OPTIMAL CONTROL APPLICATIONS AND METHODS
Optim. Control Appl. Meth. (2007)
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/oca.824

Finding initial costates in finite-horizon nonlinear-quadratic optimal control problems

Vicente Costanza*, †

Nonlinear Systems Group, INTEC (UNL-CONICET), Güemes 3450, S3000GLN Santa Fe, Argentina

SUMMARY

A procedure for obtaining the initial value of the costate in a regular, finite-horizon, nonlinear-quadratic problem is devised in dimension one. The optimal control can then be constructed from the solution to the Hamiltonian equations, integrated on-line. The initial costate is found by successively solving two first-order, quasi-linear, partial differential equations (PDEs), whose independent variables are the time-horizon duration T and the final-penalty coefficient S. These PDEs need to be integrated off-line, the solution rendering not only the initial condition for the costate sought in the particular (T, S)-situation but also additional information on the boundary values of the whole two-parameter family of control problems, that can be used for design purposes. Results are tested against exact solutions of the PDEs for linear systems and also compared with numerical solutions of the bilinear-quadratic problem obtained through a power-series' expansion approach. Bilinear systems are specially treated in their character of universal approximations of nonlinear systems with bounded controls during finite time-periods. Copyright © 2007 John Wiley & Sons, Ltd.

Received 1 September 2006; Revised 27 July 2007; Accepted 15 August 2007

KEY WORDS: nonlinear systems; optimal control; Hamiltonian equations; invariant imbedding; bilinear systems

1. INTRODUCTION

The Hamiltonian formalism has been at the core of the development of modern optimal control theory [1]. When the problem concerning an n-dimensional system and an additive cost objective is regular [2], i.e. when the Hamiltonian of the problem can be uniquely optimized by a control value u^0 depending on the remaining variables (t, x, λ), then a set of $2n$ ordinary differential equations (ODEs) with a two-point boundary-value condition, known as Hamilton's (or Hamiltonian) equations (HE), have to be solved. This is often a rather difficult numerical problem. For the

*Correspondence to: Vicente Costanza, Nonlinear Systems Group, INTEC (UNL-CONICET), Güemes 3450, S3000GLN Santa Fe, Argentina.
†E-mail: tsinoli@ceride.gov.ar

linear-quadratic regulator (LQR) with a finite horizon there exist well known methods (see for instance [3]) to transform the boundary-value problem into an initial-value one. In the infinite-horizon, bilinear-quadratic regulator and change of set-point servo, there is a recent attempt to find the missing initial condition for the costate variable, which allows to integrate the equations on-line with the underlying control process [4].

For general nonlinear systems Σ and arbitrary cost objectives $\mathscr{J}$, optimal control problems do not have a standard and effective solution strategy. The subclass of bilinear systems $\tilde{\Sigma}$ may be used as universal approximations to non-linear systems in the following sense: if the admissible values for the control variable $u(.)$ are bounded during a fixed finite period of time $[t_0, t_f]$, then a bilinear system $\tilde{\Sigma}$ can be found such that each of its state trajectories $\tilde{x}_u(.)$ differs from the trajectories $x_u(.)$ corresponding to the same control applied to the original Σ, during a time-period inside $[t_0, t_f]$, in less than an arbitrary tolerance $tol>0$ fixed *a priori* (see [5–7]). It is well known that a similar result cannot be obtained by using linear approximations, which are good only in a strictly local character, and since it is crucial to count with at least a 'sampling' period of duration where the quality of the approximation is guaranteed, then bilinear models will receive special attention in this paper. 'Suboptimal control' will mean in this context that the optimal control strategies constructed for a bilinear approximation $\tilde{\Sigma}$ and a given cost objective $\mathscr{J}$ will only approximate the solution to the optimal control problem for $(\Sigma, \mathscr{J})$. But the main advantage of using this type of approximations comes from the fact that many control situations have been clarified and solved for the class of bilinear systems, by independent methods, as in the cases of suboptimal bilinear-quadratic regulation [8–10], infinite-horizon change of set-points [4], finite-horizon tracking [11], nonlinear observer design [12], and the Kalman–Bucy filter [13].

Hamiltonian systems (modelled by a $2n$-dimensional ODE whose vector field can be expressed in terms of the partial derivatives of an underlying 'total energy' function—called 'the Hamiltonian'—constant along trajectories) are key objects in Mathematical Physics. The ODEs for the state and costate of an optimal control problem referred above constitute a Hamiltonian system from this general point of view. Bellman has contributed in both fields but was particularly interested in symplectic systems coming from Physics (see for instance [14]) when he devised a partial differential equation (PDE) for the final value of the state $x(t_f)=r(T,c)$ as a function of the duration of the process $T=t_f-t_0$, and of the final value imposed to the costate $\lambda(t_f)=c$ (one of the boundary conditions, the other being the fixed initial value of the state $x(t_0)=x_0$, see [15]). Bellman exploited in that case ideas common to the 'invariant imbedding' numerical techniques, also associated with his name.

In this article the invariant imbedding approach is generalized and proved for the nonlinear-quadratic optimal control situation, where the final value of the costate depends on the final value of the state, i.e. $c=c(r)$. The procedure followed in this proof induces another PDE for the initial value σ of the costate $\lambda(t_0)$, which was actually the main concern from the optimal control point of view. The first-order quasilinear equation for σ developed here is new. It can be integrated after the PDE for the final state ρ (independent of σ) has been solved. The 'initial' condition for σ depends on the final value of the state and the weight matrix S involved in the quadratic final-penalty $x'(T)Sx(T)$. Therefore, it seems more natural to consider here (T, S) as the independent parameters of the family of control problems under optimization. Having found the solution $\sigma(T, S)$, the HE can be integrated for each particular value of the parameters. However, the whole curve $\sigma(., S)$ can be useful in real time, as a kind of safeguard against unexpected departures of the numerical solution to HE, which carry some instabilities near equilibrium, inherent to the structure of the spectrum of their linearizations.

DOI: 10.1002/oca

The results are applicable to situations for which the initial state is known exactly (so that an open-loop solution to the optimal control problem is adequate). In most applications disturbances are present so that a state-feedback solution, provided by the usual Hamilton–Jacobi–Bellman equation (a nonlinear PDE in (t, x)), would be preferable (but more difficult to obtain).

The paper is organized as follows: in Section 2 the finite-horizon nonlinear-quadratic regulator problem is posed, the invariant imbedding approach is developed in Section 3, and illustrated in Section 4 for the linear dynamics' case in dimension one, for which exact solutions are known. In Section 5 bilinear-quadratic regulation is explored in connection with an independent method based on the generalized (x-dependent) Riccati differential equation (DRE). The results coming from both methodologies are found to be consistent, at least in dimension one, where symbolic calculations are possible. Finally, in Section 6 the invariant imbedding approach is applied to the 'optimal change of set-points' problem for bilinear systems, and its features discussed in comparison with the known series-expansion solution, developed for the same situation viewed as a tracking problem. Conclusions are exposed in Section 7.

2. SYSTEM DYNAMICS AND OBJECTIVE FUNCTIONS. THE OPTIMAL REGULATOR PROBLEM

The article applies to initialized, time-constant, finite-dimensional, deterministic, nonlinear control systems of the form

$$\dot{x} = f(x) + g(x)u = X(x, u), \quad x(0) = x_0 \tag{1}$$

with the state x evolving in some open set of $\mathbb{R}^n$, admissible control trajectories are piecewise continuous functions $u : [0, T) \to \mathbb{R}$, and f, g are smooth vector fields in their domains.

Only classical quadratic costs will be considered, for instance (for the 'regulation' case)

$$\mathscr{J}(u) = \int_0^T [x'(t)Qx(t) + Ru^2(t)]\,\mathrm{d}t + x'(T)Sx(T) \tag{2}$$

where $\mathscr{J}(u)$ is a short form for $\mathscr{J}(0, T, x_0, u(.))$, i.e. the sum of the cost generated by the admissible control strategy $u(.)$ and by the state trajectory $x(.) = \Phi(., x_0, u(.))$ (where Φ is the transition function of system (1)) measured by the integral of the Lagrangian $L(x, u) = x'Qx + Ru^2$ (Q and R constant for simplicity) plus a final penalty weighted by S. Standard assumptions on matrices $Q, S \geqslant 0$ and $R > 0$, will be imposed. This whole situation will be considered as a (T, S)-nonlinear-quadratic optimal control problem, the other parameters playing a secondary role in the sequel.

Clearly the desired state is the origin, so the problem consists on abating perturbations $x_0 \neq 0$ with optimal cost. The Hamiltonian of the problem is defined as usual

$$H(t, x, \lambda, u) \triangleq L + \lambda' X = x'Qx + Ru^2 + \lambda'[f(x) + g(x)u]$$

where λ is the adjoint or costate variable, with values in $\mathbb{R}^n$. Time constancy of the dynamics and constant coefficients in the Lagrangian imply that t does not appear explicitly, so $H(x, \lambda, u)$ will be used in what follows. H is assumed to be regular. Precisely, for each pair (x, λ) there exists

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a unique minimizing control value $u^0(x,\lambda)$ such that the 'optimal' Hamiltonian is obtained at

$$\mathscr{H}^0(x,\lambda)=H(x,\lambda,u^0(x,\lambda))$$

with u^0 a smooth function of its variables.

Then the HE associated with this problem, namely

$$\dot{x}=\left(\frac{\partial\mathscr{H}^0}{\partial\lambda}\right)'\triangleq\mathscr{F}(x,\lambda),\quad x(0)=x_0 \tag{3}$$

$$\dot{\lambda}=-\left(\frac{\partial\mathscr{H}^0}{\partial x}\right)'\triangleq-\mathscr{G}(x,\lambda),\quad \lambda(T)=2Sx(T) \tag{4}$$

can in principle be written as a $2n$-dimensional ODE in (x,λ) (see [1, 3] for the foundations of this two-point boundary-value problem). If the HE can be solved on-line, then the optimal control could be evaluated from their solution, in a state/costate feedback fashion,

$$u^*(t)=u^0(x(t),\lambda(t)) \tag{5}$$

3. THE INVARIANT IMBEDDING APPROACH

Equations (3) and (4) pose a type of mixed two-point boundary-value problem. For linear systems it can be reformulated as an initial-value matrix ODE (see for instance [3]), but no general procedure applies to nonlinear systems (see for instance [16]). In the case of bilinear systems it has been found [8] that the costate admits the formal expression:

$$\lambda(t,x)=2\pi(t,x)x$$

for some matrix-valued function $\pi(t,x)$, and recently it has been found [10] that $\pi(t,x)$ must obey a first-order PDE with boundary condition:

$$\pi(T,x)=S\quad\forall x$$

Under this approach, a matrix PDE would need to be solved for each combination of the parameters (T,S) in order to obtain $\pi^{T,S}(t,x)$. Unless specifications for a process are too strict, usually both T and S would allow for some degree of flexibility, either at the design level or induced by economical arguments and reflected on operational changes. Whatever the reasons, it is clearly useful to know how the control changes in connection to changes on the design parameters. Following this objectives, since the finite-horizon situation is essentially a two-point boundary-value problem, it is natural to revise some ideas associated with Bellman's 'invariant imbedding' approach [15].

Let us denote by $\rho(T,S)$ the final value of the state x, and by $\sigma(T,S)$ the initial value of the costate λ in the problem at hand. Let us also assume that for each combination of parameters (T,S) the two-point boundary-value problem has a unique solution, varying smoothly with smooth changes in parameters. For the system (3)–(4) there will be assumed to exist a smooth flow

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$\phi : \mathbb{R} \times O \to \mathbb{R}^n$, with O some sufficiently large open set in $\mathbb{R}^n \times \mathbb{R}^n$, such that $\phi(t, x, \lambda)$ will render the values of the state and costate at time t along the trajectory starting (when $t = 0$) at the generic value (x, λ). The following notation indicates that the two 'components' of the flow, referring to the state (denoted by ϕ_1 below) and the costate (denoted ϕ_2), each in $\mathbb{R}^n$, will be considered separately

$$\phi(t, x, \lambda) = (\phi_1(t, x, \lambda), \phi_2(t, x, \lambda)) \tag{6}$$

and, accordingly, the following identities become clear:

$$\rho(T, S) \triangleq x^{T,S}(T) = \phi_1(T, x_0, \sigma(T, S)) \tag{7}$$

$$\sigma(T, S) \triangleq \lambda^{T,S}(0) \tag{8}$$

$$\lambda^{T,S}(T) = 2S\rho(T, S) = \phi_2(T, x_0, \sigma(T, S)) \tag{9}$$

where $(x^{T,S}(.), \lambda^{T,S}(.))$ is the optimal trajectory corresponding to a fixed horizon duration T and a fixed penalty weights, and the point in (.) stands for the independent variable time $t \in [0, T]$.

By taking partial derivatives with respect to T in Equation (7) (the notation $D_1 = \partial/\partial T$, and similarly for other partial derivatives, is adopted when clarity seems necessary),

$$D_1\rho(T, S) = D_1\phi_1(T, x_0, \sigma(T, S)) + D_3\phi_1(T, x_0, \sigma(T, S))D_1\sigma(T, S) \tag{10}$$

Since the existence of the flow implies

$$D_1\phi(t, x, \lambda) = (\mathscr{F}(\phi(t, x, \lambda)), -\mathscr{G}(\phi(t, x, \lambda))) \tag{11}$$

with $\mathscr{F}$ and $\mathscr{G}$ as defined in Equations (3) and (4); then, at the final time T,

$$\begin{aligned} D_1\phi_1(T, x_0, \sigma(T, S)) &= \mathscr{F}(\phi(T, x_0, \sigma(T, S))) \\ &= \mathscr{F}(\phi_1(T, x_0, \sigma(T, S)), \phi_2(T, x_0, \sigma(T, S))) \\ &= \mathscr{F}(x(T), \lambda(T)) = \mathscr{F}(\rho(T, S), 2S\rho(T, S)) \end{aligned} \tag{12}$$

which will turn Equation (10) to be written in simplified notation as

$$\rho_T = F + \phi_{1_\lambda}\sigma_T \tag{13}$$

with the following conventions

$$F(\rho, S) \triangleq \mathscr{F}(\rho, 2S\rho) \tag{14}$$

$$\rho_T \triangleq D_1\rho(T, S) = \frac{\partial \rho}{\partial T}(T, S) \tag{15}$$

Now, the derivative of Equation (7) with respect to S gives

$$\rho_S = \phi_{1_\lambda}\sigma_S \tag{16}$$

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and repeating the procedure $(\partial/\partial T, \partial/\partial S)$ with Equation (9) renders

$$2S\rho_T = -G + \phi_{2_\lambda}\sigma_T \tag{17}$$

$$2(\rho + S\rho_S) = \phi_{2_\lambda}\sigma_S \tag{18}$$

The treatment proceeds in dimension one. Formally, from Equations (13) and (16) ϕ_{1_λ} can be eliminated; and so can ϕ_{2_λ} from Equations (17) and (18); to obtain, respectively,

$$\rho_S\sigma_T + (F - \rho_T)\sigma_S = 0 \tag{19}$$

$$2(\rho + S\rho_S)\sigma_T - (2S\rho_T + G)\sigma_S = 0 \tag{20}$$

Therefore, in order for the problem to have a nontrivial solution, the determinant of the augmented system must be null, i.e.

$$\rho\rho_T - \left(SF + \frac{G}{2}\right)\rho_S = \rho F \tag{21}$$

which is a first-order, quasi-linear PDE for ρ, that can be integrated independently from σ, with the boundary condition

$$\rho(0, S) = x_0 \tag{22}$$

Actually, the integration of Equation (21) for ρ is not so valuable in itself but for its relation to the initial value σ of the costate variable λ. After ρ is calculated numerically, then σ can be integrated from this information. For instance, from Equations (19) and (21) it follows that

$$\rho\sigma_T - \left(SF + \frac{G}{2}\right)\sigma_S = 0 \tag{23}$$

which is a linear, homogeneous, first-order PDE for σ, subject to the boundary condition

$$\sigma(0, S) = 2Sx_0 \tag{24}$$

4. SOME VALIDATIONS FROM THE ONE-DIMENSIONAL LINEAR CASE

The classical LQR problem [3, 17] with $\mathscr{F} = Ax - (W/2)\lambda, \mathscr{G} = 2Qx + A'\lambda$, where $W = BR^{-1}B'$, allows in the one-dimensional case for symbolic integration of Equation (21), since the PDE takes the simple form

$$\rho_T - (2AS - S^2W + Q)\rho_S = (A - WS)\rho \tag{25}$$

The solution of the problem is

$$\rho(T, S) = \frac{1 - \gamma(S)}{1 - \gamma(S)\mathrm{e}^{-kT}}\mathrm{e}^{-(k/2)T}x_0 \tag{26}$$

where $k/2 = \sqrt{A^2 + WQ} = -(A - W\pi_0) > 0$ (minus the closed-loop gain for the $T = \infty$ case), π_0 denoting the solution to the algebraic Riccati equation (ARE) of the problem, namely

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$p_2(S)=0$, with $p_2(S)=-(2AS-S^2W+Q)$, $R_0(S)=\int \mathrm{d}S/p_2(S)$, and $\gamma(S)=\mathrm{e}^{kR_0(S)}=(2WS-2A-k)/(2WS-2A+k)$. Note that $\rho(T,\pi_0)=\mathrm{e}^{-(k/2)T}x_0$, then it coincides with the steady-state solution to the problem, as expected.

Figures 1 and 2 show the numerical solution of Equation (25) with boundary condition as in Equation (22). In Figures 3 and 4, the evolution of two particular (fixed S-values) curves $\rho^S(.)\triangleq\rho(.,S)$ as functions of the time-horizon T is shown, combined with the state solutions obtained from the DRE for some values of the final time T and the same final penalty weights S. The two curves correspond to values of $S>\pi_0$ and $S<\pi_0$, respectively.

It is instructive to inquire into the meaning of each curve $\rho^S(.)$. For instance, it can be immediately verified that these curves obey the dynamics

$$\dot{\rho}^S = A\rho^S + B\tilde{u}^S \tag{27}$$

with some hypothetical control $\tilde{u}^S(.)\triangleq -BR^{-1}\pi^S(.)\rho^S(.)$, where

$$\pi^S(T)=\frac{1}{W}\left(A+\frac{k}{2}\frac{1+\gamma(S)\mathrm{e}^{-kT}}{1-\gamma(S)\mathrm{e}^{-kT}}\right) \tag{28}$$

It is straightforward to check that $\pi^S(.)$ is a solution to the 'opposite' DRE

$$\dot{\pi}^S=-p_2(\pi^S),\quad \pi^S(0)=S$$

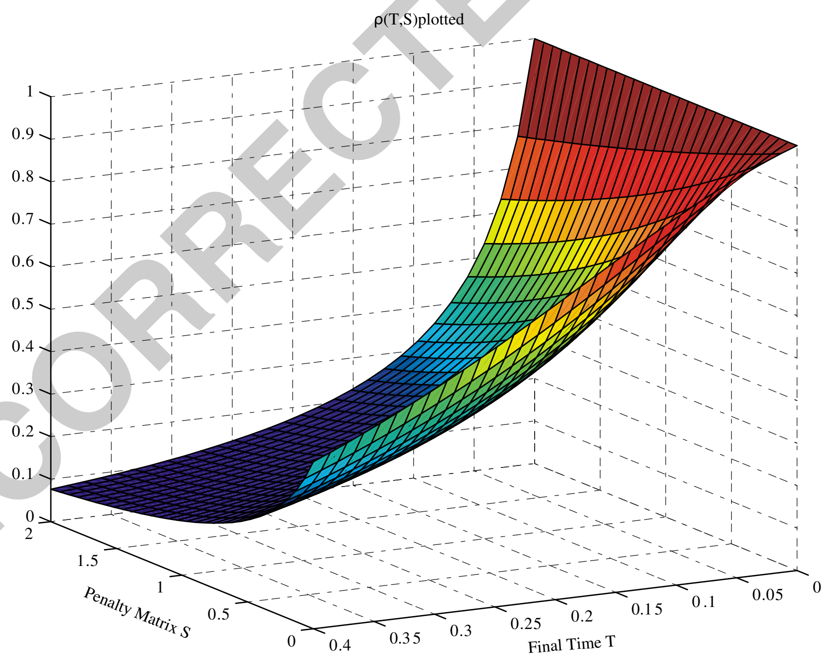

Figure 1. Solution to the PDE for $\rho(T,S)$ for the regulation case. Initial condition $x_0=1$. Other parameters: $A=-1, B=1, Q=1, R=0.1, W=B^2/R=10$.

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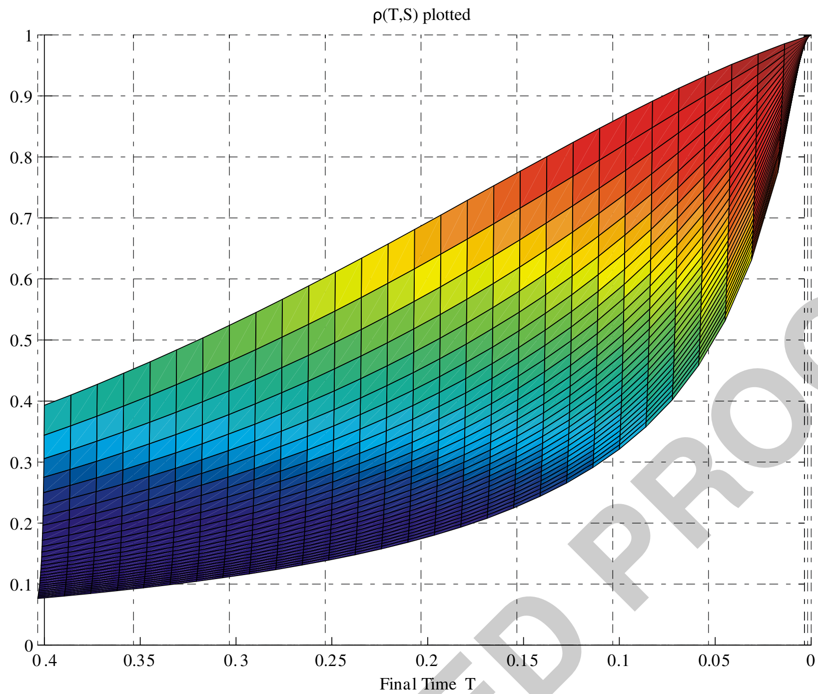

Figure 2. Projection of the graph of $\rho(T, S)$ onto the $(T; \rho)$ plane. Different curves $\rho^S(T)$, for values of S ranging from $S=0$ (uppermost) to $S=2$, can be visualized. Parameter values are as in Figure 1.

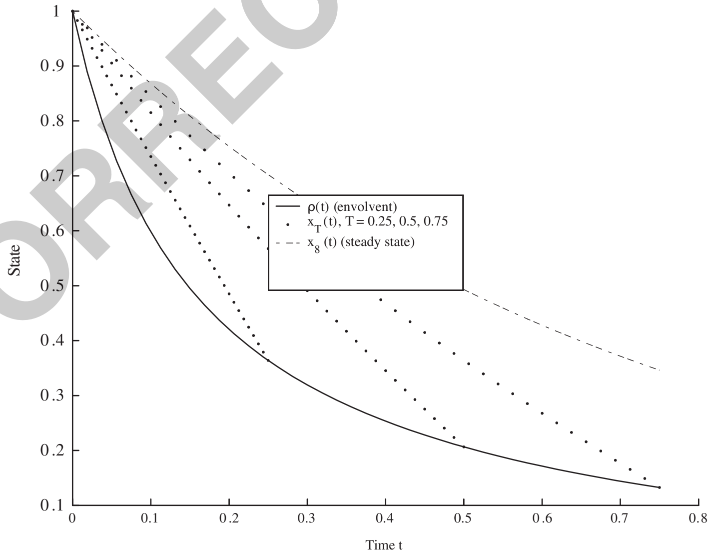

Figure 3. $\rho^S(.)=\rho(S,.)$ for a fixed value of $S=5.6>\pi_0$ and several trajectories $x^{T,S}(.)$ for different values of the final time T. Other parameters are as in Figure 1.

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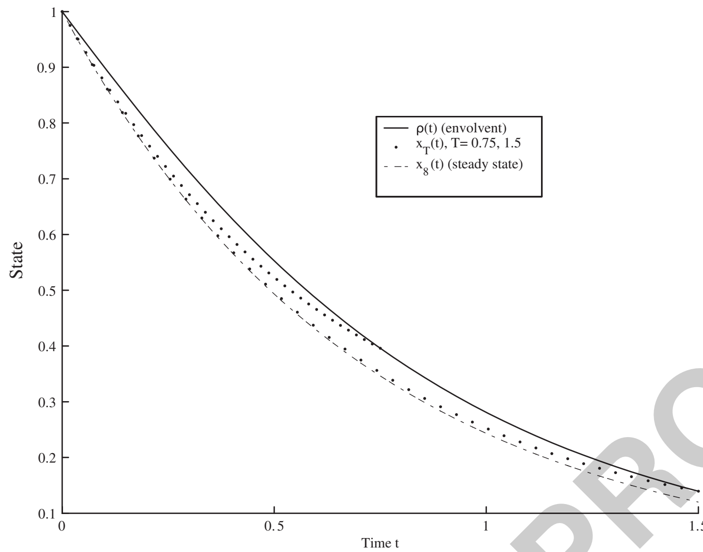

Figure 4. $\rho^S(.) = \rho(S,.)$ for a fixed value of $S = 0.01 < \pi_0$ and several trajectories $x^{T,S}(.)$ for different values of the final time T. Fixed values other than S as in Figure 3.

and that (see [3])

$$\lim_{T\to\infty} \pi^S(T) = \pi_0$$

Therefore, for fixed S, this expression for π^S contains all solutions $\pi^{T,S}(.)$ to the 'right' DRE

$$\dot{\pi} = p_2(\pi), \quad \pi(T) = S$$

that can be obtained through the simple change of variables

$$\pi^{T,S}(t) = \pi^S(T-t)$$

The relationship between π^S and the different $\pi^{T,S}$ corresponding to time-horizons of duration T is illustrated in Figure 5.

Finally, the solution to Equation (23) for the linear case, i.e. of

$$\sigma_T + p_2(S)\sigma_S = 0$$

with the boundary condition given in Equation (24), can be integrated alone since it is independent of ρ and its derivatives to give

$$\sigma(T,S) = \frac{(2A-k)\gamma(S)\mathrm{e}^{-kT} - (2A+k)}{W[\gamma(S)\mathrm{e}^{-kT} - 1]} x_0 \tag{29}$$

which clearly verifies

$$\sigma(T,S) = \lambda^{T,S}(0) = 2\pi^{T,S}(0)x^{T,S}(0) = 2\pi^S(T)x_0 \tag{30}$$

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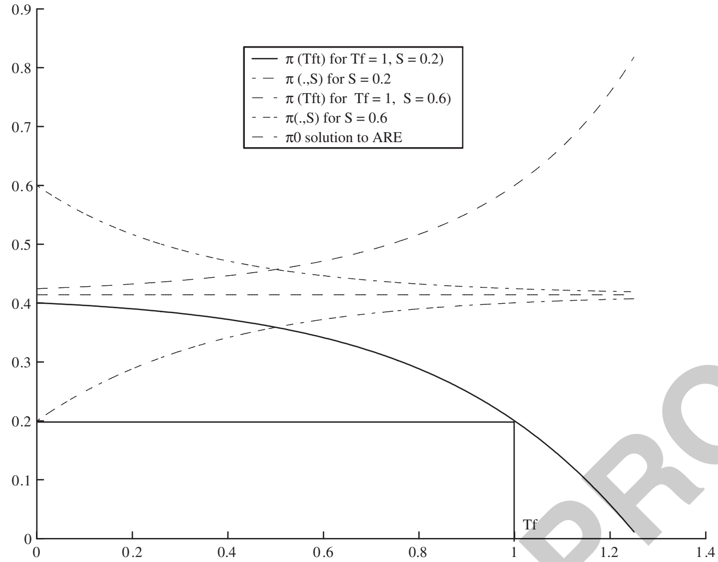

Figure 5. Curves $\pi^S(T)$ for $S=0.8>\pi_0=\sqrt{2}-1$ and $S=0.2<\pi_0$, both approaching π_0 as $T\to\infty$. For the same values of S, curves $\pi^{T_f,S}(t)$, with $T_f=1$, extended for $t>T_f$.

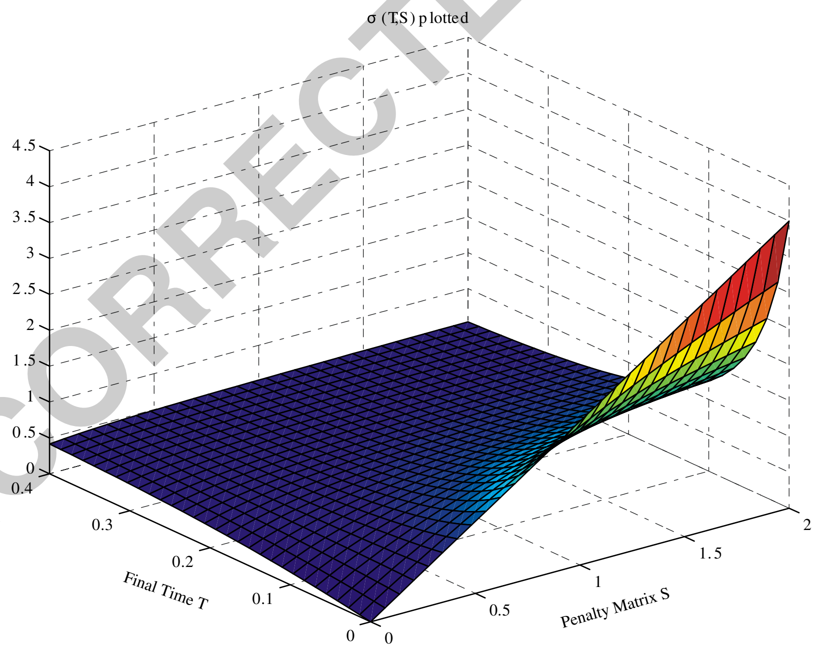

Figure 6. Exact solution for the initial state $\sigma(T,S)$ in the linear case. Parameters as in Figure 1. Limiting behavior related to the algebraic Riccati equation solution $\pi_0=\mathrm{ARE}(A, W, Q)=0.23165$.

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together with the expected

$$\lim_{T\to\infty}\sigma(T,S)=2\pi_0 x_0 \tag{31}$$

Equation (29) is illustrated in Figure 6. The limiting behavior of the costate can be checked, namely

$$\sigma(\infty,S)=2\pi_0 x_0=0.4633 \quad \forall S \tag{32}$$

5. AN APPLICATION TO THE ONE-DIMENSIONAL BILINEAR-QUADRATIC REGULATOR PROBLEM

For the bilinear case with dynamics

$$\dot{x}=Ax+Bu+Nxu \tag{33}$$

and quadratic finite-horizon cost (Equation (2)), it is known [8, 9] that the costate along the optimal trajectory passing by the state-value x at time t takes the form:

$$\lambda(t)=\left(\frac{\partial V}{\partial x}\right)'(t,x)=2\pi(t,x)x \tag{34}$$

where V is the value function for the problem starting at (t,x) with time-horizon T, i.e.

$$V(t,x)\triangleq\inf_{u(.)}\mathscr{J}(t,T,x,u(.)) \tag{35}$$

and $\pi(t,x)$ is a generalized Riccati matrix, an analogous result to the LQR problem but in this case with π depending also on the state x.

In the infinite-horizon case $\pi(t,x)$ does not depend on t, and it can be shown [4, 9] that it obeys the equation

$$0=\pi(x)\frac{(B+Nx)(B+Nx)'}{R}\pi(x)-\pi(x)A-A'\pi(x)-Q \tag{36}$$

The Hamiltonian of the problem is regular independent of the time-horizon, the control minimizing it being

$$u^0(x,\lambda)=-\frac{1}{2R}(B+Nx)'\lambda \tag{37}$$

and then the optimal Hamiltonian and the derived differential equations take the form

$$\mathscr{H}^0(x,\lambda)=x'Qx+\lambda'Ax-\frac{1}{4R}\lambda'(B+Nx)(B+Nx)'\lambda \tag{38}$$

$$\dot{x}=\left(\frac{\partial\mathscr{H}^0}{\partial\lambda}\right)'=\mathscr{F}(x,\lambda)=Ax-\frac{1}{2R}(B+Nx)(B+Nx)'\lambda,\quad x(0)=x_0 \tag{39}$$

$$\begin{aligned}\dot{\lambda}&=-\left(\frac{\partial\mathscr{H}^0}{\partial x}\right)'=-\mathscr{G}(x,\lambda)=-2Qx-A'\lambda+\frac{1}{2R}N'\lambda(B+Nx)'\lambda\\ \lambda(0)&=2\pi(0,x_0)x_0\end{aligned} \tag{40}$$

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Recently [10] it has been shown that the matrix $\pi(t,x)$ is the solution to a first-order PDE that takes the form (in simplified notation):

$$p_t+\varphi(x,p)p_x=\psi(x,p) \tag{41}$$

with boundary condition

$$p(T,x)=S \quad \forall x \tag{42}$$

In (41), the coefficient functions are

$$\varphi(x,p)\triangleq(A-W^x p)x \tag{43}$$

and

$$\psi(x,p)\triangleq W^{x,p}+pW^x p-pA-A'p-Q \tag{44}$$

where $W^x=(B+Nx)(B+Nx)'/R$; $W^{x,p}=(B+Nx)'pxN'p/R$. Note that when $p_t=0$, which corresponds to the $T=\infty$ case, then $p=p(x)$ and

$$pW^x p-pA-A'p-Q=0 \quad \forall x \tag{45}$$

(due to Equation (36)). But the term $W^{x,p}$ does not necessarily vanish, so Equation (41) takes the compact form:

$$\varphi(x,p)p_x=W^{x,p} \tag{46}$$

which can be explicitly checked in dimension one, where

$$p(x)=\frac{A+\sqrt{A^2+QW^x}}{W^x} \tag{47}$$

Finally, it is worth mentioning that for the n-dimensional bilinear system and $T<\infty$, Equation (21) takes the form

$$\rho_T+\psi(\rho,S)\rho_S=\varphi(\rho,S) \tag{48}$$

'symmetric' (in a formal sense, whose mathematical meaning is still being investigated) to (41). The PDE (48) for a bilinear system obviously reduces to its equivalent for a linear system with the same matrices A and B. However, this analogy does not extend to the role of the terms in the equations. For instance, in the linear case ψ is undoubtedly related to its ARE, but for the bilinear case the coefficient function ψ includes the extra term $W^{x,p}$. Equation (46) may then be considered as an alternative way to obtain the matrix $p(x)$ in a neighborhood of the origin, most useful in steady-state regulation problems (see [4]), coupled with the initial condition (from Equation (45))

$$p(0)=\pi_0$$

6. TRACKING AND SET-POINT CHANGES

The problem of changing set-points is frequent in engineering, mostly in the control of chemical or biological reactors. It consists of steering the system from some original state x_0, which is kept

DOI: 10.1002/oca

constant by a value u_0 of the manipulated variable (i.e. (x_0, u_0) is an equilibrium of the control system, usually called a set-point), towards another equilibrium $(\bar{x}, \bar{u})$, called the target set-point, while minimizing the deviation quadratic cost

$$\mathscr{J}(u) = \int_0^T [(x-\bar{x})' Q(x-\bar{x}) + Ru^2]\,\mathrm{d}t + (x(T)-\bar{x})' S(x(T)-\bar{x}) \tag{49}$$

In this section a bilinear system as in Equation (33) will be considered. The Hamiltonian differential equations, Riccati algebraic equation, and optimal feedback for the infinite-horizon problem have been discussed in [4, 13], where a 'relative control' $u-\bar{u}$ with respect to the equilibrium control $\bar{u}$ corresponding to $\bar{x}$ was allowed in the Lagrangian instead of the 'absolute' u (this is necessary to guarantee finiteness of $\mathscr{J}(u)$, without which optimization lacks sense). For the finite-horizon case, however, the absolute control can be considered, and then the problem takes the form of 'tracking a constant reference signal $x_r(t) \equiv \bar{x}$'. HE and boundary conditions take a slightly different form from that in the regulation case:

$$\dot{x} = Ax - \frac{(B+Nx)(B+Nx)'}{2R}\lambda, \quad x(0) = x_0 \tag{50}$$

$$\dot{\lambda} = -2Q(x-\bar{x}) - A'\lambda + \frac{N'\lambda(B+Nx)'\lambda}{2R}, \quad \lambda(T) = 2S(x(T)-\bar{x}) \tag{51}$$

since the control u^0 minimizing the Hamiltonian remains formally the same.

This problem has also been approached through power-series expansions (see [9, 11, 18] for details).

One of these approaches, inspired in the linear optimal tracking problem [2], basically proposes the following power-series expansion for the value function:

$$V(t,x) = \zeta(t) - x'\xi(t) + x'P_1(t)x + x'P_2(t)x^{[2]} + x'P_3(t)x^{[3]} + \cdots \tag{52}$$

that when replaced in the Hamilton–Jacobi–Bellman equation

$$\frac{\partial V}{\partial t}(t,x) + H\left(x, \left(\frac{\partial V}{\partial x}\right)'(t,x), u^0\left(x, \left(\frac{\partial V}{\partial x}\right)'(t,x)\right)\right) = 0 \tag{53}$$

and after collecting terms, generate the set of ODEs that the time-dependent coefficients of the series must meet in order to validate the proposal (52), namely

$$\begin{aligned}
\dot{\zeta} &= -\bar{x}'Q\bar{x} + \xi'W\xi \\
\dot{\xi} &= -2Q\bar{x} - (A - WP_1)'\xi - \tfrac{1}{2}\xi'BR^{-1}N'\xi \\
\dot{P}_1 &= -P_1A - A'P_1 + P_1WP_1 - Q + \tfrac{1}{4}N'\xi R^{-1}\xi'N - \tfrac{1}{2}P_1BR^{-1}\xi'N - \cdots \\
&\quad - \tfrac{1}{2}N'\xi R^{-1}B'P_1 - \tfrac{1}{2}\xi'BR^{-1}N'P_1 - \tfrac{1}{2}P_1NR^{-1}B'\xi + \cdots \\
\dot{P}_2 &= \cdots
\end{aligned} \tag{54}$$

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These equations are coupled and have the following final conditions:

$$\zeta(T) = \bar{x}' S \bar{x}$$
$$\xi(T) = 2S\bar{x}$$
$$P_1(T) = S$$
$$P_2(T) = P_3(T) = \cdots = 0 \text{ (of appropriate dimension)}$$

which in turn arise from the natural boundary condition to Equation (53), namely

$$V(T,x) = (x(T) - \bar{x})' S (x(T) - \bar{x}) \tag{55}$$

Usually this final-value problem needs to be integrated numerically off-line, and its solution stored in the memory of some device and afterwards employed (in real time) to construct the optimal control in feedback form:

$$u^*(t,x) = -\frac{1}{2R}(B+Nx)'\left(\frac{\partial V}{\partial x}\right)' = \frac{1}{2R}(B+Nx)'[-\xi(t) + 2P_1(t)x + \cdots]$$

In practice only a finite number of terms can be kept, and as a consequence the numerical value of the control results to be suboptimal. Also, it should be noticed that the dimensions of the matrices P_i, $i = 1, 2, \ldots,$ grow as $n \times \binom{i+1}{n}$, so improving accuracy as the evaluation of high-order terms becomes intricate. Besides, there is no flexibility with respect to T and S: if they need to be altered in response to new design requirements, then the whole calculation has to be repeated. Finally, the handling of the numerical solutions demands special care: when the control is constructed, the values of the coefficients $P_i(t)$ may be required at t-values for which they are not available. As a whole then, this approach presents inconveniences that turn desirable to look for on-line integration of HE, as has been done with the regulation case in previous sections.

Taking into account that for this problem, the final value of the costate is

$$\lambda(T) = 2S(\rho - \bar{x}) \tag{56}$$

then the procedure developed in Section 3 has to be slightly amended and the PDE for $\rho(T,S)$ takes the form

$$(\rho - \bar{x})\rho_T - \left(SF + \frac{G}{2}\right)\rho_S = (\rho - \bar{x})F \tag{57}$$

where now

$$F(\rho, S) = \mathscr{F}(\rho, 2S(\rho - \bar{x}))$$
$$G(\rho, S) = \mathscr{G}(\rho, 2S(\rho - \bar{x}))$$

and the boundary condition remains as in Equation (22).

Figure 7 illustrates the integration of Equation (57) for small values of S. The final value of the state is still far from the target $\bar{x} = 1$ in the range shown, but for fixed horizon duration T, the value of $\rho(T,S)$ increases with S, and it reaches around 0.95 for $S = 10$.

DOI: 10.1002/oca

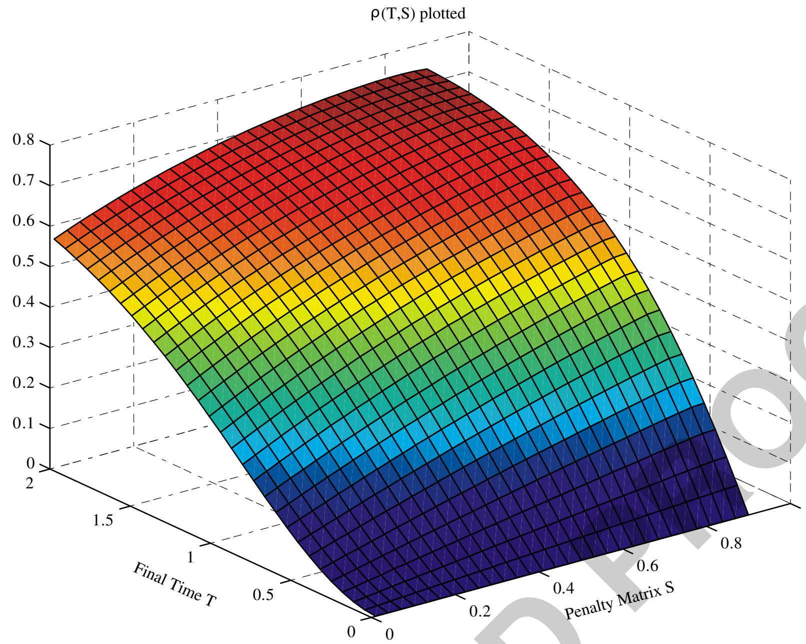

Figure 7. Solution $\rho(T, S)$ for the change of set-point problem, from $x_0 = 0$ toward $\bar{x} = 1$. Parameters are $A = -0.5$; $B = Q = R = 1$; $N = 0.5$.

The PDE for the initial costate $\sigma(T, S)$ looks formally as Equation (23), but now with an explicit dependence on the target $\bar{x}$,

$$(\rho - \bar{x})\sigma_T - \left(SF + \frac{G}{2}\right)\sigma_S = 0 \tag{58}$$

and can be integrated in parallel with Equation (57), in this case with boundary condition

$$\sigma(0, S) = 2S(x_0 - \bar{x}) \tag{59}$$

A trial integration of (simultaneously) (ρ, σ) using standard software was successfully performed. Figure 8 shows just one line of the resulting behavior $\sigma(., S)$ for $S = 1.455$.

The curve of $\sigma(., 1.455)$ has approximately the same limit of minus the curve $\check{\xi}(.)$ in Figure 9. Here $\check{\xi}(.)$ is the reverse solution of $\xi(.)$, the first-order coefficient function of time in the power-series expansion of V, that was calculated here from a 2-terms approximation to Equations (54) (the equation for ζ is irrelevant). In dimension one, this approximation reads as

$$\dot{\xi} = -2Q\bar{x} - (A - WP_1)\xi - \frac{BN}{2R}\xi^2 \tag{60}$$

$$\dot{P}_1 = -Q - 2P_1 A + WP_1^2 + \frac{N^2}{4R}\xi^2 - 2\frac{BN}{R}\xi P_1 \tag{61}$$

DOI: 10.1002/oca

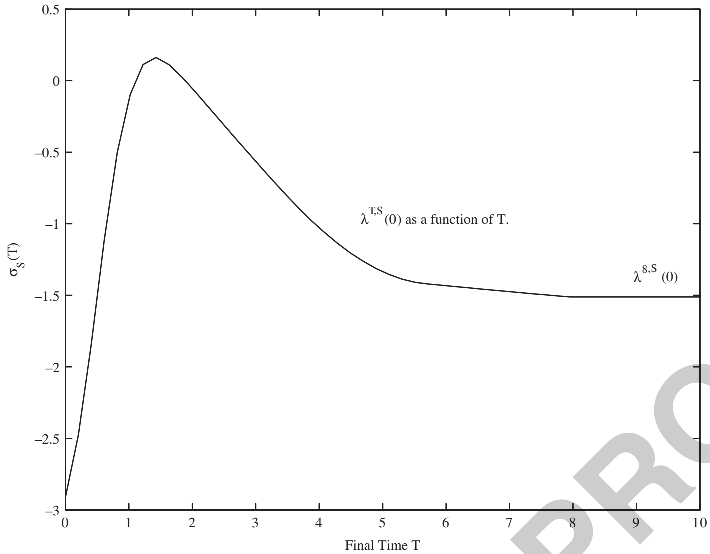

Figure 8. Curve $\sigma(.,S)$ for $S=1.455$. Extracted from the solution $\sigma(T,S)$ of the change of set-point problem with parameters as in 7.

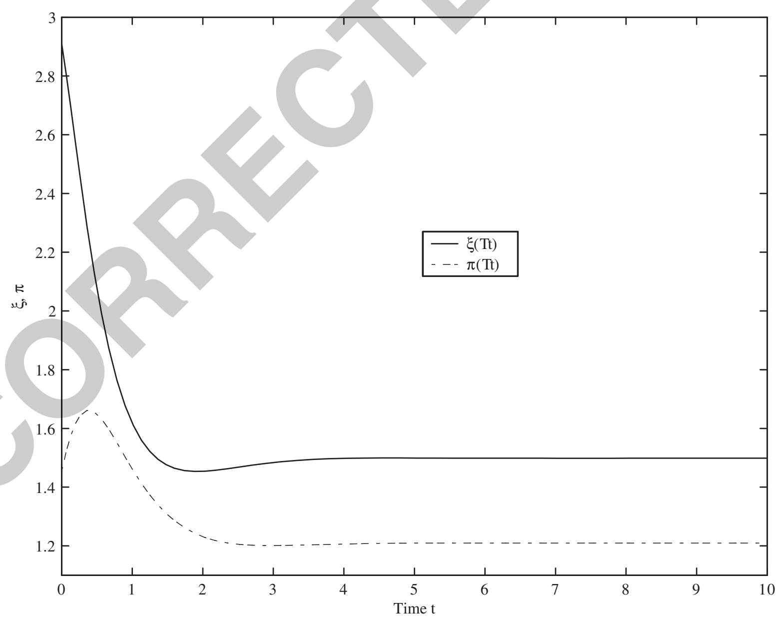

Figure 9. Approximate solutions $\check{\xi}$ and $\check{P}_1$ for the $\xi(.)$ and $P_1(.)$ coefficients in the power-series expansion of the value function $V(.,x)$, plotted in the reverse-time direction. For $x=0$, $V(.,0)=\xi(.)$.

DOI: 10.1002/oca

Equations (60) and (61) were integrated after reversing the sign of their right-hand-sides and with initial conditions: $\check{\xi}(0)=\xi(T)=2S\bar{x}=2.91$; $\check{P}_1(0)=P_1(T)=S=1.455$.

The limiting behavior of curves in Figures 8 and 9 show that the two methods (invariant imbedding and power series) are consistent, since from Equation (52):

$$\sigma(\infty, S)=\lambda^{\infty,S}(0)=\frac{\partial V}{\partial x}(0,x_0)=-\xi(t)+2P_1(t)x+\cdots|_{(t,x)=(0,0)}$$
$$=-\xi(0)\simeq-\check{\xi}(\infty)\simeq-1.5 \quad (62)$$

7. CONCLUSIONS

HE coming from optimal control theory are treated as the central object of this work. They are ODEs that can be explicitly posed when the Hamiltonian function of a given optimal control problem can be uniquely optimized with respect to u, and when the minimizing control value can be expressed as an explicit function u^0 of the remaining variables. The main obstacle in the practical use of these ODEs is that, for classical finite-horizon situations, their restrictions generate a two-point boundary problem, difficult to solve, sometimes impossible. For the LQR problem there exists a well-known procedure for recuperating initial conditions based on the linearity of the HE ODEs in the $2n$-dimensional space (x,λ). But for nonlinear systems no general procedure to transform the boundary value into an initial-value situation is acknowledged in the control literature. In this article, emphasis has been put in deriving a first-order PDE for the costate, as a key toward such transformation. This new equation resembles an equation derived by Bellman for general Hamiltonian dynamical systems through the invariant imbedding approach. The solutions to the PDE established here in dimension one allow in turn to numerically integrate the original HE on-line with the control process and to continuously construct the manipulated variable $u^*(t)=u^0(x(t),\lambda(t))$ from the state and costate values provided by this integration.

Controlling through HE has obvious advantages over optimal control methods based on the Hamilton–Jacobi–Bellman PDE previously devised for the bilinear-quadratic problem (known as power-series expansion approaches). So the integration of the quasilinear PDE proposed in this article is worth the effort, above all when standard mathematically oriented software can render accurate solutions, as has been illustrated in the examples. The PDE method solves a whole family of (T,S) problems, avoiding particular calculations and storing of information as in methods of the DRE type. Also, the equations for the final state ρ and the initial costate σ are superior (since their unknown variables are vectors) to the PDE-based method for bilinear systems (Equation (41)), whose unknown is a matrix $p(t,x)$. The analytical expression of $p(t,x)$ is known for the one-dimensional infinite-horizon problem, so it has been used here to validate the asymptotic behavior of the solutions ρ,σ associated with the bilinear-quadratic regulator problem.

Having the values of ρ,σ for a wide range of T,S values may be helpful at the control design level. From one side, the values of T,S can be reconsidered by the designer when acknowledging the final values of the state $\rho(T,S)$ that will be obtained under present conditions. And if a change in the parameter values is decided, then it will not be necessary to perform additional calculations to manage the new situation. Besides, the value of $\sigma(T,S)$ is an accurate measure of the 'marginal

DOI: 10.1002/oca

cost' of the process, i.e. it measures how much the optimal cost would change under perturbations, which can also influence the decision on T, S values.

The calculations performed here have confirmed that the invariant imbedding approach provides accurate initial conditions to Hamilton's equations, allowing to construct the control from their integration in real time. Also the PDEs developed for the final state and initial costate constitute a useful design tool, since their solutions show the influence of a whole (T, S)-family of restrictions over the performance of optimal controls.

ACKNOWLEDGEMENTS

The author would like to acknowledge Pablo S. Rivadeneira, Research Fellow at the Nonlinear Systems Group, INTEC, for his help in performing numerical calculations, editing the figures, and preparing the manuscript in general.

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DOI: 10.1002/oca