HARISH CHANDRA MODULES OF RANK ONE LIE GROUPS WITH ADMISSIBLE RESTRICTION TO SOME REDUCTIVE SUBGROUP

JORGE VARGAS

ABSTRACT. This note determine the irreducible Harish Chandra modules of a rank one semisimple Lie group with admissible restriction to some proper reductive subgroup.

1. INTRODUCTION

The problem of understanding branching laws for a unitary representation of a group G with respect to a subgroup L has received attention due in part to its important applications in Number Theory, Fourier Analysis on Homogeneous spaces and Physics. The literature is quite vast and the techniques to attack the problem rely on different areas of mathematics like algebraic geometry, combinatorics, analysis, geometry. For an update on the problem we refer to [15] and references therein. For the purpose of this note, G denotes a rank one, connected matrix simple Lie group G. Once and for all we fix a maximal compact subgroup K of G and denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ the corresponding Cartan decomposition of the Lie algebra of the group G. The Lie algebra of a group is denoted by the corresponding German lower case letter and the complexification of either a real vector space or a real connected matrix Lie group is denoted by adding the subscript $\mathbb{C}$. The aim of this note is to to list, up to equivalence, the totality of triple (G, L, V) such that L is a closed connected subgroup of K and V is a Harish-Chandra module for G which is $L-$admissible. We provide an answer in the language of associated variety of a Harish-Chandra module.

To a Harish-Chandra module (π, V), Vogan in [22], has attached an algebraic subvariety of $\mathfrak{s}_{\mathbb{C}}$. This variety is called the associated variety

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of (π, V) and denoted by $Ass(\pi)$. Based on the Langland's classification of the irreducible Harish-Chandra modules for G, Collingwood in [2] has determined the dimension of $Ass(\pi)$. We state some of our results by means of the associated variety. In order to state our results in terms of Langlands's classification of Harish-Chandra modules we refer to the tables in [2]. Let $\mathcal{N}$ denote the cone of nilpotent elements in $\mathfrak{s}_{\mathbb{C}}$. The adjoint representation of $G_{\mathbb{C}}$ restricted to $K_{\mathbb{C}}$ leaves invariant the subspace $\mathfrak{s}_{\mathbb{C}}$ as well as the cone $\mathcal{N}$. Unless $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, $\mathcal{N}$ is the closure of one orbit of $K_{\mathbb{C}}$. For example, for (π, V) the underlying Harish-Chandra module of a Discrete Series representation and $\mathfrak{g}$ isomorphic to either $\mathfrak{so}(1, 2q), q \geq 2$ or $\mathfrak{f}_{4(-20)}$ the associated variety is $\mathcal{N}$. Whereas, for $\mathfrak{sp}(1, q)$ and (π, V) a Discrete Series representation which is small in the sense of [9], the associated variety is a proper subvariety of $\mathcal{N}$. For π irreducible Harish-Chandra module, the associated variety of π is different of the trivial orbit if and only if π is infinite dimensional. To avoid cumbersome statements, in this note, we only consider infinite dimensional irreducible Harish-Chandra modules for $\mathfrak{g}$. Next, we consider a closed connected subgroup L of K together with (π, V) a Harish-Chandra module for G. Hence, the restriction of (π, V) to L decomposes as a direct sum of irreducible representations, by definition, (π, V) restricted to L is *admissible* when the multiplicity of each irreducible $L-$factor is finite. In [13], we find equivalent statements to admissibility as well as properties of $L-$admissible representations. For a noncompact closed semisimple subgroup H of G whose maximal compact subgroup is L, and an irreducible unitary representation (π, V) of G, we define (π, V) to be $H-$*admissible* whenever (π, V) restricted to H decomposes as a Discrete Hilbert sum of irreducible representations and each irreducible factor has finite multiplicity. In [6], [13] is shown that for a unitary irreducible representation of G, admissible restriction to L of the underlying Harish-Chandra module implies $H-$ admissibility. In [5] is shown that whenever (π, V) is a discrete series representation, $H-$admisibilty implies $L-$admissibility of the underlying Harish-Chandra module. Kobayashi, in [14], conjectures that for an unitary irreducible representation of G and (G, H) a symmetric pair, $H-$admissibility implies $L-$admissibility of the underlying Harish-Chandra module.

For our first result, we fix a compact connected subgroup L of K. Hence, L acts on the unit sphere of $\mathfrak{s}$. We have,

Theorem 1. *Assume L acts transitively on the unit sphere of $\mathfrak{s}$. Then, any Harish-Chandra module (π, V) for G is admissible when restricted* *to L. Conversely, if the associated variety of π is $\mathcal{N}$ and (π, V) restricted to L is admissible. Then, L acts transitively on the unit sphere of $\mathfrak{s}$.*

After we show theorem 1 we explicit the subgroups involved in the statement of the result.

The ensuing results analyzes the case the associated variety is a proper subvariety of $\mathcal{N}$. To begin with, let us recall that up to local isomorphism, the rank one groups are

$$SU(1,q), SO_0(1,q), Sp(1,q), F_{4(-20)}.$$

Respective maximal compact subgroups are

$$S(U(1) \times U(q)), SO(q), Sp(1) \times Sp(q), Spin(9).$$

With respect to the $K_{\mathbb{C}}-$orbits in $\mathcal{N}$, in [3], we find a proof that for $\mathfrak{so}(1,q), q \geq 3$ the nilpotent cone has no proper $K_{\mathbb{C}}-$invariant closed subvarieties. For either $\mathfrak{sp}(1,q), q \geq 2$ or $\mathfrak{f}_{4(-20)}$, $K_{\mathbb{C}}$ has one proper orbit and for $\mathfrak{su}(1,q)$, $K_{\mathbb{C}}$ has two proper orbits. Let L a compact connected subgroup of K.

Theorem 2. *Assume the associated variety of the Harish-Chandra module (π, V) is a proper subvariety of $\mathcal{N}$. Then,*

- *For $\mathfrak{su}(1,q), \pi_{|_L}$ is admissible iff $S[\mathbb{C}^q]^{L_{\mathbb{C}}} = \mathbb{C}$.*
- *For $\mathfrak{sp}(1,q), \pi_{|_L}$ is admissible iff L is conjugate to either $Sp(1) \times B$, with B closed subgroup of $Sp(q)$ or to $S \times Sp(q_1) \times \cdots \times Sp(q_r)$, with S closed subgroup of $Sp(1)$ and $q_1 + \cdots + q_r = q$.*

One consequence of theorem 1 and theorem 2 is: Any Harish-Chandra module for $SO(1,2q)$ (resp. $Sp(1,q)$) has admissible restriction to $U(q)$ (resp.$Sp(q)$). In section 2,3,4 we point out more examples.

For the case of $F_{4(-20)}$ it follows from theorem 1 and the classification of orthogonal groups which acts transitively on the unit sphere that every Harish-Chandra module whose associated variety is $\mathcal{N}$ does not have an admissible restriction to a proper subgroup of $Spin(9)$. In order to state a result when $Ass(\pi)$ is a proper subvariety of $\mathcal{N}$ we set up some notation. From now on, for a subgroup $Spin(k)$ of $Spin(9)$ we mean a conjugate to the inverse image of the immersion of $SO(k)$ in $SO(9)$ as an upper left block or equivalently immersed as a lower right block. For the next statement, let (π, V) be a Harish-Chandra module for $F_{4(-20)}$.

Theorem 3. *We set L to be a connected closed reductive subgroup of $Spin(9)$. Assume $Ass(\pi)$ is proper. Then π restricted to L is admissible if and only if L is conjugated to a subgroup which contains $Spin(6)$.*

Corollary 1. *Assume $Ass(\pi)$ is a proper subvariety of $\mathcal{N}$. For L simple, if π restricted to L is admissible, then L is conjugated to one of $Spin(m), m = 6, 7, 8, 9$.*

In section 4 we verify that if semisimple connected subgroup H of $F_{4(-20)}$ contains $Spin(6)$ then H is compact. The copy of $SO(6,1)$ inside $SO(8,1)$ has as one of its maximal compact subgroups the usual immersion of $SU(4)$ in $SO(8)$. Actually, $Spin(6)$ is conjugated to $SU(4)$ by an outer automorphism of $Spin(8)$ of order two. We recall that every automorphism of $Spin(8)$ is the restriction of an inner automorphism of $F_4(\mathbb{C})$. The maximal connected closed subgroups of $Spin(9)$ are:

$$Spin(n) \times Spim(m), n + m = 9, n \leq m; SU(2) \boxtimes SU(2)$$

and a homomorphic image of $SU(2)$ whose projection on $SO(9)$ acts irreducible in $\mathbb{R}^9$. We show,

Corollary 2. *Assume $Ass(\pi)$ is proper. Then π restricted to a maximal connected closed subgroup L of $Spin(9)$ is admissible if and only if L is conjugated to one of $Spin(m) \times Spim(9 - m), m = 6, 7, 8, 9$.*

The connected simple Lie subgroups of $F_4(\mathbb{C})$ has been classified by Dynkin, in order to show theorem 3 we review the list in section 4. The maximal closed, connected reductive subgroups of $F_{4(-20)}$ have been classified by Komrakov [17]. The list up to conjugation and up to covering is:

$$SU(2,1) \times SU(3), SL_2(\mathbb{R}) \times G_2, SO(8,1), Sp(2,1) \times SU(3), K.$$

Here, G_2 is a compact Lie group for the algebra $\mathfrak{g}_2$.

Theorem 4. *Let H be a proper semisimple noncompact connected subgroup so that $H \cap K$ is a maximal compact subgroup of H and π an irreducible Harish-Chandra module of $F_{4(-20)}$. Then, π restricted to $H \cap K$ is admissible if and only if the associated variety of π is a proper subvariety of $\mathcal{N}$ and H is conjugated to $SO(8,1)$.*

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2. Proof of theorem 1

For each Harish-Chandra module π, Vogan in [22] has defined the associated variety $Ass(\pi)$. This is an algebraic subset of the set of nilpotent elements $\mathcal{N}$ in $\mathfrak{s}_{\mathbb{C}}$. Besides $Ass(\pi)$ is invariant under the action of $K_{\mathbb{C}}$. Thus, $K_{\mathbb{C}}$ acts on the ring of regular functions in $Ass(\pi)$. In [22] is shown that $\mathcal{N}$ is the union of finitely many $K_{\mathbb{C}}-$orbits. Hence, $Ass(\pi)$ enjoys the same property. Meanwhile is needed we recall properties of $\mathcal{N}$.

We now show:

(2.1)*Let π be a Harish-Chandra module for G and let L denote a closed connected subgroup of K which acts transitively on the unit sphere of $\mathfrak{s}$. Then, π restricted to L is an admissible representation.*

For this, we recall several important facts, let b denote the killing form on $\mathfrak{g}$, we use the same notation for the quadratic form associated to b. Kostant-Rallis in [18] has shown:

(2.2) $\mathcal{N}$ is the zero set in $\mathfrak{s}_{\mathbb{C}}$ for the quadratic form attached to b. Moreover, the ideal of the variety $\mathcal{N}$ is the ideal spanned by the restriction to $\mathfrak{s}_{\mathbb{C}}$ of the quadratic form associated to b.

Other important fact we need, which is due to Huang-Vogan, is:

(2.3) Let L be a closed connected subgroup of K. Then, a Harish-Chandra module for G restricted to L is admissible if and only if the representation of L in the algebra of regular functions on the associated variety of the module is admissible.

The following result is useful:

(2.4) Let L be a connected complex reductive group, V a finite dimensional rational representation of L and $\mathcal{C}$ an irreducible and $L-$invariant cone in V. Then, the left regular representation of L in the ring of regular functions $\mathbb{C}[\mathcal{C}]$ of $\mathcal{C}$ is admissible if and only if $\mathbb{C}[\mathcal{C}]^L = \mathbb{C}$.

For a proof c.f. [20]. Actually, (2.4) is a consequence of the following result in algebraic geometry

(2.5) Via left multiplication each isotypic component for the representation of L in $\mathbb{C}[\mathcal{C}]$ becomes a module over the ring $\mathbb{C}[\mathcal{C}]^L$. Then, for an irreducible cone $\mathcal{C}$, each isotypic component is a finitely generated module over the ring $\mathbb{C}[\mathcal{C}]^L$. For a proof [20].

We now show (2.1) Let L be a subgroup of K which acts transitively on the unit sphere of $\mathfrak{s}$. Then $S[\mathfrak{s}_{\mathbb{C}}]^L = \mathbb{C}[b]$. In fact, if p is an invariant polynomial for L, the assumption on L implies p takes on a constant value on each sphere centered at the origin. Since, G is a rank one group, K acts transitively on each of such spheres. Thus, p is $K-$invariant polynomial. The result of Huang-Vogan (2.3) and (2.4) yields that any irreducible Harish-Chandra module for G is admissible when restricted to L and we have shown (2.1) when the associated variety is $\mathcal{N}$. To conclude the proof of direct implication in theorem 1 we consider the case $Ass(\pi)$ is a proper subvariety of $\mathcal{N}$. Since, the ring of regular functions $C[Ass(\pi)]$ of $Ass(\pi)$ is an $L-$equivariant quotient of $C[\mathcal{N}]$ and we have shown that $C[\mathcal{N}]$ is an admissible representation of L, the direct implication in theorem 1 follows from (2.3).

Remark: Another proof of the direct implication in theorem 1 is as follows, let $G = KAN$ denote an Iwasawa decomposition for G. As usual let M denote the centralizer of A in K. Because of the Casselman embedding theorem (π, V) is a subrepresentation of a minimal principal series $Ind_{MAN}^{G}(\sigma \otimes \nu)$. Hence, as $K-module$, (π, V) is a subrepresentation of $Ind_M^K(\sigma)$. Our hypothesis implies that L acts transitively on K/M. Therefore, the theorem of Mackey on restriction of an induced representation yields that the Harish-Chandra module of (π, V) has an admissible restriction to L.

Next we show:

(2.6)*Let π be an irreducible Harish-Chandra module for G whose associated variety is $\mathcal{N}$ and which has admissible restriction to L. Then, L acts transitively on the unit sphere of $\mathfrak{s}$.*

Indeed, owing to the theorem of Huang Vogan (2.3) the invariants of L in the ring of regular functions on $\mathcal{N}$ is the subspace of constant functions.

Assume L does not act transitively on the unit sphere of $\mathfrak{s}$. Thus, there exists an $L-$invariant polynomial on $\mathfrak{s}$ which separates two orbits of L in the unit sphere, however, because of our hypothesis on L and π this polynomial is the constant polynomial. Hence, L acts transitively on the unit sphere of $\mathfrak{s}$. This shows (2.4) and we have conclude the proof of theorem 1.

Remark: We like to point out, that if G is a semisimple connected Lie group so that some subgroup of K acts transitively on the unit sphere of $\mathfrak{s}$, then G has real rank equal to one.

For sake of completeness, for each $\mathfrak{g}$ we choose a Lie group G and we list the closed connected subgroups L of K which acts transitively on the unit sphere of $\mathfrak{s}$. For this, we denote by $Spin_s(2k+1)$ the image of $Spin(2k+1)$ by its spin representation.

- $SU(1,q) \supset K = U(q)$. Here $\mathfrak{s} \equiv \mathbb{R}^{2q}$. Then, L is one of: $SU(q); U(q)$ Besides, for q even $Sp(\frac{q}{2})$.
- $SO_0(1, 2q+1) \supset K = SO(2q+1)$. For $2q+1 \neq 7, L = K$. For $2q+1 = 7, L = SO(7)$ or L is the image of the seven dimension irreducible representation for G_2.

- $SO_0(1,2q) \supset K = SO(2q)$. Then L is one of: $K; SU(q); U(q)$; $Spin_s(7) \subset SO(8); Spin_s(9) \subset SO(16)$; Besides, for q even $S \times Sp(\frac{q}{2}) \subset SO(2q)$, S closed subgroup of $Sp(1)$.
- $Sp(1,q) \supset K = Sp(1) \times Sp(q)$. $\mathfrak{s} \equiv \mathbb{R}^{4q}$. $L = S \times Sp(q)$, S closed subgroup of $Sp(1)$.
- $F_{4(-20)} \supset K = Spin(9)$. $L = K$.

Theorem 1 together with the list of subgroups of K which acts transitively on the unit sphere of $\mathfrak{s}$ leads us to:

Corollary 3. *Let π be a Harish-Chandra module for G and L denote a proper subgroup of K. Then for $\mathfrak{g}, L$ listed bellow, $\pi_{|L}$ is an admissible representation of L.*

- $\mathfrak{su}(1,q)$, $L := SU(q), Sp(q/2)$
- $\mathfrak{so}(1,7)$, $L := G_2$
- $\mathfrak{so}(1,2q)$, $L := SU(q), U(q), Sp(q/2), S \times Sp(q/2)$, $Spin_s(7) \subset SO(8), Spin_s(9) \subset SO(16)$.
- $\mathfrak{sp}(1,q)$, $L := S \times Sp(q)$.

Since for $\mathfrak{so}(1,n), n \geq 3$, every non trivial $K_{\mathbb{C}}-$orbit in $\mathcal{N}$ is equal to $\mathcal{N}$ we have,

Corollary 4. *Let π be an infinite dimensional Harish-Chandra module for $\mathfrak{so}(1,n), n \geq 3$ and L a subgroup of K so that $\pi_{|L}$ is an admissible representation of L. Then,*

- *For n odd, $L = K$, or $n = 7$ and also $L = G_2$.*
- *For $n = 2q$, $L = K$ or $L = SU(q), U(q), S \times Sp(q/2)$, $Spin_s(7) \subset SO(8), Spin_s(9) \subset SO(16)$.*

3. Proof of theorem 2

In order to show theorem 2 we need to compute the associated variety of some Harish-Chandra modules for G. Whenever $\mathfrak{g} = \mathfrak{so}(1,n)$, the nilpotent cone is equal to an orbit of $K_{\mathbb{C}}$ union the origin and since, we dealt with representations whose associated variety is proper we are left to consider the algebras $\mathfrak{su}(1,q), \mathfrak{sp}(1,q), \mathfrak{f}_{4(-20)}$. For these three cases, rank of K is equal to rank of G. We fix a maximal torus T of K. Let $\Phi(\mathfrak{g},\mathfrak{t})$ denotes the root system for the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Let θ denote the Cartan involution of $\mathfrak{g}$ associated to the Cartan decomposition $\mathfrak{g} = \mathfrak{k}+\mathfrak{s}$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ denote a $\theta-$stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Then, for each one dimensional representation λ of $\mathfrak{l}$, Vogan and Zuckerman has constructed a Harish-Chandra module $A_{\mathfrak{q}}(\lambda)$ which is irreducible and nonzero whenever λ is in the good range with respect to $\mathfrak{u}$. Let $\mathfrak{u}^-$

denote the opposite algebra to $\mathfrak{u}$. For λ in the weak range with respect to $\mathfrak{u}$, in [13] we find a proof that

$$Ass(A_{\mathfrak{q}}(\lambda)) = Ad(K_{\mathbb{C}})(\mathfrak{u}^{-} \cap \mathfrak{s}_{\mathbb{C}}) \qquad (3.1).$$

Thus, $Ass(A_{\mathfrak{q}}(\lambda))$ is an irreducible $K_{\mathbb{C}}-$invariant subcone of $\mathcal{N}$. Moreover, whenever the parameter λ varies among the good range parameters, the associated variety of $A_{\mathfrak{q}}(\lambda)$ depends only on the parabolic subalgebra $\mathfrak{q}$. Thus, we may write $Ass(A_{\mathfrak{q}})$ for $Ass(A_{\mathfrak{q}}(\lambda))$.

Lemma 1. *If $Ass(\pi)$ is a proper subvariety of $\mathcal{N}$, then for a convenient data $\mathfrak{q}$ we have that $Ass(\pi) = Ass(A_{\mathfrak{q}})$.*

To show the lemma we do a case by case analysis. We recall there exists an orthogonal basis $\epsilon, \delta_1, \dots, \delta_q$ of $i\mathfrak{t}^\star$ so that

$$\Phi(\mathfrak{su}(1,q), \mathfrak{t}) = \{\pm(\epsilon - \delta_j), (\delta_r - \delta_s), r \neq s\}$$

$$\Phi(\mathfrak{sp}(1,q), \mathfrak{t}) = \{\pm(\epsilon \pm \delta_j), \pm(\delta_r \pm \delta_s), \pm 2\epsilon, \pm 2\delta_j, r \neq s, \}.$$

An orthogonal basis $\delta_1, \dots, \delta_4$ of $i\mathfrak{t}^\star$ so that

$$\Phi(\mathfrak{f}_{4(-20)}, \mathfrak{t}) = \{\pm\delta_j, (\pm\delta_r \pm \delta_s), \pm\frac{1}{2}(\delta_1 \pm \delta_2 \pm \delta_3 \pm \delta_4), r < s\}.$$

To follow, we give a system of positive roots Δ for $\Phi(\mathfrak{k}, \mathfrak{t})$ and we list the systems of positive roots for $\Phi(\mathfrak{g}, \mathfrak{t})$ which contains Δ.

• $\mathfrak{su}(1,q)$, $\Delta := \{\delta_r - \delta_s, r < s\}$, Ψ_a is associated to the lexicographic order

$$\{\delta_1 > \cdots > \delta_a > \epsilon > \delta_{a+1} > \cdots > \delta_q\}, \;\; 0 \leq a \leq q.$$

Ψ_0, Ψ_q are the holomorphic systems which contains Δ.

• $\mathfrak{sp}(1,q)$, $\Delta := \{2\epsilon, \delta_r \pm \delta_s, 2\delta_j, r < s\}$, Ψ_a associated to the lexicographic order

$$\{\delta_1 > \cdots > \delta_a > \epsilon > \delta_{a+1} > \cdots > \delta_q\}, \;\; 0 \leq a \leq q.$$

Ψ_0 is the quaternionic (small) system in the sense of Gross-Wallach [9].

• $\mathfrak{f}_{4(-20)}$, $\Delta := \{\delta_j, \delta_r \pm \delta_s, r < s, \}$. In this case, there are three systems of positive roots for $\Phi(\mathfrak{f}_{4(-20)}, \mathfrak{t})$ containing Δ.

$$\Psi_1 := \{\delta_j, (\delta_r \pm \delta_s), r < s, \frac{1}{2}(\delta_1 \pm \delta_2 \pm \delta_3 \pm \delta_4)\}.$$

whose simple roots are

$$\delta_2 - \delta_3, \delta_3 - \delta_4, \delta_4, \frac{1}{2}(\delta_1 - \delta_2 - \delta_3 - \delta_4)$$

Let $\beta := \frac{1}{2}(\delta_1 - \delta_2 - \delta_3 - \delta_4), \beta' := \frac{1}{2}(\delta_1 - \delta_2 - \delta_3 + \delta_4)$, as usual, S_γ is the reflexion about the root γ. The other systems are $\Psi_2 := S_\beta \Psi_1, \Psi_3 := S_{\beta'} \Psi_2$.

In order to complete the proof of lemma 1 we define convenient parabolic subalgebras and compute the associated variety $Ass(A_{\mathfrak{q}})$.

For either $\mathfrak{su}(1,q)$ or $\mathfrak{sp}(1,q)$ we set $\mathfrak{b}_a$ to denote the Borel subalgebra determinate by the system of positive roots Ψ_a. Formula (3.1) yields,

• $\mathfrak{su}(1,q)$. Let $\mathfrak{s}^+ := \mathfrak{b}_0 \cap \mathfrak{s}_{\mathbb{C}}$, and $\mathfrak{s}^- := \mathfrak{b}_q \cap \mathfrak{s}_{\mathbb{C}}$. Then

$$Ass(A_{\mathfrak{b}_0}) = \mathfrak{s}^-, Ass(A_{\mathfrak{b}_q}) = \mathfrak{s}^+, Ass(A_{\mathfrak{b}_a}) = \mathcal{N}, 1 \leq a < q.$$

Both, $\mathfrak{s}^{\pm}$ are $Ad(U(q))$ invariant irreducible linear subspaces of dimension q. In [3] is shown that $\mathfrak{s}^{\pm}$ are the unique proper subvarieties of $\mathcal{N}$ which are associated varieties of a Harish-Chandra module.

• $\mathfrak{sp}(1,q)$. For $1 \leq a \leq q, Ass(A_{\mathfrak{b}_a}) = \mathcal{N}$.

$$Ass(A_{\mathfrak{b}_0}) = \{v + t[Y_{2\epsilon}, v], v \in \sum_j \mathbb{C}Y_{-\epsilon\pm\delta_j}, t \in \mathbb{C}\} \qquad (3.1).$$

Here, Y_α is a nonzero root vector for the root α. Hence, $dim Ass(A_{\mathfrak{b}_0}) = 2q+1$. The last equality follows from

$$Ass(A_{\mathfrak{b}_0}) = Ad(K_{\mathbb{C}})(\sum_j \mathbb{C}Y_{-\epsilon\pm\delta_j}) = Ad(Sp(1))(\sum_j \mathbb{C}Y_{-\epsilon\pm\delta_j}) \quad (3.2).$$

Note that $\sum_j \mathbb{C}Y_{-\epsilon\pm\delta_j}$ is invariant under the action of $Texp(\mathbb{C}Y_{-2\epsilon}) \times Sp(q)$. The Bruhat decomposition for $Sp(1)$ yields the equality (3.1). We point out that the action of $Sp(q)$ on the linear subspace $\sum_j \mathbb{C}Y_{-\epsilon\pm\delta_j}$ is equivalent to usual one in $\mathbb{C}^{2q}$. In [3] is shown that the variety (3.1) is the unique proper subvariety of $\mathcal{N}$ equal to an associated variety.

• $\mathfrak{f}_{4(-20)}$. For the system Ψ_2 the long simple roots are compact and the short simple roots are noncompact. $-\beta$ is the short simple root whose node in the Dynkin diagram is an end point. Let $\mathfrak{q}_4 = \mathfrak{l}_4 + \mathfrak{u}_4$ denote the parabolic subalgebra associated to the fundamental weight corresponding to $-\beta$. Then, $\mathfrak{l}_4 \cap \mathfrak{f}_{4(-20)} \equiv \mathfrak{so}(6,1), dim\, \mathfrak{u}_4 \cap \mathfrak{k}_{\mathbb{C}} = 10, dim\, \mathfrak{u}_4 \cap \mathfrak{s}_{\mathbb{C}} = 5$ and $dim Ad(K)(\mathfrak{u}_4^- \cap \mathfrak{s}_{\mathbb{C}}) = 11$. Since in [3] there is a proof that in $\mathcal{N}$ there is only one $K_{\mathbb{C}}-$orbit of dimension 11 we conclude the proof of lemma 1. A direct computation shows that the Lie algebra of $Spin(9) \cap SO(6,1)$ is the usual immersion the algebra $\mathfrak{su}(4)$ in $\mathfrak{so}(8)$. □

We now show theorem 2.

• $\mathfrak{su}(1,q)$

In [3] is shown that the proper subvarieties of $\mathcal{N}$ which are equal to the associated varieties of some irreducible Harish-Chandra module are precisely the subvarieties $\mathfrak{s}^{\pm}$. Since, the action of $U(q)$ in $\mathfrak{s}^+$ is equivalent to the usual action of $U(q)$ in $\mathbb{C}^q$ the statement in theorem 2 about $\mathfrak{su}(1,q)$ follows from the theorem of Huang-Vogan (2.3) coupled with (3.1) Another proof is given in Kobayashi [12].

The classification of the close reductive subgroups L of $GL(q,\mathbb{C})$ such that $S[\mathbb{C}^q]^L = \mathbb{C}$ has not been accomplished, yet. A pair (L,V) where L is complex reductive groups and V finite dimensional representation of L so that L has an open orbit in V is called a prehomogeneous spaces. For a prehomogeneous space (L,V) we always have $\mathbb{C}[V]^L = \mathbb{C}$. Whenever L is a semisimple complex Lie subgroup of $GL(V)$ so that $\mathbb{C}[V]^L = \mathbb{C}$ then (L,V) is a prehomogeneous space, for a proof c.f [16]. Substantial progress on the problem of classifying prehomogeneous spaces has been accomplished by Kac, Sato, Kimura, Gerald Schwarz, Gyoja. For a reference c.f. [16] and references therein. The semisimple irreducible subgroups L of SL_q such that $(L,\mathbb{C}^q)$ is a prehomogeneous space has been classified by Kac, Sato-Kimura, Littelman, they are:

$(SL_n \boxtimes SL_m, \mathbb{C}^n \boxtimes \mathbb{C}^m), \frac{m}{2} \geq n \geq 1$; $(SL_{2m+1}, \Lambda^2(\mathbb{C}^{2m+1}))$; $(SL_{2n+1} \boxtimes SL_2, \Lambda^2(\mathbb{C}^{2n+1}) \boxtimes \mathbb{C}^2)$; $(Sp(n) \boxtimes SL_{2m+1}, \mathbb{C}^{2n} \boxtimes \mathbb{C}^{2m+1}), n \geq 2m+1$; $(Spin(10), \mathbb{C}^{16})$ half spin rep in $\mathbb{C}^{16}$; $(Sp(2) \boxtimes SL_m), m \geq 5$; $(H \boxtimes SL_m, \mathbb{C}^n \boxtimes \mathbb{C}^m), m \geq n \geq 1, H$ semisimple and which acts irreducible on $\mathbb{C}^n$.

Example of pairs $(L,\mathbb{C}^q)$ so that $\mathbb{C}[\mathbb{C}^q]^L = \mathbb{C}$ are constructed as follows: Let $L_j \subset GL(\mathbb{C}^{n_j}), j = 1, \cdots, k$ be subgroups so that $\mathbb{C}[\mathbb{C}^{n_j}]^{L_j} = \mathbb{C}$. Let $L = L_1 \times \cdots L_k$ act on $\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}$ in the obvious way. Since,

$$\mathbb{C}[\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}]^L = \mathbb{C}[\mathbb{C}^{n_1}]^{L_1} \otimes \cdots \otimes \mathbb{C}[\mathbb{C}^{n_k}]^{L_k}$$

we have that $\mathbb{C}[\mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_k}]^L = \mathbb{C}$. As consequence, a holomorphic discrete series for $SU(1,2n)$ has an admissible restriction to $S(U(n) \times U(n)), n \geq 3$.

• $\mathfrak{sp}(1,q)$ In [3], we find a proof that the orbits of $K_{\mathbb{C}}$ in $\mathcal{N}$ are: one dense orbit; one orbit of dimension $2q+1$; the trivial orbit. Thus, (3.1), (3.2) imply $Ass(A_{\mathfrak{b}_0})$ is the unique associated variety which is a proper subvariety of $\mathcal{N}$. To begin with we analyze the structure of an invariant regular functions on $Ass(A_{\mathfrak{b}_0})$. Owing to (3.1), a regular function p on $Ass(A_{\mathfrak{b}_0})$ may be written $p = \sum_k c_k(v)t^k$, where c_k are polynomials in $v \in \sum_j \mathbb{C}Y_{-\epsilon \pm \delta_j}$.

We set $T_1 := Sp(1) \cap T$. Hence, T_1 is a one dimensional torus. For $s \in T_1$, we have $Ad(s)(v + t[Y_{2\epsilon}, v]) = s^{-\epsilon}v + (s^{\epsilon}t[Y_{2\epsilon}, v])$. We consider a closed connected subgroup $B \subset Sp(q)$. Therefore, p is invariant under the action of $T_1 \times B$ if and only if for every k, c_k is a homogeneous polynomial of degree k and invariant under B. Thus, we conclude

$$\mathbb{C}[Ass(A_{\mathfrak{b}_0})]^{T_1 \times B} = \mathbb{C} \text{ if and only if } \mathbb{C}[\mathbb{C}^{2q}]^B = \mathbb{C}.$$

Similarly,

$$\mathbb{C}[Ass(A_{\mathfrak{b}_0})]^B = \mathbb{C} \text{ if and only if } \mathbb{C}[\mathbb{C}^{2q}]^B = \mathbb{C}.$$

Lemma 2. *Let B be a closed connected subgroup of $Sp(q)$. Then, $S[\mathbb{C}^{2q}]^{B_{\mathbb{C}}} = \mathbb{C}$ if and only if B is conjugated to a subgroup $Sp(q_1) \times \cdots \times Sp(q_r)$ with $q_1 + \cdots + q_r = q$.*

Proof: Since $Sp(\mathbb{C}^{2q})$ has an open orbit in $\mathbb{C}^{2q}$ the converse implication follows. For the direct implication, the hypothesis on B implies there exists $B-$irreducible linear subspaces $V_1, \cdots, V_r$ of $\mathbb{C}^{2q}$ so that

$$\mathbb{C}^{2q} = V_1 \oplus \cdots \oplus V_r.$$

Let ω denote a nondegenerate skew-symmetric form whose group of isometries is $Sp(q, \mathbb{C})$. We claim that no V_j is an isotropic subspace for ω. Otherwise, since ω is nondegenerate, $V_j \oplus V_j^\star$ would be an $B-$submodule and hence the evaluation map would give rise to an $B-$invariant element of the symmetric algebra of $\mathbb{C}^{2q}$ of positive degree. Let p_j denote the projection onto V_j along the sum of the subspaces $V_k, k \neq j$. Thus, $p_j(B)$ is an irreducible subgroup of $Sp(V_j, \omega_|)$ so that $S[V_j]^{p_j(B)} = \mathbb{C}$. From the work of [19] we may conclude $p_j(B_{\mathbb{C}}) = Sp(V_j, \omega)$. Thus, B is isomorphic to a product of symplectic groups. This concludes the proof of lemma 2 □

Lemma 2 together with the Theorem of Huang Vogan, let us conclude

(3.3) A Harish-Chandra module for $\mathfrak{sp}(1, q)$ whose associated variety is of dimension $2q + 1$ has an admissible restriction to a subgroup $L = T_1 \times B$, or to a subgroup $L = B$ if and only if B is conjugated to $Sp(q_1) \times ... \times Sp(q_r), \sum q_j = q$.

Since any two torus in $Sp(1)$ are conjugated, we obtain part of the converse implication in theorem 2 concerning to $Sp(1, q)$.

To conclude of the proof of the converse implication for $\mathfrak{sp}(1, q)$ we now show that if $Ass(\pi)$ has dimension $2q + 1$, then π restricted to $Sp(1)$ is an admissible representation.

For this we recall a result of Kostant on the minimal nonzero nilpotent orbit in $\mathfrak{s}_{\mathbb{C}}$. We state the result in a way that is valid for either $\mathfrak{sp}(1, q)$ or $\mathfrak{f}_{4(-20)}$. We fix a system of positive roots Ψ for $\Phi(\mathfrak{g}, \mathfrak{t})$.

(3.4) The minimal nonzero nilpotent orbit in $\mathfrak{s}_{\mathbb{C}}$ is equal to the orbit of any nonzero root vector. The closure of the minimal nilpotent orbit is equal to the union of the orbit with the zero orbit. Let β_M denote the maximal noncompact root in Ψ. Hence, β_M is the highest weight for the irreducible $K-$ module $\mathfrak{s}_{\mathbb{C}}$. For each non negative integer k let $V_{k\beta_M}$ denote the irreducible representation of K whose highest weight is $k\beta_M$. Then, the $K_{\mathbb{C}}-$modules structure on the ring of regular functions on the minimal nilpotent orbit is equivalent to $\oplus_{k \geq 0} V^\star_{k\beta_M}$. For a proof c.f. [10].

Back to $\mathfrak{sp}(1,q)$! Because of (3.4) the $Sp(1)\times Sp(q)-$decomposition of the ring of regular functions on the minimal nilpotent orbit is

$$\oplus_{k\geq 0} V^{\star}_{k\epsilon}\boxtimes V^{\star}_{k\delta_1}.$$

Hence, $\mathbb{C}[Ass(\mathfrak{b}_0)]$ is an admissible module over either $Sp(1)$ or $Sp(q)$. Thus, the result of Huang-Vogan let us conclude that any Harish-Chandra module whose associated variety is of dimension $2q+1$ has an admissible restriction to $Sp(1)$. This concludes the proof of the converse statement for case of $\mathfrak{sp}(1,q)$.

Remark: Since $V_{2k\epsilon}$ contains a a nonzero vector fix by T_1 we obtain that π has no admissible restriction to a proper closed connected subgroup of $Sp(1)$.

For the direct implication in theorem 2 which concerns the algebra $\mathfrak{sp}(1,q)$, let L be a subgroup of $Sp(1)\times Sp(q)$ so that some Harish-Chandra module π whose associated variety is of dimension $2q+1$ has an admissible restriction to L. After conjugation, and projecting onto the factors of K, we may assume L is a contained in one of:

$$Sp(q), T_1\times B, Sp(1)\times B.$$

In [6], [13] we find a proof of

(3.5) For a Harish-Chandra module for G, admissible restriction to L implies admissible restriction to any subgroup of K which contains L.

Hence, if L_1 denotes any of the three subgroups listed above, we have $\mathbb{C}[Ass(\pi)]^{L_1}=\mathbb{C}$. Thus, lemma 2 yields: for the first case $L=Sp(q_1)\times\cdots\times Sp(q_r)$; for the second possibility L is the product of a one dimensional torus times $L\cap Sp(q)$, after conjugation by un element of $Sp(q)$ we may assume the torus is the graph $(t,\phi(t))\, t\in T_1$ where $\phi: T_1\to T\cap Sp(q)$ is a rational morphism, then $Ad(L)$ leaves invariant the subspace $\sum_j \mathbb{C}Y_{-\epsilon\pm\delta_j}$, hence $Ad(L)\cap Sp(q)=Sp(q_1)\times\cdots\times Sp(q_r)$ and L is isomorphic to $T_1\times Sp(q_1)\times\cdots\times Sp(q_r)$; for the third case, L contains an ideal L_2 of the type $(a,\phi(a)), a\in Sp(1)$, and $\phi: Sp(1)\to Sp(q)$ a morphism. If B does not contain an $Sp(1)-$factor, then $L=Sp(1)\times B$. If the projection of L_2 into B is nontrivial, then the center of L is contained in $Sp(q)$ and we have to analyze the invariants for the $sp(1)$ factor.

4. Proof of Theorem 3 and Theorem 4

We now show that there is only one connected simple Lie group $F_{4(-20)}$ whose Lie algebra is $\mathfrak{f}_{4(-20)}$. Indeed, for $\mathfrak{f}_4$ the weight lattice agrees with the root lattice. Thus, the center of the complex simply connected Lie group of Lie algebra $\mathfrak{f}_4$ is trivial. Hence, up to isomorphism, there is only one complex simple Lie group whose Lie algebra is $\mathfrak{f}_4$. In [8] page 348 we find a proof that the analytic subgroup of $F_4(\mathbb{C})$ corresponding to $\mathfrak{f}_{4(-20)}$ is simply connected. Thus, there is up isomorphism, one connected Lie group $F_{4(-20)}$ with Lie algebra $\mathfrak{f}_{4(-20)}$. For $F_{4(-20)}$, $K \equiv Spin(9)$. The Cartan decomposition is $\mathfrak{f}_{(4(-20)} = \mathfrak{so}(9) + \mathbb{R}^{16}$ and the representation of $\mathfrak{so}(9)$ in $\mathfrak{s}$ is the spin representation. It follows from the classification of the subgroups of an orthogonal group which acts transitively on a unit sphere that if L is a subgroup of $Spin(9)$ acting transitively on the unit sphere of $\mathbb{R}^{16}$, then $L = Spin(16)$. Therefore, theorem 1 yields that for π a Harish-Chandra module for $F_{4(-20)}$ whose associated variety is $\mathcal{N}$, then π has no admissible restriction to any proper subgroup of $Spin(9)$. In [3] we find a proof there is a unique proper $Spin(9)_{\mathbb{C}}$ orbit on $\mathcal{N}$, and is of dimension 11. It is the minimal nonzero nilpotent orbit of $K_{\mathbb{C}}$ in $\mathfrak{s}_{\mathbb{C}}$. Let $\beta_M = \frac{1}{2}(\delta_1 + \delta_2 + \delta_3 + \delta_4)$. Thus, β_M is the highest weight of the spin representation of $Spin(9)$. For a dominant weight γ of $Spin(9)$ let V_γ denote the irreducible representation of highest weight γ. According to the theorem of Kostant (3.4), the left regular representation of $K_{\mathbb{C}}$ in the ring of regular functions in the closure of the minimal nilpotent orbit is equivalent to the direct sum $\sum_{k\geq 1} V^{\star}_{k\beta_M}$. Thus, the theorem of Huang-Vogan yields

(4.1) Let π be a Harish-Chandra module for $\mathfrak{f}_{4(-20)}$ whose associated variety is a proper subvariety of the nilpotent cone. Let L be a compact connected subgroup of $Spin(9)$. Then, $\pi_{|L}$ is admissible if and only if

$$V^L_{k\beta_M} = \{0\} \text{ for every } k \geq 1.$$

From now on, when we refer to $Spin(m), m = 5, 6, 7, 8, 9$ as a subgroup of $Spin(9)$ we are thinking of the immersion of $Spin(m)$ as a left upper block.

To follow we show the converse statement in theorem 3, which is a consequence of (3.5) and

(4.2)*Let π be a Harish-Chandra module for $\mathfrak{f}_{4(-20)}$ whose associated variety is a proper subvariety of $\mathcal{N}$. Then, π restricted to $Spin(6)$ is admissible.*

For this, we successively apply the theorem of Murnaghan, to

$Spin(9) \supset Spin(8) \supset Spin(7) \supset Spin(6)$ and $V_{k\beta_M}$.

Here, $k\beta_M = (\frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2})$. Thus, the set of highest weight of the irreducible $Spin(6)-$factors of $V_{k\beta_M}$ is

$$\{(\frac{k}{2}, a, b), \frac{k}{2} \geq a \geq |b|\}.$$

Therefore, for positive k the trivial representation of $Spin(6)$ does not occur in $V_{k\beta_M}$. Thus, the theorem of Huang-Vogan yields $\pi_{|Spin(6)}$ is admissible. If we go one step further to $Spin(5)$ we get that the trivial representation of $Spin(5)$ occurs in $V_{4k\beta_M}, k > 0$ which let us conclude.

(4.4)*Let* π *be a Harish-Chandra module for* $\mathfrak{f}_{4(-20)}$ *so that its associated variety is proper. Then* π *restricted to* $Spin(4) \times Spin(5)$ *is not admissible. Hence, for any subgroup L of* $Spin(9)$ *conjugated to a subgroup of* $Spin(5)$, π *restricted to L is not admissible as it follows from (3.5). In particular, for* $n = 2, 3, 4, 5$ π *restricted to* $Spin(n)$ *is not admissible.*

Here, as usual, $Spin(n) \times Spin(m), n+m = 9$ is the subgroup $Spin(n)$ times de image of $Spin(m)$ as a lower right block.
For other proof of (4.4), we verify

$$V_{k\beta_M}^{Spin(4)\times Spin(5)} \neq \{0\} \text{ for integers } k = 4s$$

This follows from the Theorem of Cartan-Helgason (theorem 8.49 in [11]). In fact, since, for the symmetric pair $(SO(9), SO(4) \times SO(5))$ a Cartan subspace is a Cartan subalgebra for $\mathfrak{so}(9)$, for the time being, we may fix $\mathfrak{t}$ equal to a Cartan subspace of the pair $(SO(9), SO(4) \times SO(5))$. The Cartan-Helgason theorem gives: $V_{k\beta_M}^{Spin(4)\times Spin(5)} \neq \{0\}$ is equivalent to $\frac{k(\beta_M,\alpha)}{2(\alpha,\alpha)}$ is a nonnegative integer for every positive root α. This is so, for $k = 4s$. Hence, π restricted to $Spin(4) \times Spin(5)$ is not an admissible representation.

Next, we study restriction to maximal subgroups of $Spin(9)$.
In [7], Dynkin has computed the maximal connected subgroups of $Spin(9)$. They are:

$$Spin(n) \times Spin(m), n + m = 9; \quad Spin(3) \boxtimes Spin(3);$$

and the image of $SU(2) = Spin(3)$ into $Spin(9)$ whose projection into $SO(9)$ is equal to the image of the nine dimensional irreducible representation of $SU(2)$.

We now show that π *restricted to the image of the irreducible representation* $(SU(2), \mathbb{R}^9)$ *is not admissible.*
We rely on a result of Birkes [1].

(4.5) Let L be a a complex connected reductive subgroup of $GL(V)$ and v a vector in V whose isotropy subgroup contains a maximal torus of L. Then, the orbit Lv is closed. In, particular, the orbit of a zero weight vector is closed.

To begin with, we construct the explicit immersion of $SU(2)$ in $Spin(9)$ as a maximal subgroup. We fix once and for all a Chevalley basis for $\mathfrak{f}_{4(-20)}$

$$H_1, H_2, H_3, H_4, Y_\alpha, \alpha \in \Phi(\mathfrak{f}_{4(-20)}, \mathfrak{t}).$$

The structure constants are as in [4]. Conjugation with respect to $\mathfrak{f}_{4(-20)}$ of Y_γ is $Y_{-\gamma}(resp. -Y_{-\gamma})$ for γ noncompact (resp. compact). We set $\delta_i(H_j) = \delta_{ij}$. In order to simplify the notation we write $Y_{++-+} := Y_{\frac{1}{2}(\delta_1+\delta_2-\delta_3+\delta_4)}$ and so on. The minimal nilpotent orbit in $\mathfrak{s}_{\mathbb{C}}$ (3.4) is the $Spin(9)_C-$orbit of Y_{++++} and hence for any noncompact root γ, any root vector Y_γ, lies in the minimal nilpotent orbit. Let

$$H = 8H_1 + 6H_2 + 4H_3 + 2H_4,\ X_+ = Y_{\delta_1-\delta_2} + Y_{\delta_2-\delta_3} + Y_{\delta_3-\delta_4} + Y_{\delta_4}.$$

We denote by X_- the conjugate of X_+. Then, H, X_+, X_- span a principal sl_2-subalgebra sl_{pr} of $\mathfrak{so}(9,\mathbb{C})$ and the ensuing representation of sl_{pr} in $\mathbb{C}^9$ is orthogonal and irreducible. It readily follows that SL_{pr} is the complexification of a copy of $SU(2)$ which, in turn, is a maximal subgroup of $Spin(9)$. Under $Ad(SL_{pr})$ the space $\mathfrak{s}_{\mathbb{C}}$ decomposes as the sum an eleven dimensional irreducible representation of highest weight vector Y_{++++} plus a five dimensional irreducible representation of highest weight vector $16Y_{++--} + Y_{+-++}$. Actually, from a simple calculation if follows that a highest weight vector for the five dimensional subrepresentation is of the form $aY_{++--} + Y_{+-++}$ for a convenient nonzero a. The table in [4] yields $a = 16$. The subspace of vector of weight zero for $ad(H)$ is spanned by the root vectors Y_{+--+}, Y_{-++-}. Both vectors belong to the minimal nilpotent orbit. Therefore (4.5) implies that the orbit $Ad(SL_{pr})Y_{+--+}$ is closed and hence can be separated of the zero orbit by a regular function invariant under $Ad(SL_{pr})$. Thus, (2.4)and (2.3) imply π restricted to $SL_{pr} \cap Spin(9) =$ "$SU(2)$" is not an admissible representation.

To follow we show π restricted to $SU(2)\boxtimes SU(2) \equiv Spin(3)\boxtimes Spin(3)$ is not admissible.

Let H, X, Y be a basis of $sl_2 := \mathfrak{sl}(2,\mathbb{C})$ so that $[H,X] = 2X, [H,Y] = -2Y, [X,Y] = H$. The irreducible representation of sl_2 in $\mathbb{C}^3$ is orthogonal. We fix a basis of weight vectors v_2, v_0, v_{-2} and quadratic form q on $\mathbb{C}^3$ invariant under the action of sl_2. Thus,

$$q(v_2, v_2) = q(v_{-2}, v_{-2}) = 0,\ q(v_0, v_0) = -q(v_2, v_{-2}) = 1.$$

This form is invariant under the action of $SU(2)$. Hence,there exists an $SU(2)-$invariant real vector subspace V of $\mathbb{C}^3$ so that q is positive definite in V and $\mathbb{C}^3 = V \otimes_{\mathbb{R}} \mathbb{C}$. We consider the quadratic form $q_1 := q \otimes q$. The maximal subgroup $SU(2) \boxtimes SU(2)$ is the image of $SU(2) \times SU(2)$ in $Spin(V \otimes V, q_1)$. Dynkin has shown that this image is a maximal subgroup. The matrix of q_1 in the ordered basis $\mathcal{B}$

$v_2 \otimes v_2, v_2 \otimes v_0, v_2 \otimes v_{-2}, v_0 \otimes v_2, v_0 \otimes v_0, v_0 \otimes v_{-2}, v_{-2} \otimes v_2, v_{-2} \otimes v_0, v_{-2} \otimes v_{-2}$

is antidiagonal and every entry in the antidiagonal is nonzero. Thus, in the ordered basis $\mathcal{B}$ the diagonal matrices

$$diag(h_1, h_2, h_3, h_4, 0, -h_4, -h_3, -h_2, -h_1)$$

gives a Cartan subalgebra for $spin(9)_{\mathbb{C}}$. Here,

$$\delta_j(diag(h_1, h_2, h_3, h_4, -h_4, -h_3, -h_2, -h_1)) = h_j.$$

On the basis $\mathcal{B}$ the matrix of $H \otimes id$ is equal to

$$diag(2, 2, 2, 0, 0, 0, -2, -2, -2)$$

and for $id \otimes H$ is

$$diag(2, 0, -2, 2, 0, -2, 2, 0, -2).$$

$H \otimes id, id \otimes H$ span a Cartan subalgebra $\mathfrak{u}$ for $sl_2 \otimes sl_2$, we denote the roots by $\pm\phi_j, j = 1, 2$ hence,

$$\phi_1(H \otimes id) = 2, \phi_1(id \otimes H) = 0, \phi_2(H \otimes id) = 0, \phi_2(id \otimes H) = 2.$$

Then, weights of the spin representation for $Spin(9)$ restricted to $\mathfrak{u}$ are

$$\frac{1}{2}(\theta_1\delta_1 + \theta_2\delta_2 + \theta_3\delta_3 + \theta_4\delta_4)_{|\mathfrak{u}} = (\theta_1 + \theta_2 + \theta_3)\frac{\phi_1}{2} + (\theta_1 - \theta_3 + \theta_4)\frac{\phi_2}{2}.$$

Here, $\theta_j \in \{1, -1\}$. Thus, the zero $\mathfrak{u}-$weight subspace has dimension zero.

Since the roots which restrict to ϕ_1 are $\delta_2, \delta_3 + \delta_4, \delta_1 - \delta_4$ and the roots that restrict to ϕ_2 are $\delta_4, \delta_1 - \delta_2, \delta_2 - \delta_3$ the vectors Y_{++++}, Y_{++-+} are dominant with respect to ϕ_1, ϕ_2. Hence, the restriction to $sl_2 \otimes sl_2$ of the spin representation of $\mathfrak{so}(9)$ decomposes as

$$\mathbb{C}^4 \boxtimes \mathbb{C}^2 \oplus \mathbb{C}^2 \boxtimes \mathbb{C}^4$$

Let a, b complex numbers and

$$v_{a,b} := Ad(exp(aY_{-\delta_1-\delta_2} + bY_{-\delta_3-\delta_4}))Y_{++++}$$

Thus, $v_{a,b}$ belongs to the minimal nilpotent orbit. We claim that when $ab \neq 0$, then $Ad(SL_2(\mathbb{C}) \boxtimes SL_2(\mathbb{C}))v_{a,b}$ is closed. To compute $v_{a,b}$ we apply the tables in [4] and obtain

$$v_{a,b} = Y_{++++} - aY_{--++} + bY_{++--} + abY_{----}$$

Hence, the $T_{\mathbb{C}}$ orbit of $v_{a,b}$ is closed as soon as $ab \neq 0$. The Bruhat decomposition of $SL_2(\mathbb{C}) \boxtimes SL_2(\mathbb{C})$ and a computation yields that the orbit is closed and we have verified that π restricted to $SU(2)\boxtimes SU(2)$ is not admissible.

Up to now, we have shown the converse implication for both theorem 3 and theorem 4, as well as corollary 2. The direct implication in theorem 3 and corollary 1 will be completed at the end of this section.

We now show the direct implication in theorem 4. For this we consider table 1 bellow, from which it follows. The maximal connected reductive subgroups H of $F_{4(-20)}$ has been obtained by Komrakov [17]. Up to conjugation and covering in the first column we list the maximal connected reductive subgroups, in the second column we compute a maximal compact subgroup and the third column indicates $H\cap K-$admissibility of Harish-Chandra modules whose associated variety is proper.

H	$H\cap K$	Adm. Res.
$Sp(1,2)\times SU(2)$	$Spin(4)\times Spin(5)$	No
$SU(3)_l\times SU(2,1)_s$	$SU(3)\times U(2)$	No
$SO(8,1)$	$Spin(8)$	Yes
$SL_2(\mathbb{R})\times G_2$	$SO(2)\times G_2$	No
$Spin(9)$	$Spin(9)$	Yes

Table 1

We are left to justify the statement for both $SL_2(\mathbb{R})\times G_2$ and $SU(3)_l\times SU(2,1)_s$. *We first consider* $SL_2(\mathbb{R})\times G_2$. For this we compute an explicit immersion of $SL_2(\mathbb{R})\times G_2$ in $F_{4(-20)}$. The complex Lie algebra $\mathfrak{f}_4$, as an $Spin(8)-$module, decomposes as the sum of the adjoint representation plus the first fundamental representation and the sum of the two spin representations. That is,

$$\mathfrak{f}_4=\mathfrak{so}(8)+(\sum_j \mathbb{C}Y_{\delta_j}+\sum_j \mathbb{C}Y_{-\delta_j})+W+W'$$

Here, W,W' are copy of the spin representations for $Spin(8)$. Actually,

$$W=\sum \mathbb{C}Y_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4},\quad W'=\sum \mathbb{C}Y_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4} \qquad (4.6)$$

The sum for W runs over the epsilon's so that $\epsilon_1\epsilon_2\epsilon_3\epsilon_4=1$ and the one for W' over the epsilon's with $\epsilon_1\epsilon_2\epsilon_3\epsilon_4=-1$.

As a module over $Spin(7)$, $\mathfrak{f}_4$ is equal to the sum of: the adjoint representation, two copies of the seven dimensional representation, the trivial representation and two copies of the spin representation. We explicit the pieces needed for future computations,

$$\mathfrak{so}(7,\mathbb{C}) := span\{H_2,H_3,H_4\} + \sum_{2\leq i\neq j\leq 4} \mathbb{C}Y_{\pm\delta_i\pm\delta_j}$$
$$+ \sum_{2\leq j\leq 4} \mathbb{C}(Y_{\delta_1+\delta_j} + Y_{-\delta_1+\delta_j}) + \sum_{2\leq j\leq 4} \mathbb{C}(Y_{\delta_1-\delta_j} + Y_{-\delta_1-\delta_j})$$

The line in $\mathfrak{so}(9)$ fixed by $\mathfrak{so}(7)$ is $\mathbb{C}(Y_{\delta_1} - Y_{-\delta_1})$. Every irreducible finite dimensional representation of G_2 is orthogonal and the lower dimensional irreducible representations are of dimension $1, 7, 14$. We fix a copy of $\mathfrak{g}_2$ in $\mathfrak{so}(7,\mathbb{C})$. Hence, there exists Z, Z' copies of the seven dimensional irreducible representation for G_2, and X, Y vectors where G_2 acts trivially so that $W = Z + \mathbb{C}X$, $W' = Z' + \mathbb{C}Y$. Since the centralizer of $\mathfrak{g}_2$ in $\mathfrak{f}_4$ is isomorphic to $\mathfrak{sl}_2$, [7], we obtain that $\mathfrak{g}_2 \times span_{\mathbb{C}}\{(Y_{\delta_1} - Y_{-\delta_1}), X, Y\}$ is a realization of $\mathfrak{g}_2 \times \mathfrak{sl}_2$ as a maximal subalgebra of $\mathfrak{f}_4$. We need more information on X, Y. For this we conjugate the copy of G_2 in $Spin(7)$ so that a Cartan subalgebra $\mathfrak{v}$ for $\mathfrak{g}_2$ is

$$\mathfrak{v} := \{h_2H_2 + h_3H_3 + h_4H_4 : h_2 + h_3 + h_4 = 0\}.$$

It follows that the subspace of $\mathfrak{s}_{\mathbb{C}}$ where $\mathfrak{v}$ acts by zero is spanned by the vectors

$$Y_{++++}, Y_{----}, Y_{-+++}, Y_{+---}.$$

Because of (4.6) the first two vectors are in W and the second two in W'. The zero weight in Z has multiplicity one and $ad(Y_{\delta_1} - Y_{-\delta_1})$ maps W in W'. All of these allows us to choose the root vectors so that

$$X = Y_{++++} + Y_{----},\ Y = Y_{-+++} - Y_{+---}$$

and

$$[Y_{\delta_1} - Y_{-\delta_1}, X] = Y, [Y_{\delta_1} - Y_{-\delta_1}, Y] = -X.$$

Hence, $X^2 + Y^2$ is invariant under $ad(Y_{\delta_1} - Y_{-\delta_1})$ and we conclude that

(4.7) $\qquad X^2 + Y^2$ is invariant under $K \cap (SL_2(\mathbb{R}) \times G_2)$.

Since $b(X^2 + Y^2, Y_{++++}) \neq 0$ we have shown that π restricted to $K \cap (SL_2(\mathbb{R}) \times G_2)$ is not an admissible representation.

We now analyze the subgroup $SU(3)_l \times SU(2,1)_s$.
The Lie algebra of this group is constructed as follows. A Cartan subalgebra is $\mathfrak{t}$ and the root system is the span of $-\delta_1 - \delta_2, \delta_2 - \delta_3, \delta_4, \frac{1}{2}(\delta_1 - \delta_2 - \delta_3 - \delta_4)$. The long roots provides the $SU(3)$ factor whereas the two short roots generate the $SU(2,1)$ factor. The decomposition of $\mathfrak{s}_{\mathbb{C}}$ as $K \cap (SU(3)_l \times SU(2,1)_s)-$module is:

$$\mathbb{C} \boxtimes \mathbb{C}^2 + \mathbb{C}^3 \boxtimes \mathbb{C}^2 + \mathbb{C} \boxtimes \mathbb{C}^2 + \mathbb{C}^3 \boxtimes \mathbb{C}^2$$

Generators for each summand are respectively

$$Y_{+--+}, Y_{+---};\ Y_{++++}, Y_{--++}, Y_{-+-+}, Y_{+++-}Y_{--+-}, Y_{-+--};$$
$$Y_{-++-}, Y_{-+++};\ Y_{++--}, Y_{----}, Y_{+-+-}, Y_{++-+}Y_{---+}, Y_{+-++}.$$

The representations of $K \cap (SU(3)_l \times SU(2,1)_s)$ in the subspaces, $\mathbb{C} \boxtimes \mathbb{C}^2, \mathbb{C} \boxtimes \mathbb{C}^2$ are contragredient. The regular function defined by either Y_{+--+} or Y_{-++-} restricted to $Ass(\pi)$ is nonconstant, since $Ass(\pi)$ is an irreducible closed subvariety, their product is nonzero. Hence, $Y_{+--+}Y_{-++-}$ determines a nonconstant regular function on $Ass(\pi)$ invariant under $K \cap (SU(3)_l \times SU(2,1)_s)$. Thus, (2.3) implies that π restricted to $K \cap (SU(3)_l \times SU(2,1)_s)$ is not admissible. This verifies table 1 and concludes the proof of the direct implication in theorem 4 and hence the proof of theorem 4 is concluded.

Next, we study admissible restriction to a reductive subgroup of $Spin(9)$ of a Harish-Chandra module whose associated variety is proper. In the following tables for each group $Spin(m)$ shown on the first line, in the first column we list a representative of each conjugacy class of its maximal connected reductive subgroups, on the second column we point if a Harish-Chandra module with proper associate variety has an admissible restriction to the subgroup on the same row.

$Spin(9)$	
$Spin(1) \times Spin(8)$	Yes
$Spin(2) \times Spin(7)$	Yes
$Spin(3) \times Spin(6)$	Yes
$Spin(4) \times Spin(5)$	No
$SU(2)$	No
$SU(2) \boxtimes SU(2)$	No

$Spin(8)$	
$Spin(1) \times Spin(7)$	Yes
$Spin(2) \times Spin(6)$	Yes
$Spin(3) \times Spin(5)$	No
$Spin(4) \times Spin(4)$	No
U(4)	No
$Spin_s(7)$	No
$SU(3)$	No
$Sp(1) \boxtimes Sp(2)$	No

$Spin(7)$	
$Spin(1) \times Spin(6)$	Yes
$Spin(2) \times Spin(5)$	No
$Spin(3) \times Spin(4)$	No
G_2	No

$Spin(6)$	
$Spin(1) \times Spin(5)$	No
$Spin(2) \times Spin(4)$	No
$Spin(3) \times Spin(3)$	No
U(3)	No

Here $U(n)$ indicates the image of the usual immersion of $U(n)$ in $SO(2n)$, G_2 is the image of the seven dimensional representation of the simple connected compact Lie group of Lie algebra $\mathfrak{g}_2$, $SU(3)$ is the image of $SU(3)$ under the adjoint representation, $SU(2)$ is the image of the irreducible representation of dimension 9 of $SU(2)$ in $Spin(9)$.

We already have verified the table for $Spin(9)$. To justify the statement on $Spin(r) \times Spin(s) \subset Spin(r+s)$ we apply (3.5), (4.2), (4.3) and (4.4). The subgroups $Spin(3) \times Spin(3), Spin(3) \times Spin(4)$ are handled via the Cartan-Helgason theorem as the subgroup $Spin(4) \times Spin(5)$. From (4.7) we deduce the statement for G_2. The analysis of the subgroups $U(3), U(4)$ is somewhat parallel. The line $\mathbb{C}Y_{++++}$ as well as the line $\mathbb{C}Y_{----}$ are invariant under $U(3), U(4)$ and the action are respectively $\frac{1}{2}(1,1,1,1), -\frac{1}{2}(1,1,1,1)$. Both vectors Y_{++++}, Y_{----} determine nonconstant regular functions on $Ass(\pi)$. The irreducibility of $Ass(\pi)$ implies that their product defines a nonconstant regular function on the associated variety. The product is invariant under $U(3), U(4)$. Thus, (2.3) implies that there is no admissible restriction to $U(3), U(4)$.

The subgroup $Sp(1) \boxtimes Sp(2) = Spin(3) \boxtimes Spin(5) \subset Spin(\mathbb{C}^2 \boxtimes \mathbb{C}^4)$ as maximal subgroup of $Spin(8)$ is handled in a blend of the technique applied to the subgroup $SU(2) \boxtimes SU(2)$ and the subgroup $SU(2)$ as maximal subgroups of $Spin(9)$. We first realize by means of a convenient basis of $\mathbb{C}^2 \otimes \mathbb{C}^4$ the action of the usual torus $\mathfrak{a}$ of $Spin(3) \boxtimes Spin(5)$. We do this in such a way that $\mathfrak{a}$ becomes a subspace of a Cartan subalgebra which consists of all the the diagonal matrices in $\mathfrak{so}(8, \mathbb{C})$. Next, we obtain the decomposition of W as $Spin(3) \boxtimes Spin(5)$ module, it is $W = \mathbb{C}^3 \boxtimes \mathbb{C} + \mathbb{C} \boxtimes \mathbb{C}^5$. Thus, the trivial weight for $\mathfrak{a}$ occurs with multiplicity two in W. It turns out that Y_{++++}, Y_{+--+} is a basis of the zero weights vectors for $\mathfrak{a}$. Thus, the theorem of Birkes (4.5) together with (2.1) yields that π restricted to $Sp(1) \boxtimes Sp(2)$ is not admissible.

We now analyze the inclusion of $SU(3)$ in $Spin(8)$. The dimension of the lower irreducible and nonequivalent representations of $SU(3)$ are $1, 3, 3, 6, 6, 8$. Either the representations of dimension three or six are not equivalent to their respective contragredient representations, the eight dimensional representation is equivalent to the adjoint representation and hence orthogonal. Dynkin has shown this image of $SU(3)$ is a maximal subgroup of $Spin(8)$. Every spin representation of $Spin(8)$ is orthogonal, which forces that any spin representation of $Spin(8)$ is irreducible when restricted to the image of $SU(3)$. Thus, the zero weight for $\mathfrak{a}$ has multiplicity two on either W or W'. Since a zero weight vector for $\mathfrak{a}$ is a sum of root vectors for $\mathfrak{t}$ and all the root vectors in $\mathfrak{s}$ are in $Ass(\pi)$ (4.2) yields an $SU(3)_{\mathbb{C}}-$closed orbit in $Ass(\pi)$ and hence, π restricted to $SU(3)$ is not admissible. This concludes the verification of the four tables.

We show the converse statement in corollary 1. For this, we list the compact simple groups together with the nontrivial morphisms into $Spin(9)$.

After the work of Dynkin it follows that the simple subgroups of $Spin(9)$ are images of

$$G_2, SU(2) \equiv Spin(3) \equiv Sp(1), SU(3),$$

$$SU(4) \equiv Spin(6), Spin(5) \equiv Sp(2), Spin(7), Spin(8), Spin(9).$$

Also, in the same paper Dynkin classified the simple subgroups of $F_4(\mathbb{C})$. They are images of one of

$$SL_2(\mathbb{C}), SL_3(\mathbb{C}), SL_4(\mathbb{C}) \equiv Spin(6,\mathbb{C}), Spin(5,\mathbb{C}) \equiv Sp(2,\mathbb{C}),$$

$$Spin(7,\mathbb{C}), Spin(8,\mathbb{C}), Spin(9,\mathbb{C}), Sp(3,\mathbb{C}), G_2(\mathbb{C}).$$

We shall verify that some groups may have nonconjugated images. We recall that any automorphism of $Spin(9)$ is inner, hence, the list of the maximal subgroups of $Spin(9)$ let us conclude that if L is a subgroup of $Spin(9)$ image of $Spin(8)$ then L is conjugated the usual immersion of $Spin(8)$ in $Spin(9)$. The group $Spin(7)$ has precisely two images into $Spin(9)$. One is $Spin(7)$ and the other is $Spin_s(7)$. In fact, the spin representation for $Spin(7)$ is orthogonal, hence its image gives rise to a subgroup $Spin_s(7)$ of $Spin(8)$. $Spin_s(7)$ is not conjugated to $Spin(7)$ because under $Spin(7)$, $\mathbb{R}^9$ decomposes as the first fundamental representation plus two copies of the trivial representation, whereas the decomposition of $\mathbb{R}^9$, as $Spin_s(7)$ module, is equal to the sum of the spin representation added to the trivial representation. The claim that, up to conjugation, these are two conjugated images of $Spin(7)$ in $Spin(9)$ follows from the fact the irreducible representations of $Spin(7)$ of dimension less than 9 are orthogonal and they are $\mathbb{R}, \mathbb{R}^7, \mathbb{R}^8$. For $Spin(6) \equiv SU(4)$ the irreducible representations for $Spin(6)$ of dimension less than 10 are: $\mathbb{R}, \mathbb{C}^4, (\mathbb{C}^4)^\star, \mathbb{R}^6$. The second and third representations are neither orthogonal nor symplectic. Hence, the morphism of $SU(4)$ in $Spin(9)$ are: $SU(4) \subset Spin(8), SU(4) \equiv Spin(6)$. We already know there is no admissible restriction to the first image, while there is admissible restriction to the second image. We claim $Spin(5)$ has two nonconjugated images in $Spin(9)$. For this we recall that the low dimensional irreducible representations of $Spin(5)$ are $\mathbb{C}, \mathbb{C}^5, \mathbb{C}^4$. The last one is a symplectic representation. The other two are orthogonal. Hence, we get the image $Spin(5) \subset SO(\mathbb{C}^4 + (\mathbb{C}^4)^\star)$. The image of this $Spin(5)$ is a subset of $SU(4) \subset Spin(8)$. Hence, there is no admissible restriction to this image of $Spin(5)$. The other image is the usual one. The images of $SU(3)$ are: the irreducible image in $Spin(8)$ and the inclusion $SU(3) \subset SO(\mathbb{C}^3 + (\mathbb{C}^3)^\star) = SO(6,\mathbb{C})$. The tables show that there is no admissible restriction to any of the two images. To finish the proof we consider $SU(2) \equiv Spin(3)$. The case of the irreducible representation of $SU(2)$ in $\mathbb{R}^9$ was considered previously. The other possibilities yields that the image of $SU(2)$ in some cases is contained in subgroups $Spin(r) \times Spin(s)$ so that there is no admissible restriction to, except for the case the image is contained in $Spin(7) \times Spin(2)$. Dynkin has shown that the irreducible image of $SU(2)$ in $\mathbb{R}^7$ is contained in G_2. The tables show there is no admissible restriction for this case. The other possibility is an inclusion of the type $SU(2) \subset SU(2r) \subset SO(\mathbb{C}^{2r} + (\mathbb{C}^{2r})^\star)$. The tables show that there is no admissible restriction and we have shown corollary 1.

To conclude the proof of the direct implication in theorem 3 we assume L is a closed connected reductive nonsimple subgroups L of $Spin(9)$ so that some Harish Chandra module with proper associate variety has an admissible restriction to L. We want to show some conjugate of L contains $Spin(6)$. Owing to the table for $Spin(9)$ we may assume L is a subgroup of one of $Spin(n) \times Spin(9-n), n = 6, 7, 8$. For L contained in $Spin(8)$ the table for $Spin(8)$ shows may assume L is a subgroup of $Spin(7)$ or $Spin(6) \times Spin(2)$. For the first case, the table for $Spin(7)$ implies L contains a copy of $Spin(6)$, for the second possibility and L semisimple we have that L is contained in $Spin(6)$, the table for $Spin(6)$ implies $L = Spin(6)$, if the center of L is of positive dimension after some work it also follows that L contains a conjugate of $Spin(6)$. For a semisimple subgroup L of $Spin(7) \times Spin(2)$ we have L is a subgroup of $Spin(7)$ and the table for $Spin(7)$ yields that L contains a conjugate of $Spin(6)$. For a reductive subgroup L of $Spin(6) \times Spin(3)$ and the projection of L into $Spin(3)$ is trivial, the table for $Spin(6)$ implies L is equal to $Spin(6)$. For a reductive subgroup L of $Spin(6) \times Spin(3)$ and the projection of L into $Spin(3)$ is non trivial we arrive in a contradiction unless $L = Spin(6) \times Spin(3)$. In fact, if L were a proper subgroup of $Spin(6) \times Spin(3)$, there would be a nontrivial smooth morphism $\phi : Spin(3) \longrightarrow Spin(6)$ so that $L = \{(\phi(a), a), a \in Spin(3)\} \dot{(} L \cap Spin(6))$ and the image of ϕ commutes with $L \cap Spin(6)$. This forces that L is contained in a conjugate of either $Spin(3) \boxtimes Spin(3)$ or $Spin(4) \times Spin(5)$, which in turn any of two, implies there is no admissible restriction to L. Also, by use of LiE and the list of maximal subgroups of $Spin(6)$ we checked that for $(L \cap Spin(6))Spin(3)$ a proper reductive subgroup of $Spin(6) \times Spin(3)$ there is no admissible restriction to L.

5. Aside on Discrete Series representations

Harish-Chandra showed that G admits representations whose matrix coefficients are square integrable with respect to Haar measure on G

if and only if a maximal torus T for K is a Cartan subgroup of G. Then, he parameterizes the equivalence classes of square integrable irreducible representations by nonsingular elements of the weight lattice of characters of T. Another way to parameterize the Harish-Chandra modules associated to Discrete Series representations is by the set of equivalence classes of $A_{\mathfrak{b}}(\lambda)$ where $\mathfrak{b}$ is a Borel subalgebra which contains $\mathfrak{t}$ and λ is a unitary character for T in the good range for $\mathfrak{b}$. Hence, (3.1) implies that the associated variety for $A_{\mathfrak{b}}(\lambda)$ depends only on $\mathfrak{b}$ and not on the character λ. Thus, (2.3) yields that admissible restricted to L of the family of Discrete Series $A_{\mathfrak{b}}(\lambda)$ depends only on the systems of positive roots Ψ corresponding to $\mathfrak{b}$. Actually, in [5] we have shown for Discrete series representation $H-$admissibility is equivalent to $H \cap K-$admissibility of the underlying Harish-Chandra module. As before, G is a connected matrix rank one Lie group. Henceforth, we assume G admits square integrable representations.

Theorem 5. *Let L be a connected compact subgroup of K. There exists a square integrable irreducible representation (π, V) whose Harish Chandra parameter λ is dominant with respect to Ψ with admissible restriction to L if and only if*

For $\mathfrak{su}(1,q)$, either Ψ is a holomorphic system and L is so that $\mathbb{C}[\mathbb{C}^q]^L = \mathbb{C}$ or Ψ is a non holomorphic system and L belongs to the class of groups which acts transitively on the unit sphere of $\mathbb{R}^{2q}$.

For G locally isomorphic to $SO(1,2q)$, Ψ is arbitrary and L is a subgroup which acts transitively on the unit sphere of $\mathbb{R}^{2q}$.

For $\mathfrak{sp}(1,q)$, and Ψ a quaternionic system, L is conjugated either to $S \times Sp(q_1) \times \cdots \times Sp(q_r)$ with $q_1 + \cdots + q_r = q$ and S subgroup of $Sp(1)$ or to $Sp(1) \times B$, B an arbitrary subgroup of $Sp(q)$. When Ψ is a non quaternionic system, L belongs to the class of subgroups that acts transitively on the unit sphere of $\mathbb{R}^{4q}$.

For $F_{4(-20)}$, Ψ is arbitrary and $L = K$.

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FAMAF-CIEM, Ciudad Universitaria, 5000 Córdoba, Argentine
E-mail address: `vargas@famaf.unc.edu.ar`