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Electrostatic potential: A new approach for some mixed boundary value problems

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A new approach to the solution of problems of electrostatics, some of them with mixed boundary conditions, is presented. The proposed scheme can be used in cases were we have a formal solution in the form of a series in Legendre polynomials and the boundary or matching conditions are given not on the whole interval $(0, \pi)$ of the polar variable, θ, but only over the interval $(0, \pi/2)$ or $(\pi/2, \pi)$. Truncation of the series after the Nth term and the projection on the subspace generated by the set of the first N even (or odd) Legendre polynomials allows us to determine the unknown coefficients of the approximate solution. The results show rapid convergence toward the exact values as we increase the number of terms, N, included in the approximate solutions. The procedure allows to solve approximately some problems whose exact solutions, we believe, are not yet known.

1. Introduction

An advanced course on Electromagnetism based on Jackson's Classical Electrodynamics [1] leaves the students with the feeling that they are able to solve almost any problem in this area. However this perception is far from been accurate; for example, if we want to find the external electrostatic potential of a cube or a finite cylindrical body, we soon realize that we lack the mathematical tools to solve such problems. Even simple academic problems, most of them with mixed boundary conditions, need rather sophisticated techniques, such as the dual integral equations [2], not always familiar to students with a basic background. The aim of this work is to contribute to increase the number of problems that the students can solve, even in an approximate way, with elementary tools. With this objective J. D. Jackson has included in the third edition of his book an introduction to finite elements analysis in electrostatics [3]. One of the authors together with Lamberti have developed a numerical procedure to find the electrostatic potential with given boundary conditions [4]. In some problems in electrostatic we can write down a formal solution as a series with unknown coefficients but the boundary or the matching conditions are given not on the whole range of the variable but only on a part of it, therefore we are unable to determine the coefficients by the use of the orthogonality property. Such is the case of many problems whose solution can be given as an expansion in Legendre polynomials but the range of the polar variable θ is only $(0, \pi/2)$ or $(\pi/2, \pi)$. We know that over these ranges the Legendre polynomials with even or odd indices are still orthogonal among themselves but the even ones are not orthogonal to the odd ones there. We can define an overlap matrix **S** whose elements are the integrals, over the restricted range of the variable, of the two polynomials with indices of different parity. By truncating the series expansion after the Nth term, imposing the matching conditions, and then projecting on one of the subspaces generated by the first even, or odd, Legendre polynomials we are left with a set of matrix equations, each of order $N \times N$, that once solved gives us an approximate solution. In order to show the simplicity and the accuracy of the procedure we will apply it to solve four problems of electrostatics. Two of them have known solution and will be used to compare our results with the exact ones, but we have not been able to find the solution for the others in the available literature.

2. Examples

2.1. Charged conducting disk

This problem has been treated in the first edition of Jackson's Classical Electrodynamics. Assume that the disk, of radius a, is on the $\{x, y\}$ plane with center at the origin of the coordinate system (see Fig. 1) and held at a potential V. We propose an expression for the electrostatic potential for the outer region $r \geq a$, $\theta \in (0, \pi)$, $\phi_o(r, \theta)$, and one for the inner region $r \leq a, \theta \in (0, \pi/2)$, $\phi_i(r, \theta)$, given by

$$\begin{aligned} \phi_o(r,\theta) &= \sum_{j=0}^{\infty} \frac{B_j P_j(\cos(\theta))}{r^{j+1}}, \quad j = 0,2,4.., \\ \phi_i(r,\theta) &= \sum_{j=0}^{\infty} A_j P_j(\cos(\theta)) r^j + \sum_{\alpha=1}^{\infty} C_\alpha P_\alpha(\cos(\theta)) r^\alpha, \\ & \qquad j = 0,2,4...., \quad \alpha = 1,3,5,.... \end{aligned} \tag{1}$$

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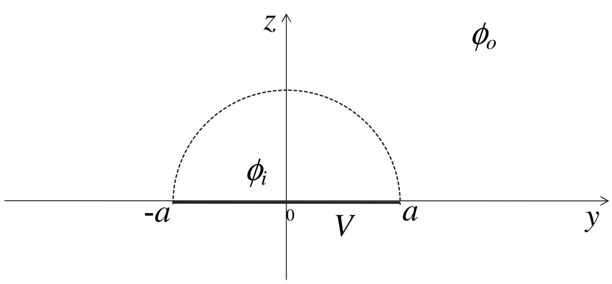

Fig. 1. Electrified disk of radius a. The dotted line is the common frontier of the two regions where are defined the potentials ϕ_i and ϕ_o.

The form chosen for $\phi_o(r,\theta)$ makes sure that $\phi_o(r,\theta)=\phi_o(r,\pi-\theta)$ then $\frac{\partial\phi_o}{\partial\theta}|_{\theta=\pi/2} = 0$. Since $\phi_i(r,\theta)$ is defined in the restricted interval $\theta\in(0,\pi/2)$ it has not defined parity. For the region $r\le a$, $\theta\in(\pi/2,\pi)$ the potential can be obtained from $\phi_i(r,\theta)$ by symmetry.

From now on we will assume that latin indices are even and that greek ones are odd.

Defining the basis row vectors with the even, P_n, and the odd, P_α, Legendre polynomials

$$\begin{aligned}\mathbf{F}(\theta)&\equiv(P_0(\cos(\theta)),P_2(\cos(\theta)),P_4(\cos(\theta)),\ldots.)\\ \mathbf{G}(\theta)&\equiv(P_1(\cos(\theta)),P_3(\cos(\theta)),P_5(\cos(\theta)),\ldots.)\end{aligned} \tag{2}$$

the column vectors **A**, **B**, **C** and the infinite diagonal matrices $\mathbf{R}_1(r)$, $\mathbf{R}_2(r)$, $\mathbf{R}_3(r)$, $\mathbf{R}_4(r)$, by

$$\begin{aligned}&\mathbf{A}^T\equiv(A_0,A_2,A_4,\ldots),\mathbf{B}^T\equiv(B_0,B_2,B_4,\ldots),\mathbf{C}^T\equiv(C_1,C_3,C_5,\ldots)\\ &\mathbf{R}_1(r)\equiv\mathrm{diag}(1,r^2,r^4,r^6,\ldots)\\ &\mathbf{R}_2(r)\equiv\mathrm{diag}(r,r^3,r^5,r^7,\ldots)\\ &\mathbf{R}_3(r)\equiv\mathrm{diag}(r^{-1},r^{-3},r^{-5},\ldots)\\ &\mathbf{R}_4(r)\equiv\mathrm{diag}(r^{-2},r^{-4},r^{-6},\ldots)\end{aligned} \tag{3}$$

respectively, the potentials given in Eq. (1) can be written in a matrix form

$$\begin{aligned}\phi_i(r,\theta) &= \mathbf{FR}_1(r)\mathbf{A}+\mathbf{GR}_2(r)\mathbf{C} \quad r\le a,\theta\in\left(0,\tfrac{\pi}{2}\right)\\ \phi_o(r,\theta) &= \mathbf{FR}_3(r)\mathbf{B} \quad\quad r\ge a,\theta\in(0,\pi)\end{aligned} \tag{4}$$

We have used the superscript T to indicate the transpose of the column vectors. We also define the overlap matrices:

$$\begin{aligned}\mathbf{Q}&\equiv\int_0^{\frac{\pi}{2}}\mathbf{F}^T\mathbf{F}\,\sin(\theta)\mathrm{d}\theta = \mathrm{diag}\left(1,\frac{1}{5},\frac{1}{9},\ldots.\right)\\ \mathbf{S}&\equiv\int_0^{\frac{\pi}{2}}\mathbf{F}^T\mathbf{G}\,\sin(\theta)\mathrm{d}\theta\end{aligned} \tag{5}$$

The elements of the overlap matrix **S** are [5]

$$\mathbf{S}_{j\alpha} = \frac{(-1)^{\frac{j+\alpha-1}{2}}j!\alpha!}{2^{j+\alpha-1}(\alpha-j)(j+\alpha+1)\left[\left(\frac{j}{2}\right)!\left(\frac{\alpha-1}{2}\right)!\right]^2} \tag{6}$$

Imposing the continuity of the potential and its normal derivative at $r=a$ for $0\le\theta\le\pi/2$ we have

$$\begin{aligned}\mathbf{FR}_1\mathbf{A}+\mathbf{GR}_2\mathbf{C} &= \mathbf{FR}_3\mathbf{B}\\ \mathbf{FR}_1'\mathbf{A}+\mathbf{GR}_2'\mathbf{C} &= \mathbf{FR}_3'\mathbf{B}\end{aligned} \tag{7}$$

where the matrices $\mathbf{R}_i$ and their derivatives $\mathbf{R}'_i$ are evaluated at $r=a$. From Eq. (7) after multiplying from the left by $\mathbf{F}^T\sin(\theta)$ and integrating over the variable $\theta\in(0,\pi/2)$, i.e. projecting on the subspace generated by the even Legendre polynomials, we arrive at the two matrix equations

$$\begin{aligned}\mathbf{QR}_1\mathbf{A}+\mathbf{SR}_2\mathbf{C} &= \mathbf{QR}_3\mathbf{B}\\ \mathbf{QR}_1'\mathbf{A}+\mathbf{SR}_2'\mathbf{C} &= \mathbf{QR}_3'\mathbf{B}\end{aligned} \tag{8}$$

The potential $\phi_i(r,\theta)$ satisfies the condition:

$$\phi_i\left(r,\theta=\frac{\pi}{2}\right) = V\Rightarrow A_0 = V, A_2 = A_4 = \ldots.. = 0 \tag{9}$$

Truncating the series expansion after the Nth term we are left with a set of two $N\times N$ matrix equations with two unknown vectors **B** and **C**. Once the coefficients are known the approximate potentials are obtained. In Fig. 2 we show the difference between the external potential $\phi_o(r=a,\ \theta)$ and the internal potential $\phi_i(r=a,\theta)$ as a function of θ for two values of N. As we can see this difference decreases as we increase N insuring the continuity of the potential as N goes to infinity. In order to have an idea of the accuracy of the approximate solution, we have calculated the capacity of the disk for different values of N, see Fig. 3. As can be seen a reasonable approximate value of the capacity can be obtained with very little effort, $q/V = 2\pi\varepsilon_0 a$ for $N=1$, against the exact value $8\epsilon_0 a$. The approximated charge density, $\sigma(r/a)$, on the disk as function of the radial coordinate is compared with the exact one in Fig. 4 for different values of N. This figure show clearly the convergence of the approximate solution to the exact one as we increase the value of N.

2.2. Conducting plane with a circular hole of radius a

The origin of coordinates is taken at the center of the hole and the z axis normal to the plane, the unitary vector $\mathbf{e}_3$ is along this direction. We also assume that far from the hole there is an electric field in the z direction having different values on either side of the

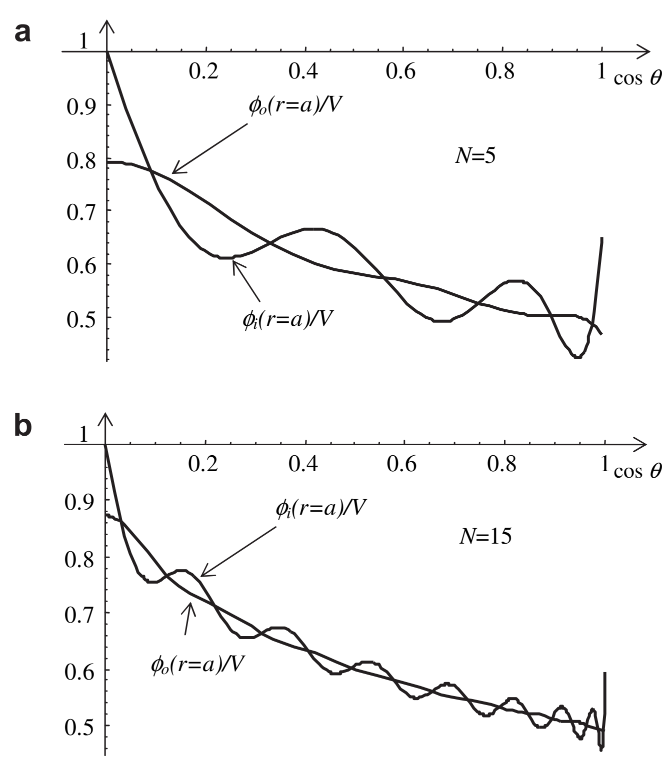

Fig. 2. Behavior of the approximate potential $\phi_i(r=a,\theta)$ and $\phi_o(r=a,\theta)$ at the common boundary as a function of $\cos(\theta)$ for two values of N. a- $N=5$; b- $N=15$.

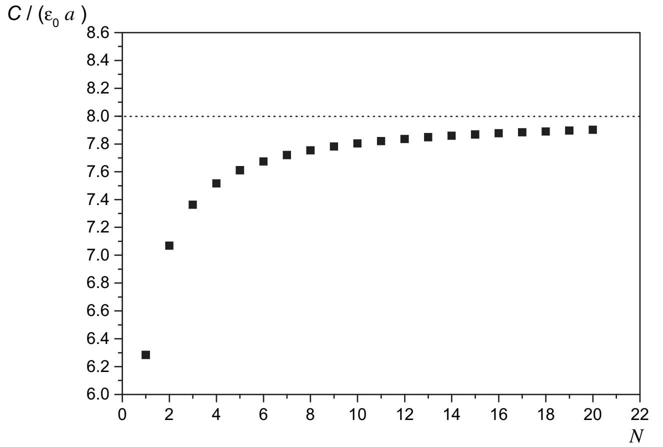

Fig. 3. The approximate electrical capacity C, divided by $\epsilon_0 a$, of the electrified disk of radius a as a function of N. The exact value is 8.

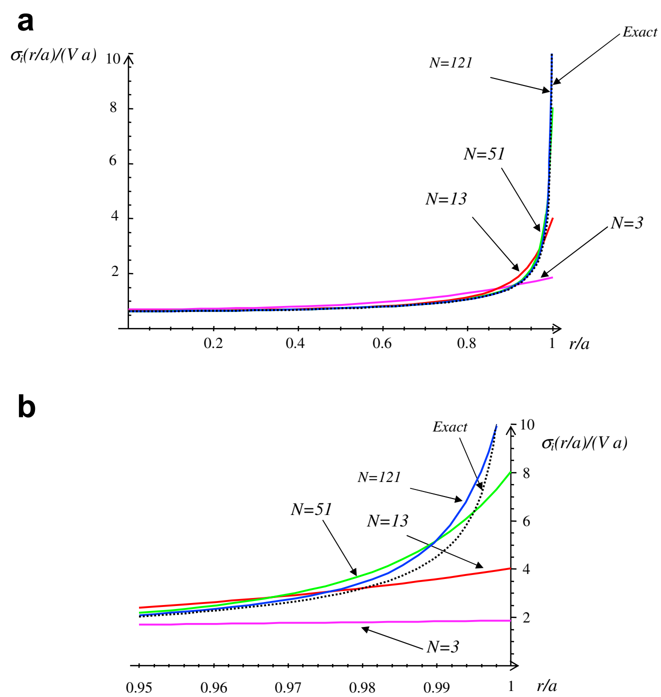

Fig. 4. Charge density, $\sigma(r/a)$, of an electrified disk of radius a, hold at potential V, for different values of N, also the exact density, dotted line, as a function of the distance to the center of the disk. (a) For the full range, $r \in (0, a)$ and (b) for the expanded region $r \in (0.95a, a)$.

grounded plane, see Fig. 5. Let ϕ_1, ϕ_2 and ϕ_i be the electrostatic potentials for the regions: $r \geq a,\ \theta \in (0, \pi/2),\ r \geq a,\ \theta \in (\pi/2, \pi)$ and $r \leq a$, $\theta \in (0, \pi)$ respectively. Defining:

$$\phi_0^+ \equiv -E^+ z, \quad \phi_0^- \equiv E^- z \tag{10}$$

the general expression for the potential that satisfies the boundary condition on the plane outside the hole and far from it, are

$$\begin{aligned}\phi_1(r,\theta) &= \phi_0^+ + \textstyle\sum_{\alpha=1}^{\infty} H_\alpha P_\alpha(\cos(\theta)) r^{-\alpha-1} \\ &\equiv \mathbf{GR}_2(r)\mathbf{I}^+ + \mathbf{GR}_4(r)\mathbf{H} \\ \phi_2(r,\theta) &= \phi_0^- + \sum_{\alpha=1}^{\infty} D_\alpha P_\alpha(\cos(\theta)) r^{-\alpha-1} \\ &\equiv \mathbf{GR}_2(r)\mathbf{I}^- + \mathbf{GR}_4(r)\mathbf{D} \\ \phi_i(r,\theta) &= \sum_{j=0}^{\infty} A_j P_j(\cos(\theta)) r^j + \textstyle\sum_{\alpha=1}^{\infty} C_\alpha P_\alpha(\cos(\theta)) r^\alpha \\ &\equiv \mathbf{FR}_1(r)\mathbf{A} + \mathbf{GR}_2(r)\mathbf{C}\end{aligned} \tag{11}$$

where $\mathbf{F}$, $\mathbf{G}$, $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{R}_i(r)$ are the same as before, $\mathbf{D}$ and $\mathbf{H}$ are two new row vectors with odd subindices. We have written the asymptotic potentials ϕ_0^+ and ϕ_0^- using the two column vectors $\mathbf{I}^+$ and $\mathbf{I}^-$ whose components are $I^+{}_\alpha = -E^+\delta_{\alpha,1}$ and $I^-{}_\alpha = E^-\delta_{\alpha,1}$ respectively, $\delta_{\alpha,1}$ is the so-called Kronecker delta. Imposing the continuity of the potential and its normal derivative at the boundary $r = a$, multiplying from the left by $\mathbf{F}^T \sin(\theta)$, integrating over θ, and taking into account that

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} \mathbf{F}^T\mathbf{F}\sin(\theta)d\theta &= \int_0^{\frac{\pi}{2}} \mathbf{F}^T\mathbf{F}\sin(\theta)\mathrm{d}\theta = \mathbf{Q} \\ \int_{\frac{\pi}{2}}^{\pi} \mathbf{F}^T\mathbf{G}\sin(\theta)d\theta &= -\int_0^{\frac{\pi}{2}} \mathbf{F}^T\mathbf{G}\sin(\theta)\mathrm{d}\theta = -\mathbf{S},\end{aligned} \tag{12}$$

we get the following set of matrix equations

$$\begin{aligned}\mathbf{QR}_1\mathbf{A} + \mathbf{SR}_2\mathbf{C} &= \mathbf{SR}_2\mathbf{I}^+ + \mathbf{SR}_4\mathbf{H} \\ \mathbf{QR}_1'\mathbf{A} + \mathbf{SR}_2'\mathbf{C} &= \mathbf{SR}_2'\mathbf{I}^+ + \mathbf{SR}_4'\mathbf{H} \\ \mathbf{QR}_1\mathbf{A} - \mathbf{SR}_2\mathbf{C} &= -\mathbf{SR}_2\mathbf{I}^- - \mathbf{SR}_4\mathbf{D} \\ \mathbf{QR}_1'\mathbf{A} - \mathbf{SR}_2'\mathbf{C} &= \mathbf{SR}_2'\mathbf{I}^- - \mathbf{SR}_4'\mathbf{D}\end{aligned} \tag{13}$$

where the matrices $\mathbf{R}_i$ and their derivatives $\mathbf{R}'_i$ should be evaluated at $r = a$. The Eq. (13) are an inhomogeneous set of matrix equations whose solution gives us the coefficient $\mathbf{A}$, $\mathbf{C}$, $\mathbf{H}$ and $\mathbf{D}$ of the series expansion for the potentials. The roughest approximation, i.e. $N = 1$ leads us to

$$\begin{aligned}&\phi_1(r,\theta) \approx -E^+ z - \tfrac{(E^+ + E^-)}{4r^2} a^3 \cos(\theta), \quad r \geq a, \theta \in \left(0, \tfrac{\pi}{2}\right) \\ &\phi_2(r,\theta) \approx E^- z + \tfrac{(E^+ + E^-)}{4r^2} a^3 \cos(\theta), \quad r \geq a, \theta \in \left(\tfrac{\pi}{2}, \pi\right) \\ &\phi_i(r,\theta) \approx -\frac{3a\left(E^+ + E^-\right)}{8} - \frac{\left(E^+ - E^-\right)}{2} r\cos(\theta), \quad r \leq a, \theta \in (0, \pi)\end{aligned}$$

The second term in ϕ_1 (ϕ_2) corresponds to the potential of an electric dipole $\mathbf{p} = -\pi\epsilon_0 a^3(E^+ + E^-)\mathbf{e}_3$ ($\mathbf{p} = \pi\epsilon_0 a^3(E^+ + E^-)\mathbf{e}_3$), located at the center of the hole. For the exact solution (6) the dipolar contribution to the potential comes from an electric dipole given by

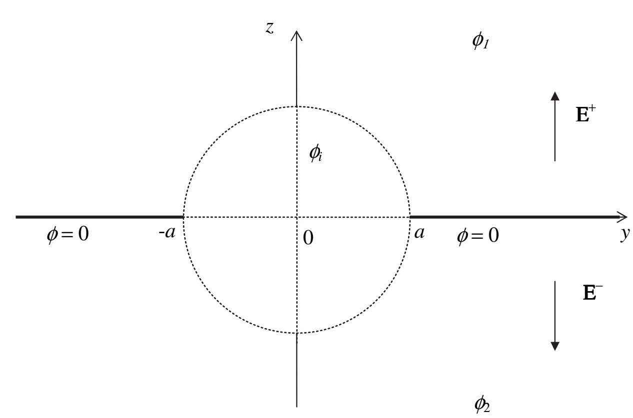

Fig. 5. An infinite grounded conducting plane with a circular hole of radius a cut in it. There is constant electric field far from the hole, normal to the plane of value $\mathbf{E}^+$ and $\mathbf{E}^-$, for $z > 0$ and $z < 0$ respectively.

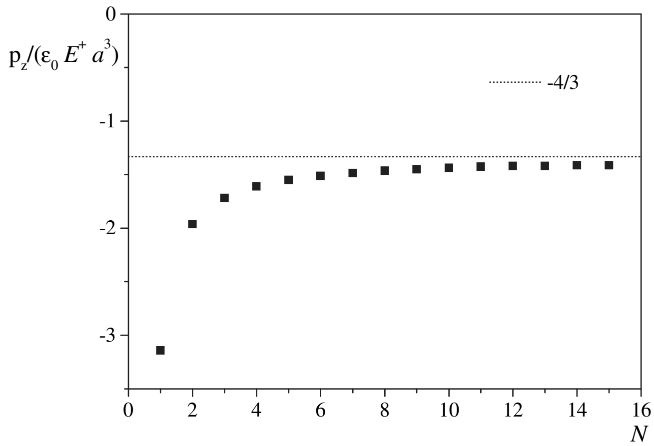

Fig. 6. The z component of the dipole moment $\mathbf{P}_z$ divided by $\epsilon_0 E^+ a^3$, of the charge distribution around the hole as a function of N. We have taking $E^- = 0$. The dotted line is the exact value $\mathbf{P}^e{}_z/(\epsilon_0 E^+ a^3) = -4/3$.

$\mathbf{p}^e = -4/3\varepsilon_0 a^3(E^+ + E^-)\mathbf{e}_3$. The potential at the center of the hole, in this approximation, is $\phi_i(0,\theta) \approx -\frac{3a(E^+ + E^-)}{8} = -0.375a(E^+ + E^-)$, while the exact result is $\phi^e(0,\theta) = -a/\pi(E^+ + E^-) = -0.318a$ $(E^+ + E^-)$, i.e. an approximation within the 20%.

The magnitude of the above mentioned electric dipole, i.e. the coefficient H_1, as a function of N is shown in Fig. 6, and the potential along the radial coordinate inside the hole, $\phi_i(r/a, \theta = \pi/a)$, in Fig. 7. In both figures we have taking $E^- = 0$.

In Fig. 8 we compare the behavior of the external potential, $\phi_1(r = a, \theta)$, and the internal potential, $\phi_i(r = a, \theta)$, at the common boundary as a function of $\cos(\theta)$ for $N = 14$.

It should be emphasized that the exact solution of the last two problems requires a knowledge of dual integral equations and much work as can be seen in Jackson's book [6].

2.3. Hemispherical hole drilled in a flat conductor

The problem of a hemispherical boss on a conducting plane in an external constant electric field is a classical problem equivalent to the one of a spherical grounded conductor in a uniform electric field [8]. However, we have not found the solution for the analogous problem with a hole instead of a boss that we will now consider here. The boundary of the conductor is part of the $\{x, y\}$ plane and the surface of the hole given by $r = a, \theta \in (\pi/2, \pi)$, which are kept at potential zero, see Fig. 9. We assume that far

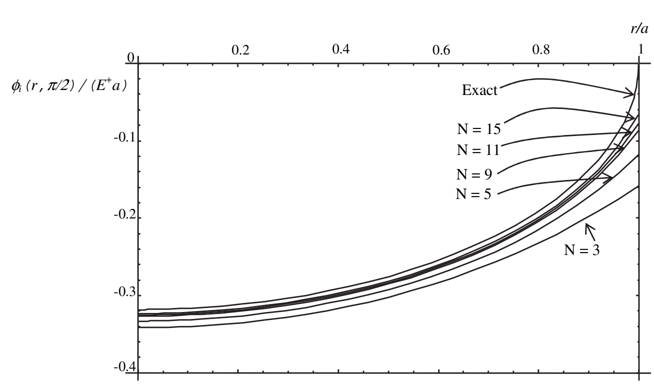

Fig. 7. The potential $\phi_i(r/a)$ on the plane $\{x, y\}$ as a function of the radial coordinated r/a for different values of the parameter N.

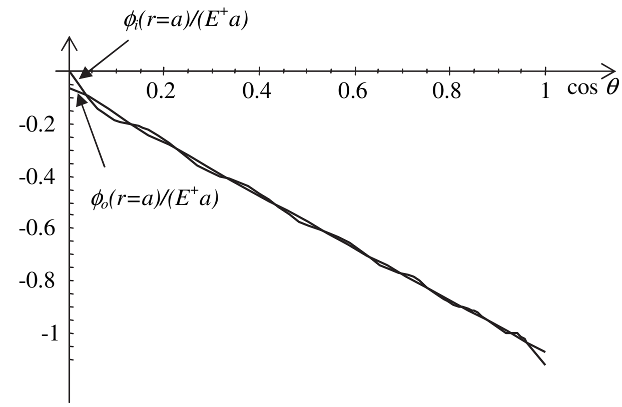

Fig. 8. Behavior of the approximate potential $\phi_i(r = a, \theta)$ and $\phi_1(r = a, \theta)$ at the common boundary as a function of $\cos(\theta)$ for $\theta \in (0, \pi/2)$, for $N = 14$.

from the hole there is an electric field in the z direction $\mathbf{E}_0 = E_0\mathbf{e}_3$. We propose an electric potential for points inside the sphere $r \le a$, $\phi_i(r, \theta)$, and one outside the sphere, $\phi_o(r, \theta)$, that can be written as follows:

$$\begin{aligned}\phi_i &= \sum_{j=0}^{\infty} A_j r^j P_j(\cos(\theta)) + \sum_{\alpha=1}^{\infty} C_\alpha r^\alpha P_\alpha(\cos(\theta)) \\ &\equiv \mathbf{FR}_1(r)\mathbf{A} + \mathbf{GR}_2(r)\mathbf{C}, \quad r \le a, \theta \in (0, \pi) \\ \phi_o &= -E_0 r P_1(\cos(\theta)) + \sum_{\alpha=1}^{\infty} D_\alpha r^{-\alpha-1} P_\alpha(\cos(\theta)) \\ &\equiv \mathbf{GR}_2(r)\mathbf{I}^+ + \mathbf{GR}_4(r)\mathbf{D}, \quad r \ge a, \quad \theta \in \left(0, \tfrac{\pi}{2}\right)\end{aligned} \tag{14}$$

where the components of $\mathbf{I}^+$ are $I_\alpha^+ = -E_0\delta_{\alpha,1}$. Using the continuity of the potential and its derivative at the common frontier, the boundary condition for ϕ_i, the definition of the matrices $\mathbf{Q}$ and $\mathbf{S}$, Eq. (5), and their property, Eq. (12), we get the following matrix equations

$$\begin{aligned}\mathbf{QR}_1\mathbf{A} - \mathbf{SR}_2\mathbf{C} &= 0 \\ \mathbf{QR}_1\mathbf{A} + \mathbf{SR}_2\mathbf{C} &= \mathbf{SR}_2\mathbf{I}^+ + \mathbf{SR}_4\mathbf{D} \\ \mathbf{QR}_1'\mathbf{A} + \mathbf{SR}_2'\mathbf{C} &= \mathbf{SR}_2'\mathbf{I}^+ + \mathbf{SR}_4'\mathbf{D}\end{aligned} \tag{15}$$

Solving this set of matrix equations for the column vectors $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{D}$ as functions of $\mathbf{I}^+$ we get the potential and from it the charge density on the plane or on the wall of the hole. In Fig. 10 we show the external potential $\phi_o(r = a, \theta)$ and the internal potential $\phi_i(r = a, \theta)$ as a function of $\cos(\theta)$ for $N = 15$ and radial component of the

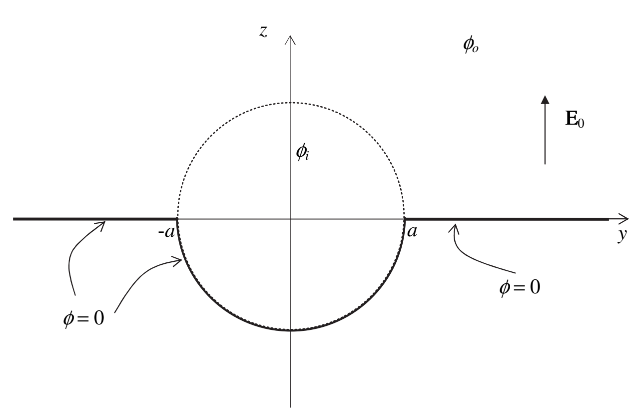

Fig. 9. A hemispherical hole of radius a drilled on the flat surface, $z = 0$, of a homogeneous infinite conducting body that filled the semi-space $z \le 0$. A constant electric field $\mathbf{E}_0$ exists far away from the conductor.

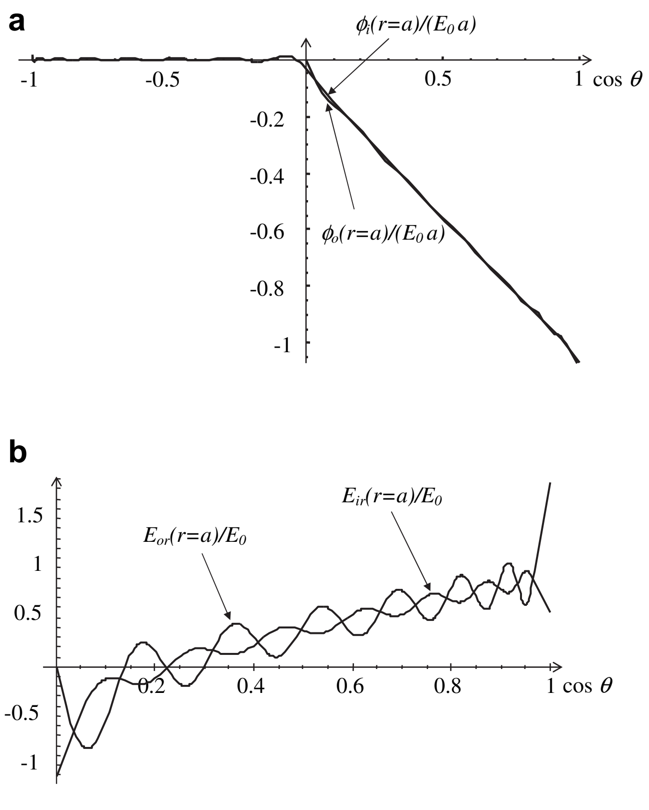

Fig. 10. (a) Behavior of the approximate potential $\phi_i(r=a,\ \theta)$ and $\phi_o(r=a,\ \theta)$ as a function of $\cos(\theta)$ for $\theta \in (0,\ \pi)$. (b) The approximate radial component of the electric fields, E_{or}/E_0 and E_{ir}/E_0 as a function of $\cos(\theta)$, at the common boundary $r=a$. Both for $N=15$.

electric fields evaluated at common boundary of the two regions. Fig. 11 shows the charge per unit polar angle θ, $\eta(\theta)$, on the wall of the hole for $N=15$ and $N=20$. The oscillations are due to the finite number of terms taken in the approximate solution. The charge density on the plane is given by $\sigma_o(r) = -\varepsilon_0 \partial\phi_o/\partial z|_{z=0}$; the charge density on the plane if the hole were not there is $\sigma_p(r) = \epsilon_0 E_0$. We define $\zeta(r) \equiv (\sigma_o(r) - \sigma_p(r))2\pi r$, i.e. the excess of charge per unit radial coordinate, it is shown in Fig. 12. This excess of charge is mainly concentrated close to the rim of the hole. The exact solution satisfies that the charge inside the hole plus the excess of charge on the plane surface must be equal to the charge on the lid of the hole if it were absent. It can be shown that the approximate solution

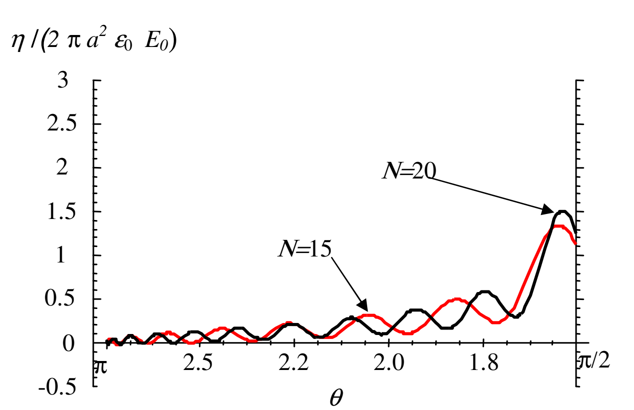

Fig. 11. The electrical charge per unit polar angle, $\eta(\theta)$, on the wall of the hole, for $N=15$ and $N=20$, as a function of θ.

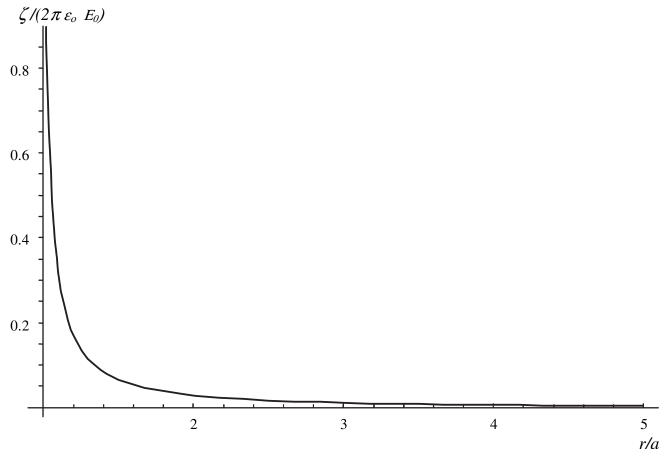

Fig. 12. The excess of electrical charge per unit radial coordinate, $\zeta(r/a)$, on the plane $z=0$, with $N=17$, for $r>a$.

given here, for any value of N, satisfies this requirement. In fact, this equality is built in the zero component of the last matrix equation given in Eq. (15) if one takes into account the values of $S_{0,\alpha}$.

2.4. Electrical resistance of a semi-infinite conductor

We assume that we have an isotropic homogeneous conductor with conductivity σ in the region given by the following conditions: $r \geq a,\ \ \theta \in (0, \pi/2)$ and $r \leq a,\ \theta \in (0,\ \pi)$. The hemispherical cup of radius a and $\theta \in (\pi/2, \pi)$ is one of the electrodes which is kept at potential V. The other electrode is held at potential zero and it is supposed to be at infinity, see Fig. 13. Once the system has reached a stationary regime the total current through one of the electrodes will be used to calculate the electrical resistance. The stationary character of the problem allows as to define an electrostatic potential, $\mathbf{E} = -\nabla\phi$, and write the conditions $\nabla.\mathbf{J} = 0$ for all points inside the conductor, where $\mathbf{J}$ is the current density. From Ohm microscopic law $\mathbf{J} = \sigma\mathbf{E}$ then we have $\nabla.\mathbf{E} = -\Delta\phi = 0$. Further more the z component of the current density should vanish at the plane $z=0$ for $r \geq a$, that is $\partial\phi/\partial z = 0$, because no current flows across this part of the plane. This condition on the normal component of the current density does not rule out the possibility of a static surface charge density there.

We should mention that the calculation of the resistance with an hemispherical electrode buried in the conductor instead of protruding, as in our case, can easily be calculated giving the value $R_b = 1/2\pi\sigma a$ [7]. The present is a typical problem with mixed boundary conditions. The electrostatic potential, for points inside the sphere of radius a, $\phi_i(r, \theta)$ and for points outside the sphere, $\phi_o(r, \theta)$, can be written as

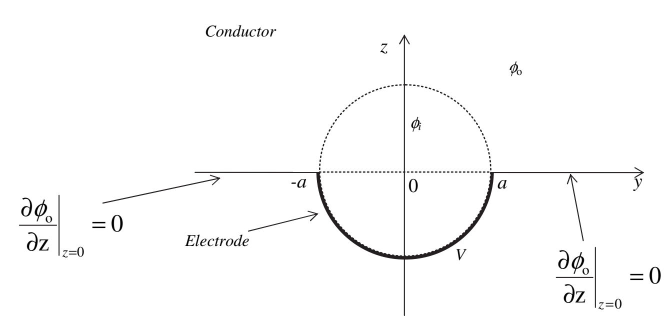

Fig. 13. The electrode, $r=a$, $\theta \in (\pi/2, \pi)$, hold at potential V. The normal derivative of ϕ_o on the surface $z=0$ for $r>a$, vanishes.

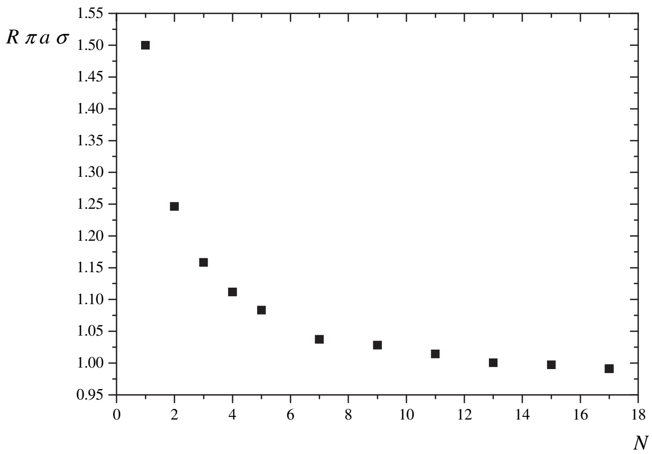

Fig. 14. The electrical resistance, R, of the system shown in Fig. 13, time $\pi a\sigma$, as a function of the parameter N.

$$\begin{aligned}\phi_i(r,\theta) &= \sum_{n=0}^{\infty} A_n r^n P_n(\cos(\theta)) + \sum_{\alpha=1}^{\infty} C_\alpha r^\alpha P_\alpha(\cos(\theta)) \\ &\equiv \mathbf{FR}_1(r)\mathbf{A} + \mathbf{GR}_2(r)\mathbf{C}, \quad r \le a, \theta \in (0,\pi) \\ \phi_o(r,\theta) &= \sum_{n=0}^{\infty} B_n r^{-n-1} P_n(\cos(\theta)) \\ &\equiv \mathbf{FR}_3(r)\mathbf{B}, \quad r \ge a, \theta \in \left(0,\tfrac{\pi}{2}\right)\end{aligned} \tag{16}$$

Proceeding as in the last example we obtain the following 3 matrix equations:

$$\begin{aligned}\mathbf{QR}_1\mathbf{A} + \mathbf{SR}_2\mathbf{C} &= \mathbf{QR}_3\mathbf{B} \\ \mathbf{QR}_1'\mathbf{A} + \mathbf{SR}_2'\mathbf{C} &= \mathbf{QR}_3'\mathbf{B} \\ \mathbf{QR}_1\mathbf{A} - \mathbf{SR}_2\mathbf{C} &= \mathbf{QI}\end{aligned} \tag{17}$$

where we have used the column vector **I** whose components are $I_n = V\delta_{n,0}$. To have an idea of the approximate value of the resistance we take again $N=1$, obtaining in this case $R \approx 3/2\pi\sigma a = 3R_b$. The values of the approximate resistance obtained as a function of N is shown in Fig. 14. We expect that the asymptotic value $\lim_{N\to\infty} R$ be greater than R_b.

3. Conclusions

We have presented a procedure that allows us to solve problems of electrostatics, some of them with mixed boundary conditions, using elementary tools of linear algebra. The results obtained show a rapid convergence toward the exact values in cases that we have to compare with. It is shown that we can obtain reasonable approximate solution with very little effort, and it can be improved increasing the number of terms included in the expansion. In this work we have chosen to project on the subspace generated by the first N even Legendre polynomials, if we had chosen the space generated by the odds ones we would have a slightly different approximate solution. The difference between these two solutions goes to zero as we increase the number of terms, N, in the expansion. Finally we want to mention that we have successfully use this scheme in a course on Classical Electromagnetism at our University.

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References

[1] J.D. Jackson, Classical Electrodynamics, third ed., John Wiley and Sons, Inc., New York, 1998.
[2] I.N. Sneddon, Mixed Boundary Value Problem in Potential Theory. North Holland, Amsterdam, 1966.
[3] Reference 1, pag. 79.
[4] W. Lamberti, D. Prato, Am. J. Phys. 59 (1991) 1.
[5] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, fifth ed., Academic Press, San Diego, 1994, p. 828.
[6] Reference 1, pag. 133.
[7] B. Avants, D. Soodak, G. Ruppeiner, Am. J. Phys. 67 (1999) 593.
[8] J. Jeans, The Mathematical Theory of Electricity and Magnetism, fifth ed., University Press, Cambridge, 1958, p. 194.