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On one-dimensional superlinear indefinite problems involving the ϕ-Laplacian

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Abstract. Let $\Omega := (a, b) \subset \mathbb{R}$, $m \in L^1(\Omega)$ and $\phi : \mathbb{R} \to \mathbb{R}$ be an odd increasing homeomorphism. We consider the existence of *positive* solutions for problems of the form

$$\begin{cases} -\phi(u')' = m(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : [0, \infty) \to [0, \infty)$ is a continuous function which is, roughly speaking, superlinear with respect to ϕ. Our approach combines the Guo-Krasnoselskiĭ fixed-point theorem with some estimates on related nonlinear problems. We mention that our results are new even in the case $m \geq 0$.

Mathematics Subject Classifications. 34B15, 34B18, 35J25.

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1. Introduction

Let $\Omega := (a, b) \subset \mathbb{R}$, $m \in L^1(\Omega)$ with $m^+ \not\equiv 0$, and let $\phi : \mathbb{R} \to \mathbb{R}$ be an odd increasing homeomorphism. In this paper, we proceed with the investigation of *positive* solutions of the problem

$$\begin{cases} -\phi(u')' = m(x)f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f : [0, \infty) \to [0, \infty)$ is a continuous function. The existence of positive solutions for problems as (1.1) involving the so-called ϕ-Laplacian has been intensively studied over the last 20 years. We cite, among many others, the articles [1,3,8,12,14–16,18,22,23]. These problems arise in a number of applications such as reaction–diffusion systems, population biology, glaciology, nonlinear elasticity, combustion theory, non-Newtonian fluids, etc., see for instance [5,6,9,17]. We remark that they also appear naturally in the study of

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radial solutions for nonlinear equations in annular domains (see e.g., [12,20] and its references).

In particular, the problem (1.1) with either sublinear or superlinear nonlinearities was considered in [2, Corollary 3.4], [19, Theorem 1.1] and [21, Theorem 2], but with some rather strong hypothesis on m and ϕ (for the special case $\phi(x) = x$ with sublinear or superlinear nonlinearities we refer to [10] and [7] respectively, and the references therein). More precisely, in [2] it was assumed that $m \in \mathcal{C}(\overline{\Omega})$ with $\min_{\overline{\Omega}} m > 0$, while in [19,21] it was required that $m \geq 0$ in Ω and $m \not\equiv 0$ on any subinterval of Ω. Regarding the conditions on ϕ, the restrictions imposed in all of these papers do not allow neither exponential-like nor logarithmic-like nonlinearities. On the other hand, in [13], we recently studied the sublinear case, for any nonnegative (nontrivial) m and also weakening the conditions on ϕ.

Our aim in this article is to deal with the superlinear case. Here, we shall even allow m to *change sign* in Ω as long as its negative part is small, and we shall also be able to treat nonlinearities ϕ that are not covered by [2,19, 21]. We shall rely on the well-known Guo-Krasnoselskiĭ fixed point theorem combined with some estimates derived in [13].

In the next section, we compile some auxiliary results, while in Sect. 3, we state and prove our main theorem. Finally, at the end of the paper, we give several examples illustrating our results and their relation with the ones in [2,19,21] (see also Remark 3.2).

2. Preliminaries

Let $\phi : \mathbb{R} \to \mathbb{R}$ be an odd increasing homeomorphism and $h \in L^1(\Omega)$. We start considering the problem

$$\begin{cases} -\phi(v')' = h(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Remark 2.1. For every $h \in L^1(\Omega)$, (2.1) has a unique solution $v \in \mathcal{C}^1(\overline{\Omega})$ such that $\phi(v')$ is absolutely continuous and that the equation holds pointwise, *a.e.* $x \in \Omega$. In fact,

$$v(x) = \int_a^x \phi^{-1}\left(c_h - \int_a^y h(t)\,\mathrm{d}t\right) \mathrm{d}y, \tag{2.2}$$

where c_h is the unique constant such that $v(b) = 0$. Furthermore, the solution operator $\mathcal{S}_\phi : L^1(\Omega) \to \mathcal{C}^1(\overline{\Omega})$ is completely continuous and nondecreasing, see [4, Lemma 2.1] and [13, Lemma 2.2]. In addition,

$$0 \leq c_h \leq \int_a^b h \tag{2.3}$$

whenever $h \geq 0$ in Ω (see [13, (2.5)]).

Let us now introduce some notation. For $h \in L^1(\Omega)$ with $0 \leq h \not\equiv 0$, define

$$\mathcal{A}_h := \{x \in \Omega : h(y) = 0 \, a.e. \;\; y \in (a, x)\},$$
$$\mathcal{B}_h := \{x \in \Omega : h(y) = 0 \, a.e. \;\; y \in (x, b)\},$$

and

$$\alpha_h := \begin{cases} \sup \mathcal{A}_h & \text{if } \mathcal{A}_h \neq \emptyset, \\ a & \text{if } \mathcal{A}_h = \emptyset, \end{cases} \qquad \beta_h := \begin{cases} \inf \mathcal{B}_h & \text{if } \mathcal{B}_h \neq \emptyset, \\ b & \text{if } \mathcal{B}_h = \emptyset, \end{cases}$$

$$\underline{\theta}_h := \min\left\{ \frac{1}{\beta_h - a}, \frac{1}{b - \alpha_h} \right\}, \quad \overline{\theta}_h := \frac{\alpha_h + \beta_h}{2}.$$

We note that since $h \not\equiv 0$, $\underline{\theta}_h$ is well defined and $\alpha_h < \beta_h$ (and hence, $\overline{\theta}_h \in (\alpha_h, \beta_h)$). Observe also that if $g \in \mathcal{C}(\overline{\Omega})$ with $g > 0$ in Ω, then $\underline{\theta}_h = \underline{\theta}_{hg}$ and $\overline{\theta}_h = \overline{\theta}_{hg}$. We also set

$$\delta_\Omega(x) := \operatorname{dist}(x, \partial\Omega) = \min(x - a, b - x).$$

We shall employ the following estimates several times in the sequel. For the proof, see [13, Lemma 2.3 and its proof].

Lemma 2.2. *Let $0 \leq h \in L^1(\Omega)$ with $h \not\equiv 0$. Then, in $\overline{\Omega}$ it holds that*

$$\underline{\theta}_h \left\| \mathcal{S}_\phi(h) \right\|_\infty \delta_\Omega \leq \mathcal{S}_\phi(h) \leq \phi^{-1}\left(\int_a^b h \right) \delta_\Omega. \tag{2.4}$$

In addition,

$$\left\| \mathcal{S}_\phi(h) \right\|_\infty \geq \min\left\{ \int_a^{\overline{\theta}_h} \phi^{-1}\left(\int_y^{\overline{\theta}_h} h \right) dy, \int_{\overline{\theta}_h}^b \phi^{-1}\left(\int_{\overline{\theta}_h}^y h \right) dy \right\} > 0. \tag{2.5}$$

We conclude this section with a particular case of the well-known Guo-Krasnoselskiĭ fixed-point theorem (e.g., [11, Theorem 2.3.4]).

Lemma 2.3. *Let X be a Banach space and let $\mathcal{K}$ be a cone in X. Let $\Omega_1, \Omega_2 \subset X$ be two open sets with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose that $T : \mathcal{K} \cap (\overline{\Omega}_2 \diagdown \Omega_1) \to \mathcal{K}$ is a completely continuous operator and*

$$\|Tv\| \geq \|v\| \quad \text{for } v \in \mathcal{K} \cap \partial\Omega_2, \qquad \|Tv\| \leq \|v\| \quad \text{for } v \in \mathcal{K} \cap \partial\Omega_1.$$

Then, T has a fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \diagdown \Omega_1)$.

3. Main results

Let us introduce the following assumptions on ϕ and f:

H1. There exist $\underline{t}, \overline{t} > 0$ and two increasing homeomorphisms $\varphi : [0, \underline{t}] \to [0, \varphi(\underline{t})]$ and $\psi : [\overline{t}, \infty) \to [\psi(\overline{t}), \infty)$, such that

$$\phi(tx) \geq \varphi(t)\phi(x) \quad \text{for all} \quad t \in [0, \underline{t}],\ x \geq 0, \tag{3.1}$$

$$\phi(tx) \leq \psi(t)\phi(x) \quad \text{for all} \quad t \in [\overline{t}, \infty),\ x \geq 0. \tag{3.2}$$

F1. There exist $t_0, k_1, k_2, q_1, q_2 > 0$ such that

$$f(t) \geq k_1 t^{q_1} \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad f(t) \leq k_2 t^{q_2} \quad \text{for} \quad t \in [0, t_0]. \tag{3.3}$$

We shall write as usual $m = m^+ - m^-$ with $m^\pm := \max(\pm m, 0)$. Let us also set $\mathcal{C}_0^1(\overline{\Omega}) := \{u \in \mathcal{C}^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ and denote the interior of its positive cone by

$$\mathcal{P}^\circ := \left\{ u \in \mathcal{C}_0^1(\overline{\Omega}) : u > 0 \text{ in } \ \Omega \, \text{and} \, u'(b) < 0 < u'(a) \right\}.$$

Theorem 3.1. *Let $m \in L^1(\Omega)$ with $m^+ \not\equiv 0$. Assume H1 and F1 with*

$$\underline{\lim}_{t\to\infty} \frac{t^{q_1}}{\psi(t)} = \infty \quad \textit{and} \quad \overline{\lim}_{t\to 0^+} \frac{t^{q_2}}{\varphi(t)} = 0. \tag{3.4}$$

Then, there exists $\delta_0 > 0$ such that for all $\delta \in [0, \delta_0]$ the problem

$$\begin{cases} -\phi(u')' = (m^+(x) - \delta m^-(x)) f(u) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

admits a solution $u \in \mathcal{P}^\circ$.

Remark 3.2. (i) A quick look at the final two paragraphs in the proof of Theorem 3.1 shows that one can replace the last conditions in (3.3) and (3.4) by

$$f \text{ is increasing in } (0, t_0) \text{ for some } t_0 > 0 \quad \text{and} \quad \overline{\lim}_{t\to 0^+} \frac{f(t)}{\varphi(t)} = 0.$$

(ii) In the particular case of the p-Laplacian, i.e., $\phi(t) = |t|^{p-2}\, t$ with $p > 1$, clearly H1 (with $\varphi(t) = \psi(t) = t^{p-1}$) is true. Moreover, (3.4) is valid if and only if $q_1, q_2 > p - 1$, so that here we get the standard growth condition that characterizes superlinear problems.

Proof. Let $\underline{t}, \overline{t}, t_0, k_1, k_2, q_1, q_2 > 0$ and φ, ψ be as in H1 or F1, as appropriate. For $\delta \geq 0$, let $m_\delta := m^+ - \delta m^-$ and $\mathcal{K}$ be the cone given by

$$\mathcal{K} := \left\{ v \in \mathcal{C}(\overline{\Omega}) : v \geq \frac{\theta_{m^+}}{2} \|v\|_\infty \delta_\Omega \text{ in } \overline{\Omega} \right\}, \tag{3.5}$$

and for $v \in \mathcal{K}$ define $\mathcal{T}_\delta v := \mathcal{S}_\phi(m_\delta f(v))$. Observe that, $\mathcal{C}_0^1(\overline{\Omega}) \cap (\mathcal{K} \setminus \{0\}) \subset \mathcal{P}^\circ$.

Let $B_R(0)$ be the open ball in $\mathcal{C}(\overline{\Omega})$ with radius R and centered at 0. We shall first show that, for any fixed $R > r > 0, \mathcal{T}_\delta : \mathcal{K} \cap \left(\overline{B_R(0)} \setminus B_r(0) \right) \to \mathcal{K}$ if δ is small enough. Indeed, let $u_\delta := \mathcal{T}_\delta v$ and $C := \max_{t\in[0,R]} f(t)$. Integrating over (a, x) and recalling that $\mathcal{S}_\phi$ and ϕ are nondecreasing we obtain

$$\phi(u_\delta'(x)) - \phi(u_0'(x)) \leq \int_a^x \phi(u_\delta')' - \phi(u_0')' = \delta \int_a^x m^- f(v) \leq \delta C \int_a^b m^-,$$

and integrating over (x, b) and arguing as above we have $\phi(u_0'(x)) - \phi(u_\delta'(x)) \leq \delta C \int_a^b m^-$. Then, letting $x \to a^+$ we infer that

$$|\phi(u_\delta'(a)) - \phi(u_0'(a))| \leq \delta C \int_a^b m^-. \tag{3.6}$$

Now, by (2.2), $u_\delta'(x) = \phi^{-1}(c_{m_\delta f(v)} - \int_a^x m_\delta f(v))$ with $c_{m_\delta f(v)} = \phi(u_\delta'(a))$. Note also that, (2.3) and (3.6) yield that, for all $x \in \overline{\Omega}$,

$$\left| c_{m_\delta f(v)} - \int_a^x m_\delta f(v) \right| \leq |\phi(u_\delta'(a))| + \int_a^b |m_\delta|\, f(v) \leq 2C \int_a^b |m_\delta|$$

and

$$\left| c_{m_\delta f(v)} - \int_a^x m_\delta f(v) - \left(c_{m^+ f(v)} - \int_a^x m^+ f(v) \right) \right| \leq 2\delta C \int_a^b m^-.$$

Hence, since ϕ^{-1} is uniformly continuous on compact intervals, it follows that, given any $\varepsilon > 0$, $\|u'_\delta - u'_0\|_{\mathcal{C}(\overline{\Omega})} < \varepsilon$ if $\delta = \delta(\varepsilon, C, m^-) > 0$ is sufficiently small. Moreover, taking this into account we deduce that there exists some $\delta_\varepsilon > 0$ (depending only on ε, C and m^-) such that

$$\|u_\delta - u_0\|_{\mathcal{C}^1(\overline{\Omega})} < \varepsilon \quad \text{for all} \quad \delta \in [0, \delta_\varepsilon]. \tag{3.7}$$

On the other side, from the first condition in F1, for every $0 \not\equiv v \in \mathcal{K}$ we have that $f(v) > 0$ in Ω. Thus, employing the first inequality in (2.4) we get

$$u_0 \geq \underline{\theta}_{m^+ f(v)} \|u_0\|_\infty \delta_\Omega = \underline{\theta}_{m^+} \|u_0\|_\infty \delta_\Omega \quad \text{in } \overline{\Omega}.$$

Also, since $v \in \mathcal{K} \setminus B_r(0)$, from (2.5) and F1 we deduce that $\|u_0\|_\infty \geq c$ for some $c > 0$ depending on r but not on v. Thus, we may choose $0 < \eta \leq \frac{\underline{\theta}_{m^+}}{2} \|u_0\|_\infty$. So, having in mind (3.7) and that $u_0 \in \mathcal{P}^\circ$ and $\mathcal{S}_\phi$ is nondecreasing,

$$u_\delta \geq u_0 - \eta\delta_\Omega \geq \left(\underline{\theta}_{m^+} \|u_0\|_\infty - \eta\right) \delta_\Omega \geq \frac{\underline{\theta}_{m^+}}{2} \|u_0\|_\infty \delta_\Omega \geq \frac{\underline{\theta}_{m^+}}{2} \|u_\delta\|_\infty \delta_\Omega \quad \text{in } \overline{\Omega}$$

for all $\delta \in [0, \delta_0]$, for some $\delta_0 > 0$ (not depending on v). Therefore, for such δ, $\mathcal{T}_\delta(v) \in \mathcal{K}$ as asserted. Furthermore, taking into account (2.5), making $\delta_0 > 0$ smaller if necessary and reasoning as above we may also derive that

$$\|u_\delta\|_\infty \geq \frac{1}{2} \min \left\{ \int_a^{\overline{\theta}_{m^+}} \phi^{-1} \left(\int_y^{\overline{\theta}_{m^+}} m^+ f(v) \right) \mathrm{d}y, \int_{\overline{\theta}_{m^+}}^b \phi^{-1} \left(\int_{\overline{\theta}_{m^+}}^y m^+ f(v) \right) \mathrm{d}y \right\}. \tag{3.8}$$

Note also that, since $v \to m_\delta f(v)$ is continuous from $\mathcal{C}(\overline{\Omega})$ into $L^1(\Omega)$, Remark 2.1 yields that $\mathcal{T}_\delta$ is completely continuous.

On the other hand, (3.2) implies that $t\phi^{-1}(x) \leq \phi^{-1}(\psi(t)x)$ for all $t \in [\overline{t}, \infty)$ and $x \geq 0$. In addition, since $\psi : [\overline{t}, \infty) \to [\psi(\overline{t}), \infty)$ is an homeomorphism, $\psi^{-1}(t) \geq \overline{t}$ for all $t \geq \psi(\overline{t})$. Hence,

$$\psi^{-1}(t)\phi^{-1}(x) \leq \phi^{-1}(tx) \quad \text{for all} \quad t \in [\psi(\overline{t}), \infty), \quad x \geq 0. \tag{3.9}$$

Let us now define

$$\mathcal{M}_1 := \int_a^{\overline{\theta}_{m^+}} \phi^{-1} \left(k_1 \left(\frac{\underline{\theta}_{m^+}}{2} \right)^{q_1} \int_y^{\overline{\theta}_{m^+}} m^+ \delta_\Omega^{q_1} \right) \mathrm{d}y,$$
$$\mathcal{M}_2 := \int_{\overline{\theta}_{m^+}}^b \phi^{-1} \left(k_1 \left(\frac{\underline{\theta}_{m^+}}{2} \right)^{q_1} \int_{\overline{\theta}_{m^+}}^y m^+ \delta_\Omega^{q_1} \right) \mathrm{d}y,$$
$$\mathcal{M} := \frac{2}{\min\{(\mathcal{M}_1, \mathcal{M}_2)\}}.$$

By the first condition in (3.4), there exists $t_\psi \geq \max\{\overline{t}, \psi(\overline{t})\}$ such that $t^{q_1} \geq \mathcal{M}^{q_1}\psi(t)$ for all $t \geq t_\psi$. Recalling that $\psi : [\overline{t}, \infty) \to [\psi(\overline{t}), \infty)$ is an homeomorphism, there also exists some $\widetilde{t} \geq t_\psi$ such that $\psi^{-1}(\widetilde{t}) \geq t_\psi$. Hence, $\psi^{-1}(\widetilde{t}) \geq \mathcal{M}\widetilde{t}^{1/q_1}$. Let us fix $R := \widetilde{t}^{1/q_1}$. Then, $R^{q_1} \geq \psi(\overline{t})$ and

$$\psi^{-1}(R^{q_1}) \geq \mathcal{M}R. \tag{3.10}$$

We claim that $\|\mathcal{T}_\delta v\|_\infty \geq \|v\|_\infty$ for $v \in \mathcal{K} \cap \partial B_R(0)$. Indeed, since ϕ^{-1} is increasing, taking into account (3.8), F1, (3.5), (3.9) and (3.10) we see that

$$\begin{aligned}
\|\mathcal{T}_\delta v\|_\infty &\geq \frac{1}{2}\min\left\{\int_a^{\overline{\theta}_{m^+}} \phi^{-1}\left(k_1\int_y^{\overline{\theta}_{m^+}} m^+ v^{q_1}\right)\mathrm{d}y, \int_{\overline{\theta}_{m^+}}^b \phi^{-1}\left(k_1\int_{\overline{\theta}_{m^+}}^y m^+ v^{q_1}\right)\mathrm{d}y\right\}\\
&\geq \frac{1}{2}\min\left\{\int_a^{\overline{\theta}_{m^+}} \phi^{-1}\left(k_1\left(\frac{\underline{\theta}_{m^+}}{2}\|v\|_\infty\right)^{q_1}\int_y^{\overline{\theta}_{m^+}} m^+\delta_\Omega^{q_1}\right)\mathrm{d}y,\right.\\
&\qquad \left.\int_{\overline{\theta}_{m^+}}^b \phi^{-1}\left(k_1\left(\frac{\underline{\theta}_{m^+}}{2}\|v\|_\infty\right)^{q_1}\int_{\overline{\theta}_{m^+}}^y m^+\delta_\Omega^{q_1}\right)\mathrm{d}y\right\}\\
&\geq \frac{1}{2}\psi^{-1}\left(\|v\|_\infty^{q_1}\right)\min\left\{\mathcal{M}_1, \mathcal{M}_2\right\}\\
&\geq \|v\|_\infty.
\end{aligned}$$

We notice next that (3.1) says that $t\phi^{-1}(x) \geq \phi^{-1}(\varphi(t)x)$ for all $t \in [0, \underline{t}]$ and $x \geq 0$, and therefore,

$$\varphi^{-1}(r)\phi^{-1}(x) \geq \phi^{-1}(rx) \quad \text{for all } r \in [0, \varphi(\underline{t})],\ x \geq 0. \tag{3.11}$$

Let

$$\epsilon := \left(\frac{2}{(b-a)\,\phi^{-1}(k_2\int_a^b m^+)}\right)^{q_2} > 0.$$

The second condition in (3.4) tells us that there exists $t_\varphi \leq \underline{t}$ such that $t^{q_2} \leq \epsilon\varphi(t)$ for all $t \leq t_\varphi$. Thus, $\varphi^{-1}(t) \leq (\epsilon t)^{1/q_2}$ for $t \leq \varphi(t_\varphi)$ and hence

$$\varphi^{-1}(r^{q_2}) \leq \epsilon^{1/q_2} r \quad \textit{for all } r \leq \varphi(t_\varphi)^{1/q_2}. \tag{3.12}$$

We choose now $r := \min\left\{t_0, \varphi\left(t_\varphi\right)^{1/q_2}, R/2\right\}$. Having in mind that $\mathcal{S}_\phi$ and ϕ^{-1} are nonincreasing, the second inequality in (2.4), F1, (3.11) and (3.12) we find that for $v \in \mathcal{K} \cap \partial B_r(0)$,

$$\begin{aligned}
0 &\leq \mathcal{T}_\delta v = \mathcal{S}_\phi(m_\delta f(v)) \leq \mathcal{S}_\phi(m^+ f(v)) \leq \phi^{-1}\left(\int_a^b m^+ f(v)\right)\delta_\Omega\\
&\leq \phi^{-1}\left(k_2\|v\|_\infty^{q_2}\int_a^b m^+\right)\delta_\Omega \leq \varphi^{-1}\left(\|v\|_\infty^{q_2}\right)\phi^{-1}\left(k_2\int_a^b m^+\right)\frac{b-a}{2}\\
&\leq \|v\|_\infty.
\end{aligned}$$

Thus, $\|\mathcal{T}_\delta v\|_\infty \leq \|v\|_\infty$ for such v. Now, from a direct application of Lemma 2.3, the theorem follows. □

Let us write as above $m_\delta = m^+ - \delta m^-$. As an immediate consequence of Theorem 3.2 we have the following corollary.

Corollary 3.3. *Let m, ϕ and f be as in Theorem 3.2, and let $\lambda > 0$. Then, there exists $\delta_0 = \delta_0(\lambda) > 0$ such that for all $\delta \in [0, \delta_0]$ the problem*

$$\begin{cases} -\phi(u')' = \lambda m_\delta(x) f(u) & \textit{in } \Omega,\\ u = 0 & \textit{on } \partial\Omega,\end{cases}$$

admits a solution $u = u_\lambda \in \mathcal{P}^\circ$. Moreover, u_λ can be chosen such that

$$\lim_{\lambda\to 0^+}\|u_\lambda\|_{\mathcal{C}(\overline{\Omega})} = \infty. \tag{3.13}$$

Proof. The existence assertion is clear. Suppose now (3.13) does not hold. Then, recalling that $\mathcal{S}_\phi$ is nondecreasing and Lemma 2.2 we get

$$0 \leq u_\lambda = \mathcal{S}_\phi(\lambda m_\delta f(u_\lambda)) \leq \mathcal{S}_\phi(\lambda m^+ f(u_\lambda)) \leq \phi^{-1}\left(\lambda \int_a^b m^+ f(u_\lambda)\right)\delta_\Omega \to 0$$

uniformly in Ω as $\lambda \to 0^+$. In other words, $u_\lambda \to 0$ in $\mathcal{C}(\overline{\Omega})$. But this is not possible because in the proof of the above theorem we can choose r uniformly away from 0 for all λ close to 0 (see the last two paragraphs of the aforementioned proof). □

Examples. We assume without loss of generality that $x \geq 0$.

Let $\eta : (0,\infty) \to (0,\infty)$ be a continuous and nonincreasing function with $\lim_{x\to 0^+} x^p \eta(x) = 0$ for some $p > 0$. Define

$$\phi(x) := x^p \eta(x) \quad \text{for } x > 0, \quad \phi(0) := 0.$$

Then, clearly ϕ fulfills H1 with $\underline{t} = \overline{t} = 1$ and $\varphi(t) = \psi(t) = t^p$. Moreover, (3.4) holds if and only if $q_1, q_2 > p$.

Note that, the above paragraph implies that if $\phi : [0,\infty) \to [0,\infty)$ is an increasing homeomorphism such that $\phi(x)/x^p$ is nonincreasing for some $p > 0$, then ϕ satisfies H1.

Let us exhibit next a few particular cases. We notice that in all the examples below it is easy to check that $\phi : [0,\infty) \to [0,\infty)$ is indeed an increasing homeomorphism.

(e1) Let

$$\phi(x) := x^{p_1} + x^{p_2}, \quad p_1 \geq p_2 > 0.$$

Since $\phi(x)/x^{p_1}$ is nonincreasing, we see that H1 holds.

(e2) Let

$$\phi(x) := \frac{x^{p_1}}{1 + x^{p_2}}, \quad p_1 > p_2 > 0.$$

Then, $\phi(x)/x^{p_1}$ is decreasing, and therefore, H1 is valid.

(e3) Let

$$\phi(x) := (\ln(x+1))^p, \quad p > 0.$$

A few computations show that $\phi(x)/x^p$ is decreasing and thus H1 is true. Let us observe that since

$$\lim_{x\to\infty} \frac{\phi(tx)}{\phi(x)} = 1 \quad \text{for all } t > 0, \tag{3.14}$$

it is easy to check that ϕ *does not* satisfy the conditions in [2,19,21].

(e4) Let

$$\phi(x) := \operatorname{arsinh} x.$$

Then, $\phi(x)/x$ is decreasing and so H1 holds. Since here ϕ also satisfies (3.14), ϕ *does not* meet the assumptions in [2,19,21].

(e5) The functions

$$\phi(x) := x(|\ln x| + 1) \quad \text{and} \quad \phi(x) := x - \ln(x+1)$$

satisfy H1 since in both cases $\phi(x)/x^2$ is decreasing.

(e6) Let $\beta : [0,\infty) \to (0,\infty)$ be continuous and concave. Then, for any $p>0$, $\phi(x) := x^p\beta(x)$ fullfils H1 with $\varphi(t) = \psi(t) = t^{p+1}$. Indeed, this follows from the fact that $\beta(x)/x$ is nonincreasing.

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