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Dynamics of damped oscillations: physical pendulum

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Abstract
The frictional force between a physical damped pendulum and the medium is usually assumed to be proportional to the pendulum velocity. In this work, we investigate how the pendulum motion will be affected when the drag force is modeled using power-laws bigger than the usual 1 or 2, and we will show that such assumption leads to contradictions with experimental observations. For this purpose, a more general model of a damped pendulum is introduced, assuming a power-law with integer exponents in the damping term of the equation of motion, and also in the non-harmonic regime. A Runge–Kutta solver is implemented to compute the numerical solutions for the first five powers, showing that the linear drag has the fastest decay to rest, and that bigger exponents have long-time fluctuation around the equilibrium position, which have no correlation (as is expected) with experimental results.

Keywords: damped oscillations, physical pendulum, non-conservative systems

(Some figures may appear in colour only in the online journal)

1 Introduction

The study of oscillatory motion is one of the essential topics in engineering and physics science, since this allows us to understand how to apply the fundamental laws of classical mechanics to simple physical systems. In basic physics courses, the behavior of different types of oscillators and pendulums are analyzed, and systems such as the simple harmonic oscillator and the physical pendulum are studied in great depth. The most used textbooks,

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such as Halliday *et al* [1] and Serway and Jewett [2], develop these issues in a concise way, enriching the physics of these systems with some effects, such as damping of the medium, and the inclusion of external driving forces. However, these authors solve the equation of motion of the pendulum in the limit of small oscillations, i.e. they work in the so-called harmonic approximation.

In this context, we can start by remembering that the periodic motion exhibited by a pendulum is harmonic only for small oscillation angles. Beyond this limit, the equation of motion is nonlinear. Although an integral expression exists for the period of the nonlinear pendulum, it is usually not discussed in introductory physics courses because it is not possible to evaluate this integral exactly in terms of elementary functions. In other words, by elementary functions we are referring to functions that are studied in introductory calculus courses. For this reason, different approximate expressions for the dependence of the period on the angular amplitude of the pendulum have been developed by many authors. For example, a simple expression beyond the harmonic limit is given for a mass–string system in [3, 4]. Furthermore, the difficulty of writing the pendulum solution in terms of elementary functions is a problem that continues in the non-harmonic regime.

As was mentioned, the rich physics of pendulum dynamics is highlighted when different correction terms are included in the equation of motion. Corrections, such as the friction with the medium, effects such as the stretching of the pendulum string, friction at the pivot of the system, among others, are discussed in some theoretical, and experimental papers [5–8]. However, in this article we focus on the damped physical pendulum where the only external force included is the frictional force or drag with the medium.

Although the damped oscillations are of great importance in classical, and also quantum, mechanics [9, 10] in the aforementioned literature, there is limited information oriented to investigating the physical properties of pendulums with frictional forces in the non-harmonic approximation. Generally speaking, when a body moves in a fluid with a velocity v, the frictional force will depend on v. This dependence can be a very complicated function, but there are many practical situations where the viscous damping force will be proportional to some power of v. For objects moving at low speeds, where turbulence is negligible, the medium resistance is approximately proportional to its velocity. Also, in other applications, the drag may have terms proportional to the square or higher powers of the velocity beyond the Newtonian and Stokes drag, i.e. in general the resistance of the medium may be modeled using power-laws of the form $F_{\mathrm{fric}} \propto v^n$.

There are many applications, not necessarily harmonic oscillators or pendulums, where the frictional force is modeled using power-law damping [11–14]. Now, for the particular case of the physical pendulum, such assumptions are usually neglected since exponents greater than 2, or non-power-law for the drag resistance, have no connection with experimental observation. However, in this work we want to verify this linear relation, showing what happens to the pendulum if such mathematical freedom is allowed, and exploring some numerical techniques along the way. For this purpose, we will investigate the influence of the integer exponent n in the equation of motion of the damped physical pendulum. Additionally, we will give a mathematical approach where a generalized model is introduced, assuming the frictional force $F_{\mathrm{fric}} \propto v^n$ in the equation of motion of the physical pendulum. Different exponents are analyzed, showing that integers greater than two are not appropriate exponents for drag modeling, since the system will oscillate for a long time before stopping in the equilibrium position. Thus, the behavior presented by the system with this generalization is not correlated with the experimental observations. Additionally, we find that the fastest decay to the equilibrium position is for the linear exponent. To carry out this study, we introduce the Runge–Kutta method to solve the nonlinear differential equations that arise naturally when

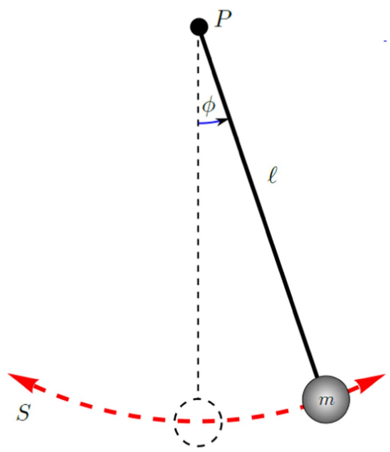

Figure 1. Simple pendulum of mass m.

the classical mechanical laws are applied to this generalized damped pendulum. Finally, we give some foundations and basic techniques used in the numerical analysis of systems of differential equations.

This article is organized as follows. A brief review of oscillations is given in section 2. Then, in section 3, we introduce a generalized damped pendulum, which introduces an integer exponent into the angular velocity of the differential equation. In section 4, a numerical analysis is performed, showing some comparative graphs for different values of the damping coefficient, and also taking different powers in the numerical solutions. In this section, we also analyze the stability for the numerical solutions obtained and the convergence rates of the numerical method implemented. Finally, we close the work by giving some final remarks and conclusions.

2 Review of oscillations

In this section, we will introduce some foundations for damped and undamped oscillations. A simple way to introduce this topic is by first considering the most studied case in basic physic courses, which is the simple pendulum. A simple pendulum consists of a mass m hanging from a string of length ℓ and negligible mass, fixed at a pivot point P. When the mass is displaced by an initial angle and is released, the pendulum will swing back and forth with periodic motion around an equilibrium position located at the lowest point of the trajectory, describing a length of arc S, as shown in figure 1.

When the initial position is small, the amplitude of the pendulum has no effect on the period, and all the description will be analogous to simple harmonic motion. Now, in more realistic situations, inevitably the pendulum loses energy due to frictional forces and, as a consequence, its amplitude decreases with time. To explain the gradual energy loss, different kinds of dissipative forces can be incorporated into the system. However, in the most practical situation, a non-conservative force, which is proportional to the pendulum speed, is assumed. This force can be written as

$$F_{\text{frict}} = -bv, \tag{1}$$

where b is a positive constant that depends on the medium, materials, and also on the body shape. Here, the minus indicates that the force is opposite to the relative movement of the mass with the medium. Now, it is possible to write equation (1) in terms of the angular velocity using $v = \ell \frac{\mathrm{d}\phi}{\mathrm{d}t}$, thus we have

$$F_{\text{frict}} = -b\ell \frac{\mathrm{d}\phi}{\mathrm{d}t}. \tag{2}$$

Now, using the Newton's second law, we can find the equation of motion of a damped pendulum,

$$m\frac{\mathrm{d}^2 S}{\mathrm{d}t^2} = -mg \sin\phi - b\ell \frac{\mathrm{d}\phi}{\mathrm{d}t}. \tag{3}$$

Taking into account that $S = \ell\phi$, we can write the following differential equation,

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}\phi}{\mathrm{d}t} + \frac{g}{\ell}\sin\phi = 0,\ \ddot{\phi} + \gamma\dot{\phi} + \omega_0^2 \sin\phi = 0, \tag{4}$$

where the dots means $\frac{\mathrm{d}}{\mathrm{d}t}$, $\gamma = b/m$ is the damping coefficient and $\omega_0^2 = g/\ell$ is the angular frequency of the undamped motion in the harmonic regime. Alternatively, one can arrive at equation (4) using torques together with Newton's law for rotations. Remember that torques, or moments of force, are defined by the following cross product,

$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{5}$$

Then, the Newton law equation can be written as follows,

$$\sum \vec{\tau} = I\vec{\alpha}, \tag{6}$$

where $\alpha = \ddot{\phi}$ is the angular acceleration, and I is the moment of inertia of the body. In the special case of a simple pendulum, the moment of inertia is given by

$$I = m\ell^2. \tag{7}$$

Now, equation (2) introduces the friction force in the system, so the friction torque of the non-vacuum medium can be expressed in the following way,

$$\tau_{\text{fric}} = F_{\text{fric}}\ell, = -\frac{b}{m} m\ell^2\dot{\phi}, = -\frac{b}{m} I\dot{\phi}. \tag{8}$$

Finally, equation (8) can be written in terms of the damping coefficient γ as follows,

$$\tau_{\text{fric}} = -\gamma I\dot{\phi}. \tag{9}$$

This equation can be used to describe the damping of any rigid body of moment of inertia I that oscillates around an equilibrium point.

Finally, we can introduce a more general pendulum, usually called a physical pendulum. A physical pendulum consists of a rigid body that undergoes a rotation about a fixed axis passing through a fixed point P (the pivot). Now, assuming a rigid body of inertia I and mass m, we can write the torque of the weight mg as follows,

$$\tau_{\text{w}} = -mg\ell \sin\phi, \tag{10}$$

this torque acts at the center of mass of the rigid body located at a distance ℓ from the pivot. Finally, combining the last equation together with equations (6) and (9) it is possible to write,

$$\ddot{\phi} + \gamma\dot{\phi} + \omega_0^2 \sin\phi = 0, \tag{11}$$

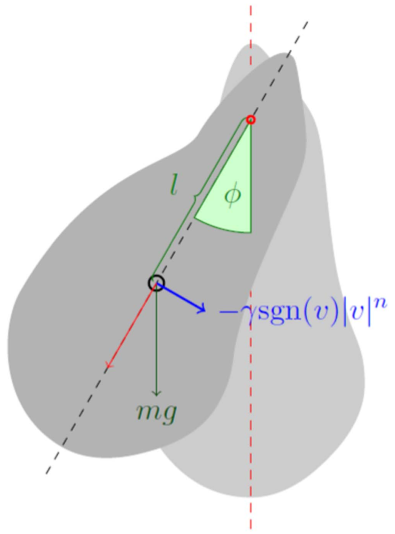

Figure 2. Physical pendulum.

where the angular frequency $\omega_0^2 = mg\ell/I$ corresponds to the frequency of the undamped motion for small amplitudes [15]. In the following section we will generalize the last equations and we will study some numerical solutions.

3 Generalized damped pendulum

Now we are ready to describe the motion of a more general damped pendulum introducing a modification in the equation of motion (11). The simple pendulum introduced in the previous section, can be considered as the simplest case of the generalized physical pendulum. Figure 2 shows a representation of a physical pendulum.

The damped physical pendulum is also a non-conservative system where an external torque works on it. As we mentioned above, in some situations one can assume that the friction is given by equation (9). However, in order to get a more general case, we will introduce the following generalized friction torque given by

$$\tau_{\text{fric}} = \begin{cases} -\gamma I\dot{\phi}^n, & \text{for odd } n, \\ -\gamma I \ \text{sgn}(\dot{\phi})\dot{\phi}^n, & \text{for even } n, \end{cases} \tag{12}$$

where n is assumed to be a positive integer exponent and sgn is the sign or signum function, which takes the values $1,\ 0,\ -1$ for $\dot{\phi} > 0$, $\dot{\phi} = 0$ and $\dot{\phi} < 0$, respectively. Since the drag is always directed against the direction of the velocity, the use of the signum function is very important in order to get an adequate modeling of the frictional force. In this way, the position, and the angular velocity, change their signs correctly from positive to negative (and vice versa) when the pendulum passes through the equilibrium point. In this schema, the damping constant γ needs to have dimensions of $\sec^{n-2}$ in order to be dimensionally consistent with the torque units. Now, both cases can be combined into one equation introducing a new parameter ϵ_n defined by

$$\epsilon_n = \begin{cases} 1, & \text{for odd } n, \\ \text{sgn}(\dot{\phi}), & \text{for even } n \end{cases}. \tag{13}$$

Finally, the friction torque can be written as

$$\tau_{\text{fric}} = -\gamma I \epsilon_n \dot{\phi}^n. \tag{14}$$

A more general approach can be obtained using the following equation,

$$\tau_{\text{fric}} = -\gamma I \dot{\phi}\, |\dot{\phi}|^{n-1}. \tag{15}$$

This alternative drag equation agrees with (14) in the case of integer exponents, however in equation (15) n can be assumed as a non-integer power, but such situations will not be analyzed in this article. Also, numerically, it is cheaper to use the function ϵ_n than the product $\dot{\phi}|\phi|^{n-1}$ since the factor ϵ_n can be introduced as an auxiliary int-function, which returns a constant value. For these reasons, and keeping in mind the objectives of this article, equation (14) for the frictional drag is more suitable than equation (15). Then, using (14) and (6) we can write,

$$\sum \vec{\tau} = I\vec{\alpha}, \; -mg\ell \sin\phi - \gamma I \epsilon_n \dot{\phi}^n = I\alpha,$$

and finally we get the following differential equation,

$$\ddot{\phi} + \gamma\epsilon_n \dot{\phi}^n + \omega_0^2 \sin\phi = 0, \tag{16}$$

where ω_0 is the same angular frequency introduced in the previous section. It is important to note that the non-conservative force described in this article is generated by the medium viscosity, usually considered as air, although other kind of frictional forces could also be introduced.

Now, we want to analyze some solutions of the last differential equation, and to study the motion of the physical pendulum for different n keeping $\gamma = \text{constant}$. Since equation (16) has no analytical solutions, we will solve this differential equation using numerical methods. Now, in order to implement the numerical method, it is necessary to write (16) as a system of ordinary differential equations (ODEs) introducing the following change of variables,

$$x_0 = \phi, \qquad x_1 = \dot{\phi}. \tag{17}$$

Using these new variables, the second order equation (16) can be written as a system of two first order differential equations,

$$\dot{x}_0 = x_1, \tag{18}$$

$$\dot{x}_1 = -\gamma\epsilon_n x_1^n - \omega_0^2 \sin x_0. \tag{19}$$

Alternatively, one can introduce a vector field $\vec{F}$, and write the system in the following way,

$$\dot{\vec{x}} = \vec{F}, \tag{20}$$

where $\vec{x} = [x_0, x_1]$, and the vector field $\vec{F} = [x_1, -\gamma\epsilon_n x_1^n - \omega_0^2 \sin x_0]$. In the next section, we will solve this system of equations using the Runge–Kutta method. The Runge–Kutta solver is a handle method to find the solution of the ODE system, the flexibility and easy implementation of this algorithm makes it a powerful tool to study the desired system given by equation (20).

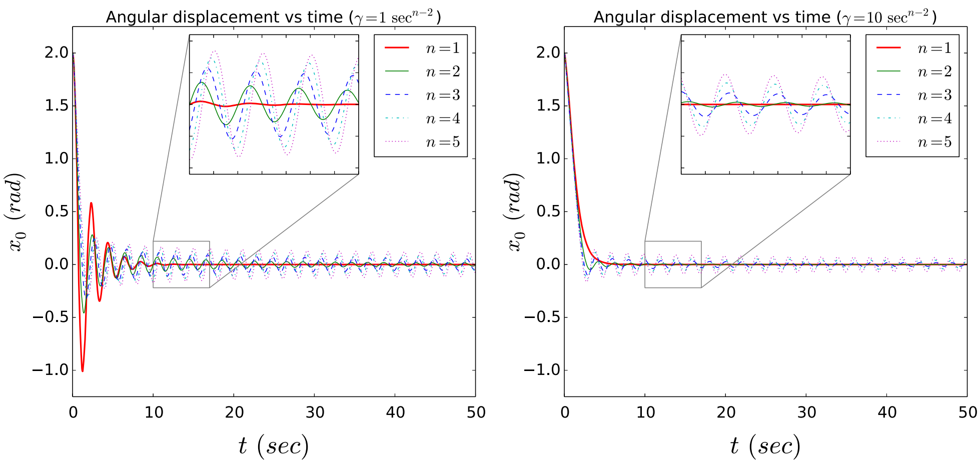

Figure 3. The angular displacement versus time for the first five powers. The left plot is for a small value of the damping constant γ, while the right one corresponds to an overdamped oscillation for $n = 1$.

4 Numerical analysis

In this section, we focus on a numerical study of equations (18)–(19) assuming a constant damping coefficient and considering the first five integer exponents. In our numerical study a Runge–Kutta of fourth order (RK4) is implemented in order to integrate the ODE system. However, in this article, we will not focus on the development of the numerical algorithm since there is much open and commercial software that includes numerical packages to solve systems of ODEs, RK4 being one of the most extended methods. A complete description of the RK family can be found in [16]. Additionally, there are many available codes written in C/C++, Fortan, and Python, which can be used to verify our results. Our C++ implementation can be downloaded from the following repository: https://github.com/GonzaQuiro/RK4.git.

All the numerical solutions are found assuming the following initial conditions:$x_0(0) = 2$, $x_1(0) = 0\ \mathrm{s}^{-1}$, and the parameters $m = 0.2$ kg, $\ell = 1$ m, $I = 0.2$ kg m^2, for two different damping coefficients $\gamma = 1$ and $\gamma = 10\ \mathrm{s}^{n-2}$. Also the step-size (the integration step of the RK4 method) is fixed to $h = 0.001$ s. Figure 3 shows the angular displacement of the physical pendulum for different n.

It is clear that the frictional torque will bring the system to the equilibrium position. However, the oscillations for $n > 1$ fall to zero very slowly as one can see in figure 3. In the 'stability analysis' section, we prove that the amplitude falls to zero when $t \to \infty$ using some stability criteria in order to support this affirmation and also discard errors in the numerical implementation. Furthermore, it is possible to observe that as n takes higher values, the oscillation amplitudes are greater, i.e. keeping $\gamma =$ constant the highest decay is for the lowest power. Additionally, for $n = 1$ and $\gamma = 10\ \mathrm{s}^{-1}$, the pendulum decays to the equilibrium position without oscillations, and this particular case corresponds to an overdamped pendulum [1, 2]. Now, when higher exponents are considered in the system, will be necessary bigger damping constants to overdamped the system as we show in figure 4,

In our generalized approach, if the curve for a particular integer n' falls to zero without oscillations, then every smaller exponent satisfying the condition $n \leqslant n'$ will also decay

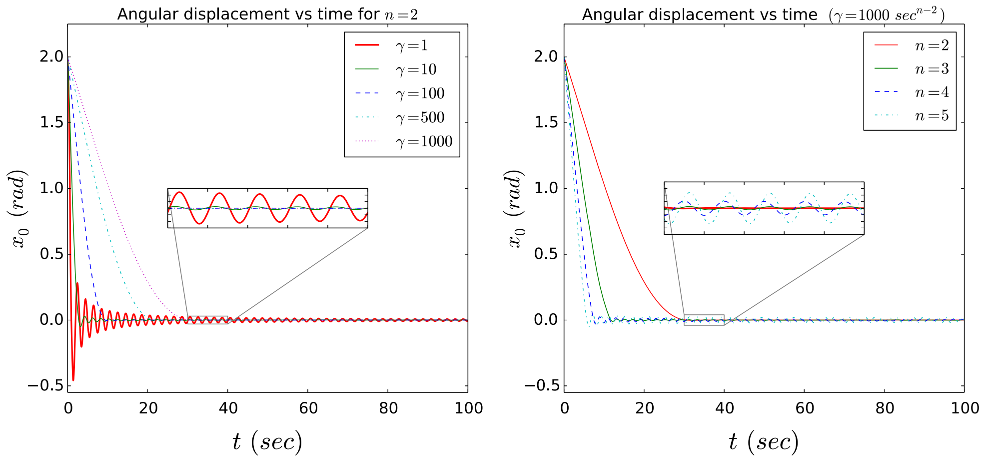

Figure 4. The angular displacement for $n = 2$ (left) for different γ and the behavior for five different powers (right) while the damping constant is $\gamma = 1000\ \mathrm{s}^{n-2}$.

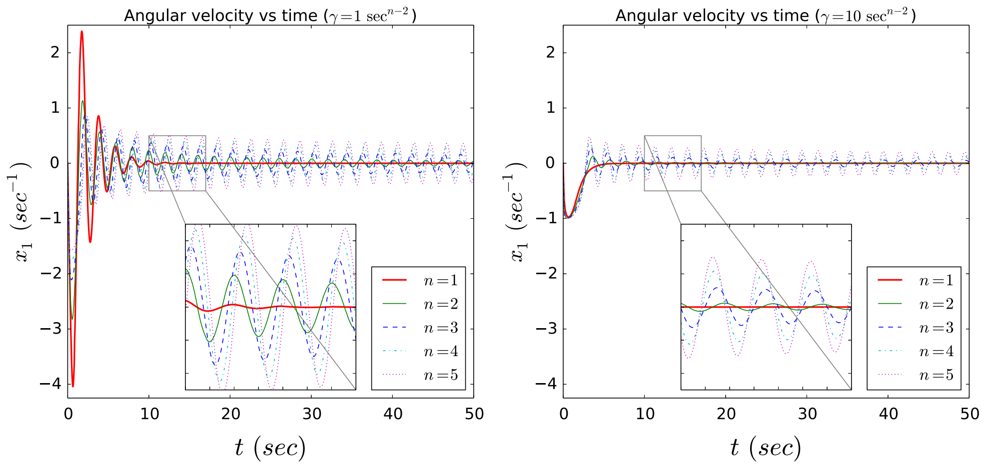

Figure 5. The angular velocity as a time function for $\gamma = 1\ \mathrm{s}^{n-2}$, and $\gamma = 10\ \mathrm{s}^{n-2}$.

without oscillations. In fact, for $n \neq 1$ the pendulum oscillates with small amplitudes around the equilibrium, even for the integer $n = 2$ small fluctuation around the point $x_0 = 0$ can be observed, but these small oscillations decrease as the gamma factor increases.

On the other hand, a similar behavior is observed when the angular velocity is plotted, and figure 5 show the curves $\dot{\phi}$ versus t for $\gamma = 1$ and $\gamma = 10\ \mathrm{s}^{n-2}$.

Again, for $n = 1$ the pendulum moves around the origin a few times before stopping completely, while for $n > 1$ the oscillations around the equilibrium point persist for long periods of time. Note that, beyond $n = 2$, all the powers will present similar oscillations reaching the rest at infinity. In this sense a non-lineal assumption in the damping term will affect the dynamical of the system, however the behavior shown by these exponents are not correlated with the experimental observations. This leads to the conclusion that the linear assumption is a good approximation to the usual pendulum drag.

An important observation can be made from figures 3 and 5: the ordering of the curves at short times is opposite to long times. That is, the amplitude is higher for lower exponents at short times, but this behavior is reversed at long times. This behavior can be understood by analyzing the frictional equation (14). At short times, when $\dot{\phi} > 1$, the effect of the frictional torque on the system is more significant for higher n since

$$|\gamma I \epsilon_{n+1} \dot{\phi}^{n+1}| > |\gamma I \epsilon_n \dot{\phi}^n| \qquad n \geqslant 1. \tag{21}$$

Thus, the effect of the friction on the system is the lowest for $n = 1$ causing the system to oscillate with the greatest amplitude, followed by $n = 2$, 3 and so on. On the other hand, for long times when the pendulum is in the small oscillation regime, and the velocity satisfies $\dot{\phi} < 1$, the previous equation takes the form

$$|\gamma I \epsilon_{n+1} \dot{\phi}^{n+1}| < |\gamma I \epsilon_n \dot{\phi}^n| \qquad n \geqslant 1. \tag{22}$$

For these cases the exponent $n = 1$ has the most significant contribution in the differential equation compared with $n > 1$, which explains the behavior of the curves in the figures.

In other words, for higher exponents when $\dot{\phi} < 1$, the pendulum oscillates around the equilibrium point for a very long time, these oscillations have bigger amplitudes when bigger integers are considered since the drag force decreases as n increases. In general terms, the condition $\phi < 1$ or $\phi > 1$ and the integer exponent have a direct impact on the equation of motion and also on the dynamics of the system.

4.1 Energy of the generalized damped pendulum

We want to analyze the energy for the physical damped pendulum, for when the mechanical energy of the system is introduced, and the trajectories on the phase portrait for the previously discussed solutions are plotted. The mechanical energy (E) is a scalar quantity which is defined by the sum of the gravitational potential energy (U), and the kinetic energy (T) as follows,

$$E = T + U = \frac{1}{2} I \dot{\phi}^2 + mg\ell(1 - \cos\phi). \tag{23}$$

Now, the damped pendulum is a system where the drag force dissipates energy (23), the falls for $\gamma = 1$, 10 s^{n-2} of the previous solutions can be observed in figure 6.

In figure 6, one can see the same behavior mentioned in the previous section for $n \neq 1$, where the energy falls to zero so slowly that the system oscillates with small amplitudes around the equilibrium point for a long time. However, it is possible to show that the oscillations will go to zero as $t \to \infty$ for every $n > 1$. This phenomenon becomes clear in the phase diagram, i.e. in a plot of the angular velocity versus the angular position (see figure 7).

In both graphs, the usual solution is for $n = 1$ to go to zero in few integration steps, while for greater exponents the fluctuation around $\phi = 0$ are bigger in amplitude and also energy.

4.2 Stability analysis

In this section we will analyze the stability of the solutions around the critical point located at $\phi = 0$. Although in figures 6 and 7 it seems that there are closed trajectories around the point (0, 0), it is possible to show using the Bendixon criterion [17] that there is no limit cycle (closed trajectories) in the phase space.

The Bendixon criterion states that, if the divergence of the vector field $\vec{F}$, $\nabla \cdot \vec{F}$, is not identically zero and does not change sign in the domain of $\vec{F}$, then (20) has no closed orbit. Therefore, computing the divergence of equation (20) we get

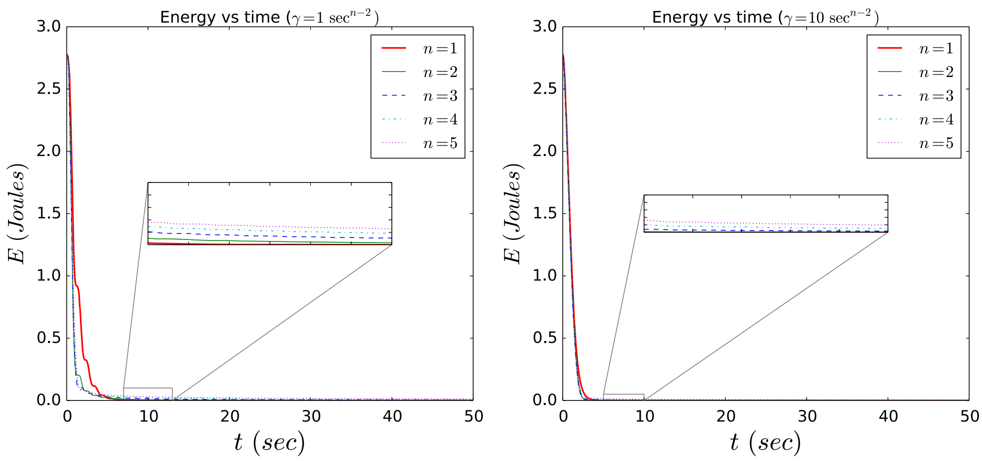

Figure 6. The energy decay for a damped physical pendulum for the first five integer powers.

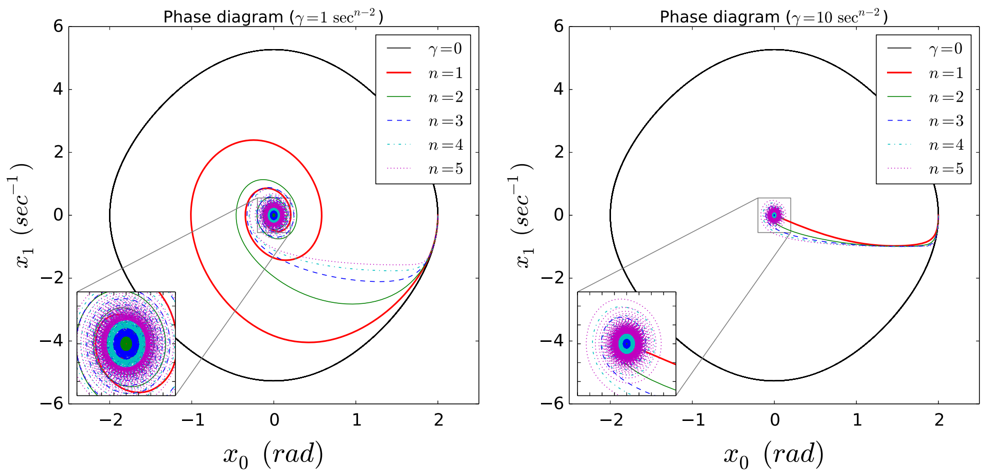

Figure 7. Phase diagrams for the damped pendulum. The closed circle corresponds to the undamped pendulum ($\gamma = 0$).

$$\nabla \cdot \vec{F} = -n\epsilon_n \gamma x_1^{n-1}. \tag{24}$$

Now, the rhs of the last equation is negative for odd n since the power $n - 1$ is always even and consequently the factor x_1^{n-1} is always positive. In addition, for even n, the difference $n - 1$ is odd, so the factor x_1^{n-1} can take negative and positive values, but the product $\epsilon_n x_1^{n-1}$ is always positive. Thus, the Bendixon criterion shows that there are no closed orbits around the critical point (0, 0), which means that the trajectories for $n \geqslant 1$ will fall to the attractor after giving a finite number of laps around the origin.

We can verify this affirmation by studying the relations between the Lyapunov functions and the stability of the systems. There are many methods to compute the Lyapunov coefficients, and these usually have local and global versions of Lyapunov's direct method. The local versions are concerned with the stability properties in the neighborhood of the equilibrium point and usually involve a locally positive definite function. In order to assert global asymptotic stability of the system, we assume that there exists a scalar function $E(\vec{x})$ with continuous first order derivatives such that [18],

$$E(\vec{x}) > 0; \qquad \text{is positive definite } \dot{E}(\vec{x}) < 0; \qquad \text{is negative definite}$$

then the equilibrium at the origin is globally asymptotically stable.

Now, from equation (23), together with equations (18)–(19), we can introduce the following function,

$$E(x_0, x_1) = \frac{1}{2}Ix_1^2 + mgl(1 - \cos x_0). \tag{25}$$

This function represents the total energy of the pendulum, composed of the sum of the potential energy and the kinetic energy. This function is a positive definite function, since $0 \leqslant (1 - \cos x_0) \leqslant 2$, and the other terms are quadratics, i.e. $E(x_0, x_1) > 0$. Now, the time derivative of equation (25) can be written as

$$\dot{E} = -\gamma I \epsilon_n \dot{x}_1^{n+1} \leqslant 0. \tag{26}$$

Finally, we have from equations (25) and (26) that $E(x_0, x_1) > 0$ and $\dot{E}(x_0, x_1) < 0$, so in agreement with the previous theorem, we conclude that the origin is a stable equilibrium position for all $n \geqslant 1$. Thus, the origin is an attractor for the system, i.e. a point where any solution ends up.

4.3 Numerical errors and convergence rates

In this section, the numerical errors and the convergence rates of the numerical results presented in the previous sections are studied. Initially, we compare the exact and the numerical solution for $n = 1$ with $\gamma = 0.1\ \mathrm{s}^{-1}$ in the regime of the small oscillations, i.e. assuming a small angular amplitude $x_0 = 0.1$ and setting again the initial velocity $x_1 = 0$.

In the last figure we plot the solution of (20) using the harmonic approximation $\sin(x_0) \approx x_0$, and the well known solution [15],

$$\hat{x}_0(t) = A\mathrm{e}^{-\frac{\gamma}{2}t}\cos(\omega t + \alpha), \tag{27}$$

where the hat is introduced to distinguish between the numerical solution $x_0(t)$ and the exact solution $\hat{x}_0(t)$. In addition, α is an initial phase and ω is the frequency of the oscillation which are given by

$$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}, \tag{28}$$

$$\tan\alpha = -\frac{\gamma}{2\omega} - \frac{\hat{x}_1(0)}{\hat{x}_0(0)\omega}. \tag{29}$$

It is possible to compute all the constants using the initial conditions and the parameter mentioned above to find,

$$\omega = 3.130095845, \tag{30}$$

$$\alpha = -0.01597259328, \tag{31}$$

$$A = 0.1000127575, \tag{32}$$

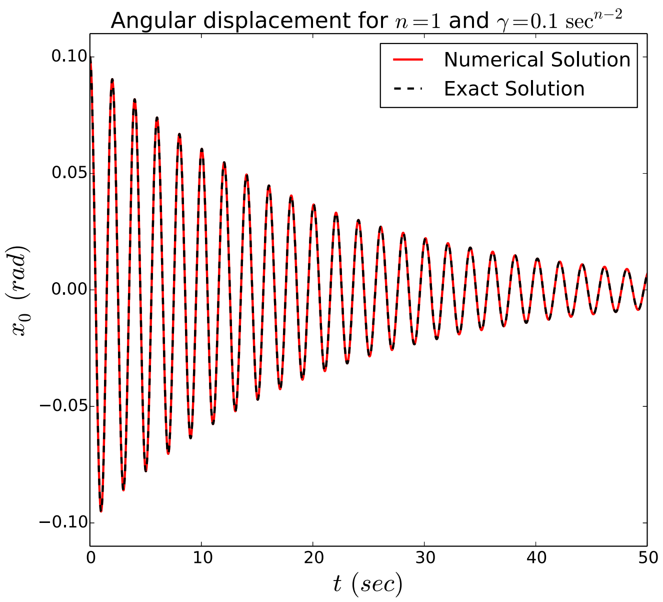

Figure 8. Matching between the exact and numerical solution for the underdamped pendulum in the harmonic approximation.

Therefore, we can write the exact solution as follows,

$$\hat{x}_0(t) = 0.1000127575\mathrm{e}^{-0.05t}\cos(3.130095845t - 0.01597259328). \tag{33}$$

Both solutions are plotted in figure 8, showing that they splice perfectly.

Now, we introduce the precision quotient Q to measure the convergence order of the RK4 implementation. The precision quotient Q is found by computing the ratio of the differences between the numerical solution with step sizes h, $h/2$ and $h/4$. The coefficient Q is usually defined as follows,

$$Q(t) = \frac{\sqrt{(x_0^{(h)} - x_0^{\left(\frac{h}{2}\right)})^2 + (x_1^{(h)} - x_1^{\left(\frac{h}{2}\right)})^2}}{\sqrt{(x_0^{\left(\frac{h}{2}\right)} - x_0^{\left(\frac{h}{4}\right)})^2 + (x_1^{\left(\frac{h}{2}\right)} - x_1^{\left(\frac{h}{4}\right)})^2}}. \tag{34}$$

For a well implemented numerical method it is expected that the coefficient Q has fluctuations around 2^m, or tend to this value as time increases. This means that step by step the ratio should be close to 2^m. In equation (34) m is the order of the numerical method, in the special case of the RK4 solver $m = 4$, thus $Q(t)$ must go to 16 in short, mid, or long times. It is important to note that the rhs. of (34) is a time-dependent function since $x_0 = x_0(t)$ and $x_1 = x_1(t)$. Now, the numerical results are presented in figure 9.

The last figure shows the convergence ratio for $2 \leqslant n \leqslant 5$, and the exponents $n \geqslant 2$ clearly are straight lines with a constant value of 16. However, the exponent $n = 2$ has large fluctuations around 1. This behavior arises when the numerical solutions for the steps h, $h/2$, $h/4$ are closer to each other. In this case, the numerical solutions tend to the exact solution such that the quotient Q cannot measure the precision of the method at all times, only in the initial instants. In other words, in these situations, the quotient starts around 16 with jumps to huge values, as one can see in figure 9 where Q is around 16 and goes to 50 in a few

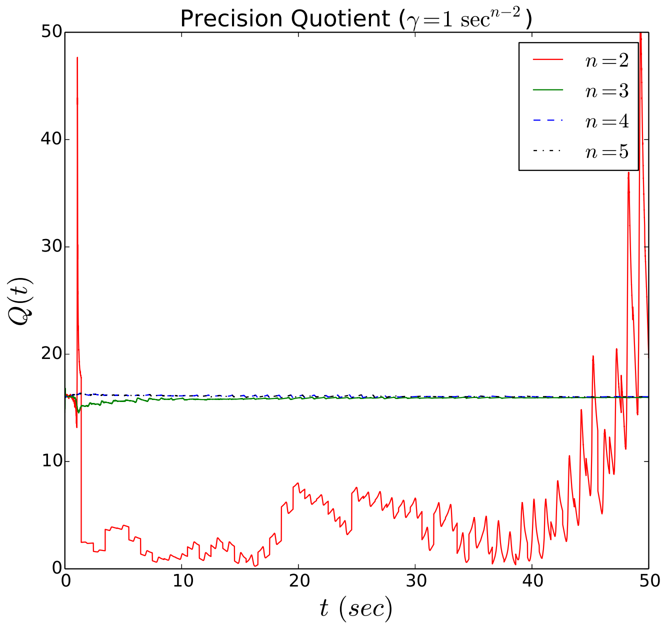

Figure 9. Numerical convergence test for the cases $1 < n \leqslant 5$ keeping $\gamma = 1$ and using the initial conditions (2, 0).

seconds, then decreases to values close to one, i.e. $Q \approx 1$. Finally, based on figures 8 and 9, we can confirm that our numerical implementation works adequately.

5 Final remarks

A more general model of a damped pendulum was introduced to analyze how the assumption of power-laws in the damping term affect the dynamic of the system. This model considers an integer exponent n in the frictional force of the differential equation and, in the case of even powers, includes the signum function to model adequately the medium drag. From this assumption, we can show that the behavior of the pendulum for $n > 2$ does not correspond to the observed motion in the laboratory.

In order to solve the generalized pendulum, some numerical solutions of the generalized differential equation in the non-harmonic regime were computed. The numerical solutions were found by implementing the traditional RK4 solver, our main results were presented in a series of plots, and the stability and the convergence of the numerical method was studied. Our analysis shows that the amplitudes and the mechanical energy fall to zero when t goes to ∞. Furthermore, the curves show small oscillations or fluctuation with low amplitudes for $n > 2$ around the equilibrium point. These oscillations will remain even with γ extremely large, and the oscillations have bigger amplitudes when bigger integers are considered. Finally, it was possible to conclude that $n = 1$, i.e. the linear power, has the fastest decay to the equilibrium position.

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