigma_2(\omega_1, \omega_2, \omega_3) = x_1x_2 + x_2^2$ in this case is the sum of the two $\mathbb{Z}_2$-class polynomials $\bar{\beta}_1$ and $\bar{\beta}_3$; hence $\mathrm{w}_2(M_{\Gamma_{-a2}}) = 0$.

Problem 3.5. Find a method (or extend the method here) to compute the second Stiefel–Whitney class for flat manifolds with holonomy group $\mathbb{Z}_2^k$ that are not of diagonal type.

4. Cohomology and spectral properties

In this section we will exhibit several families of flat manifolds of diagonal type with a variety of spectral and cohomological properties.

Two manifolds are said to be p-isospectral if they have the same spectrum with respect to on the Hodge Laplacian Δ_p acting on p-forms. In this case both manifolds must have the same Betti numbers, since $\mathrm{betti}_p(M)$ equals the multiplicity of the zero eigenvalue of Δ_p. It is not very easy to exhibit p-isospectral manifolds for all $0 \leq p \leq n$ having different cohomological properties. In this article we will exhibit several simple examples of Sunada-isospectral manifolds in dimension $n = 4$, $F \equiv \mathbb{Z}_2^2$, having different $\mathbb{Z}_2$-cohomology; actually, all H^i's are different for $1 \leq i \leq 3$ (in Appendix A see, for instance, manifolds $M_{7'}$ and M_{10}, or M_8 and $M_{9'}$). Much more difficult to construct is a pair, M, M', p-isospectral for all p, where M and M' have the same H^1 and different H^2 with $\mathbb{Z}_2$-coefficients. In the following subsection, we will use the formulas in Theorem 2.7 to produce such an example.

4.1. An illustrative example

We will first construct a pair of 5-manifolds with holonomy group $\mathbb{Z}_2^4$ having the same first $\mathbb{Z}_2$-cohomology, but different second $\mathbb{Z}_2$-cohomology, that are p-isospectral for all p. In both cases the second Stiefel–Whitney class does not vanish.

The groups are taken from the CARAT (Aachen) list (see [22]). Namely, they are the first and fourth in this list, there denoted by #g1, `1-th torsion-free group in` $\mathbb{Z}$`-class group 218.1.1`, and #g4, `4-th torsion-free group in` $\mathbb{Z}$`-class group 218.1.1`.

Their generators are described in column notation in the following tables:

#g1 :

B_1	B_2	B_3	B_4
1	1	-1	$1_{\frac{1}{2}}$
$1_{\frac{1}{2}}$	1	1	-1
$1_{\frac{1}{2}}$	1	-1	-1
1	-1	$1_{\frac{1}{2}}$	1
-1	$1_{\frac{1}{2}}$	1	1

#g4 :

B_1	B_2	B_3	B_4
1	1	-1	$1_{\frac{1}{2}}$
1	$1_{\frac{1}{2}}$	1	-1
$1_{\frac{1}{2}}$	1	-1	-1
1	-1	$1_{\frac{1}{2}}$	1
-1	$1_{\frac{1}{2}}$	1	1

The computations were first carried out with the aid of a computer program and then also checked by hand. We have

(a) the differential $d_2^{0,1} : \mathbb{Z}_2^5 \to \mathbb{Z}_2^{10}$ has rank 5 for both groups,

(b) the differential $d_2^{1,1} : \mathbb{Z}_2^{20} \to \mathbb{Z}_2^{20}$ has rank 18 for #g1 and 19 for #g4,
(c) the differential $d_2^{0,2} :: \mathbb{Z}_2^{10} \to \mathbb{Z}_2^{50}$ has (maximal) rank 10 for both groups.
The $\mathbb{Z}_2$-class polynomials $\bar\beta_i$, $1 \le i \le 5$ are given by

$$\bar\beta_1 = \bar\beta_1' = x_3x_4 + x_4^2$$
$$\bar\beta_2 = x_1x_4 + x_1^2 \qquad \bar\beta_2' = x_2x_4 + x_2^2$$
$$\bar\beta_3 = \bar\beta_3' = x_1x_3 + x_1x_4 + x_1^2$$
$$\bar\beta_4 = \bar\beta_4' = x_2x_3 + x_3^2$$
$$\bar\beta_5 = \bar\beta_5' = x_1x_2 + x_2^2.$$

Since they are linearly independent for both groups, rank $d_2^{0,1} = 5$, which shows (a). Moreover, Lemma 2.6 applies, so $d_2^{0,2}$ has maximal rank, which implies (c).

To verify (b) is more complicated, since one has to compute the rank of a 20×20 matrix with $\mathbb{Z}_2$-entries. Actually the elements in the image of $d_2^{1,1}$ are the third-degree polynomials in four variables given by the products of the $\mathbb{Z}_2$-class polynomials by the monomials x_ℓ, namely

$$x_\ell\bar\beta_i, \qquad x_\ell\bar\beta_i' \quad \text{for } \ell = 1, \dots, 4,\ i = 1, \dots, 5.$$

Now, for #g1, we have the relations

$$\begin{cases} x_1\bar\beta_4 + x_2\bar\beta_2 + x_2\bar\beta_3 + x_3\bar\beta_2 + x_3\bar\beta_3 = 0, \\ x_1\bar\beta_2 + x_1\bar\beta_3 + x_3\bar\beta_2 + x_4\bar\beta_2 + x_4\bar\beta_3 = 0. \end{cases}$$

A computation shows that 18 out of the 20 polynomials $x_\ell\bar\beta_i$ are linearly independent.

In the case of #g4 we have only one relation

$$x_1\bar\beta_2' + x_2\bar\beta_2' + x_2\bar\beta_5' + x_4\bar\beta_5' = 0,$$

and another long computation shows that 19 polynomials out of the 20 polynomials $x_\ell\bar\beta_i'$ are linearly independent. Thus (b) follows.

Now, applying formulas (2.10) and (2.12) in Theorem 2.7, we obtain that the first $\mathbb{Z}_2$-cohomology group has dimension 4 for both groups, while the second has dimension 7 for the first group, and 6 for the second.

We now show that the second Stiefel–Whitney class $w_2 \in H^2(M, \mathbb{Z}_2)$ does not vanish. To see that this is not zero, by Theorem 3.3, we need to verify that w_2 is not a sum of $\mathbb{Z}_2$-class polynomials.

The two weights are, for both groups,

$$\omega_1 = x_3, \qquad \omega_2 = x_4, \qquad \omega_3 = x_3 + x_4, \qquad \omega_4 = x_2, \qquad \omega_5 = x_1$$

so the second symmetric polynomial in the ω_j's is given in both cases by the homogeneous polynomial

$$\sigma_2 = x_1x_2 + x_3x_4 + x_3^2 + x_4^2.$$

Suppose σ_2 is a sum S of $\mathbb{Z}_2$-class polynomials. Then S must contain the $\mathbb{Z}_2$-class polynomials $\bar\beta_1 = \bar\beta_1'$ and $\bar\beta_4 = \bar\beta_4'$, otherwise the monomials x_3^2 and x_4^2 would not appear. But then S must contain the monomial x_2x_3 which does not appear either in σ_2 or in any other $\mathbb{Z}_2$-class polynomial, so it cannot be cancelled out by adding $\mathbb{Z}_2$-class polynomials.

Concerning the spectral properties of the corresponding manifolds, they are both very similar. Indeed, a computation using the definition in (B.1) shows that the Sunada numbers of the two groups coincide; therefore both manifolds are p-isospectral for $0 \le p \le n$. In fact, one shows that the only non-vanishing Sunada numbers in this case are $c_{1,1} = c_{2,1} = c_{2,2} = c_{3,1} = 3$ and $c_{3,2} = c_{4,1} = c_{4,2} = c_{5,0} = 1$. Furthermore, it is not difficult to check by the methods in [23] that both manifolds are length isospectral; that is, they have the same lengths of closed geodesics counted with multiplicities.

Problem 4.1. In [23, Ex. 3.4], a pair of Sunada isospectral manifolds with the same lengths of closed geodesics but different multiplicities is given. A straightforward computation shows that these manifolds have different cohomology over $\mathbb{Z}_2$. It would be of interest to find a pair of manifolds with the same spectral properties, but where they both have the same $\mathbb{Z}_2$-cohomology.

4.2. Hantzsche–Wendt manifolds

A *Hantzsche–Wendt group* [4] is an n-dimensional diagonal *orientable* Bieberbach group Γ with $F \cong \mathbb{Z}_2^{n-1}$. The point (or holonomy) group is $\{B : Be_i = \pm e_i, 1 \le i \le n, \det B = 1\}$ and necessarily n must be odd. The corresponding manifold $M_\Gamma = \Gamma \setminus \mathbb{R}^n$ is called a *Hantzsche–Wendt* (or HW) manifold. Although the holonomy action is fixed, by varying the translation vectors and one gets a large class of non-homeomorphic flat manifolds (see [4]). They all are rational homology spheres, but their $\mathbb{Z}_2$-cohomology is far from being zero and can be different for different manifolds.

It is convenient to choose generators of F, B_i, with $B_i(e_i) = e_i$ and $B_ie_j = -e_j$, if $j \ne i$, for $1 \le i \le n-1$.

In column notation, an HW group has the following form.

B_1	B_2	B_3	$\dots$	B_{n-1}
$1_{\frac{1}{2}}$	-1_*	-1_*	$\dots$	-1_*
-1_*	$1_{\frac{1}{2}}$	-1_*	$\dots$	-1_*
-1_*	-1_*	$1_{\frac{1}{2}}$	$\dots$	-1_*
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
-1_*	-1_*	-1_*	$\dots$	$1_{\frac{1}{2}}$
-1_*	-1_*	-1_*	$\dots$	-1_*

Here, each $*$ can be either 0 or $\frac{1}{2}$.

In dimension $n = 3$ there is just one HW manifold, namely the didicosm (see Examples 1.1, 2.8 and 3.4).

In dimension $n = 5$ there are two isomorphism classes. The corresponding Bieberbach groups can be described as follows:

Γ_1^5 :

B_1	B_2	B_3	B_4
$1_{\frac{1}{2}}$	$-1_{\frac{1}{2}}$	-1	-1
-1	$1_{\frac{1}{2}}$	$-1_{\frac{1}{2}}$	-1
-1	-1	$1_{\frac{1}{2}}$	$-1_{\frac{1}{2}}$
-1	-1	-1	$1_{\frac{1}{2}}$
$-1_{\frac{1}{2}}$	-1	-1	-1

Γ_2^5 :

B_1	B_2	B_3	B_4
$1_{\frac{1}{2}}$	$-1_{\frac{1}{2}}$	-1	-1
-1	$1_{\frac{1}{2}}$	$-1_{\frac{1}{2}}$	-1
-1	-1	$1_{\frac{1}{2}}$	$-1_{\frac{1}{2}}$
-1	-1	-1	$1_{\frac{1}{2}}$
-1	$-1_{\frac{1}{2}}$	-1	-1

Using Theorem 2.7, one can verify that in both cases

$$\dim H^1(M, \mathbb{Z}_2) = 4 \quad \text{and} \quad \dim H^2(M, \mathbb{Z}_2) = 6.$$

We now look at the second Stiefel–Whitney class.

The 2-weights are

$$\begin{aligned} &\omega_1 = x_2 + x_3 + x_4, \qquad \omega_2 = x_1 + x_3 + x_4, \qquad \omega_3 = x_1 + x_2 + x_4, \\ &\omega_4 = x_1 + x_2 + x_3, \qquad \omega_5 = x_1 + x_2 + x_3 + x_4. \end{aligned}$$

Thus $\sigma_2(\omega_1, \dots, \omega_5) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$.

The $\mathbb{Z}_2$-class polynomials are given by

$$\begin{aligned} &x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4, \\ &x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_2x_4 + x_3x_4, \\ &x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3^2 + x_3x_4, \\ &x_1x_4 + x_2x_4 + x_3x_4 + x_4^2, \\ &\text{together with } \begin{cases} x_1x_2 + x_1x_3 + x_1x_4 & \text{for } \Gamma_1^5, \\ x_1x_2 + x_2x_3 + x_2x_4 & \text{for } \Gamma_2^5. \end{cases} \end{aligned}$$

Now, assume we have a linear system that one gets by imposing the condition that $\sigma_2 = \sum_{\ell<m} x_\ell x_m$ is equal to a linear combination (with coefficients 0 or 1) of $\mathbb{Z}_2$-class polynomials. Since the first polynomial is the only one containing x_1^2, the first coefficient must be zero, and so on, up to the fourth coefficient. So, only the fifth coefficient is left, but clearly neither $x_1x_2 + x_1x_3 + x_1x_4$ nor $x_1x_2 + x_2x_3 + x_2x_4$ can be equal to $\sum_{\ell<m} x_\ell x_m$.

Therefore, it follows that the second Stiefel–Whitney class does not vanish in either case.

By using the methods above, we give a very simple proof of the following consequence of a result in [6], where the first homology with integer coefficients is computed.

Proposition 4.2. *For any n-dimensional HW manifold,* $\dim H^1(M, \mathbb{Z}_2) = n - 1$.

Proof. Let M_n be an HW manifold of dimension n. The result follows from (2.10) and the fact that the differential $d_2^{0,1}$ has maximal rank n, since the $\mathbb{Z}_2$-class polynomials are linearly independent. To see this, it is enough to observe that the first $\mathbb{Z}_2$-class polynomial is the only one containing the monomial x_1^2, the second is the only one containing x_2^2, and so on up to x_{n-1}^2. The last $\mathbb{Z}_2$-class polynomial is nonzero and does not contain any monomial of the form x_j^2, so it is necessarily independent from the others. □

Problem 4.3. (1) Determine the $\mathbb{Z}_2$-cohomology of an arbitrary HW manifold. Determine the structure of the cohomology ring.

(2) Identify the HW manifolds having most regular $\mathbb{Z}_2$-cohomology rings, that is, having smallest possible dimensions.

(3) Connect some properties of the cohomology with some properties of the associated directed graphs as in [4].

(4) Interpret the isospectrality of HW manifolds in terms of some properties of the graphs.

4.3. The family $\mathcal{K}_n$

We now consider a family $\mathcal{K}_n$ (see [9] and also [3]) which is a subfamily of the *generalized Hantzsche–Wendt manifolds*. The GHW manifolds consist of those Bieberbach groups in dimension n having holonomy group $\mathbb{Z}_2^{n-1}$. These groups form a very large class that includes the examples constructed in [10] and also the HW groups discussed in the previous subsection as a minority. For instance, in dimension $n = 4$ there are 12 (resp. 0) GHW (resp. HW) groups, while if $n = 5$ there are 123 (resp. 2) GHW (resp. HW) groups. Note that a GHW group is of type HW if and only if it is orientable. We refer to [3,9,24] for more facts on these groups. For instance, in [9] it is shown that all GHW manifolds are isospectral on forms; that is, for the full Hodge–Laplacian acting on p-forms for all $0 \le p \le n$. This also holds for the family $\mathcal{K}_n$.

In this subsection we will obtain cohomological results on the family $\mathcal{K}_n$ and as a byproduct, we will construct isospectral 4-manifolds M, M', having the same $\mathbb{Z}_2$-cohomology but where $w_2(M) = 0$ and $w_2(M') \neq 0$.
We first define the family $\mathcal{K}_n$. Let Γ_n be the Bieberbach group having holonomy group $\mathbb{Z}_2^{n-1}$ and lattice $\mathbb{Z}^n$, with generators given as follows:

B_1	B_2	B_3	$\dots$	$\dots$	B_{n-1}
-1	$1_{\frac{1}{2}a_{12}}$	$1_{\frac{1}{2}a_{13}}$	$\dots$	$\dots$	$1_{\frac{1}{2}a_{1\,n-1}}$
$1_{\frac{1}{2}}$	-1	$1_{\frac{1}{2}a_{23}}$	$\dots$	$\dots$	$1_{\frac{1}{2}a_{2\,n-1}}$
1	$1_{\frac{1}{2}}$	-1	$\dots$	$\dots$	$1_{\frac{1}{2}a_{3\,n-1}}$
1	1	$1_{\frac{1}{2}}$	$\ddots$	$\dots$	$1_{\frac{1}{2}a_{4\,n-1}}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\vdots$
1	1	1	$\dots$	$\ddots$	-1
1	1	1	$\dots$	$\dots$	$1_{\frac{1}{2}}$

where $B_1, \dots B_{n-1}$ are generators of $\mathbb{Z}_2^{n-1}$ and the coefficients a_{ij} are either 1 or 0. We note that if the coefficients are all equal to zero, we get the examples given by Lee and Szczarba in [10].

Denote by M_n the compact flat manifold with Bieberbach group Γ_n.

Our first goal in this subsection is to prove the following theorem.

Theorem 4.4. *If M_n is a GHW manifold in the family $\mathcal{K}_n$, then*
For $n = 4$

(a) *if $a_{12} = a_{13} = 1$, then $w_2(M_3) = 0$ (for any value of a_{23});*
(b) *if either a_{12} or a_{13} is zero, then $w_2(M_3) \neq 0$ (for any value of a_{23}).*
(c) *there exist isospectral pairs of manifolds in $\mathcal{K}_4$ such that one has vanishing w_2 and the other not.*

For $n \ge 5$: $w_2(M_n) \neq 0$ for any choice of the coefficients a_{ij}.

Proof. For $n = 4$ we write $x := a_{12}$, $y := a_{13}$, $z := a_{23}$, so that we denote the groups by (x, y, z):

B_1	B_2	B_3
-1	$1_{\frac{x}{2}}$	$1_{\frac{y}{2}}$
$1_{\frac{1}{2}}$	-1	$1_{\frac{z}{2}}$
1	$1_{\frac{1}{2}}$	-1
1	1	$1_{\frac{1}{2}}$

The $\mathbb{Z}_2$-class polynomials are

$$\begin{gathered} x(x_1x_2 + x_2^2) + y(x_3^2 + x_1x_3), \\ x_1^2 + x_1x_2 + z(x_2x_3 + x_3^2), \\ x_2^2 + x_2x_3, \\ x_3^2. \end{gathered}$$

We now look at the second Stiefel–Whitney class.

Let $x_i : \mathbb{Z}_2^{n-1} \to \mathbb{Z}_2$ be the homomorphism sending B_i to 1 and all B_j with $j \neq i$ to 0. Then $i^*x'_1 = x_1, \dots, i^*x'_{n-1} = x_{n-1}$, $i^*x'_n = 0$. Thus the 2-weights are $\omega_1 = x_1, \dots, \omega_{n-1} = x_{n-1}$, $\omega_n = 0$ and $\sigma_2(\omega_1, \dots, \omega_n) = \sigma_2(x_1, \dots, x_{n-1})$.

We consider the linear system obtained by letting $\sum_{\ell<m} x_\ell x_m$ be equal to a $\mathbb{Z}_2$-linear combination of $\mathbb{Z}_2$-class polynomials:

$$x_1x_2 + x_1x_3 + x_2x_3 = \lambda_1 x_3^2 + \lambda_2(x_2^2 + x_2x_3) + \lambda_3[x_1^2 + x_1x_2 + z(x_2x_3 + x_3^2)] + \lambda_4[x(x_1x_2 + x_2^2) + y(x_3^2 + x_1x_3)].$$

Table 1
Family $\mathcal{K}_4$.

(x, y, z)	$\text{betti}_1^{\mathbb{Z}_2} = \text{betti}_3^{\mathbb{Z}_2}$	$\text{betti}_2^{\mathbb{Z}_2}$	w_2	Sunada n.	Isospectral pairs
$(0, 0, 0)$	4	6	$\neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 0 \end{pmatrix}$	
$(1, 0, 0)$	3	4	$\neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$	♣
$(1, 1, 0)$	3	4	0	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$	♡
$(0, 1, 0)$	3	4	$\neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$	♣
$(0, 0, 1)$	4	6	$\neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}$	
$(1, 0, 1)$	3	4	$\neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$	♡
$(0, 1, 1)$	3	4	$\neq 0$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$	
$(1, 1, 1)$	3	4	0	$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	

By comparing the x_1^2 components we get that $\lambda_3 = 0$. Taking this into account, we have

$$\begin{array}{ll} \lambda_2 + x\lambda_4 = 0 & \text{(from the } x_2^2\text{-component)} \\ \lambda_1 + y\lambda_4 = 0 & \text{(from the } x_3^2\text{-component)} \\ x\lambda_4 = 1 & \text{(from the } x_1x_2\text{-component)} \\ y\lambda_4 = 1 & \text{(from the } x_1x_2\text{-component)} \\ \lambda_2 = 1 & \text{(from the } x_2x_3\text{-component).} \end{array}$$

Hence, we get

$$\lambda_1 = \lambda_2 = 1, \qquad \lambda_3 = 0, \qquad x\lambda_4 = 1, \qquad y\lambda_4 = 1.$$

Thus, there are two cases to consider.

(1) If $x = y = 1$, then the system has the solution $\lambda_1 = \lambda_2 = \lambda_4 = 1$, $\lambda_3 = 0$. So, $x_1x_2 + x_1x_3 + x_2x_3$ is a sum of $\mathbb{Z}_2$-class polynomials and $w_2 = 0$ (for any choice of a_{32}).
(2) If x or y is zero, then the system has no solution. Therefore $w_2 \neq 0$ (for any z).

The computation of the $\mathbb{Z}_2$-cohomology can be done as usual. We shall list the results in Table 1.

Note that the manifolds $(1, 1, 0)$ and $(1, 0, 1)$, marked with ♡, have the same $\mathbb{Z}_2$-cohomology, but, as we have just shown, they have different second Stiefel–Whitney classes.

Furthermore, we claim that they are isospectral, as we shall see, by computing the Sunada numbers $c_{s,t}$. The groups have the form

B_1	B_2	B_3
-1	$1_{\frac{1}{2}}$	$1_{\frac{1}{2}}$
$1_{\frac{1}{2}}$	-1	1
1	$1_{\frac{1}{2}}$	-1
1	1	$1_{\frac{1}{2}}$

B'_1	B'_2	B'_3
-1	$1_{\frac{1}{2}}$	1
$1_{\frac{1}{2}}$	-1	$1_{\frac{1}{2}}$
1	$1_{\frac{1}{2}}$	-1
1	1	$1_{\frac{1}{2}}$

We note that B_1 has exactly one $\frac{1}{2}$ in its fixed space, spanned by e_2, e_3, e_4; hence B_1 contributes 1 to $c_{3,1}$. Now, B_2 has exactly two $\frac{1}{2}$'s in its fixed space, spanned by e_1, e_3, e_4; hence B_2 contributes 1 to $c_{3,2}$. In the same way, B_3 contributes 1 to $c_{3,2}$. If we look at B_1B_2, the fixed space is spanned by e_3, e_4, of dimension 2, and it has $\frac{1}{2}$ in its third coordinate; hence it contributes 1 to $c_{2,1}$. Taking into account the contributions of B_1B_3 to $c_{2,2}$, of B_2B_3 to $c_{2,1}$, of $B_1B_2B_3$ to $c_{1,1}$ and of Id, we can see that $c_{3,2} = c_{2,1} = 2$, $c_{3,1} = c_{2,2} = c_{1,1} = c_{4,0} = 1$, and the remaining numbers $c_{s,t}$ are all equal to zero.

The analogous calculation for the second group yields the same values for the $c_{s,t}$; hence the corresponding manifolds are isospectral. We leave this verification to the reader.

We look now at the second Stiefel–Whitney class w_2 of $\mathcal{K}_n$ for $n = 5$.

B_1	B_2	B_3	B_4
-1	$1_{\frac{1}{2}a_{12}}$	$1_{\frac{1}{2}a_{13}}$	$1_{\frac{1}{2}a_{14}}$
$1_{\frac{1}{2}}$	-1	$1_{\frac{1}{2}a_{23}}$	$1_{\frac{1}{2}a_{24}}$
1	$1_{\frac{1}{2}}$	-1	$1_{\frac{1}{2}a_{34}}$
1	1	$1_{\frac{1}{2}}$	-1
1	1	1	$1_{\frac{1}{2}}$

where the coefficients a_{ij} are either 1 or 0. The $\mathbb{Z}_2$-class polynomials are

$$\begin{aligned}
P_1(x_1,\ldots,x_4) &= a_{12}(x_1x_2 + x_2^2) + a_{13}(x_1x_3 + x_3^2) + a_{14}(x_1x_4 + x_4^2),\\
P_2(x_1,\ldots,x_4) &= x_1^2 + x_1x_2 + a_{23}(x_2x_3 + x_3^2) + a_{24}(x_2x_4 + x_4^2),\\
P_3(x_1,\ldots,x_4) &= x_2^2 + x_2x_3 + a_{34}(x_3x_4 + x_4^2),\\
P_4(x_1,\ldots,x_4) &= x_3^2 + x_3x_4,\\
P_5(x_1,\ldots,x_4) &= x_4^2.
\end{aligned}$$

Let us solve the linear system obtained by assuming that $\sum_{\ell<m} x_\ell x_m$ is in the kernel of r^*, i.e.,

$$\begin{aligned}
x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 &= \lambda_1P_1(x_1,\ldots,x_4) + \lambda_2P_2(x_1,\ldots,x_4) + \lambda_3P_3(x_1,\ldots,x_4)\\
&\quad + \lambda_4P_4(x_1,\ldots,x_4) + \lambda_5P_5(x_1,\ldots,x_4).
\end{aligned}$$

Since $P_2(x_1,\ldots,x_4)$ is the only polynomial containing x_1^2, $\lambda_2 = 0$. Taking this into account, we see that there is no monomial x_2x_4 on the right-hand side. Thus, there are no solutions, for any values of the coefficients. So the second Stiefel–Whitney class (given by $r^*(\sum_{\ell<m} x_\ell x_m)$) does not vanish.

With entirely similar arguments, one shows that, in the general case of dimension n, $w_2 \neq 0$, for any value of the coefficients. □

To conclude the subsection we consider the case of the manifold $M_n = LS$, i.e. the case when all a_{ij}'s equal 0, as in the examples given by Lee and Szczarba. We will compute the second cohomology group explicitly.

Theorem 4.5.

$$\dim H^1(LS,\mathbb{Z}_2) = n, \qquad \dim H^2(LS,\mathbb{Z}_2) = \binom{n}{2}. \tag{4.1}$$

Proof. In this case, the $\mathbb{Z}_2$-class polynomials are

$$\bar\beta_1 = 0,\ldots,\bar\beta_i = x_i^2 + x_ix_{i+1},\ldots,\bar\beta_n = x_{n-1}^2.$$

Thus rank $d_2^{0,1} = n-1$, and applying formula (2.10) in Theorem 2.7 we get

$$\dim H^1(LS,\mathbb{Z}_2) = n - (n-1) + n - 1 = n.$$

Relative to the second assertion, we note that we are in the hypotheses of (2.12), so we just need to compute rank $d_2^{1,1}$. It is not difficult to see that all polynomials $x_j\bar\beta_i$ are linearly independent. Thus rank $d_2^{1,1} = (n-1)^2$. Hence

$$\dim H^2(M,\mathbb{Z}_2) = \binom{n}{2} - (n-1) + (n-1)n - (n-1)^2 = \binom{n}{2}. \quad \square$$

4.4. GHW manifolds in dimension 5

In this subsection we give the cohomology with $\mathbb{Z}_2$-coefficients of all GHW 5-manifolds obtained by using Theorem 2.7, and with the help of a computer program we will also list all the isospectral pairs.

By definition, these manifolds have holonomy group $\mathbb{Z}_2^4$; they are all of diagonal type and correspond to three different holonomy representations that will be described below.

They are given in Table 2 following the enumeration in [22] and listed according to their first and second $\mathbb{Z}_2$-cohomology groups. By Poincaré duality, this yields all the cohomological information, since for $\mathbb{Z}_2$-coefficients all manifolds behave as orientable ones.

Table 2
Cohomology classes of GHW manifolds in dimension 5.

$\text{betti}_1^{\mathbb{Z}_2}$	$\text{betti}_2^{\mathbb{Z}_2}$	List of manifolds
4	5	7, 14, 16, 21, 58, 61, 67, 69, 74, 77, 84, 85, 104, 105, 106, 107, 112, 115, 117, 118, 121, 122
4	6	2, 3, 4, 6, 8, 9, 10, 11, 13, 17, 18, 20, 22, 23, 36, 37, 38, 39, 40, 41, 44, 45, 50, 51, 52, 53, 57, 59, 60, 62, 65, 66, 68, 71, 73, 75, 76, 78, 80, 81, 82, 83, 90, 91, 92, 93, 96, 97, 98, 100, 101, 102, 109, 111, 113, 114, 116, 119
4	7	1, 5, 12, 15, 19, 28, 29, 30, 31, 42, 43, 46, 47, 48, 49, 54, 55, 56, 63, 64, 70, 72, 79, 86, 87, 88, 89, 94, 95, 99, 103, 108, 110, 120, 123
5	10	24, 25, 26, 27, 32, 33, 34, 35

We conclude the subsection by listing all the isospectral pairs among these 123 manifolds, obtained by comparison of the Sunada numbers, calculated with the help of a computer program. We will write $1 = 4 = 6$ to mean that the manifolds numbered 1, 4 and 6 are isospectral, while, if a number is missing, it means that the manifold is not isospectral to any other manifold in the list.

$$1 = 4 = 6 \quad 2 = 9 \quad 10 = 13 \quad 11 = 12 = 18 = 20 \quad 15 = 19 \quad 16 = 17 \quad 26 = 32 \quad 28 = 56$$
$$29 = 48 \quad 37 = 49 = 88 = 108 \quad 38 = 72 \quad 41 = 42 = 44 = 57 = 60 = 64 = 80$$
$$43 = 45 = 46 = 52 = 59 = 62 = 96 = 98 \quad 53 = 97 \quad 55 = 110 \quad 63 = 99 \quad 66 = 68 = 84 = 104$$
$$67 = 69 = 74 = 83 = 90 \quad 70 = 92 = 109 = 120 \quad 71 = 87 = 100 \quad 73 = 76 = 86 = 89$$
$$75 = 78 = 94 = 102 \quad 77 = 93 = 106 = 117 \quad 79 = 103 \quad 81 = 116 \quad 82 = 112 \quad 85 = 105$$
$$101 = 111 \quad 114 = 118 \quad 115 = 121.$$

Observe that the first 21 groups have the same integral holonomy representation, they are non-orientable and, furthermore, $\text{betti}_1 = 0$. The holonomy representation is given by

B_1	B_2	B_3	B_4
−1	1	1	1
−1	−1	1	1
1	−1	1	1
1	1	−1	1
1	1	1	−1

There are two additional holonomy representations. The groups 22 and 23 correspond to the two HW 5-manifolds (sometimes called 'cyclic' and '$(n-1)$-cyclic', by the shape of the corresponding graphs; see Section 4.2). On the other hand, the groups numbered from 24 to 123 have the same integral holonomy representation; they have $\text{betti}_1 = 1$ and give non-orientable manifolds.

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Appendix A. $\mathbb{Z}_2$-cohomology of four-dimensional Bieberbach groups with diagonal holonomy

In this appendix we consider the diagonal manifolds of dimension 4 having canonical lattice of translations $\mathbb{Z}^4$ and holonomy group ${\mathbb{Z}_2}^k$, with $k = 2, 3$. We give the first and second (hence all) $\mathbb{Z}_2$-cohomology groups, together with the p-isospectral classes for $0 \le p \le 4$, and the $\mathbb{Z}_2$-class polynomials $\bar{\beta}_\ell$ in each case. We include in tables the full list of manifolds divided into families having different holonomy representations, and following the enumeration given in [25], in slightly modified column notation.

The tables show several isospectral, or p-isospectral pairs for $1 \le p \le 3$, having different $\mathbb{Z}_2$-cohomology groups. We note that, for all pairs, if both manifolds in a pair have different $\mathbb{Z}_2$-cohomology, then all H^i, for $1 \le i \le 3$, are different from each other. Thus it is not possible to obtain an example as in Section 4.1 of two manifolds having the same H^1 and different H^2.

We recall also that some of the p-isospectral manifolds in the tables have different lengths of closed geodesics (see [23]).

We remark also that in some instances there are several isometry classes corresponding to the same diffeomorphism class. In our case, for simplicity, we have chosen only one representative in each class (with the only exception of M_{19} and $M_{19'}$) in such a way that all isospectral classes are present, but not necessarily all p-isospectral classes (for a complete list see [25]).

Note: p-isospectralities are expressed by the coincidence of symbols, such as ♣, ◇, □, ⊕, etc., that are included in the tables.

Family $\mathcal{F}_1$: $F \cong \mathbb{Z}_2^2$, $\text{betti}_1^{\mathbb{Z}} = 2$, $\text{betti}_2^{\mathbb{Z}} = 1$.

M_Γ		$M_{7'}$			M_8			$M_{9'}$			M_{10}			M_{11}		
B_1	B_2	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$
1	1		$\frac{1}{2}$	x_2^2			0		$\frac{1}{2}$	x_2^2		$\frac{1}{2}$	x_2^2		$\frac{1}{2}$	x_2^2
1	1		$\frac{1}{2}$	x_2^2		$\frac{1}{2}$	x_2^2	$\frac{1}{2}$	$\frac{1}{2}$	$x_1^2 + x_2^2$	$\frac{1}{2}$		x_1^2	$\frac{1}{2}$		x_1^2
1	-1	$\frac{1}{2}$		$x_1^2 + x_{12}$	$\frac{1}{2}$		$x_1^2 + x_{12}$			0			0	$\frac{1}{2}$	0	$x_1^2 + x_{12}$
-1	1			0		$\frac{1}{2}$	$x_{12} + x_2^2$			0		$\frac{1}{2}$	$x_{12} + x_2^2$		$\frac{1}{2}$	$x_{12} + x_2^2$
$\text{betti}_1^{\mathbb{Z}_2}$		4			3			4			3			3		
$\text{betti}_2^{\mathbb{Z}_2}$		6			4			6			4			4		
isosp.		♣			◇			◇			♣					
2-isosp.		■			□			□			■			■		
1, 3-isosp.		○			○			○			○					

Family $\mathcal{F}_3$: $F \cong \mathbb{Z}_2^2$, $\text{betti}_1^{\mathbb{Z}} = 1$, $\text{betti}_2^{\mathbb{Z}} = 1$.

M_Γ		M_{18}			M_{19}			$M_{19'}$			M_{20}			M_{21}		
B_1	B_2	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$
1	-1			0			0	$\frac{1}{2}$		$x_1^2 + x_{12}$						0
1	-1				$\frac{1}{2}$		$x_1^2 + x_{12}$	$\frac{1}{2}$		$x_1^2 + x_{12}$	$\frac{1}{2}$		$x_1^2 + x_{12}$	$\frac{1}{2}$		$x_1^2 + x_{12}$
-1	-1		$\frac{1}{2}$	x_{12}								$\frac{1}{2}$	x_{12}		$\frac{1}{2}$	x_{12}
1	1	$\frac{1}{2}$	$\frac{1}{2}$	$x_1^2 + x_2^2$		$\frac{1}{2}$	x_2^2		$\frac{1}{2}$	x_2^2		$\frac{1}{2}$	x_2^2	$\frac{1}{2}$	$\frac{1}{2}$	$x_1^2 + x_2^2$
$\text{betti}_1^{\mathbb{Z}_2}$		4			4			4			3			3		
$\text{betti}_2^{\mathbb{Z}_2}$		6			6			6			4			4		
isosp.		♠			♠			♡						♡		
2-isosp.		□			□			□			■			□		
1, 3-isosp.		⊕			⊕			•			⊕			•		

Family $\mathcal{F}_4$: $F \cong \mathbb{Z}_2^2$, $\text{betti}_1^{\mathbb{Z}} = 1$, $\text{betti}_2^{\mathbb{Z}} = 0$.

M_Γ		M_{24}			M_{25}			M_{26}			M_{27}		
B_1	B_2	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$	b_1b_2		$\bar{\beta}$
-1	1			0		$\frac{1}{2}$	$x_{12} + x_2^2$		$\frac{1}{2}$	$x_{12} + x_2^2$		$\frac{1}{2}$	$x_{12} + x_2^2$
-1	-1		$\frac{1}{2}$	x_{12}		$\frac{1}{2}$	x_{12}		$\frac{1}{2}$	x_{12}		$\frac{1}{2}$	x_{12}
1	-1							$\frac{1}{2}$		$x_1^2 + x_{12}$	$\frac{1}{2}$		$x_1^2 + x_{12}$
1	1	$\frac{1}{2}$	$\frac{1}{2}$	$x_1^2 + x_2^2$	$\frac{1}{2}$		x_1^2			0		$\frac{1}{2}$	x_2^2
$\text{betti}_1^{\mathbb{Z}_2}$		4			3			3			3		
$\text{betti}_2^{\mathbb{Z}_2}$		6			4			4			4		
isosp.		✠						✠					
2-isosp.		⊞						⊞					
1, 3-isosp.		⊗			⊗			⊗			⊗		

Family $\mathcal{F}_5$: $F \cong \mathbb{Z}_2^3$, $\mathrm{betti}_1^{\mathbb{Z}} = 1, \mathrm{betti}_2^{\mathbb{Z}} = 0$.

M_Γ			M_{33}				M_{34}				M_{35}				M_{36}			
B_1	B_2	B_3	$b_1b_2b_3$			$\bar{\beta}$	$b_1b_2b_3$			$\bar{\beta}$	$b_1b_2b_3$			$\bar{\beta}$	$b_1b_2b_3$			$\bar{\beta}$
-1	1	-1			$\frac{1}{2}$	x_{13}			$\frac{1}{2}$	x_{13}			$\frac{1}{2}$	x_{13}				0
1	1	-1		$\frac{1}{2}$		$x_2^2 + x_{23}$		$\frac{1}{2}$		$x_2^2 + x_{23}$		$\frac{1}{2}$		$x_2^2 + x_{23}$		$\frac{1}{2}$		$x_2^2 + x_{23}$
1	-1	1				0			$\frac{1}{2}$	$x_{23} + x_3^2$			$\frac{1}{2}$	$x_{23} + x_3^2$	$\frac{1}{2}$			$x_1^2 + x_{12}$
1	1	1	$\frac{1}{2}$		$\frac{1}{2}$	$x_1^2 + x_3^2$	$\frac{1}{2}$		$\frac{1}{2}$	$x_1^2 + x_3^2$	$\frac{1}{2}$	$\frac{1}{2}$		$x_1^2 + x_2^2$			$\frac{1}{2}$	x_3^2
$\mathrm{betti}_1^{\mathbb{Z}_2}$			4				3				3				4			
$\mathrm{betti}_2^{\mathbb{Z}_2}$			6				4				4				6			
isosp.							$\blacklozenge$				$\bigstar$							
2-isosp.			$\boxtimes$				$\boxminus$				$\boxdot$				$\boxdot$			
1, 3-isosp.							$\odot$				$\oslash$				$\odot$			

M_Γ			M_{37}				M_{38}				M_{39}			
B_1	B_2	B_3	$b_1b_2b_3$			$\bar{\beta}$	$b_1b_2b_3$			$\bar{\beta}$	$b_1b_2b_3$			$\bar{\beta}$
-1	1	-1			$\frac{1}{2}$	x_{13}			$\frac{1}{2}$	x_{13}		$\frac{1}{2}$		$x_{12} + x_2^2 + x_{23}$
1	1	-1		$\frac{1}{2}$		$x_2^2 + x_{23}$		$\frac{1}{2}$		$x_2^2 + x_{23}$		$\frac{1}{2}$		$x_2^2 + x_{23}$
1	-1	1	$\frac{1}{2}$			$x_1^2 + x_{12}$	$\frac{1}{2}$		$\frac{1}{2}$	$x_1^2 + x_{12} + x_{23} + x_3^2$	$\frac{1}{2}$			$x_1^2 + x_{12}$
1	1	1			$\frac{1}{2}$	x_3^2			$\frac{1}{2}$	x_3^2			$\frac{1}{2}$	x_3^2
$\mathrm{betti}_1^{\mathbb{Z}_2}$			3				3				3			
$\mathrm{betti}_2^{\mathbb{Z}_2}$			4				4				4			
isosp.							$\blacklozenge$				$\blacktriangleright$			
2-isosp.			$\boxtimes$				$\boxminus$				$\boxtimes$			
1, 3-isosp.							$\odot$				$\ominus$			

M_Γ			M_{40}				M_{41}				M_{42}			
B_1	B_2	B_3	$b_1b_2b_3$			$\bar{\beta}$	$b_1b_2b_3$			$\bar{\beta}$	$b_1b_2b_3$			$\bar{\beta}$
-1	1	-1		$\frac{1}{2}$	$\frac{1}{2}$	$x_{12}+x_2^2+x_{13}+x_{23}$		$\frac{1}{2}$	$\frac{1}{2}$	$x_{12}+x_2^2+x_{13}+x_{23}$			$\frac{1}{2}$	x_{13}
1	1	-1		$\frac{1}{2}$		$x_2^2 + x_{23}$		$\frac{1}{2}$		$x_2^2 + x_{23}$		$\frac{1}{2}$		$x_2^2 + x_{23}$
1	-1	1	$\frac{1}{2}$			$x_1^2 + x_{12}$	$\frac{1}{2}$		$\frac{1}{2}$	$x_1^2+x_{12}+x_{23}+x_3^2$	$\frac{1}{2}$		$\frac{1}{2}$	$x_1^2+x_{12}+x_{23}+x_3^2$
1	1	1			$\frac{1}{2}$	x_3^2			$\frac{1}{2}$	x_3^2			$\frac{1}{2}$	$x_1^2 + x_2^2$
$\mathrm{betti}_1^{\mathbb{Z}_2}$			3				3				3			
$\mathrm{betti}_2^{\mathbb{Z}_2}$			4				4				4			
isosp.			$\bigstar$				$\blacktriangleright$							
2-isosp.			$\boxdot$				$\boxtimes$				$\boxdot$			
1, 3-isosp.			$\oslash$				$\ominus$							

Family $\mathcal{F}_6$: $F \cong \mathbb{Z}_2^3$, $\text{betti}_1^{\mathbb{Z}} = 0, \text{betti}_2^{\mathbb{Z}} = 0.$

M_Γ			M_{43}		M_{44}	
B_1	B_2	B_3	$b_1b_2b_3$	$\bar{\beta}$	$b_1b_2b_3$	$\bar{\beta}$
1	−1	−1	$\ \ \frac{1}{2}$	x_{23}	$\ \ \frac{1}{2}$	x_{23}
1	1	−1	$\ \frac{1}{2}\ $	$x_2^2 + x_{23}$	$\frac{1}{2}\ \frac{1}{2}\ $	$x_1^2 + x_{13} + x_2^2 + x_{23}$
1	−1	1	$\frac{1}{2}\ \ $	$x_1^2 + x_{12}$	$\frac{1}{2}\ \ $	$x_1^2 + x_{12}$
−1	−1	1	$\ \ \frac{1}{2}$	$x_{13} + x_{23} + x_3^2$	$\ \frac{1}{2}\ \frac{1}{2}$	$x_{12} + x_{13} + x_{23} + x_3^2$
$\text{betti}_1^{\mathbb{Z}_2}$			3		3	
$\text{betti}_2^{\mathbb{Z}_2}$			4		4	
isosp.						
2-isosp.			$\boxtimes$		$\boxdot$	
1, 3-isosp.						

Appendix B. Isospectrality and Sunada numbers

In the examples of isospectral manifolds occurring in this article, we have dealt with flat manifolds of diagonal type, and in this context isospectrality is equivalent to a combinatorial condition, as shown in [11, Thm. 3.12(d)] and [23, Thm. 4.5]. We will now briefly review this condition that is rather useful and easy to apply.

If G is a finite group and F and F' are subgroups of G, we say that F and F' are *almost conjugate* in G if there is a bijection $\varphi : F \leftrightarrow F'$ such that $\varphi(x) = g_x x g_x^{-1}$, for some $g_x \in G$. Equivalently, $\#([g] \cap F) = \#([g] \cap F')$ for each $g \in G$, where $[g]$ denotes the G-conjugacy class of g.

Theorem B.1 ([26]). *Let M be a compact Riemannian manifold. If $G \subset \text{Isom}(M)$ and F, F' are almost conjugate subgroups of G acting freely on M, then $F \setminus M$ and $F' \setminus M$ are isospectral. Moreover, they are p-isospectral for every $0 \le p \le n$ (i.e., with respect to the Laplacian acting on p-forms).*

This theorem applies in the context of compact flat manifolds by taking $M = \Lambda\backslash\mathbb{R}^n, F = \Lambda\backslash\Gamma, F' = \Lambda\backslash\Gamma'$, with a suitable choice of a finite subgroup $G \subset \text{Isom}(M)$. Then $F, F' \subset \text{Isom}(M)$ act freely on M, and, furthermore, $F \setminus M \approx \Gamma \setminus \mathbb{R}^n = M_\Gamma$, $F' \setminus M \approx \Gamma' \setminus \mathbb{R}^n = M_{\Gamma'}$.

In the case of manifolds of diagonal type, a simple combinatorial condition that is equivalent to the Sunada condition was given in [4, Prop. 3.3] and [11, Prop. 3.5], consisting of the equality of the *Sunada numbers* $c_{s,t}$. We recall that if $0 \le t \le s \le n$, $c_{s,t}$ is the number of elements in the holonomy group $F = \Lambda \setminus \Gamma$ of M_Γ, having exactly s 1's in the diagonal (or column) and $t\ \frac{1}{2}$'s coming with those 1's. Equivalently,

$$c_{s,t} := \#\left\{B : BL_b \in \Gamma,\ \dim\ker(B - Id) = s \text{ and } b \text{ has exactly } t \text{ coordinates in } \frac{1}{2}\mathbb{Z} \smallsetminus \mathbb{Z} \text{ within } \ker(B - Id).\right\} \tag{B.1}$$

Note that $c_{n,0}(\Gamma) = 1$ and $\sum_{s,t} c_{s,t}(\Gamma) = |F|$.

The equality of the Sunada numbers of two flat manifolds of diagonal type is equivalent to the conditions in the hypothesis of Sunada's theorem (see [11, Prop. 3.5]). Thus, by Theorem B.1, the equality of the Sunada numbers implies that the two manifolds are p-isospectral for all p.

We refer to [27] for a recent survey on Sunada isospectrality technique and its applications.

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