Contents lists available at ScienceDirect
Journal of Algebra
www.elsevier.com/locate/jalgebra

Cohomology of vertex algebras

José I. Liberati

Ciem – CONICET (FAMAF), Medina Allende y Haya de la Torre, Ciudad Universitaria, (5000) Córdoba, Argentina

ARTICLE INFO

Article history:
Received 10 July 2016
Available online 20 October 2016
Communicated by Volodymyr Mazorchuk

Keywords:
Vertex algebra
Cohomology

ABSTRACT

Let V be a vertex algebra and M a V-module. We define the first and second cohomology of V with coefficients in M, and we show that the second cohomology $H^2(V,M)$ corresponds bijectively to the set of equivalence classes of square-zero extensions of V by M. In the case that $M = V$, we show that the second cohomology $H^2(V,V)$ corresponds bijectively to the set of equivalence classes of first order deformations of V.

1. Introduction

This is the second of a series of papers (see [11]) trying to extend certain restricted definitions and constructions developed for vertex operator algebras to the general framework of vertex algebras without assuming any grading condition neither on the vertex algebra nor on the modules involved, and we make a strong emphasis on the commutative associative algebra point of view instead of the Lie theoretical point of view.

In this work we define the first and second cohomology of a vertex algebra V with coefficients in a V-module M, and we show that the second cohomology $H^2(V,M)$ corresponds bijectively to the set of equivalence classes of square-zero extensions of V by M.

E-mail address: joseliberati@gmail.com.

http://dx.doi.org/10.1016/j.jalgebra.2016.09.031

In the case that $M = V$, we show that the second cohomology $H^2(V, V)$ corresponds bijectively to the set of equivalence classes of first order deformations of V. If we restrict it to a vertex algebra given by an associative commutative algebra, then we clearly obtain the Harrison cohomology.

In [6,7], Huang developed the cohomology theory of graded vertex algebras using analytical methods and complex variables. In the present paper we develop the cohomology theory for vertex algebras (without grading conditions) using algebraic methods and formal variables, obtaining a very simplified, clear and nice theory.

In the definition of $H^2(V, M)$, Huang used two complex variables. In fact, in the proofs of the theorems that relate the second cohomology with extensions and deformations [7], Huang passed from two complex variables to one formal variable. We directly use one formal variable (cf. [2] versus [5]). So, if we add grading conditions to our definitions and constructions, then we obtain a simpler algebraic version of the results in [7].

As it was pointed out in [6], Borcherds [4] also proposed a cohomology theory for general vertex algebras by using his categorical formulation of vertex algebra and an analogy with the Hochschild homology of associative algebras. However, the subtle details of this cohomology theory were not carried out and the basic properties that the cohomology theory must have were not discussed.

We keep using the more comfortable notation introduced in [11], where the map $Y(a, z)b$ is replaced by $a_{\dot{z}}\, b$ (see section 2 for the detail).

This paper is organized as follows. In section 2, we introduce the basic definitions and notations. In section 3, we define the first and second cohomology of a vertex algebra V with coefficients in a module M. In section 4, we show that the second cohomology $H^2(V, M)$ corresponds bijectively to the set of equivalence classes of square-zero extensions of V by M. In section 5, in the case that $M = V$, we show that the second cohomology $H^2(V, V)$ corresponds bijectively to the set of equivalence classes of first order deformations of V.

Unless otherwise specified, all vector spaces, linear maps and tensor products are considered over an algebraically closed field $\mathbf{k}$ of characteristic 0.

2. Definitions and notation

In order to make a self-contained paper, in this section we present the notion of vertex algebra and their modules. Our presentation and notation differ from the usual one because we want to emphasize the point of view that vertex algebras are analog of associative commutative algebras with unit.

Throughout this work, we define $(x + y)^n$ for $n \in \mathbb{Z}$ (in particular, for $n < 0$) to be the formal series

$$(x + y)^n = \sum_{k \in \mathbb{Z}_+} \binom{n}{k} x^{n-k} y^k,$$

where $\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}$.

Please cite this article in press as: J.I. Liberati, Cohomology of vertex algebras, J. Algebra (2017), http://dx.doi.org/10.1016/j.jalgebra.2016.09.031

Definition 2.1. A *vertex algebra* is a quadruple $(V, {}_{\dot{z}}\,, \mathbf{1}, d)$ that consists of a vector space V equipped with a linear map

$$\begin{aligned} {}_{\dot{z}}\ : V \otimes V &\longrightarrow V((z)) \\ a \otimes b &\longmapsto a_{\dot{z}}\, b, \end{aligned} \tag{2.1}$$

a distinguished vector $\mathbf{1}$ and $d \in \mathrm{End}(V)$ satisfying the following axioms ($a, b, c \in V$):

- *Unit*:
$$\mathbf{1}_{\dot{z}}\, a = a \quad \text{and} \quad a_{\dot{z}}\, \mathbf{1} = e^{zd} a;$$
- *Translation-Derivation*:
$$(da)_{\dot{z}}\, b = \frac{d}{dz}\,(a_{\dot{z}}\, b), \quad d(a_{\dot{z}}\, b) = (da)_{\dot{z}}\, b + a_{\dot{z}}\,(db);$$
- *Commutativity*:
$$a_{\dot{z}}\, b = e^{zd}(b_{-\dot{z}}\ a);$$
- *Associativity*: For any $a, b, c \in V$, there exist $l \in \mathbb{N}$ such that
$$(z+w)^l\ \ (a_{\dot{z}}\, b)_{\dot{w}}\, c = (z+w)^l\ \ a_{z \dot{+} w}\,(b_{\dot{w}}\, c), \tag{2.2}$$
in $V[[z^{\pm 1}, w^{\pm 1}]]$.

Observe that the standard notation $Y(a, z)\, b$ for the z-product in (2.1) has been changed. We adopted this notation following the practical idea of the λ-bracket in the notion of Lie conformal algebra (also called vertex Lie algebra in the literature), see [8]. In fact, we have been using this notation since 2011 (see the undergraduate thesis of my student in [12]).

The commutativity axiom is known in the literature as skew-symmetry (see [9,8]). We will use the word 'commutativity' to emphasize the point of view of a vertex algebra as a generalization of an associative commutative algebra with unit (and a derivation), as in [1,10].

An equivalent definition can be obtained by replacing the associativity axiom by the *associator formula* (which is equivalent to what is known in the literature as the iterate formula (see [9], pp. 54–55) or the n-product identity (see [1])):

$$(a_{\dot{z}}\, b)_{\dot{y}}\, c - a_{z \dot{+} y}\,(b_{\dot{y}}\, c) = b_{\dot{y}}\,(a_{y \dot{+} z}\, c - a_{z \dot{+} y}\, c), \tag{2.3}$$

for $a, b, c \in V$. Observe that in the last term of the associator formula we can not use linearity to write it as a difference of $b_{\dot{w}}\big(a_{w \dot{+} z}\, c\big)$ and $b_{\dot{w}}\big(a_{z \dot{+} w}\, c\big)$, because neither of

Please cite this article in press as: J.I. Liberati, Cohomology of vertex algebras, J. Algebra (2017), http://dx.doi.org/10.1016/j.jalgebra.2016.09.031

these expressions in general exists (see [9], p. 55). This alternative definition of vertex algebra using the (iterate formula or) the associator formula is essentially the original definition given by Borcherds [3], but in our case it is written using the generating series in z instead of the n-products.

It is well known (see [9]) that in the two equivalent definitions that we presented, some of the axioms can be obtained from the others, but we prefer to make emphasis on the properties of d and the explicit formula for the multiplication by the unit.

Definition 2.2. A (left) *module* over a vertex algebra V is a vector space M equipped with an endomorphism d of M and a linear map

$$\begin{aligned} V\otimes M &\longrightarrow M((z)) \\ (a,u) &\longmapsto a\,{}_{\dot z}^{M}\,u \end{aligned}$$

satisfying the following axioms ($a \in V$ and $u \in M$):

- *Unit*:

$$\mathbf{1}{}_{\dot z}^{M}\,u = u;$$

- *Translation-Derivation*:

$$(da){}_{\dot z}^{M}\,u = \frac{d}{dz}(a{}_{\dot z}^{M}\,u), \quad d(a{}_{\dot z}^{M}\,u) = (da){}_{\dot z}^{M}\,u + a{}_{\dot z}^{M}(d\,u);$$

- *Associativity*: For any $a, b \in V$ and $u \in M$, there exist $l \in \mathbb{N}$ such that

$$(z+w)^l \;\, (a_{\dot z}\,b){}_{\dot w}^{M}\,u \;\, = (z+w)^l \;\, a{}_{z\dot{+}w}^{\;M}\,(b{}_{\dot w}^{M}\,u)\;, \tag{2.4}$$

  in $M[[z^{\pm 1}, w^{\pm 1}]]$.

Sometimes, if everything is clear, we shall use $a_{\dot z}\,u$ instead of $a{}_{\dot z}^{M}u$. Obviously, V is a module over V. We follow [1], in the definition of module, because we need to work with this $\mathbf{k}[d]$-module structure (similar to the situation of Lie conformal algebras [8]).

Any V-module M satisfies the *weak commutativity* or *locality*: for all $a, b \in V$, there exist $k \in \mathbb{N}$ such that

$$(x-y)^k\, a_{\dot x}\,(b_{\dot y}\,u) = (x-y)^k\, b_{\dot y}\,(a_{\dot x}\,u) \quad \text{for all } u \in M. \tag{2.5}$$

For V-modules M and N, a *V-homomorphism* or a *homomorphism of V-modules* from M to N is a linear map $\varphi : M \to N$ such that for $a \in V$ and $u \in M$

$$\varphi(a_{\dot z}\,u) = a_{\dot z}\,\varphi(u) \quad \text{and} \quad \varphi(du) = d\varphi(u).$$

A subspace W of a vertex algebra V is called an *ideal of* V if $a_{\dot{z}}\, b \in W$ for all $a \in V$ and $b \in W$.

3. Definition of lower cohomologies

Let V be a vertex algebra, and M a (left) V-module. We define the *right action* of V on M by

$$m_{\dot{z}}\, a = e^{zd}(a_{-\dot{z}}\ m),$$

for $a \in V$ and $m \in M$. A linear map $f : V \to M$ is called a *vertex derivation* if

$$f(a_{\dot{z}}\, b) = a_{\dot{z}}\, f(b) + f(a)_{\dot{z}}\, b$$

for $a, b \in V$. We denote by $\mathrm{VDer}(V, M)$ the space of all such derivations.

Now we consider

$$0 \longrightarrow C^0(V, M) \xrightarrow{\delta_0} C^1(V, M) \xrightarrow{\delta_1} C^2(V, M)$$

with $C^0(V, M) = M$ and $\delta_0 \equiv 0$, therefore $H^0(V, M) = \mathrm{Ker}\delta_0 = M$. We define

$$C^1(V, M) = \{\ g \in \mathrm{Hom}_{\mathbf{k}}(V, M)\ :\ g(da) = dg(a)\ \text{ and }\ \ g(\mathbf{1}) = 0\ \}$$

and we set for $g \in C^1(V, M)$

$$(\delta_1\, g)_z(a, b) = a_{\dot{z}}\, g(b) - g(a_{\dot{z}}\, b) + g(a)_{\dot{z}}\, b$$

with $a, b \in V$. Hence, $H^1(V, M) = \mathrm{Ker}\delta_1 = \mathrm{VDer}(V, M)$. We define $C^2(V, M)$ as the space of linear functions $f_z : V \otimes V \to M((z))$ satisfying (for all $a, b \in V$)

∗ *Unit*:

$$f_z(a, \mathbf{1}) = f_z(\mathbf{1}, b) = 0 \tag{3.1}$$

∗ *Translation-Derivation*:

$$\frac{d}{dz} f_z(a, b) = f_z(da, b), \qquad d(f_z(a, b)) = f_z(da, b) + f_z(a, db), \tag{3.2}$$

∗ *Symmetry*:

$$f_z(a, b) = e^{zd} f_{-z}(b, a). \tag{3.3}$$

Please cite this article in press as: J.I. Liberati, Cohomology of vertex algebras, J. Algebra (2017), http://dx.doi.org/10.1016/j.jalgebra.2016.09.031

Now, let $Z^2(V,M)$ be the space of functions $f_z \in C^2(V,M)$ that satisfy that for all $a,b,c \in V$ there exists $n \in \mathbb{N}$ such that

$$(x+z)^n \Big[f_z(a_{\dot x}\, b, c) + f_x(a,b)_{\dot z}\, c \Big] = (x+z)^n \Big[a_{x \dot + z}\, f_z(b,c) + f_{x+z}(a, b_{\dot z}\, c) \Big] \tag{3.4}$$

and define $H^2(V,M) = Z^2(V,M)/\mathrm{Im}\delta_1$. If V is an associative commutative algebra, it is clear that we obtain the Harrison cohomology.

The associativity condition (2.2) of a vertex algebra produce the condition (3.4). But if we impose the associator formula (2.3), we shall see after the proof of Theorem 4.2 that (3.4) could be replaced by

$$\begin{aligned} 0 = {} & f_z(a_{\dot x}\, b, c) + f_x(a,b)_{\dot z}\, c - a_{x \dot + z}\, f_z(b,c) - f_{x+z}(a, b_{\dot z}\, c) \\ & - b_{\dot z}\, (f_{z+x}(a,c) - f_{x+z}(a,c)) - f_z(b, a_{z \dot + x}\, c - a_{x \dot + z}\, c). \end{aligned} \tag{3.5}$$

for all $a,b,c \in V$, and the RHS of (3.5) could be the definition of $(\delta_2 f)_{z,x}(a,b,c)$. In this case $H^2(V,M) = \mathrm{Ker}\delta_2/\mathrm{Im}\delta_1$, but we do not know how to define higher cohomology.

Proposition 3.1. *$H^2(V,M)$ is well defined, that is* $\mathrm{Im}\delta_1 \subseteq Z^2(V,M)$.

Proof. Let $g : V \to M$ such that $dg(a) = g(da)$ and $g(\mathbf{1}) = 0$. We define $f_z : V \otimes V \to M((z))$ by $f_z(a,b) = (\delta_1\, g)_z(a,b) = a_{\dot z}\, g(b) - g(a_{\dot z}\, b) + g(a)_{\dot z}\, b$. Now,

$$f_z(\mathbf{1}, b) = \mathbf{1}_{\dot z}\, g(b) - g(\mathbf{1}_{\dot z}\, b) + g(\mathbf{1})_{\dot z}\, b = g(b) - g(b) = 0,$$

and

$$f_z(a, \mathbf{1}) = a_{\dot z}\, g(\mathbf{1}) - g(a_{\dot z}\, \mathbf{1}) + g(a)_{\dot z}\, \mathbf{1} = a_{\dot z}\, g(\mathbf{1}) - g(e^{zd}a) + e^{zd} g(a) = 0,$$

therefore f_z satisfies (3.1). Now we prove that it satisfies (3.2):

$$\begin{aligned} \frac{d}{dz} f_z(a,b) = {} & da_{\dot z}\, g(b) - g(da_{\dot z}\, b) + de^{zd}(b_{-\dot z}\ g(a)) - e^{zd}((db)_{-\dot z}\ g(a)) \\ = {} & da_{\dot z}\, g(b) - g(da_{\dot z}\, b) + e^{zd}(b_{-\dot z}\ dg(a)) = f_z(da, b), \end{aligned}$$

and

$$\begin{aligned} d(f_z(a,b)) = {} & da_{\dot z}\, g(b) + a_{\dot z}\, dg(b) - g(da_{\dot z}\, b) - g(a_{\dot z}\, db) \\ & + e^{zd}((db)_{-\dot z}\ g(a)) + e^{zd}(b_{-\dot z}\ dg(a)) \\ = {} & f_z(da, b) + f_z(a, db). \end{aligned}$$

The symmetry (3.3) follows by

$$\begin{aligned} e^{zd} f_{-z}(b,a) &= e^{zd}\left(b_{-\dot z}\ g(a)\right) - e^{zd} g(b_{-\dot z}\ a) + e^{zd}\left(e^{-zd}(a_{\dot z}\, g(b))\right) \\ &= g(a)_{\dot z}\, b - g(e^{zd} b_{-\dot z}\ a) + a_{\dot z}\, g(b) = f_z(a,b). \end{aligned}$$

Now we should check (3.4):

$$
\begin{aligned}
f_z(a_{\dot{x}}\, b, c) + f_x(a,b)_{\dot{z}}\, c - a_{x\dot{+}z}\, f_z(b,c) - f_{x+z}(a, b_{\dot{z}}\, c) \\
= (a_{\dot{x}}\, b)_{\dot{z}}\, g(c) - a_{x\dot{+}z}\,(b_{\dot{z}}\, g(c)) - g((a_{\dot{x}}\, b)_{\dot{z}}\, c) + g(a_{x\dot{+}z}\,(b_{\dot{z}}\, c)) \\
- a_{x\dot{+}z}\,(g(b)_{\dot{z}}\, c) + (a_{\dot{x}}\, g(b))_{\dot{z}}\, c + (g(a)_{\dot{x}}\, b)_{\dot{z}}\, c - g(a)_{x\dot{+}z}\,(b_{\dot{z}}\, c)
\end{aligned}
\tag{3.6}
$$

using the associativity (2.4) of M, the first two terms in the RHS cancel after the multiplication by $(x+z)^n$ for some $n \in \mathbb{N}$. Similarly with the third and fourth terms, by using the associativity (2.2) of V. Now consider the fifth and sixth terms in (3.6). Using that $e^{zd} a_{\dot{x}}\, w = a_{x\dot{+}z}\,(e^{zd} w)$, we have

$$
\begin{aligned}
(a_{\dot{x}}\, g(b))_{\dot{z}}\, c - a_{x\dot{+}z}\,(g(b)_{\dot{z}}\, c) &= e^{zd}(c_{-\dot{z}}\ \ (a_{\dot{x}}\, g(b))) - a_{x\dot{+}z}\,(e^{zd}(c_{-\dot{z}}\ \ g(b))) \\
&= e^{zd}[c_{-\dot{z}}\ \ (a_{\dot{x}}\, g(b)) - a_{\dot{x}}\,(c_{-\dot{z}}\ \ g(b))]
\end{aligned}
$$

which is zero after the multiplication by $(x+z)^n$ for some $n \in \mathbb{N}$, due to locality (2.5) in the action of V on M. Finally, consider the seventh and eighth terms in (3.6). On one hand, we have

$$
\begin{aligned}
(g(a)_{\dot{x}}\, b)_{\dot{z}}\, c &= e^{zd}(c_{-\dot{z}}\ \ (g(a)_{\dot{x}}\, b)) = e^{zd}(c_{-\dot{z}}\ \ (e^{xd}(b_{-\dot{x}}\ \ g(a)))) \\
&= e^{(z+x)d}\big(c_{-\dot{z}-x}\,(b_{-\dot{x}}\ \ g(a))\big),
\end{aligned}
$$

and on the other hand, we have

$$
\begin{aligned}
g(a)_{x\dot{+}z}\,(b_{\dot{z}}\, c) &= e^{xd}\Big((b_{\dot{z}}\, c)_{-\dot{x}}\ \ (e^{zd} g(a))\Big) = e^{xd}\Big((e^{zd}(c_{-\dot{z}}\ \ b))_{-\dot{x}}\ \ (e^{zd} g(a))\Big) \\
&= e^{(x+z)d}((c_{-\dot{z}}\ \ b)_{-\dot{x}}\ \ g(a)),
\end{aligned}
$$

then using the associativity (2.4), both terms are equal after the multiplication by $(x+z)^n$ for some $n \in \mathbb{N}$, finishing the proof. $\square$

4. Second cohomology and square-zero extensions

Definition 4.1. (a) Let V be a vertex algebra. A *square-zero ideal of V* is an ideal W of V such that for any $a, b \in W$, $a_{\dot{x}}\, b = 0$.

(b) Let V be a vertex algebra and M a V-module. A *square-zero extension* (Λ, f, g) *of V by M* is a vertex algebra Λ together with a surjective homomorphism $f : \Lambda \to V$ of vertex algebras such that $\ker f$ is a square-zero ideal of Λ (so that $\ker f$ has the structure of a V-module) and an injective homomorphism g of V-modules from M to Λ such that $g(M) = \ker f$.

(c) Two square-zero extensions (Λ_1, f_1, g_1) and (Λ_2, f_2, g_2) of V by M are *equivalent* if there exists an isomorphism of vertex algebras $h : \Lambda_1 \to \Lambda_2$ such that the diagram

ARTICLE IN PRESS
JID:YJABR AID:15951 /FLA [m1L; v1.190; Prn:4/11/2016; 14:54] P.8 (1-14)

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "g_1"'] \arrow[d, "1_M"'] & \Lambda_1 \arrow[r, "f_1"'] \arrow[d, "h"'] & V \arrow[r] \arrow[d, "1_V"] & 0 \\ 0 \arrow[r] & M \arrow[r, "g_2"'] & \Lambda_2 \arrow[r, "f_2"'] & V \arrow[r] & 0, \end{tikzcd}$$

is commutative.

Now we have the following result:

Theorem 4.2. *Let V be a vertex algebra and M a V-module. Then the set of the equivalence classes of square-zero extensions of V by M corresponds bijectively to $H^2(V,M)$.*

Proof. Let (Λ, f, g) be a square-zero extension of V by M. Then there is an injective linear map $\Gamma : V \to \Lambda$ such that the linear map $h : V \oplus M \to \Lambda$ given by $h(a,u) = \Gamma(a) + g(u)$ is a linear isomorphism. By definition, the restriction of h to M is the isomorphism g from M to $\ker f$. Then the vertex algebra structure and the V-module structure on Λ give a vertex algebra structure and a V-module structure on $V \oplus M$ such that the embedding $i_2 : M \to V \oplus M$ and the projection $p_1 : V \oplus M \to V$ are homomorphisms of vertex algebras. Moreover, $\ker p_1$ is a square-zero ideal of $V \oplus M$, i_2 is an injective homomorphism such that $i_2(M) = \ker p_1$ and the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "i_2"] \arrow[d, "1_M"'] & V \oplus M \arrow[r, "p_1"] \arrow[d, "h"'] & V \arrow[r] \arrow[d, "1_V"] & 0 \\ 0 \arrow[r] & M \arrow[r, "g"'] & \Lambda \arrow[r, "f"'] & V \arrow[r] & 0 \end{tikzcd}$$

of V-modules is commutative. So we obtain a square-zero extension $(V \oplus M, p_1, i_2)$ equivalent to (Λ, f, g). We need only consider a square-zero extension of V by M of the particular form $(V \oplus M, p_1, i_2)$. Note that the difference between two such square-zero extensions are in the ${}_{\dot z}$-product maps. So we use $(V \oplus M, {}^{V\oplus M}_{\ \ \dot z}, p_1, i_2)$ to denote such a square-zero extension.

We now write down the ${}_{\dot z}$-product map for $V \oplus M$ explicitly. Since $(V \oplus M, {}^{V\oplus M}_{\ \ \dot z}, p_1, i_2)$ is a square-zero extension of V, there exists a linear map $\psi_z : V \otimes V \to M((z))$ such that

$$(a,u) {}^{V\oplus M}_{\ \ \dot z} (b,v) = (a_{\dot z}\, b, a_{\dot z}\, v + u_{\dot z}\, b + \psi_z(a,b)) \tag{4.1}$$

for $a, b \in V$ and $u, v \in M$.

Now we shall prove that $V \oplus M$ with ${}^{V\oplus M}_{\ \ \dot z}$, the vacuum vector $\mathbf{1}_{V\oplus M} = (\mathbf{1}, 0)$ and $d_{V\oplus M}(a,u) = (d_V a, d_M u)$, is a vertex algebra if and only if $\psi_z \in Z^2(V,M)$. In order to simplify the proof, observe that in Proposition 4.8.1 in [9], they showed that $V \oplus M$ with

Please cite this article in press as: J.I. Liberati, Cohomology of vertex algebras, J. Algebra (2017), http://dx.doi.org/10.1016/j.jalgebra.2016.09.031

$\mathbf{1}_{V\oplus M}$, $d_{V\oplus M}$ and $\overset{V\oplus M}{{}_{\dot z}}$ corresponding to $\psi_z \equiv 0$, is a vertex algebra. Therefore, when we check the axioms, we know that all the terms without ψ_z satisfy the corresponding equation. So, in order to prove that $V\oplus M$ with $\overset{V\oplus M}{{}_{\dot z}}$ given by (4.1) is a vertex algebra, we only need to see the terms with ψ_z. For example, the element $(\mathbf{1},0)$ satisfies (for $a,b\in V$ and $u,v\in M$)

$$(\mathbf{1},0)\overset{V\oplus M}{{}_{\dot z}}(b,v)=(\mathbf{1}_{\dot z}\,b,\mathbf{1}_{\dot z}\,v+\psi_z(\mathbf{1},b))=(b,v)$$

and

$$\begin{aligned}(a,u)\overset{V\oplus M}{{}_{\dot z}}(\mathbf{1},0)&=(a_{\dot z}\,\mathbf{1},u_{\dot z}\,\mathbf{1}+\psi_z(a,\mathbf{1}))=(e^{zd_V}a,e^{zd_M}(\mathbf{1}_{-\dot z}\ u)+\psi_z(a,\mathbf{1}))\\&=e^{zd_{V\oplus M}}(a,u)\end{aligned}$$

if and only if $\psi_z(\mathbf{1},b)=0=\psi_z(a,\mathbf{1})$ for all $a,b\in V$. From now on, we use $d=d_V=d_M$. A simple computation shows that $V\oplus M$ satisfies the translation-derivation properties if and only if for all $a,b\in V$

$$\frac{d}{dz}\psi_z(a,b)=\psi_z(da,b),\qquad \text{and}\quad d(\psi_z(a,b))=\psi_z(da,b)+\psi_z(a,db).$$

Now consider the commutativity axiom, that is:

$$\begin{aligned}e^{zd_{V\oplus M}}(b,v)\overset{V\oplus M}{{}_{-\dot z}}(a,u)&=e^{zd_{V\oplus M}}(b_{-\dot z}\ a,b_{-\dot z}\ u+e^{-zd}a_{\dot z}\,v+\psi_{-z}(b,a))\\&=(e^{zd}b_{-\dot z}\ a,e^{zd}b_{-\dot z}\ u+a_{\dot z}\,v+e^{zd}\psi_{-z}(b,a))\end{aligned}$$

and $(a,u)\overset{V\oplus M}{{}_{\dot z}}(b,v)=(a_{\dot z}\,b,a_{\dot z}\,v+e^{zd}b_{-\dot z}\ u+\psi_z(a,b))$. Therefore, ψ_z must satisfy $\psi_z(a,b)=e^{zd}\psi_{-z}(b,a)$.

Similarly, expanding

$$\Big((a,u)\overset{V\oplus M}{{}_{\dot x}}(b,v)\Big)\overset{V\oplus M}{{}_{\dot z}}(c,w)\qquad\text{and}\qquad (a,u)\overset{V\oplus M}{{}_{x\dot+z}}\Big((b,v)\overset{V\oplus M}{{}_{\dot z}}(c,w)\Big),$$

and taking the terms with ψ_z, it is easy to see that the associativity axiom (2.2) holds if and only if for all $a,b,c\in V$ there exists $n\in\mathbb{N}$ such that

$$(x+z)^n\Big[\psi_z(a_{\dot x}\,b,c)+\psi_x(a,b)_{\dot z}\,c\Big]=(x+z)^n\Big[a_{x\dot+z}\,\psi_z(b,c)+\psi_{x+z}(a,b_{\dot z}\,c)\Big],\qquad(4.2)$$

proving that $V\oplus M$ is a vertex algebra if and only if $\psi_z\in Z^2(V,M)$, and together with the projection $p_1:V\oplus M\to V$ and the embedding $i_2:M\to V\oplus M$, $V\oplus M$ is a square-zero extension of V by M.

Next we prove that two elements of $Z^2(V,M)$ obtained in this way differ by an element of $\delta_1C^1(V,M)$ if and only if the corresponding square-zero extensions of V by M are equivalent.

Please cite this article in press as: J.I. Liberati, Cohomology of vertex algebras, J. Algebra (2017), http://dx.doi.org/10.1016/j.jalgebra.2016.09.031

Let $\psi, \phi \in Z^2(V, M)$ be two such elements obtained from square-zero extensions $(V \oplus M, {}_{\dot z}^{(1)}, p_1, i_2)$ and $(V \oplus M, {}_{\dot z}^{(2)}, p_1, i_2)$, respectively. Assume that $\psi = \phi + \delta_1(g)$ where $g \in C^1(V, M)$.

We now define a linear map $h : V \oplus M \to V \oplus M$ by

$$h(a, u) = (a, u + g(a))$$

for $a \in V$ and $u \in M$. Then h is a linear isomorphism and it satisfies (for $a, b \in V$ and $u, v \in M$)

$$\begin{aligned}
h\left((a, u)\, {}_{\dot z}^{(1)} (b, v)\right) &= (a_{\dot z}\, b, a_{\dot z}\, v + u_{\dot z}\, b + \psi_z(a, b) + g(a_{\dot z}\, b)) \\
&= (a_{\dot z}\, b, a_{\dot z}\, v + u_{\dot z}\, b + \phi_z(a, b) + a_{\dot z}\, g(b) + g(a)_{\dot z}\, b) \qquad (4.3) \\
&= (a, u + g(a))\, {}_{\dot z}^{(2)} (b, v + g(b)) \\
&= h(a, u)\, {}_{\dot z}^{(2)} h(b, v).
\end{aligned}$$

Thus h is an isomorphism of vertex algebras from $(V \oplus M, {}_{\dot z}^{(1)}, (\mathbf{1}, 0))$ to $(V \oplus M, {}_{\dot z}^{(2)}, (\mathbf{1}, 0))$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "i_2"] \arrow[d, "1_M"'] & V \oplus M \arrow[r, "p_1"] \arrow[d, "h"'] & V \arrow[r] \arrow[d, "1_V"] & 0 \\
0 \arrow[r] & M \arrow[r, "i_2"] & V \oplus M \arrow[r, "p_1"] & V \arrow[r] & 0
\end{tikzcd} \qquad (4.4)$$

is commutative. Thus the two square-zero extensions of V by M are equivalent.

Conversely, let $(V \oplus M, {}_{\dot z}^{(1)}, p_1, i_2)$ and $(V \oplus M, {}_{\dot z}^{(2)}, p_1, i_2)$ be two equivalent square-zero extensions of V by M. So there exists an isomorphism $h : V \oplus M \to V \oplus M$ of vertex algebras such that (4.4) is commutative. Let $h(a, u) = (f(a, u), g(a, u))$ for $a \in V$ and $u \in M$. Then by (4.4), we have $f(a, u) = a$ and $g(0, u) = u$. Since h is linear, we have $g(a, u) = g(a, 0) + g(0, u) = u + g(a, 0)$. So $h(a, u) = (a, u + g(a, 0))$. Taking $g(a)$ to be $g(a, 0)$, we see that there exists a linear map $g : V \to M$ such that $h(a, u) = (a, u + g(a))$. Using that $d\, h(a, 0) = h(d(a, 0))$ and $h(\mathbf{1}, 0) = (\mathbf{1}, 0)$, it is clear that $g(da) = dg(a)$ and $g(\mathbf{1}) = 0$. Thus, $g \in C^1(V, M)$.

Let ψ and ϕ be elements of $Z^2(V, M)$ obtained from $(V \oplus M, {}_{\dot z}^{(1)}, p_1, i_2)$ and $(V \oplus M, {}_{\dot z}^{(2)}, p_1, i_2)$, respectively. Then, since h is a homomorphism of vertex algebras, (4.3) holds for $a, b \in V$ and $u, v \in M$. So the two expressions in the middle of (4.3) are equal. Thus, we have $\psi = \phi + \delta_1(g)$. Therefore, ψ and ϕ differ by an element of $\delta_1 C^1(V, M)$. $\square$

In the proof of Theorem 4.2 (by taking the terms with ψ_z), we saw in (4.2) that the associativity axiom holds in $(V \oplus M, {}_{\dot z}^{V \oplus M}, (\mathbf{1}, 0))$ if and only if for all $a, b, c \in V$ there exists $n \in \mathbb{N}$ such that

$$(x+z)^n \Big[\psi_z(a_{\dot x}\, b, c) + \psi_x(a,b)_{\dot z}\, c\Big] = (x+z)^n \Big[a_{x\dot+z}\, \psi_z(b,c) + \psi_{x+z}(a, b_{\dot z}\, c)\Big].$$

Recall that we can replace the associativity axiom (2.2) in the definition of vertex algebra, by the associator formula (2.3). By taking the terms with ψ_z, it is possible to prove that the associator formula holds in $(V \oplus M, {}^{V\oplus M}_{\quad\dot z}, (\mathbf{1}, 0))$ if and only if for all $a, b, c \in V$ we have

$$\begin{aligned} 0 = \psi_z(a_{\dot x}\, b, c) &+ \psi_x(a,b)_{\dot z}\, c - a_{x\dot+z}\, \psi_z(b,c) - \psi_{x+z}(a, b_{\dot z}\, c) \\ &- b_{\dot z}\, (\psi_{z+x}(a,c) - \psi_{x+z}(a,c)) - \psi_z(b, a_{z\dot+x}\, c - a_{x\dot+z}\, c). \end{aligned}$$

We avoid replacing the associativity or associator formula by the Jacobi identity because we want to make emphasis that a vertex algebra is a generalization of an associative commutative algebra, and that the cohomology must be a generalization of Harrison cohomology.

5. Second cohomology and first order deformations

Definition 5.1. (a) Let t be a formal variable and let $(V, {}_{\dot z}\,, \mathbf{1}, d)$ be a vertex algebra. A *first order deformation of* V is a family of z-products of the form

$$a \underset{z}{*} b = a_{\dot z}\, b + t\, f_z(a,b)$$

with $a, b \in V$, where $f_z : V \otimes V \to V((z))$ is a linear map (independent of t), such that $(V, \underset{z}{*}, \mathbf{1}, d)$ is a family of vertex algebras up to the first order in t (i.e. modulo t^2). More precisely, the quadruple $(V, \underset{z}{*}, \mathbf{1}, d)$ satisfies the following conditions:

∗ *Unit*:

$$\mathbf{1} \underset{z}{*} a = a \quad \text{ and } \quad a \underset{z}{*} \mathbf{1} = e^{zd} a; \tag{5.1}$$

∗ *Translation-Derivation*:

$$(da) \underset{z}{*} b = \frac{d}{dz}\,(a \underset{z}{*} b), \quad d(a \underset{z}{*} b) = (da) \underset{z}{*} b + a \underset{z}{*} (db); \tag{5.2}$$

∗ *Commutativity*:

$$a \underset{z}{*} b = e^{zd}(b \underset{-z}{*} a); \tag{5.3}$$

∗ *Associativity up to the first order in t*: For any $a, b, c \in V$, there exist $l \in \mathbb{N}$ such that

$$(z+w)^l \ \ (a \underset{z}{*} b) \underset{w}{*} c = (z+w)^l \ \ a \underset{z+w}{*} (b \underset{w}{*} c) \quad \mod t^2. \tag{5.4}$$

Please cite this article in press as: J.I. Liberati, Cohomology of vertex algebras, J. Algebra (2017), http://dx.doi.org/10.1016/j.jalgebra.2016.09.031

(b) Two first order deformations $\overset{(1)}{\underset{z}{*}}$ and $\overset{(2)}{\underset{z}{*}}$ of $(V, \underset{z}{\cdot}, \mathbf{1}, d)$ are *equivalent* if there exists a family $\phi_t : V \to V[t]$, of linear maps of the form $\phi_t = 1_V + tg$ where $g : V \to V$ is a linear map such that

$$\phi_t(a \overset{(1)}{\underset{z}{*}} b) = \phi_t(a) \overset{(2)}{\underset{z}{*}} \phi_t(b) \mod t^2$$

for $a, b \in V$.

We have:

Theorem 5.2. *Let V be a vertex algebra. Then the set of the equivalence classes of first order deformations of V corresponds bijectively to $H^2(V, V)$.*

Proof. Let $\underset{z}{*}$ be a first order deformation of V. By definition, there exists a linear map $f_z : V \otimes V \to V((z))$ such that

$$a \underset{z}{*} b = a \underset{z}{\cdot} b + t\, f_z(a, b) \tag{5.5}$$

for $a, b \in V$, and $(V, \underset{z}{*}, \mathbf{1}, d)$ is a family of vertex algebras up to the first order in t.

The unit properties (5.1) for $(V, \underset{z}{*}, \mathbf{1}, d)$ gives

$$\mathbf{1} \underset{z}{*} a = \mathbf{1} \underset{z}{\cdot} a + t\, f_z(\mathbf{1}, a) = a$$

and

$$a \underset{z}{*} \mathbf{1} = a \underset{z}{\cdot} \mathbf{1} + t\, f_z(a, \mathbf{1}) = e^{zd} a$$

for $a \in V$. So, they are equivalent to

$$f_z(a, \mathbf{1}) = 0 = f_z(\mathbf{1}, a) \quad \text{for all } a \in V. \tag{5.6}$$

Similarly, the coefficient in t^0 of the Translation-Derivation properties (5.2) corresponds exactly to the Translation-Derivation properties of $(V, \underset{z}{\cdot}, \mathbf{1}, d)$, and the coefficient of t^1 in (5.2) corresponds exactly to the following properties on f_z:

$$\frac{d}{dz} f_z(a, b) = f_z(da, b), \quad \text{and} \quad d(f_z(a, b)) = f_z(da, b) + f_z(a, db). \tag{5.7}$$

Now the coefficient in t^0 of the Commutativity property (5.3) corresponds exactly to the Commutativity property of $(V, \underset{z}{\cdot}, \mathbf{1}, d)$, and the coefficient of t^1 in (5.3) corresponds exactly to the following property on f_z:

$$f_z(a, b) = e^{zd} f_{-z}(b, a). \tag{5.8}$$

In the same way, using (5.5), we take the expansions modulo t^2 of the expressions

$$(a \underset{z}{*} b) \underset{w}{*} c \quad \text{and} \quad a \underset{z+w}{*} (b \underset{w}{*} c)$$

and we consider the coefficients of t^0 and t^1 of them. By a direct computation, we can see that the coefficient of t^0 of the associativity property (5.4) corresponds exactly to the associativity property of $\dot{z}$, and the coefficient of t^1 corresponds exactly to the following property: for all $a, b, c \in V$ there exists $n \in \mathbb{N}$ such that

$$(w+z)^n \Big[f_w(a_{\dot{z}}\, b, c) + f_z(a,b)_{\dot{w}}\, c \Big] = (w+z)^n \Big[a_{z\dot{+}w}\, f_w(b,c) + f_{z+w}(a, b_{\dot{w}}\, c) \Big]. \tag{5.9}$$

Therefore, using (5.6), (5.7), (5.8) and (5.9), we have seen that

$$a \underset{z}{*} b = a_{\dot{z}}\, b + t\, f_z(a,b)$$

is a first order deformation of V if and only if $f_z \in Z^2(V,V)$.

Now we prove that two first order deformations of V are equivalent if and only if the difference between the corresponding elements in $Z^2(V,V)$ is in $\mathrm{Im}\delta_1$.

Consider two first order deformations of V given by

$$a \overset{(1)}{\underset{z}{*}} b = a_{\dot{z}}\, b + t\, \psi_z(a,b) \quad \text{and} \quad a \overset{(2)}{\underset{z}{*}} b = a_{\dot{z}}\, b + t\, \phi_z(a,b),$$

where ψ_z and ϕ_z are in $Z^2(V,V)$. They are equivalent if and only if there exists $f_t = 1_V + t\, g$ where $g : V \to V$ is a linear map such that

$$f_t(a \overset{(1)}{\underset{z}{*}} b) = f_t(a) \overset{(2)}{\underset{z}{*}} f_t(b) \mod t^2 \tag{5.10}$$

for $a, b \in V$. Now since

$$f_t(a \overset{(1)}{\underset{z}{*}} b) = f_t(a_{\dot{z}}\, b + t\, \psi_z(a,b)) = a_{\dot{z}}\, b + t\, \psi_z(a,b) + t\, g(a_{\dot{z}}\, b) \mod t^2$$

and

$$f_t(a) \overset{(2)}{\underset{z}{*}} f_t(b) = (a + t\, g(a)) \overset{(2)}{\underset{z}{*}} (b + t\, g(b)) = a_{\dot{z}}\, b + t\, a_{\dot{z}}\, g(b) + t\, g(a)_{\dot{z}}\, b + t\, \phi_z(a,b) \mod t^2$$

then (5.10) is equivalent to

$$\psi_z(a,b) - \phi_z(a,b) = a_{\dot{z}}\, g(b) - g(a_{\dot{z}}\, b) + g(a)_{\dot{z}}\, b$$

for all $a, b \in V$. Therefore, it is equivalent to $\psi_z - \phi_z = (\delta_1 g)_z$, finishing the proof. □

Acknowledgments

The author would like to thank M.V. Postiguillo and C. Bortni for their constant help and support throughout this work. Special thanks to C. Boyallian.

References

[1] B. Bakalov, V.G. Kac, Field algebras, Int. Math. Res. Not. IMRN (3) (2003) 123–159.
[2] B. Bakalov, V.G. Kac, A. Voronov, Cohomology of conformal algebras, Comm. Math. Phys. 200 (3) (1999) 561–598, MR1675121.
[3] R.E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Natl. Acad. Sci. USA 83 (1986) 3068–3071.
[4] R.E. Borcherds, Vertex algebras, in: Topological Field Theory, Primitive Forms and Related Topics, Kyoto, 1996, in: Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 35–77.
[5] A. De Sole, V. Kac, Lie conformal algebra cohomology and the variational complex, Comm. Math. Phys. 292 (3) (2009) 667–719, MR2551791.
[6] Y.Z. Huang, A cohomology theory of grading-restricted vertex algebras, Comm. Math. Phys. 327 (1) (2014) 279–307, MR3177939.
[7] Y.Z. Huang, First and second cohomologies of grading-restricted vertex algebras, Comm. Math. Phys. 327 (1) (2014) 261–278, MR3177938.
[8] V.G. Kac, Vertex Algebras for Beginners, second edition, Univ. Lecture Ser., vol. 10, American Mathematical Society, Providence, RI, 1996, 1998.
[9] J. Lepowsky, H. Li, Introduction to Vertex Operator Algebras and Their Representations, Progr. Math., vol. 227, Birkhäuser Boston, Inc., Boston, MA, 2004, xiv+318 pp.
[10] H. Li, Axiomatic G_1-vertex algebras, Commun. Contemp. Math. 5 (2) (2003) 281–327.
[11] J. Liberati, Tensor product of modules over a vertex algebra, available at http://arxiv.org/abs/1609.07551.
[12] F. Orosz, Algebra de Vértices, Undergraduate Thesis – Advisor: J. Liberati – Universidad Nacional de Córdoba, Facultad de Matemática, Astronomía y Física, http://www.famaf.unc.edu.ar/institucional/biblioteca/trabajos/601/16294.pdf.