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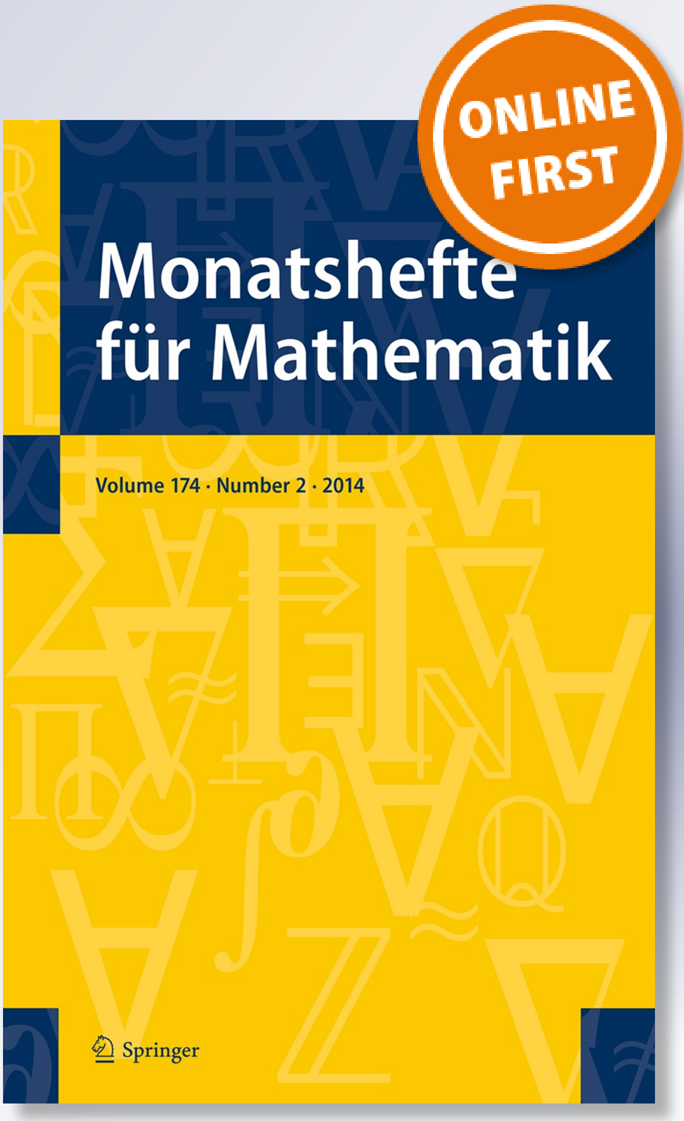

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# Some totally geodesic submanifolds of the nonlinear Grassmannian of a compact symmetric space

**Marcos Salvai**

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**Abstract** Let $M$ and $N$ be two connected smooth manifolds, where $M$ is compact and oriented and $N$ is Riemannian. Let $\mathcal{E}$ be the Fréchet manifold of all embeddings of $M$ in $N$, endowed with the canonical weak Riemannian metric. Let $\sim$ be the equivalence relation on $\mathcal{E}$ defined by $f \sim g$ if and only if $f = g \circ \phi$ for some orientation preserving diffeomorphism $\phi$ of $M$. The Fréchet manifold $\mathcal{S} = \mathcal{E}/_\sim$ of equivalence classes, which may be thought of as the set of submanifolds of $N$ diffeomorphic to $M$ and is called the nonlinear Grassmannian (or Chow manifold) of $N$ of type $M$, inherits from $\mathcal{E}$ a weak Riemannian structure. We consider the following particular case: $N$ is a compact irreducible symmetric space and $M$ is a reflective submanifold of $N$ (that is, a connected component of the set of fixed points of an involutive isometry of $N$). Let $\mathcal{C}$ be the set of submanifolds of $N$ which are congruent to $M$. We prove that the natural inclusion of $\mathcal{C}$ in $\mathcal{S}$ is totally geodesic.

**Keywords** Manifold of embeddings · Geodesic · Symmetric space · Reflective submanifold

**Mathematics Subject Classification (2000)** 22C05 · 53C22 · 53C35 · 58B20 · 58D10 · 58E10

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## 1 Introduction and statement of the result

### 1.1 Manifolds of embeddings

Let $M$, $N$ be connected smooth manifolds (from now on, by smooth we mean $\mathcal{C}^\infty$). The set $\mathcal{E}$ of all embeddings of $M$ into $N$ is a Fréchet manifold (Theorem 44.1 in [4]; see also [8]). If $M$ is compact and oriented and $N$ is Riemannian, then $\mathcal{E}$ has a canonical weak Riemannian metric defined as follows: If $f \in \mathcal{E}$ and $u, v \in T_f\mathcal{E}$ (that is, $u, v$ are smooth vector fields along $f$), then

$$\langle u, v\rangle = \int_M \langle u(x), v(x)\rangle\, \omega_f(x), \tag{1}$$

where $\omega_f$ is the volume element of the Riemannian metric on $M$ induced by $f$.

Let $\sim$ be the equivalence relation on $\mathcal{E}$ defined by $f \sim g$ if and only if $f = g \circ \phi$ for some orientation preserving diffeomorphism $\phi$ of $M$. The set $\mathcal{S} = \mathcal{E}/_\sim$ of equivalence classes is called the nonlinear Grassmannian (or Chow manifold) of $N$ of type $M$. It is a Fréchet manifold with a weak Riemannian metric in such a way that the associated projection $\Pi : \mathcal{E} \to \mathcal{S}$ is a principal bundle with structure group $\text{Diff}_+(M)$, and a Riemannian submersion. Cf. [6], where much more general metrics on $\mathcal{S}$ are considered.

For any $f \in \mathcal{E}$ we have the decomposition $T_f\mathcal{E} = \mathcal{H}_f \oplus \mathcal{V}_f$ in horizontal and vertical subspaces at $f$, where $\mathcal{V}_f = \text{Ker}\left(d\Pi_f\right)$ and $\mathcal{H}_f$ is the orthogonal complement of $\mathcal{V}_f$. They consist of all the smooth vector fields along $f$ which are tangent to $f(M)$, respectively, normal, at each point of $M$.

### 1.2 Reflective submanifolds

A reflective submanifold $M$ of a Riemannian manifold $N$ is a connected component of the set of fixed points of an involutive isometry of $N$. In particular, $M$ is closed and totally geodesic in $N$. Reflective submanifolds of symmetric spaces have been extensively studied by Leung in a series of papers beginning with [5] (see also Chapter 9 of [2]). For instance, every complete totally geodesic connected submanifold of a simply connected space form is reflective. Also, the reflective submanifolds of $\mathbb{C}P^n$ are exactly, up to isometry, $\mathbb{C}P^k$ $(1 \leq k < n)$ and $\mathbb{R}P^n$ (canonical embedding). In particular, $\mathbb{R}P^1$, that is, a geodesic, is not a reflective submanifold of $\mathbb{C}P^n$ if $n \geq 2$.

### 1.3 The nonlinear Grassmannian of a compact symmetric space

Let $N$ be a compact connected symmetric space and let $G$ be the identity component of the isometry group of $N$. Let $o \in N$ and let $K$ be the isotropy subgroup at $o$. We have the canonical projection $\pi : G \to N$, $\pi(g) = g(o)$. For the sake of simplicity, we assume further that $G$ is semisimple and $\pi$ is a Riemannian submersion, where $G$

is endowed with the Riemannian metric defined at the identity by the opposite of the Killing form.

Let $M$ be a reflective submanifold of $N$ and let $\mathcal{E}, \mathcal{S}$ be the spaces associated to $M, N$ as in SubSect. 1.1. We may suppose that $o \in M$.

Let $H = \{g \in G \mid g(M) = M\}$. Since $M$ is closed in $G$, $H$ is a closed subgroup, and hence a Lie subgroup, of $G$.

Let $\mathcal{C}$ be the set of submanifolds of $N$ which are $G$-congruent to $M$, that is, $\mathcal{C} = \{g(M) \mid g \in G\}$. We may identify $\mathcal{C} \cong G/H$.

Now we can state the main result of the paper: $\mathcal{C}$ is totally geodesic in $\mathcal{S}$. More precisely,

**Theorem 1** *Let $\iota : M \to N$ be the inclusion and let*

$$F : \mathcal{C} \cong G/H \to \mathcal{S}, \qquad F(gH) = \Pi \circ g \circ \iota$$

(*a well-defined map*). *Then $(\mathcal{C}, F)$ is a totally geodesic submanifold of $\mathcal{S}$.*

*Remark 2* (*a*) Geodesics in $\mathcal{S}$ are not good from the metric point of view, since (1) induces on $\mathcal{S}$ a vanishing geodesic distance [1,7]. Nevertheless, they are distinguished curves which deserve being studied. For instance, in the case $M = S^1$, $N = S^3$, they played a role in a characterization of the Hopf fibrations of $S^3$ [10].

(*b*) We do not know whether the Riemannian metric induced on $\mathcal{C}$ from $\mathcal{S}$ is normal with respect to $G$ (i.e., whether the canonical projection $\tilde{\pi} : G \to \mathcal{C}$ is a Riemannian submersion for some bi-invariant Riemannian metric on $G$), but at least in the simplest case it is:

**Proposition 3** *Let $M$ be a reflective submanifold of $S^n$, that is, $M$ is a great sphere. Then the metric on $\mathcal{C}$ induced from $\mathcal{S}$ is normal.*

*Proof* Let $\{e_i \mid i = 0, \ldots, n\}$ be the canonical basis of $\mathbb{R}^{n+1}$ and suppose $M = S^n \cap \operatorname{span}\{e_i \mid i = 0, \ldots, m\} \cong S^m$. Given $0 \leq i < j \leq n$ and $t \in \mathbb{R}$, let $R_t^{i,j} \in SO(n+1) = G$ be the rotation fixing $e_k$ for $k \neq i, j$ and satisfying $R_t^{i,j} e_i = (\cos t) e_i + (\sin t) e_j$. Let $E^{i,j} = \frac{d}{dt}\big|_0 R_t^{i,j}$. Then $\left\{E^{i,j} \mid 0 \leq i < j \leq m\right\}$ is an orthonormal basis of $so(n+1)$ with respect to a negative multiple of the Killing form. Now we follow the notatios of SubSect. 1.3. If we take $o = e_0$ and call $\mathfrak{h}$ the Lie algebra of $H$, then $\mathfrak{h}^\perp = \operatorname{span}\left\{E^{i,j} \mid 0 \leq i \leq m < j \leq n\right\}$. We have to check that

$$d\pi_I : \mathfrak{h}^\perp \subset so(n+1) = T_I G \to T_M \mathcal{C}$$

satisfies $\|d\pi_I(X)\| = c|X|$ for all $X \in \mathfrak{h}^\perp$ and some constant $c$.

The vector fields corresponding to $E^{i,j}$ along the inclusion $\iota : S^m \to S^n$ are

$$V^{i,j}(q) = \frac{d}{dt}\bigg|_0 R_t^{i,j}(q) = \frac{d}{dt}\bigg|_0 (\cos t) x_i(q) e_i + (\sin t) x_i(q) e_j = x_i(q) e_j,$$

where $q \in S^m$. Now we apply the definition (1). We compute

$$\left|V^{i,j}(q)\right|^2 = x_i^2(q), \qquad \left\langle V^{i,j}(q), V^{k,\ell}(q)\right\rangle = \delta_{j\ell} x_i(q) x_k(q).$$

Since $y_i =_{\text{def}} x_i|_{S^m}$ $(i = 0, \ldots, m)$ are elements of the canonical orthogonal basis of spherical harmonics on $S^m$, we have $\langle V^{i,j}, V^{k,j}\rangle = 0$ if $i \neq k$. Besides,

$$(m+1) \int_{S^m} y_i^2(q)\, \omega_\iota(q) = \sum_{s=0}^{m} \int_{S^m} y_s^2(q)\, \omega_\iota(q) = \int_{S^m} \omega_\iota(q) = \text{vol}\ \left(S^m\right).$$

Therefore, $\left\| V^{i,j} \right\|^2 = \frac{1}{m+1} \text{vol}\ (S^m)$ and so one can take the square root of this number as $c$ above.

## 2 Proof of the main result

### 2.1 The structure of $\mathcal{C}$

Naitoh proved that if $M, N$ are as in SubSect. 1.3, then $(G, H)$ is a symmetric pair. We recall here the more recent and general version by Tasaki. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra of $G$ associated to the point $o \in N$.

**Theorem 4** ([9,13]) *Let $\mathfrak{h}$ be the Lie algebra of $H$ and let $m_- \subset \mathfrak{p}$ be such that $d\pi_e \mathfrak{m}_- = T_o M$. Then $\mathfrak{k} = \mathfrak{k}_+ + \mathfrak{k}_-$ and $\mathfrak{p} = \mathfrak{m}_+ + \mathfrak{m}_-$ in such a way that*

$$\mathfrak{h} = \mathfrak{k}_- + \mathfrak{m}_- \ \ \textit{and}\ \ T_M \mathcal{C} \cong \mathfrak{h}^\perp = \mathfrak{k}_+ + \mathfrak{m}_+. \tag{2}$$

*Moreover,*

$$\left[\mathfrak{h}, \mathfrak{h}^\perp\right] \subset \mathfrak{h}^\perp \ \ \textit{and}\ \ \left[\mathfrak{h}^\perp, \mathfrak{h}^\perp\right] \subset \mathfrak{h}. \tag{3}$$

### 2.2 The evolution equation for geodesics

Let $M, N$ be as in SubSect. 1.1. Kainz obtained in [3] a necessary and sufficient condition for a curve $f : I \to \mathcal{E}$ to be a geodesic, where $I$ is an interval of the real line.

In the very particular case when $f(t)$ is a totally geodesic embedding and $f'(t)$ is a normal vector field along $f(t)$ for all $t \in I$, the condition simplifies as follows [7, SubSect. 4.2]: $f$ is a geodesic if and only if

$$\left.\frac{D}{dt}\right|_{t_o} f'(t)(x) \in d\left(f(t_o)\right)_x (T_x M) \tag{4}$$

for all $t_o$ and all $x \in M$, where $\frac{D}{dt}$ denotes covariant derivative along the curve $I \ni t \mapsto f(t)(x)$.

## 2.3 The acceleration of an orbit in a normal space

When applying the criterion above to our case, we will need an expression for the covariant acceleration of the orbit of a one-parameter group of isometries.

Let $G$ be a connected Lie group endowed with a bi-invariant Riemannian metric and let $K$ be a closed connected Lie subgroup of $G$ with Lie algebra $\mathfrak{k}$. Consider on $P = G/K$ the Riemannian metric such that the canonical projection $\pi : G \to P$ is a Riemannian submersion (the normal metric on $P$). In these conditions, the geodesics of $G$ are one-parameter subgroups; in particular, the fibers are totally geodesic.

**Lemma 5** *Let G and P be as above, and let $\beta$ be the curve in P defined by $\beta = \pi \circ \alpha$, where $\alpha(t) = \exp t(U + V)$, with $U \in \mathfrak{k}$ and $V \in \mathfrak{k}^{\perp}$. Then*

$$\frac{D\dot{\beta}}{dt}(0) = d\pi_e [U, V]_e.$$

Before proving the lemma we recall from [11] some definitions and statements about submersions and parallel transport.

Let $\pi : B \to P$ be a Riemannian submersion with totally geodesic fibers. For $E \in TB$, let $\mathcal{H}E$ and $\mathcal{V}E$ denote the horizontal and vertical parts of $E$, respectively. The O'Neill tensor field $A$ on $B$, of type (0, 2), is defined by

$$A_E F = \mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F) + \mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F).$$

Let $E$ be a vector field along a curve $\alpha$ in $B$. By the main result in [11],

$$\mathcal{H}(E') = \mathcal{L}\left((d\pi(E))'\right) + A_{\mathcal{H}E}(\mathcal{V}\dot{\alpha}) + A_{\mathcal{H}\dot{\alpha}}(\mathcal{V}E), \tag{5}$$

where the prime denotes covariant derivative (along $\alpha$ or $\pi \circ \alpha$, accordingly) and, if $W$ is a vector field along $\pi \circ \alpha$, then $\mathcal{L}(W)$ is the horizontal vector field along $\alpha$ projecting to $F$.

*Proof of Lemma 5* We consider the Riemannian submersion $\pi : G \to P$ and apply Eq. (5) to $E = \dot{\alpha} = U \circ \alpha + V \circ \alpha$, whose covariant derivative vanishes since $\alpha$ is a geodesic of $G$ (the metric is bi-invariant). We obtain

$$0 = \mathcal{L}(\dot{\beta}') + 2A_{\mathcal{H}\dot{\alpha}}(\mathcal{V}\dot{\alpha}). \tag{6}$$

Hence, by definition of the tensor $A$ and using that $\nabla_V U = \frac{1}{2}[V, U]$ since the metric on $G$ is bi-invariant, one has

$$\mathcal{L}(\dot{\beta}') = -2\mathcal{H}(\nabla_{\mathcal{H}\dot{\alpha}}\mathcal{V}\dot{\alpha}) = -2\mathcal{H}((\nabla_V U) \circ \alpha) = \mathcal{H}(([U, V]) \circ \alpha).$$

Applying $d\pi$ and evaluating at $t = 0$, one gets the desired formula for $\dot{\beta}'(0)$. □

*Proof of Theorem 1* We consider on $\mathcal{C} = G/H$ the metric induced from $\mathcal{S}$ (which in principle may not be normal). Let $\tilde{F} : G/H \to \mathcal{E}$ be defined by $\tilde{F}(gH) = g \circ \iota$, that is, the following diagram is commutative.

$$\begin{array}{ccc} G/H & \xrightarrow{\tilde{F}} & \mathcal{E} \\ & \searrow_{F} & \downarrow \Pi \\ & & \mathcal{S} \end{array}$$

Given a geodesic $\gamma$ in $\mathcal{C}$, we will prove that $\tilde{F} \circ \gamma$ is a horizontal geodesic in $\mathcal{E}$. Hence, $F \circ \gamma$ is a geodesic in $\mathcal{S}$ and so $(\mathcal{C}, F)$ is totally geodesic, as desired. Since $F$ is $G$-equivariant and the action of $G$ preserves the metrics on $\mathcal{C}$ and $\mathcal{E}$ and also the vertical and horizontal distributions on $\mathcal{E}$ (see above their description in terms of vector fields along the embeddings), it suffices to prove the assertion only for $\gamma$ with $\gamma(0) = H$.

Now, by Theorem 4, since the metric of $\mathcal{C}$ is $G$ -invariant, the geodesics of $\mathcal{C}$ are the same as the geodesics of $\mathcal{C}$ endowed with the normal metric (see Exercise 10(b) on page 330 of [12]). Hence, $\gamma(t) = \tilde{\pi} e^{tX}$, for some $X \in \mathfrak{h}^{\perp}$. We call $f = \tilde{F} \circ \gamma : \mathbb{R} \to \mathcal{E}$. Now we check that we can apply the criterion of Kainz. First, $f(t)$ is totally geodesic for any $t \in \mathbb{R}$, since $f(t) = e^{tX} f(0)$, with $e^{tX}$ an isometry of $N$ and $f(0) = \iota : M \to N$, which is totally geodesic since it is reflective. Secondly, the vector field $f'(t)$ along $f(t)$ is normal to $f(t)$ . Again by $e^{tX}$-invariance, one can take $t = 0$. Let $q \in M$. Since $M$ is a totally geodesic submanifold of the symmetric space $G/H$ through $o$, $q = e^{Y}.o$ for some $Y \in \mathfrak{m}_{-}$. We compute

$$f'(0)(q) = \left.\frac{d}{dt}\right|_0 e^{tX} q = \left.\frac{d}{dt}\right|_0 e^{tX} e^{Y}.o = \left.\frac{d}{dt}\right|_0 e^{Y} e^{tZ}.o = \left(de^{Y}\right)_o d\pi_o(Z),$$

where

$$Z = \mathrm{Ad}\left(e^{-Y}\right) X = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\mathrm{ad}_Y)^n X.$$

Now, $[Y, X] \in [\mathfrak{m}_{-}, \mathfrak{h}^{\perp}] \subset \mathfrak{h}^{\perp}$ by (3). Hence, $Z \in \mathfrak{h}^{\perp}$ and so $d\pi_e(Z) \perp T_o M$. Therefore $f'(0)(q) \perp \left(de^{Y}\right)_o d\pi(\mathfrak{m}_{-}) = T_q M$ (this last equality is well-known to hold for totally geodesic submanifolds of a symmetric space).

Now we can use Kainz evolution Eq. (4). Again by $e^{Y}$-invariance, without loss of generality we may check it only at $q = o$. Let $c(t) = e^{sX}.o$ and suppose, by Theorem 4, that $X = U + V$, with $U \in \mathfrak{k}_{+}$ and $V \in \mathfrak{m}_{+}$. We can apply Lemma 5 to $G/K$, with $M$ in the role of $P$:

$$\left.\frac{D}{dt}\right|_0 c'(t) = d\pi_e[U, V] \in d\pi_e[\mathfrak{k}_{+}, \mathfrak{m}_{+}],$$

which belongs to $\mathfrak{p}$ (since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$) and also to $\mathfrak{h}$, since $[\mathfrak{h}^{\perp}, \mathfrak{h}^{\perp}] \subset \mathfrak{h}$ by (3). Therefore, by (2), $\left.\frac{D}{dt}\right|_0 c'(t) \in d\pi_e(\mathfrak{m}_{-}) \in T_o M$, as desired. □

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