AUTHOR QUERY FORM

ELSEVIER	**Journal:** PLA **Article Number:** 19506	**Please e-mail or fax your responses and any corrections to:** **E-mail: corrections.esnl@elsevier.vtex.lt** **Fax: +1 61 9699 6735**

Dear Author,

Any queries or remarks that have arisen during the processing of your manuscript are listed below and highlighted by flags in the proof. Please check your proof carefully and mark all corrections at the appropriate place in the proof (e.g., by using on-screen annotation in the PDF file) or compile them in a separate list.

For correction or revision of any artwork, please consult http://www.elsevier.com/artworkinstructions

Electronic file usage
Sometimes we are unable to process the electronic file of your article and/or artwork. If this is the case, we have proceeded by:

☐ Scanning (parts of) your article ☐ Rekeying (parts of) your article ☐ Scanning the artwork

Articles in Special Issues: Please ensure that the words 'this issue' are added (in the list and text) to any references to other articles in this Special Issue.

Uncited references: References that occur in the reference list but not in the text – please position each reference in the text or delete it from the list.

Missing references: References listed below were noted in the text but are missing from the reference list – please make the list complete or remove the references from the text.

Location in article	Query / remark Please insert your reply or correction at the corresponding line in the proof
Q1	Please cite uncited reference (Ref. [11]) or delete it. (p. 6/ line 24)
Q2	Please verify if Ref. [20] is correct. (p. 6/ line 45)

Thank you for your assistance.

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Discrete-continuous bispectral operators and rational Darboux transformations

Carina Boyallian *, Sofía Portillo

Ciem-FAMAF, Universidad Nacional de Córdoba, Ciudad Universitaria, 5000 Córdoba, Argentina

ARTICLE INFO

Article history:
Received 15 September 2009
Received in revised form 7 December 2009
Accepted 11 January 2010
Available online xxxx
Communicated by A.P. Fordy

Keywords:
Bispectral problem
Darboux transformations

ABSTRACT

In this Letter we construct examples of discrete-continuous bispectral operators obtained by rational Darboux transformations applied to a regular pseudo-difference operator with constant coefficients. Moreover, we give an explicit procedure to write down the differential operators involved in the bispectral situation corresponding to the pseudo-difference operator obtained by the Darboux process.

1. Introduction

The bispectral problem, as originally formulated by Duistermaat and Grünbaum [6], is the description of all situations where a pair of differential operators in the variables x and z has a common eigenfunction $\psi(x,z)$, namely

$$L(x,\partial_x)\psi(x,z) = \lambda(z)\psi(x,z), \tag{1.1}$$

$$B(z,\partial_z)\psi(x,z) = \theta(z)\psi(x,z). \tag{1.2}$$

For simplicity we say that L, B or ψ are bispectral if the situation above holds.

The results in [6] have interesting connections with a variety of topics that goes from Korteweg–de Vries equation to different areas of pure mathematics such as automorphism and ideal structure of Weyl algebra in one variable [2,5], representations of $W_{1+\infty}$ algebra [3], Calogero–Moser system [20], Huygens' principle [4], traces for intertwiners of (quantized) simple Lie algebras [7,8], etc.

In [6] all bispectral differential operators $L(x,\partial_x)$ of order two were classified. In this paper they used very explicitly the Darboux transformation, mapping a given second order bispectral differential operator into another one, which is also bispectral.

Wilson in [19] approached the problem from the point of view of commutative algebra of differential operators, and this work was translated to the language of Darboux transformations in [17], where an explicit formula for the bispectral operators was found.

Grünbaum and Haine first considered in [10] a discrete–differential version of the above problem when the variable x runs over the integers $\mathbb{Z}$ and accordingly one replaces the differential operator $L(x,\partial_x)$ by a pseudo-difference operator

$$L(l,\Lambda) = \sum_{i=p}^{q} b_i(l)\Lambda^i, \tag{1.3}$$

where $-p$ and q are non-negative integers, $b_i(l)$ are functions on $\mathbb{Z}$, and Λ^i is the operator that acts on a function $f:\mathbb{Z}\to\mathbb{C}$ by simply shifting the variable of the function in i:

$$\Lambda^i(f)(l) = f(l+i). \tag{1.4}$$

There is also a doubly infinite discrete–discrete version of the bispectral problem considered, for instance, in [9].

As an extension of Wilson's work [19], Haine and Iliev conjectured that [15] gives all the rank one solutions.

* Corresponding author. Tel.: +54 351 4334050 ext. 317; fax: +54 351 4334054.
E-mail addresses: boyallia@mate.uncor.edu (C. Boyallian), portillo@mate.uncor.edu (S. Portillo).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2010.01.027

In [13] they construct families of bispectral difference operators of the form $a(n)\Lambda + b(n) + c(n)\Lambda^{-1}$, where Λ is the shift operator, obtained as discrete Darboux transformations from appropriate extensions of Jacobi operators. They also conjecture that along with operators previously constructed in [12] and [16] they exhaust all bispectral regular operators of the form above.

In this Letter we construct examples of discrete-continuous bispectral operators obtained by rational Darboux transformations applied to regular pseudo-difference operators with constant coefficients following the scheme developed in [17] for the continuous–continuous case. Moreover, this allows us to give an explicit procedure to write down the bispectral differential operator corresponding to the pseudo-difference operator obtained by the Darboux process.

The Letter is organized as follows: In the first section we review some results about factorization of pseudo-difference operators, define Darboux transformations and introduce the notion of rational Darboux transformation. Next, we prove a discrete analog of Reach's Lemma [18] and using this we state the main result of the Letter, in which proof is hidden the procedure that allows us to construct the differential operator B involved in the bispectrality. Finally, we apply our results to some example.

2. Darboux transformation

In this section we review some facts about Darboux transformation and introduce the notion of rational Darboux transformation.

Given $\Phi^{(1)}, \Phi^{(2)}, \dots, \Phi^{(k)}$ functions in a discrete variable $l \in \mathbb{Z}$, the *discrete Wronskian* or *Casorati determinant* $W_k(l)$ is defined as

$$W_k(l) = W\left[\Phi^{(1)}, \Phi^{(2)}, \dots, \Phi^{(k)}\right](l) = \det\left(\Phi^{(j)}(i+l-1)\right)_{1 \leqslant i,j \leqslant k}. \tag{2.1}$$

We also use the following important notation throughout the Letter: $\tilde{\Lambda}^k$ shifts the immediate expression following the symbol only; i.e. for a function f on a discrete variable l,

$$\left(\tilde{\Lambda}^k f\right)(l) = f(l+k).$$

Note that $(\tilde{\Lambda}^k f)(l) \neq f(l+k)\Lambda^k$. Observe that any function on a discrete variable $l \in \mathbb{Z}$ can be viewed as a bi-infinite vector simply by writing down the value of the function in l in the l-th entry of the vector. Thus, it makes sense to act on such vectors by operators of the form (1.3), hence we can view operators (1.3) as bi-infinite matrices with a finite number of non-zero diagonals. This point of view is used in the following lemma. It was proved by Adler and van Moerbeke in [1] (cf. Lemma 6.1, p. 508).

Lemma 2.1. *Consider the pseudo-difference operator*

$$L = a_{-r}\Lambda^{-r} + a_{-r+1}\Lambda^{-r+1} + \dots + a_{n-r-1}\Lambda^{n-r-1} + \Lambda^{n-r} \tag{2.2}$$

with $n \geqslant 2$, $r \geqslant 0$, $n \geqslant r$, diagonal matrices a_j, with leading term $a_{-r}(j) \neq 0$ for j sufficiently small. Then any choice of basis $\Phi^{(1)}, \dots, \Phi^{(n)} \in \operatorname{Ker} L$ leads to a factorization of L:

$$L = \left(I - \Lambda^{-1}\left(\tilde{\Lambda}^{-r+1}\beta_n\right)\right)\left(I - \Lambda^{-1}\left(\tilde{\Lambda}^{-r+2}\beta_{n-1}\right)\right)\cdots\left(I - \Lambda^{-1}(\beta_{n-r+1})\right)(\Lambda - \beta_{n-r}I)(\Lambda - \beta_{n-r-1}I)\cdots(\Lambda - \beta_1 I) \tag{2.3}$$

with

$$\beta_k(l) = \frac{\alpha_k(l+1)}{\alpha_k(l)}, \quad \alpha_k(l) = \frac{W_k(l)}{W_{k-1}(l)}, \; W_k(l) = W\left[\Phi^{(1)}, \Phi^{(2)}, \dots, \Phi^{(k)}\right](l).$$

Remark 2.2. Once you choose a basis for $\operatorname{Ker} L$ with L like in (2.2), observe that $\operatorname{Ker} L = \operatorname{Ker}(\Lambda^r L)$ and $\alpha_1(l)$ in Lemma 2.1 coincides with the first element of the chosen basis.

Let L be a pseudo-difference operator of the form

$$L = a_{-r}\Lambda^{-r} + a_{-r+1}\Lambda^{-r+1} + \dots + a_{n-r-1}\Lambda^{n-r-1} + \Lambda^{n-r}$$

with $n \geqslant 2$, $r \geqslant 0$, $n \geqslant r$, a_j are functions in the discrete variable, and the leading term $a_{-r}(l) \neq 0$ for l sufficiently small. We will call such monic pseudo-difference operators *regular*.

Consider a regular pseudo-difference operator L in the discrete variable l with n and r as in Lemma 2.1, and a non-zero eigenfunction $\psi(l,\lambda)$ for which $L\psi(l,\lambda) = \lambda\psi(l,\lambda)$. By Lemma 2.1 we may write L in the form $L = RS + \lambda$ for some pseudo-difference operator R and $S = (\Lambda - \beta_1 I)$, where $\beta_1 = \frac{\tilde{\Lambda}(\psi(l,\lambda))}{\psi(l,\lambda)}$. The operator

$$\tilde{L} = SR + \lambda$$

is called a *Darboux transformation* of L. Observe that this definition coincides with the definition used in [1] applied to the pseudo-difference operator $L - \lambda$, namely applying the factorization in Lemma 2.1 to such operator and moving the factor most to the right, all the way to the left.

Note that if $L\psi(l,\lambda) = \lambda\psi(l,\lambda)$, then $\tilde{\psi}(l,\lambda) = (\Lambda - \frac{\tilde{\Lambda}(\psi(l,\lambda))}{\psi(l,\lambda)} I)\psi(l,\lambda)$ satisfies $\tilde{L}\tilde{\psi}(l,\lambda) = \lambda\tilde{\psi}(l,\lambda)$, so if one knows the eigenfunctions for L, then one obtains the eigenfunctions for $\tilde{L}$ just applying $(\Lambda - \frac{\tilde{\Lambda}(\psi(l,\lambda))}{\psi(l,\lambda)} I)$ to them. We have the following lemma which is the discrete analogous of Proposition 2.1 in [17]. We omit the proof since it is word by word identical.

Proposition 2.3. *Let L and L_0 be regular pseudo-difference operators, with L obtained from L_0 by a sequence of N Darboux transformations. Then there exists a monic difference operator U of order N such that $LU = UL_0$. Furthermore, $\operatorname{Ker} U$ is L_0-invariant.*

Conversely, if L and L_0 are monic pseudo-difference operators and there exists a difference operator U of order N such that $LU = UL_0$, then L may be obtained from L_0 by a sequence of N (or possibly fewer) Darboux transformation.

Recall that given a difference operator $U=\sum_{j=0}^{n} a_j \Lambda^j$ of degree n, with constant coefficient a_k, the space of solutions of the homogeneous equation $\sum_{j=0}^{n} a_j \Lambda^j = 0$, has a basis of n-independent solutions of the form

$$\lambda_1^j, j\lambda_1^j, \ldots, j^{\alpha_1-1}\lambda_1^j, \lambda_2^j, \ldots, j^{\alpha_2-1}\lambda_2^j, \ldots, \lambda_k^j, \ldots, j^{\alpha_k-1}\lambda_k^j,$$

where $\lambda_1, \ldots, \lambda_k$ are the roots of it underlying characteristic polynomial and $\alpha_1, \ldots, \alpha_k$ are its corresponding multiplicities, namely $\sum_{j=0}^{n} a_j x^j = \prod_{i=1}^{k}(x-\lambda_i)^{\alpha_i}$. Due to Proposition 2.3 we introduce the following definition.

Definition 2.4. Let L_0 be a regular pseudo-difference operator and L be a pseudo-difference operator obtained from L_0 by a Darboux transformation. We say that this Darboux transformation is *rational* if the corresponding intertwining difference operator U between L_0 and L (given by Proposition 2.3), has kernel generated by functions of the form $\langle p_k(l)e^{l\gamma_k}, k=1,\ldots,d\rangle$ where p_k are polynomials in one variable, γ_k are constants and d is the order of U.

This definition is motivated by the "one-point" condition considered in Remark 3.5 in [17]. Here, the author based his results in G. Wilson ideas who extended the bispectral problem to a more general setting of commutative rings of differential operators.

In [15], they take care of the difference version of Wilson's results, and similar notions appeared. Moreover, as it was pointed out by the referee, our result is in fact equivalent to [15]. The proof of this fact has been sketched in the recent paper [14] were they give the explicit form of the kernel of U and the relation is sketched between the Darboux approach and the methods of [15].

In this work we will take a Darboux transformation approach and this is done in the following section.

3. Bispectral operators

In this section we prove that the operators obtained by a rational Darboux transformation applied to a regular pseudo-difference operator with constant coefficients are solutions of the discrete-continuous bispectral problem. Moreover, we give an explicit procedure to write down the differential operators involved in the bispectral situation corresponding to the pseudo-difference operator obtained by the Darboux process. First, we recall some definitions.

Let f be a function defined in a discrete parameter l and consider the *discrete derivative* operator

$$(\Delta f)(l) = (\Lambda - I)\big(f(l)\big). \tag{3.1}$$

It is a straightforward verification that Δ satisfies the following Leibnitz Rule: $(\Delta fg)(l) = f(l)\Delta g(l) + \Delta f(l)\tilde{\Lambda} g(l)$, where f and g are functions in a discrete parameter l.

Recall that the analogous of indefinite integral for the difference calculus is given by the notion of *indefinite sum* (or summation). We denote by $\Sigma f(l)$ any function such that

$$\Delta\big(\Sigma f(l)\big) = f(l). \tag{3.2}$$

If $z(l) = \Sigma y(l)$, then any other indefinite sum of $y(l)$ differs from $z(l)$ by a constant. Moreover, if y and z are functions of the discrete variable l, the indefinite sum is a linear operator that satisfies the following property,

$$\Sigma\big(y(l)\Delta z(l)\big) = y(l)z(l) - \Sigma\big(\tilde{\Lambda} z(l)\Delta y(l)\big)$$

which are a discrete analog of the integration by parts. We will call them *summation by parts*.

Remark 3.1. Given $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(k)}$ functions in a discrete variable $l \in \mathbb{Z}$, denote by

$$W_\Delta[\Phi_1, \ldots, \Phi_k](l) = \det\begin{pmatrix} \Phi_1(l) & \ldots & \Phi_k(l) \\ \Delta(\Phi_1(l)) & \ldots & \Delta(\Phi_k(l)) \\ \vdots & \ddots & \vdots \\ \Delta^{k-1}(\Phi_1(l)) & \ldots & \Delta^{k-1}(\Phi_k(l)) \end{pmatrix}. \tag{3.3}$$

An important remark is that using the definition of Δ and elimination by rows, by properties of the determinant, it is straightforward to check that $W_\Delta[\Phi_1, \ldots, \Phi_k](l) = W[\Phi_1, \ldots, \Phi_k](l)$ introduced in (2.1).

The following technical lemma is crucial for the proof of the main result of this Letter. It is the discrete analog of Lemma 4 in [18].

Lemma 3.2. *Let $f_0, f_1, \ldots, f_{n+1}$ functions which depend on a discrete parameter l. Then*

$$W_\Delta[f_1, \ldots, f_n, F](l) = \tilde{\Lambda}\big(\Sigma\big[f_0(l)W_\Delta[f_1, \ldots, f_n](l)\big]\big)W_\Delta[f_1, \ldots, f_{n+1}](l), \tag{3.4}$$

where $F(l) = \sum_{j=1}^{n+1}(-1)^{n+1-j} f_j(l)\Sigma[f_0(l)W_\Delta[f_1, \ldots, \hat{f}_j, \ldots, f_{n+1}](l)]$ and $\hat{f}_j$ means that f_j is removed.

Proof. First we observe that

$$\det\begin{pmatrix} f_1 & \dots & f_n & f_{n+1} \\ \Delta f_1 & \dots & \Delta f_n & \Delta f_{n+1} \\ \vdots & & \vdots & \vdots \\ \Delta^{n-1} f_1 & \dots & \Delta^{n-1} f_n & \Delta^{n-1} f_{n+1} \\ \Delta^k f_1 & \dots & \Delta^k f_n & \Delta^k f_{n+1} \end{pmatrix} = 0, \tag{3.5}$$

for $k = 0, 1, \dots, n-1$. Thus, expanding this identity by the last row,

$$\sum_{j=1}^{n+1} (-1)^{n+1-j} \Delta^k\big(f_j(l)\big) W_\Delta[f_1, \dots, \hat{f}_j, \dots, f_{n+1}](l) = 0 \tag{3.6}$$

with $k = 0, 1, \dots, n-1$. We need to compute $\Delta F(l)$, $\Delta^2 F(l), \dots, \Delta^n F(l)$. By the Leibnitz Rule satisfied by Δ we have

$$\Delta F(l) = \sum_{j=1}^{n+1} (-1)^{n+1-j} \tilde{\Lambda}\big(f_j(l)\big)\big[f_0(l) W_\Delta[f_1, \dots, \hat{f}_j, \dots, f_{n+1}](l)\big]$$
$$+ \sum_{j=1}^{n+1} (-1)^{n+1-j} \Delta\big(f_j(l)\big) \Sigma\big[f_0(l) W_\Delta[f_1, \dots, \hat{f}_j, \dots, f_{n+1}](l)\big].$$

By the definition of Δ and properties of the determinant, it is immediate that (3.6) is valid with Δ replaced by Λ. Thus, using Remark 3.1, the first summand of the right-hand side above is 0.

Similarly, by an inductive process we have that

$$\Delta^k F(l) = \sum_{j=1}^{n+1} \Delta^k\big(f_j(l)\big) \Sigma\big[f_0(l) W_\Delta[f_1, \dots, \hat{f}_j, \dots, f_{n+1}](l)\big],$$

for $k = 1, \dots, n-1$. Now, using that $\Lambda = \Delta + \mathrm{Id}$, we have that

$$\Delta^n F(l) = \sum_{j=1}^{n+1} (-1)^{n+1-j} \tilde{\Lambda}\big(\Delta^{n-1}\big(f_j(l)\big)\big)\big[f_0(l) W_\Delta[f_1, \dots, \hat{f}_j, \dots, f_{n+1}](l)\big]$$
$$+ \sum_{j=1}^{n+1} (-1)^{n+1-j} \Delta^n\big(f_j(l)\big) \Sigma\big[f_0(l) W_\Delta[f_1, \dots, \hat{f}_j, \dots, f_{n+1}](l)\big]$$
$$= \sum_{j=1}^{n+1} (-1)^{n+1-j} \Delta^n\big(f_j(l)\big)\big[f_0(l) W_\Delta[f_1, \dots, \hat{f}_j, \dots, f_{n+1}](l)\big]$$
$$+ \sum_{j=1}^{n+1} (-1)^{n+1-j} \Delta^{n-1}\big(f_j(l)\big)\big[f_0(l) W_\Delta[f_1, \dots, \hat{f}_j, \dots, f_{n+1}](l)\big]$$
$$+ \sum_{j=1}^{n+1} (-1)^{n+1-j} \Delta^n\big(f_j(l)\big) \Sigma\big[f_0(l) W_\Delta[f_1, \dots, \hat{f}_j, \dots, f_{n+1}](l)\big].$$

The second term in the last equality is zero by (3.6), then we can rewrite $\Delta^n F(l)$ as

$$\Delta^n F(l) = \sum_{j=1}^{n+1} (-1)^{n+1-j} \Delta^n\big(f_j(l)\big) \tilde{\Lambda}\big(\Sigma\big[f_0(l) W_\Delta[f_1, \dots, \hat{f}_j, \dots, f_{n+1}](l)\big]\big).$$

Replacing these computation in $W_\Delta[f_1, \dots, f_n, F]$, and using again the equality (3.6), by column elimination, most terms disappear, and we obtain

$$W_\Delta[f_1, \dots, f_n, F](l) = \tilde{\Lambda}\big(\Sigma\big[f_0(l) W_\Delta[f_1, \dots, f_n](l)\big]\big) W_\Delta[f_1, \dots, f_n, f_{n+1}](l),$$

finishing the proof. □

Now we are ready to aim the main result of this Letter. Let

$$L_0 = a_{-r}\Lambda^{-r} + a_{-r+1}\Lambda^{-r+1} + \dots + a_{n-r-1}\Lambda^{n-r-1} + \Lambda^{n-r}$$

be a regular pseudo-difference operator with constant coefficients and let $g(w) = \sum_{i=-r}^{n-r} a_i w^i$ be its underlying Laurent polynomial. Set $f(l,t) = e^{lt}$, with l a discrete parameter and t a continuous one. Observe that

$$L_0 f(l,t) = \Theta(t) f(l,t),$$

where $\Theta(t) = g(e^t)$. Thus, f is an eigenfunction of L_0, whose associated eigenvalue is $\Theta(t)$. Furthermore, if $B_0 = h(\frac{d}{dt})$ is a differential operator with constant coefficients (namely, h is a polynomial in one variable), we have that

$$B_0 f(l,t) = \alpha(l) f(l,t),$$

where $\alpha(l) = h(l)$. Thus, we have that L_0 is a bispectral operator.

Now, applying rational Darboux transformations on L_0, we construct new bispectral operators, which is our goal.

Theorem 3.3. *Let L_0 be a regular pseudo-difference operator with constant coefficients, and L be a pseudo-difference operator obtained from L_0 by a rational Darboux transformation. Then L is again bispectral.*

Proof. Since L is obtained from L_0 using rational Darboux transformations, we have, by definition, that there exists an intertwining monic difference operator U of order m such that

$$LU = UL_0$$

and $\operatorname{Ker} U = \langle f_k(l) = p_k(l)e^{l\lambda_k}, k = 1,\dots,m\rangle$, where p_k are polynomials in one variable and λ_k are constant.

Define $\Psi(l,t) = Ue^{lt}$, $f_0(l) = p(l)\prod_k e^{-l\lambda_k}$, for some polynomial p, and $f_{m+1}(l) = e^{lt}$. Observe that

$$\Psi(l,t) = \frac{W[f_1,\dots,f_m,e^{lt}]}{W[f_1,\dots,f_m]}$$

and

$$L\Psi(l,t) = LUe^{lt} = UL_0e^{lt} = U\Theta(t)e^{lt} = \Theta(t)\Psi(l,t),$$

thus Ψ is an eigenfunction associated to the operator L, with the same eigenvalue $\Theta(t)$ associated to L_0. Now, applying Lemma 3.2 combined with Remark 3.1 to the setting above with $F = \sum_{k=1}^{m+1}(-1)^{m+1-k} f_k(l)\Sigma[f_0(l)W[f_1,\dots,\hat{f}_k,\dots,f_{m+1}](l)]$, we have that

$$W[f_1,\dots,f_m,F] = \tilde{\Lambda}\big(\Sigma\big[f_0(l)W[f_1,\dots,f_m](l)\big]\big)W[f_1,\dots,f_m,f_{m+1}]. \tag{3.7}$$

Note that using properties of the determinant and summation by parts, F can be written as $F = [\Sigma_{k=1}^{r+1} R_k(e^t)l^k]e^{lt}$, which is nothing but e^{lt} times a polynomial in l, with rational functions on e^t as coefficients.

Observe that given a polynomial Q, we have that $Q(l)e^{lt} = Q(\frac{d}{dt})e^{lt}$. Using this, we can write F as an ordinary differential operator, say B, applied to e^{lt}. Thus, $W[f_1,\dots,f_n,B(e^{lt})] = \tilde{\Lambda}(\Sigma[f_0(l)W[f_1,\dots,f_m](l)])W[f_1,\dots,f_n,e^{lt}]$, and B can be extracted from the Wronskian. Thus we have that

$$B\left(\frac{d}{dt}\right)\Psi(l,t) = \alpha(l)\Psi(l,t),$$

with $\alpha(l) = \tilde{\Lambda}(\Sigma[f_0(l)W[f_1,\dots,f_m](l)])$, finishing the proof. □

Remark 3.4. Observe that the proof of the theorem above gives a procedure to compute the bispectral differential operator $B = B_p$ associated to L for an arbitrary polynomial p. Let us check this in the following example.

Example 3.5. We will give the explicit expression for the corresponding differential equation of the bispectral pseudo-difference operator of degree 1 obtained after applying one Darboux transformation to

$$L_0 = \Lambda - 2I + \Lambda^{-1},$$

which is considered in [15].

Using Lemma 2.1, we have that $L_0 = (I - \Lambda^{-1}\frac{l}{l-1})(\Lambda - \frac{l+1}{l})$. By performing one Darboux transformation we obtain

$$L = \left(\Lambda - \frac{l+1}{l}\right)\left(I - \Lambda^{-1}\frac{l}{l-1}\right),$$

and the intertwining difference operator in Proposition 2.3 is $U = (\Lambda - \frac{l+1}{l})$ whose kernel is generated by $f_1(l) = \frac{l}{2}$. Thus, this Darboux transformation is rational, and L is again bispectral.

Denote by $x^{(r)} = x(x-1)\dots(x-r+1)$. As in Theorem 3.3 set $\Psi(l,t) = Ue^{lt}$, $f_0(l) = p(l)$, with $p(l) = \sum_{j=0}^N p_j l^{(j)}$ an arbitrary polynomial, $f_1(l) = \frac{l}{2}$ and $f_2(l) = e^{lt}$.

Observe that in this case

$$F = -f_1(l)\Sigma\big(f_0(l)W[f_2](l)\big) + f_2(l)\Sigma\big(f_0(l)W[f_1](l)\big) = -\frac{l}{2}\sum_{j=0}^N p_j\Sigma\big(l^{(j)}e^{lt}\big) + e^{lt}\sum_{j=0}^N p_j\Sigma\left(l^{(j)}\frac{l}{2}\right).$$

It is easy to check by induction, using summation by parts, that

$$\Sigma\big(l^{(j)}e^{lt}\big) = \frac{e^{lt}}{e^t-1}\sum_{r=0}^j(-1)^r\left(\frac{e^t}{e^t-1}\right)^r\frac{j!}{(j-r)!}l^{(j-r)},$$

Please cite this article in press as: C. Boyallian, S. Portillo, Physics Letters A (2010), doi:10.1016/j.physleta.2010.01.027

and, using that $\Delta = \Lambda - I$,

$$\Sigma\big(l^{(j)}l\big) = \left(1 - \frac{1}{j+1}\right) l^{(j+1)} + \frac{l^{(j+2)}}{j+2}.$$

Thus, F can be rewritten as

$$F(l) = \left(\sum_{j=0}^{N} \frac{p_j}{2}\left[\frac{-1}{(e^t-1)}\sum_{r=0}^{j}(-1)^r\left(\frac{e^t}{e^t-1}\right)^r \frac{j!}{(j-r)!} l^{(j-r)}l + \left(1-\frac{1}{j-1}\right)l^{(j+1)} + \frac{l^{(j+2)}}{j+2}\right]\right)e^{lt},$$

and B_p, the bispectral differential operator associated to L is given by

$$B_p\left(\frac{d}{dt}\right) = \sum_{j=0}^{N} \frac{p_j}{2}\left[\frac{-1}{(e^t-1)}\sum_{r=0}^{j}(-1)^r\left(\frac{e^t}{e^t-1}\right)^r \frac{j!}{(j-r)!}\left(\frac{d}{dt}\right)^{(j-r)}\frac{d}{dt} + \left(1-\frac{1}{j-1}\right)\left(\frac{d}{dt}\right)^{(j+1)} + \frac{1}{j+2}\left(\frac{d}{dt}\right)^{(j+2)}\right],$$

for an arbitrary polynomial p.

Acknowledgement

We are very grateful to J. Liberati for introducing us to this problem and for enlightening suggestions and discussions.

Uncited references

[11]

References

[1] M. Adler, P. van Moerbeke, Int. Math. Res. Not. 10 (1998) 489.
[2] B. Bakalov, E. Horozov, M. Yakimov, Phys. Lett. A 222 (1–2) (1996) 59.
[3] B. Bakalov, E. Horozov, M. Yakimov, Duke Math. 190 (2) (1998) 41.
[4] Yu. Berest, Huygens' principle and the bispectral problem, in: The Bispectral Problem, Montreal, PQ, 1997, in: CRM Proc. Lecture Notes, vol. 14, AMS, RI, 1998, pp. 11–30.
[5] Yu. Berest, G. Wilson, Int. Math. Res. Not. 2 (1999) 105.
[6] J.J. Duistermaat, F.A. Grünbaum, Commun. Math. Phys. 103 (1986) 177.
[7] P. Etingoff, A. Varchenko, Traces of intertwiners for quantum groups and difference equations I, math/9907181.
[8] G. Felder, Y. Markiv, V. Tarasov, Differential equations compatible with KZ equations, math/0001184.
[9] F.A. Grünbaum, Ramanujan J. 5 (2001) 263.
[10] F.A. Grünbaum, L. Haine, A theorem of Bochner, in: Algebraic Aspects of Integrable Systems, in: Progr. Nonlinear Differential Equations Appl., vol. 26, Birkhäuser Boston, Boston, MA, 1997, pp. 143–172.
[11] F.A. Grünbaum, L. Haine, Methods Appl. Anal. 6 (2) (1999) 209.
[12] F.A. Grünbaum, L. Haine, E. Horozov, J. Comput. Appl. Math. 106 (2) (1999) 271.
[13] F. Grünbaum, M. Yakimov, Discrete bispectral Darboux transformations from Jacobi operators, math/0012191v1.
[14] L. Haine, E. Horozov, P. Iliev, Glasg. Math. J. 51 (A) (2009) 95, arXiv:0807.2888.
[15] L. Haine, P. Iliev, Int. Math. Res. Not. 6 (2000) 281.
[16] L. Haine, P. Iliev, J. Phys. A: Math. Gen. 34 (11) (2001) 2445.
[17] J. Liberati, Lett. Math. Phys. 41 (1997) 321.
[18] M. Reach, Commun. Math. Phys. 119 (1988) 385.
[19] G. Wilson, J. Reine Angew. Math. 442 (1993) 177.
[20] G. Wilson, Invent. Math. 133 (1) (1998) 1.