

Derivations of certain algebras defined by étale descent

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Abstract We give an explicit description of the Lie algebra of derivations for a class of infinite dimensional algebras which are given by étale descent. The algebras under consideration are twisted forms of central algebras over rings, and include the multiloop algebras that appear in the construction of extended affine Lie algebras.

Keywords Twisted form · Derivation · Centroid · Étale descent · Multiloop algebra

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1 Introduction

Many interesting infinite dimensional Lie algebras can be thought as being “finite dimensional” when viewed, not as algebras over the given base field, but rather as algebras over their centroids. From this point of view, the algebras in question look like twisted forms of simpler objects with which one is familiar. The quintessential example of this type of behaviour is given by the affine Kac–Moody Lie algebras.

An affine Kac–Moody Lie algebra \mathcal{L} (derived modulo its centre) has centroid $R = \mathbb{C}[t^{\pm 1}]$, and there exists a unique finite dimensional simple Lie algebra \mathfrak{g} (whose type is called the absolute type of \mathcal{L}) such that

$$\mathcal{L} \otimes_R R' \simeq (\mathfrak{g} \otimes_{\mathbb{C}} R) \otimes_R R'$$

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with $R \rightarrow R'$ faithfully flat and étale (one can in fact take $R' = \mathbb{C}[t^{\pm 1/m}]$ for a suitable $m \geq 1$). In other words, as R -algebras, \mathcal{L} and $\mathfrak{g} \otimes_{\mathbb{C}} R$ are locally isomorphic for the étale topology on $\text{Spec}(R)$. Since $\mathbf{Aut}(\mathfrak{g})$ is smooth, Grothendieck's descent theory allows us to compute the isomorphism classes of such algebras by means of the pointed sets $H_{\text{ét}}^1(R, \mathbf{Aut}(\mathfrak{g}))$. In fact, as we vary \mathfrak{g} over the nine Cartan-Killing types $A_\ell, B_\ell, \dots, E_8$ we obtain 16 classes in the resulting $H_{\text{ét}}^1$, and these correspond precisely to the isomorphism classes of the affine algebras [22].

Extended affine Lie algebras (EALAs for short) are natural and rather elegant “higher nullity” analogues of the affine algebras (see [1, 18, 19] for details. For a beautiful survey of the theory of EALAs, see [20]). A reasonable understanding of how these algebras fit within the cohomological language of forms is now beginning to emerge [2, 3, 12, 13, 22].

Both the affine algebras and their EALA descendants have connections with Physics, and it is here where central extensions play a crucial role. In the affine case for example, it is not the complex Lie algebra \mathcal{L} that is of interest to physicists, but rather its one-dimensional universal central extension $\widehat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C}c$. This presents an interesting duality: \mathcal{L} can be viewed as a twisted form when thought as an algebra over $R = \mathbb{C}[t^{\pm 1}]$, but not when viewed as a complex Lie algebra. By contrast, as an R -algebra, \mathcal{L} is centrally closed, but as a \mathbb{C} -algebra it is not. The relevant central extension $\widehat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C}c$ exists over \mathbb{C} but not over R .¹

It is thus somehow surprising that natural central extensions of twisted forms of Lie algebras can be obtained solely from their defining descent data [23].² What is missing from this natural descent construction of central extensions is a good understanding of when they are universal. This brings us to the current work of E. Neher.

Just as the affine algebras are built out of loop algebras by adding central extensions and derivations, Neher has shown how to build EALAs out of basic objects called Lie tori [18, 19].³ Furthermore, in work in progress [21], Neher has also shown how to relate his construction of (universal) central extensions to the one given by descent in [23]. To do this, however, a particular explicit description of the algebra of derivations of multiloop algebras is needed.

The structure of the algebra of derivations of a multiloop algebra has recently been determined by Azam [6]. Azam's proof, which is motivated by earlier work of Benkart and Moody [8], is rather involved and depends on a delicate induction reasoning. In this note we give an explicit description of the algebra of derivations for a large class of algebras defined by étale descent. Our methods are quite transparent and, when applied to the particular case of multiloop algebras, yield a new concise and conceptual proof of the Benkart–Moody–Azam result.

2 Centroids of algebras and their derivations

Throughout this note k will denote a ring (commutative and unital) and $k\text{-alg}$ the category of associative commutative and unital k -algebras. Fix an object R of $k\text{-alg}$.

Let \mathcal{L} be an R -algebra (not necessarily associative, commutative or unital). Recall that the *centroid* of \mathcal{L} consists of the endomorphisms of the R -module \mathcal{L} that commute with left and

¹ In fact, $\widehat{\mathcal{L}}$ is not even an R -algebra in any meaningful way.

² Some interesting difficulties arise from the fact that central extensions bring cyclic homology into the picture, but there is no étale descent for cyclic homology.

³ The multiloop algebras appear as the centreless cores of the EALAs that one is trying to build. They are Lie tori as well as a very special type of twisted form. It is in this way that the connection between EALAs and Galois cohomology emerges.

right multiplication by elements of \mathcal{L} . That is,

$$\text{Ctd}_R(\mathcal{L}) = \{\chi \in \text{End}_R(\mathcal{L}) : \chi(xy) = \chi(x)y = x\chi(y) \forall x, y \in \mathcal{L}\}.$$

for all $x, y \in \mathcal{L}$. The centroid is a subalgebra of the (associative and unital) R -algebra $\text{End}_R(\mathcal{L})$. For each $r \in R$ the homothety $\chi_r : x \mapsto rx$ belongs to $\text{Ctd}_R(\mathcal{L})$. This yields an R -algebra homomorphism

$$\chi_{\mathcal{L},R} : R \rightarrow \text{Ctd}_R(\mathcal{L}), \tag{2.1}$$

which is injective if and only if \mathcal{L} is faithful. Recall that \mathcal{L} is called *central* if the map $\chi_{\mathcal{L},R}$ is an isomorphism, and *perfect* if \mathcal{L} is spanned as a k -module (in fact as an abelian group) by the set $\{xy : x, y \in \mathcal{L}\}$. By restriction of scalars we can view \mathcal{L} also as a k -algebra. At the centroid level, this yields the (in general proper) inclusion

$$\text{Ctd}_R(\mathcal{L}) \subset \text{Ctd}_k(\mathcal{L}). \tag{2.2}$$

Perfectness, on the other hand, is independent of whether we view \mathcal{L} as an algebra over R or k .

For convenience we recall the following simple yet useful fact (see Sect. 4 of [4], [9, 15] for details and more general results on centroids).

Lemma 2.3 *If \mathcal{L} is perfect the centroid $\text{Ctd}_R(\mathcal{L})$ is commutative and the inclusion $\text{Ctd}_R(\mathcal{L}) \subset \text{Ctd}_k(\mathcal{L})$ is an equality.*

This will be the situation that will be considered in our work. In particular $\text{Ctd}_R(\mathcal{L})$ is an object of k -alg and \mathcal{L} can naturally be viewed as an algebra over the (commutative) ring $\text{Ctd}_R(\mathcal{L})$.

We finish this section by describing the main problem that we want to study. By restriction of scalars we may view \mathcal{L} as a k -algebra. We then have a natural k -Lie algebra homomorphism $\eta_{\mathcal{L}} : \text{Der}_k(\mathcal{L}) \rightarrow \text{Der}_k(\text{Ctd}_k(\mathcal{L}))$ given by

$$\eta_{\mathcal{L}}(\delta)(\chi) = [\delta, \chi] = \delta \circ \chi - \chi \circ \delta \tag{2.4}$$

for all $\delta \in \text{Der}_k(\mathcal{L})$ and $\chi \in \text{Ctd}_k(\mathcal{L})$. Assume, furthermore, that \mathcal{L} is such that the natural map

$$\chi_{\mathcal{L}} : R \xrightarrow{\chi_{\mathcal{L},R}} \text{Ctd}_R(\mathcal{L}) \subset \text{Ctd}_k(\mathcal{L}) \tag{2.5}$$

obtained by composing (2.1) and (2.2) is an isomorphism. Then $\eta_{\mathcal{L}}$ induces a k -Lie algebra homomorphism (also denoted $\eta_{\mathcal{L}}$)

$$\eta_{\mathcal{L}} : \text{Der}_k(\mathcal{L}) \rightarrow \text{Der}_k(R). \tag{2.6}$$

For future reference let us observe that the isomorphism $\chi_{\mathcal{L}} : R \rightarrow \text{Ctd}_k(\mathcal{L})$ under consideration is given by $r \mapsto \chi_r$ where $\chi_r : x \mapsto rx$. Thus for $\delta \in \text{Der}_k(\mathcal{L})$ our map (2.6) is determined by the identity

$$\eta_{\mathcal{L}}(\delta)(r) = t \iff [\delta, \chi_r] = \chi_t \quad \text{for all } r, t \in R. \tag{2.7}$$

One could say that the main objective of our work is to identify a useful class of algebras for which the map $\eta_{\mathcal{L}}$ is well understood. In Sect. 3 we will discuss a class of algebras for which the map $\chi_{\mathcal{L}} : R \rightarrow \text{Ctd}_k(\mathcal{L})$ is an isomorphism, while in Sect. 4 we describe a class of algebras for which the map $\eta_{\mathcal{L}}$ has a natural section. This leads to an explicit description of the algebra of derivations of the k -algebra \mathcal{L} . Finally, in Sect. 5 we apply the results of the two previous sections to study the case of multiloop algebras in detail.

3 Twisted forms of algebras

In what follows A will denote an algebra (not necessarily associative...) over k . For each object S in k -alg we will find it at times convenient to denote the resulting S -algebra $A \otimes_k S$ by A_S .

Lemma 3.1 *Assume that the k -algebra A is finitely presented as a k -module, and that $k \rightarrow R$ is flat. Then the canonical map*

$$\nu_{A,k,R} : \text{Ctd}_k(A) \otimes_k R \rightarrow \text{Ctd}_R(A \otimes_k R) = \text{Ctd}_R(A_R)$$

is an R -algebra isomorphism.⁴

Proof The map in question is the restriction to $\text{Ctd}_k(A) \otimes_k R$ of the canonical map $\text{End}_k(A) \otimes_k R \rightarrow \text{End}_R(A \otimes_k R) = \text{End}_R(A_R)$. Let

$$\beta = \beta_{A,k} : \text{End}_k(A) \rightarrow \text{Hom}_k(A \otimes_k A, A \oplus A)$$

be the unique k -linear map satisfying

$$\beta_{A,k}(f)(a_1 \otimes a_2) = (f(a_1 a_2) - f(a_1) a_2, f(a_1 a_2) - a_1 f(a_2)).$$

By definition $\text{Ctd}_k(A) = \ker(\beta_{A,k})$. We have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ctd}_k(A) \otimes_k R & \longrightarrow & \text{End}_k(A) \otimes_k R & \xrightarrow{\beta_{A,k} \otimes 1} & \text{Hom}_k(A \otimes_k A, A \oplus A) \otimes_k R \\ & & \downarrow \nu_{A,k,R} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ctd}_R(A_R) & \longrightarrow & \text{End}_R(A_R) & \xrightarrow{\beta_{A_R,R}} & \text{Hom}_R(A_R \otimes_R A_R, A_R \oplus A_R). \end{array}$$

The top row is exact because $k \rightarrow R$ is flat, the middle vertical arrow is bijective because A is finitely presented, while the right vertical map is injective because $A \otimes_k A$ is of finite type [7, Ch. 1, Sect. 2.10, Prop. 11]. From this it readily follows that $\nu_{A,k,R}$ is an isomorphism. \square

The k -group functor of automorphisms of A will be denoted by $\mathbf{Aut}(A)$. Thus

$$\begin{aligned} \mathbf{Aut}(A) : k\text{-alg} &\rightarrow \text{Grp} \\ S &\mapsto \mathbf{Aut}(A)(S) = \text{Aut}_{S\text{-alg}}(A_S), \end{aligned}$$

where this last is the group of automorphisms of the S -algebra A_S . If as a k -module A is projective of finite type, then $\mathbf{Aut}(A)$ is an affine group scheme over $\text{Spec}(k)$.

Recall that a *twisted form* of the R -algebra A_R for the fpqc topology on $\text{Spec}(R)$ is an R -algebra \mathcal{L} such that

$$\mathcal{L} \otimes_R R' \simeq_{R'\text{-alg}} A_R \otimes_R R' \tag{3.2}$$

for some faithfully flat extension $R \rightarrow R'$. Given a form \mathcal{L} as above for which (3.2) holds, we say that \mathcal{L} is *trivialized* by R' . The R -isomorphism classes of such algebras can be computed by means of cocycles, just as one does in Galois cohomology [25]:

$$\text{Isomorphism classes of } R'/R\text{-forms of } A_R \longleftrightarrow H^1(R'/R, \mathbf{Aut}(A)). \tag{3.3}$$

⁴ The assumptions we have made on A and R are natural within the context of the present work, but the main example that we have in mind is of course when k is a field and A is finite dimensional. In the case when k is a field, many other examples when the map $\nu_{A,k,R}$ is an isomorphism can be found in [6, 5, 9].

Since we will need the explicit description of this correspondence for our work, we will briefly recall the basic relevant facts.⁵ Let $R'' = R' \otimes_R R'$ and $R''' = R' \otimes_R R' \otimes_R R'$. We have the standard R -algebra homomorphisms $p_i : R' \rightarrow R''$, $i = 1, 2$ and $p_{ij} : R'' \rightarrow R'''$, $1 \leq i < j \leq 3$ corresponding to the projections on the i th and (i, j) th components respectively (see [28], Sect. 7.6 for details). These yield group homomorphisms (also denoted by p_i and p_{ij})

$$\begin{aligned} \mathbf{Aut}(A)(R') &\xrightarrow{p_i} \mathbf{Aut}(A)(R''), \\ \mathbf{Aut}(A)(R'') &\xrightarrow{p_{ij}} \mathbf{Aut}(A)(R'''). \end{aligned}$$

An R'/R -cocycle with values in $\mathbf{Aut}(A)$ is an element $u \in \mathbf{Aut}(A)(R'')$ such that $p_{13}(u) = p_{23}(u)p_{12}(u)$. If $g \in \mathbf{Aut}(A)(R')$ then $g \cdot u = p_2(g)u p_1(g)^{-1}$ is also a cocycle. This defines an action of the group $\mathbf{Aut}(A)(R')$ on the set of cocycles, and we define $H^1(R'/R, \mathbf{Aut}(A))$ to be the quotient set (whose elements are thus equivalence classes of cocycles) defined by this action. $H^1(R'/R, \mathbf{Aut}(A))$ is a pointed set whose distinguished element is the class of the cocycle $1 \in \mathbf{Aut}(A)(R'')$.

The correspondence (3.3) is given by attaching to a cocycle u the R -algebra

$$\mathcal{L}_u = \{x \in A \otimes_k R' : u p_1^A(x) = p_2^A(x)\},$$

where $p_i^A = \text{id}_A \otimes p_i : A \otimes_k R' \rightarrow A \otimes_k R' \otimes_R R'$. If $\mu : R' \otimes_R R' \rightarrow R'$ is the map corresponding to the multiplication of the ring R' , then the diagram

$$\begin{array}{ccc} A \otimes_k R' \otimes_R R' & & \\ \cup & \searrow^{1 \otimes \mu} & \\ & & A \otimes_k R' \\ \mathcal{L}_u \otimes_R R' & \nearrow & \end{array}$$

induces an isomorphism $\mathcal{L}_u \otimes_R R' \simeq A \otimes_k R'$.

Lemma 3.4 *Let \mathcal{L} be a twisted form of A_R for the fpqc topology on $\text{Spec}(R)$. Assume A is perfect and central as a k -algebra, and finitely presented as a k -module. Then*

- (1) \mathcal{L} is perfect. In particular $\text{Ctd}_R(\mathcal{L})$ is commutative and the inclusion $\text{Ctd}_R(\mathcal{L}) \subset \text{Ctd}_k(\mathcal{L})$ is an equality.
- (2) As an R -module \mathcal{L} is faithful and finitely presented.
- (3) The canonical map $\chi_{\mathcal{L}} : R \rightarrow \text{Ctd}_k(\mathcal{L})$ is an isomorphism.

Proof (1) Since perfectness is preserved by base change $A_{R'}$ is perfect. A straightforward faithfully flat descent argument [13, Lemma 4.6.1] yields that \mathcal{L} is perfect. The rest of (1) now follows from Lemma 2.3.

(2) and (3) By Lemma 3.1 the canonical map $R' \rightarrow \text{Ctd}_{R'}(A \otimes_k R')$ is an isomorphism. By reasoning as in [13, Lemma 4.6.2,3] we see that (2) holds, and also that the canonical map $\chi_{\mathcal{L},R} : R \rightarrow \text{Ctd}_k(\mathcal{L})$ is an isomorphism. Now (3) follows from (1). \square

Our next objective is to show that for a large class of twisted forms (that include multiloop algebras) the Lie algebra homomorphism $\eta_{\mathcal{L}}$ defined in (2.6) admits a natural section. One of the crucial assumptions is that the faithfully flat trivializing base change $R \rightarrow R'$ be étale.

⁵ The general theory of descent we are using can be found within [26,27]. The formalism of torsors is clearly presented in [11,17]. An accessible account that is (almost) sufficient for our needs can be found in [16,28].

4 Derivations of certain algebras defined by étale descent

Assume that $R \rightarrow R'$ is a faithfully flat and étale morphism in k -alg. Let $d \in \text{Der}_k(R)$. We view d naturally as an element of $\text{Der}_k(R, R')$ via $R \rightarrow R'$. Since $R \rightarrow R'$ is faithfully flat we can (and henceforth will) naturally identify R with a k -subalgebra of R' . After this identification the assumption that $R \rightarrow R'$ is étale yields the existence of a unique $d' \in \text{Der}_k(R')$ extending d [14] Cor. 20.5.8. Similarly d' extends to two derivations of $R'' = R' \otimes_R R'$ via the two morphisms $p_i : R' \rightrightarrows R''$. If we denote these by d''_1 and d''_2 , then both d''_1 and d''_2 extend d under $R \rightarrow R' \rightrightarrows R''$. Since these two composite maps coincide and the resulting map $R \rightarrow R''$ is étale, it follows that $d''_1 = d''_2 = d''$ for some unique $d'' \in \text{Der}_k(R'')$. In particular $d'(s) \otimes 1 = d''(s \otimes 1)$ and $1 \otimes d'(s) = d''(1 \otimes s)$ for all $s \in R'$. That is

$$p_i \circ d' = d'' \circ p_i. \tag{4.1}$$

Let $R''_0 = \{s \in R'' : d''(s) = 0 \text{ for all } d \in \text{Der}_k(R)\}$. It is clear that R''_0 is a k -subalgebra of R'' . This yields a group homomorphism $\mathbf{Aut}(A)(R''_0) \rightarrow \mathbf{Aut}(A)(R'')$ for any k -algebra A . An element $u \in \mathbf{Aut}(A)(R'')$ is in the image of this map if and only if $u(A \otimes 1 \otimes 1) \subset A \otimes_k R''_0$.

Theorem 4.2 *Let A be a k -algebra which is finitely presented as a k -module. Let $R \rightarrow R'$ be a faithfully flat and étale extension in k -alg with $k \rightarrow R$ flat. Consider the twisted form \mathcal{L}_u of A_R corresponding to a cocycle $u \in \mathbf{Aut}(A)(R'')$. Assume that the following two conditions hold.*

- (i) *The canonical map $\chi_{\mathcal{L}} : R \rightarrow \text{Ctd}_k(\mathcal{L}_u)$ is an isomorphism.⁶*
- (ii) *The cocycle u belongs to the image of $\mathbf{Aut}(A)(R''_0)$ in $\mathbf{Aut}(A)(R'')$.*

Then the Lie algebra homomorphism

$$\eta_{\mathcal{L}_u} : \text{Der}_k(\mathcal{L}_u) \rightarrow \text{Der}_k(R)$$

described in (2.6) admits a natural section ρ . Furthermore

$$\text{Der}_k(\mathcal{L}_u) = \text{Der}_R(\mathcal{L}_u) \rtimes \rho(\text{Der}_k(R)).$$

In particular, if $k \rightarrow R$ is étale then $\text{Der}_k(\mathcal{L}_u) = \text{Der}_R(\mathcal{L}_u)$.

Proof Let $d \in \text{Der}_k(R)$. The map $\text{id}_A \otimes d'' : A \otimes_k R'' \rightarrow A \otimes_k R''$ is clearly a derivation of $A \otimes_k R''$ as a k -algebra. The key point is that

$$\text{id}_A \otimes d'' \text{ commutes with the action of } u. \tag{4.3}$$

Indeed, if $x \in A$ and we write $u^{-1}(x \otimes 1 \otimes 1) = \sum x_i \otimes s_i$ for some $x_i \in A$ and $s_i \in R''_0$, then using that u is R'' -linear and $d''(s_i) = 0$ we see that for all $s \in R''$ we have

$$\begin{aligned} u \circ (\text{id}_A \otimes d'') \circ u^{-1}(x \otimes s) &= u \circ (\text{id}_A \otimes d'') \left(\sum x_i \otimes s_i s \right) \\ &= u \left(\sum x_i \otimes s_i d''(s) \right) \\ &= x \otimes d''(s) = (\text{id}_A \otimes d'')(x \otimes s). \end{aligned}$$

From (4.1) we obtain

$$p_i^A \circ (\text{id}_A \otimes d') = (\text{id}_A \otimes d'') \circ p_i^A. \tag{4.4}$$

⁶ For example if A is perfect and central.

From (4.3) and (4.4) it follows that $\text{id}_A \otimes d' \in \text{Der}_k(A \otimes_k R')$ stabilizes \mathcal{L}_u . Indeed if $y \in \mathcal{L}_u$ then

$$\begin{aligned} up_1^A((\text{id}_A \otimes d')(y)) &= (u \circ (\text{id}_A \otimes d'') \circ p_1^A)(y) = ((\text{id}_A \otimes d'') \circ u \circ p_1^A)(y) \\ &= (\text{id}_A \otimes d'') (p_2^A(y)) = p_2^A((\text{id}_A \otimes d')(y)). \end{aligned}$$

This shows that there exists a k -linear map $\rho : \text{Der}_k(R) \rightarrow \text{Der}_k(\mathcal{L}_u)$ given by

$$\rho : d \mapsto (\text{id}_A \otimes d')|_{\mathcal{L}_u}. \tag{4.5}$$

It is clear that ρ is a Lie algebra homomorphism. To verify that ρ is a section of $\eta = \eta_{\mathcal{L}_u}$ in (2.6) we observe that for all $d \in \text{Der}_k(R)$, $y = \sum x_i \otimes s_i \in \mathcal{L}_u$ and $r \in R$ we have

$$[\text{id}_A \otimes d', \chi_r] \left(\sum x_i \otimes s_i \right) = \sum x_i \otimes d'(r)s_i = d'(r)y = d(r)y = \chi_{d(r)}(y).$$

According to (2.7) this shows that $\eta(\rho(d)) = d$ as desired.

Since by assumption $\text{Ctd}_k(\mathcal{L}_u)$ consists of the homotheties χ_r for $r \in R$, it is clear from the definition of η that $\ker(\eta) = \text{Der}_R(\mathcal{L}_u)$. This shows that $\text{Der}_R(\mathcal{L}_u)$ is an ideal of $\text{Der}_k(\mathcal{L}_u)$. If moreover $\delta \in \text{Der}_k(\mathcal{L}_u)$, then $\delta - \rho(\eta(\delta)) \in \ker(\eta)$. Thus $\text{Der}_k(\mathcal{L}_u) = \text{Der}_R(\mathcal{L}_u) + \rho(\text{Der}_k(R))$. That this sum is direct is easy to see. This completes the proof of our result. \square

Before making a few relevant observations and remarks pertaining to this last result, we should point out that the assumption made on the cocycle u is quite restrictive and certainly not necessary for the thesis of the Theorem to hold. One is indeed fortunate that many interesting cases fall under this assumption.

Remark 4.6 It is important to observe that the definition of the section ρ given by the Theorem, and the resulting identification of $\text{Der}_k(R)$ with a subalgebra of $\text{Der}_k(\mathcal{L}_u)$, are completely natural and explicit: $d \in \text{Der}_k(R)$ extends uniquely to a derivation $d' \in \text{Der}_k(R')$. Then $\text{id}_A \otimes d'$ is a derivation of the k -algebra $A \otimes_k R'$ and $\rho(d)$ is nothing but the restriction of $\text{id}_A \otimes d'$ to $\mathcal{L}_u \subset A \otimes_k R'$.

Note also that the isomorphism $R \simeq \text{Ctd}_k(\mathcal{L}_u)$ is given by restricting the scalar action of R in $A \otimes_k R'$ to \mathcal{L}_u . With this identification $\text{Ctd}_k(\mathcal{L}_u) = \{\chi_r : r \in R\}$, and the natural action of $\text{Der}_k(R)$ on $\text{Ctd}_k(\mathcal{L}_u)$ is then given by $d(\chi_r) = \chi_{d(r)}$.

Remark 4.7 The most natural type of twisted forms to which the Theorem applies are those given by constant cocycles, namely when u belongs to the image of the map $\mathbf{Aut}(A)(k) \rightarrow \mathbf{Aut}(A)(R')$. This is the case of multiloop algebras, as we will see in the next section.

Remark 4.8 Let $g \in \mathbf{Aut}(A)(R')$, and set $v = p_2(g)up_1(g)^{-1}$. Since $\mathcal{L}_u \simeq \mathcal{L}_v$ as R -algebras, the Lie algebra homomorphism $\eta_{\mathcal{L}_v} : \text{Der}_k(\mathcal{L}_v) \rightarrow \text{Der}_k(R)$ corresponding to \mathcal{L}_v admits a section, but the cocycle v need not satisfy the assumption of the Theorem. An easy calculation shows that

$$\text{Der}_k(\mathcal{L}_u) = \text{Der}_R(\mathcal{L}_u) \rtimes \rho_g(\text{Der}_k(R)),$$

where ρ_g is given by $\rho_g(d) = g(\text{id}_A \otimes d')g^{-1}$ for all $d \in \text{Der}_k(R)$.

Example 4.9 Assume that k is a field of characteristic 0, and that A is a finite dimensional central simple Lie algebra over k . To abide by standard notational conventions we will denote A by \mathfrak{g} .

Every derivation of the R -Lie algebra \mathfrak{g}_R is inner. This follows from theorem 1.1 of [8], and also by the following direct reasoning: If $\delta \in \text{Der}_R(\mathfrak{g}_R)$ we may view the restriction of δ to \mathfrak{g} as a derivation $\delta_{\mathfrak{g}}$ of \mathfrak{g} with values in \mathfrak{g}_R (via the adjoint representation). Since \mathfrak{g} is finite dimensional there exists a finite dimensional submodule M of \mathfrak{g}_R such that $\delta_{\mathfrak{g}}$ takes values in M . By Whitehead’s lemma there exists $x \in M \subset \mathfrak{g}_R$ such that $\delta_{\mathfrak{g}}(y) = [x, y]$. This shows that δ and $\text{ad}_{\mathfrak{g}_R}(x)$ agree on $\mathfrak{g} \otimes 1$. By R -linearity $\delta = \text{ad}_{\mathfrak{g}_R}(x)$.

Assume now that \mathcal{L} is a twisted form of \mathfrak{g}_R . Choose a faithfully flat base change $R \rightarrow R'$ that trivializes \mathcal{L} . Since $\mathcal{L} \otimes_R R' \simeq \mathfrak{g} \otimes_k R'$ and every derivation of $\mathfrak{g} \otimes_k R'$ is inner, we conclude that $(\text{Der}_R(\mathcal{L})/\text{IDer}(\mathcal{L})) \otimes_R R' = 0$. By faithfully flat descent $\text{IDer}(\mathcal{L}) = \text{Der}_R(\mathcal{L})$.

Finally, if $\mathcal{L} = \mathcal{L}_u$ for a cocycle u as in Theorem 4.2 then

$$\text{Der}_k(\mathcal{L}_u) = \text{IDer}(\mathcal{L}_u) \rtimes \rho(\text{Der}_k(R)).$$

Example 4.10 Assume $R = k[t]$ where k is a field. Let A be an algebra as in Theorem 4.2, and assume that the connected component of the identity of the algebraic group $\mathbf{Aut}(A)$ is reductive. Let \mathcal{L} be a twisted form of A_R which is trivialized by the base change $k[t] \rightarrow k_s[t]$, where k_s is the separable closure of k .⁷ From the work of Raghunathan and Ramanathan [24] we know that the natural map $H_{\text{ét}}^1(k, \mathbf{Aut}(A)) \rightarrow H_{\text{ét}}^1(k[t], \mathbf{Aut}(A))$ is bijective. Thus $\mathcal{L} \simeq \mathcal{L}_u$ as R -algebras (a fortiori also as k -algebras) for some constant cocycle u . We have

$$\text{Der}_k(\mathcal{L}) \simeq \text{Der}_k(\mathcal{L}_u) = \text{Der}_R(\mathcal{L}_u) \rtimes \rho\left(R \frac{d}{dt}\right).$$

The action of $R \frac{d}{dt} = \text{Der}_k(R)$ on \mathcal{L} , however, is not explicit. Of course if \mathcal{L} is given to us in the form $\mathcal{L} = \mathcal{L}_v$, then we can apply the considerations described in Remark 4.8.

For the Laurent polynomial ring $k[t^{\pm 1}]$ the situation is much more delicate. The natural map $H_{\text{ét}}^1(k[t^{\pm 1}], \mathbf{Aut}(A)) \rightarrow H_{\text{ét}}^1(k((t)), \mathbf{Aut}(A))$ is bijective whenever $\mathbf{Aut}(A)$ is reductive and the characteristic of k is good [10]. If, for example, the image of the class of u in $H_{\text{ét}}^1(k((t)), \mathbf{Aut}(A))$ is constant (a problem that in theory can be studied by Bruhat-Tits methods), then Theorem 4.2 can be applied.

5 The Galois case. Applications to multiloop algebras

Throughout this section k is assumed to be a field, and A will denote a k -algebra that satisfies assumption (i) of Theorem 4.2.⁸ In this situation the canonical maps $\mathbf{Aut}(A)(k) \rightarrow \mathbf{Aut}(A)(R'_0) \rightarrow \mathbf{Aut}(A)(R'')$ are all injective, and we identify the first two groups with their respective images. We denote $\mathbf{Aut}(A)(k)$ by $\text{Aut}_k(A)$.

Assume that our form \mathcal{L}_u is trivialized by a (finite) Galois extension R' of R (see [16, 28] or, ultimately and inevitably, [26]). Recall then that $R \rightarrow R'$ is faithfully flat, and that if $\Gamma \subset \text{Aut}_R(R')$ denotes the Galois group of the extension then the map

$$R' \otimes_R R' \rightarrow R' \times \cdots \times R' \quad (|\Gamma|\text{-times})$$

given by

$$a \otimes b \mapsto (\gamma(a)b)_{\gamma \in \Gamma}$$

is an isomorphism of R' -algebras (with R' acting by multiplication on the second component of $R' \otimes_R R'$). Under the resulting identification of $\mathbf{Aut}(A)(R'')$ with $\mathbf{Aut}(A)(R') \times \cdots \times$

⁷ By a theorem of Steinberg this assumption is superfluous if k is of characteristic 0.

⁸ For example A finite dimensional and central simple.

$\mathbf{Aut}(A)(R')$ our cocycle u corresponds to a $|\Gamma|$ -tuple $(u_\gamma)_{\gamma \in \Gamma}$ which satisfies the usual cocycle condition $u_{\gamma\mu} = u_\gamma \cdot u_\mu$, with Γ acting on $\mathbf{Aut}(A)(R')$ in the natural way.

Remark 5.1 At the level of Galois cocycles the assumption that u be an element of $\mathbf{Aut}(A)(R'_0)$ translates into the following condition: For all $\gamma \in \Gamma$ we have $u_\gamma \in \mathbf{Aut}(A)(R'_0)$ where $R'_0 = \{s \in R' : d'(s) = 0 \text{ for all } d \in \text{Der}_k(R)\}$.

Note that this condition is automatically satisfied whenever the u_γ are obtained from automorphisms of the k -algebra A , i.e. $u_\gamma = v_\gamma \otimes 1$ for some $v_\gamma \in \text{Aut}_k(A)$. The action of Γ is in this case trivial, and the cocycle condition simply states that $\gamma \mapsto v_\gamma$ is a group homomorphism from Γ to $\text{Aut}_k(A)$. This situation arises in the case of multiloop algebras, as we now explain.

We will assume henceforth that $R = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$. Fix an n -tuple $\mathbf{m} = (m_1, \dots, m_n)$ of positive integers, and set $R_{\mathbf{m}} = R' = k[t_1^{\pm \frac{1}{m_1}}, \dots, t_n^{\pm \frac{1}{m_n}}]$. We assume in what follows that the m_i are relatively prime to the characteristic of k . The natural map $R \rightarrow R'$ is then faithfully flat and étale.

Let $m = \prod_{1 \leq i \leq n} m_i$. Assume k contains a primitive m th root of unity ξ_m , and set $\xi_{m_i} = \xi_m^{\prod_{j \neq i} m_j}$. Then $R \rightarrow R'$ is Galois with Galois group $\Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \dots \times \mathbb{Z}/m_n\mathbb{Z}$, where for each $\mathbf{e} = (e_1, \dots, e_n) \in \mathbb{Z}^n$ the corresponding element $\bar{\mathbf{e}} = (\bar{e}_1, \dots, \bar{e}_n) \in \Gamma$ acts on R' via

$$\bar{\mathbf{e}} \cdot t_i^{\frac{1}{m_i}} = \xi_{m_i}^{e_i} t_i^{\frac{1}{m_i}}.$$

Multiloop algebras arise under these assumptions by considering an n -tuple $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ of commuting elements of $\text{Aut}_k(A)$ satisfying $\sigma_i^{m_i} = 1$. For each n -tuple $(i_1, \dots, i_n) \in \mathbb{Z}^n$ we consider the simultaneous eigenspace

$$A_{i_1 \dots i_n} = \{x \in A : \sigma_j(x) = \xi_{m_j}^{i_j} x \text{ for all } 1 \leq j \leq n\}.$$

Then $A = \sum A_{i_1 \dots i_n}$, and $A = \bigoplus A_{i_1 \dots i_n}$ if we restrict the sum to those n -tuples (i_1, \dots, i_n) for which $0 \leq i_j < m_j$.

The map $\Gamma \rightarrow \text{Aut}_k(A)$ given by

$$\bar{\mathbf{e}} = (\bar{e}_1, \dots, \bar{e}_n) \mapsto \sigma_1^{-e_1} \dots \sigma_n^{-e_n} = v_{\bar{\mathbf{e}}}$$

is a group homomorphism whose corresponding cocycle $u = (u_{\bar{\mathbf{e}}})_{\bar{\mathbf{e}} \in \Gamma}$ with $u_{\bar{\mathbf{e}}} = v_{\bar{\mathbf{e}}} \otimes 1$ is constant (see Remark 4.7), hence satisfies assumption (ii) of Theorem 4.2. The corresponding form \mathcal{L}_u is the *multiloop algebra* commonly denoted by $L(A, \boldsymbol{\sigma})$,

$$\mathcal{L}_u = L(A, \boldsymbol{\sigma}) = \bigoplus_{(i_1, \dots, i_n) \in \mathbb{Z}^n} A_{i_1 \dots i_n} \otimes t_1^{\frac{i_1}{m_1}} \dots t_n^{\frac{i_n}{m_n}} \subset A \otimes_k R_{\mathbf{m}}.$$

We have

$$\text{Der}_k(L(A, \boldsymbol{\sigma})) = \text{Der}_R(L(A, \boldsymbol{\sigma})) \rtimes \rho(\text{Der}_k(R)). \tag{5.2}$$

We have $\text{Der}_k(R) = R \frac{\partial}{\partial t_1} \oplus \dots \oplus R \frac{\partial}{\partial t_n}$, and the (unique) way in which the elements of $\text{Der}_k(R)$ extend to $\text{Der}_k(R_{\mathbf{m}})$ is clear. The explicit action of $\text{Der}_k(R)$ on $L(A, \boldsymbol{\sigma})$ and on $\text{Ctd}_k(L(A, \boldsymbol{\sigma}))$ is now as described in Remark 4.6.

Finally, if k is algebraically closed of characteristic 0 and $A = \mathfrak{g}$ is a simple finite dimensional Lie algebra, then the multiloop algebras $L(\mathfrak{g}, \boldsymbol{\sigma})$ arise naturally in modern infinite dimensional Lie theory as we have explained in the Introduction (for example, if $n = 1$,

the $L(\mathfrak{g}, \sigma)$ are the derived algebras of the affine Kac–Moody Lie algebras, modulo their centres). By Remark 4.9

$$\mathrm{Der}_k(L(\mathfrak{g}, \sigma)) = \mathrm{IDer}(L(\mathfrak{g}, \sigma)) \rtimes \rho(\mathrm{Der}_k(R)). \quad (5.3)$$

Remark 5.4 The analogue of Theorem 4.2 for automorphisms instead of derivations *fails*. We do have an exact sequence

$$1 \rightarrow \mathrm{Aut}_R(\mathcal{L}_u) \rightarrow \mathrm{Aut}_k(\mathcal{L}_u) \xrightarrow{\eta} \mathrm{Aut}_k(R)$$

for all cocycles u . The homomorphism η need not be surjective, and even when it is, and if u satisfies the assumption of the Theorem, the resulting exact sequence need not split. This situation takes place, for example, when $A = M_{2 \times 2}(\mathbb{C})$, $R = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$, and \mathcal{L}_u is the standard quaternion algebra over R (see [13, example 4.11]).

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