Spherical functions associated with the three-dimensional sphere

Inés Pacharoni · Juan Tirao · Ignacio Zurrián

Abstract In this paper, we determine all irreducible spherical functions Φ of any K-type associated with the pair $(G, K) = (\mathrm{SO}(4), \mathrm{SO}(3))$. This is accomplished by associating with Φ a vector-valued function $H = H(u)$ of a real variable u, which is analytic at $u = 0$ and whose components are solutions of two coupled systems of ordinary differential equations. By an appropriate conjugation involving Hahn polynomials, we uncouple one of the systems. Then, this is taken to an uncoupled system of hypergeometric equations, leading to a vector-valued solution $P = P(u)$, whose entries are Gegenbauer's polynomials. Afterward, we identify those simultaneous solutions and use the representation theory of SO(4) to characterize all irreducible spherical functions. The functions $P = P(u)$ corresponding to the irreducible spherical functions of a fixed K-type π_ℓ are appropriately packaged into a sequence of matrix-valued polynomials $(P_w)_{w \geq 0}$ of size $(\ell + 1) \times (\ell + 1)$. Finally, we prove that $\widetilde{P}_w = P_0^{-1} P_w$ is a sequence of matrix orthogonal polynomials with respect to a weight matrix W. Moreover, we show that W admits a second-order symmetric hypergeometric operator $\widetilde{D}$ and a first-order symmetric differential operator $\widetilde{E}$.

Keywords Matrix-valued spherical functions · Matrix orthogonal polynomials · The matrix hypergeometric operator · Three-dimensional sphere

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1 Introduction

The theory of spherical functions dates back to the classical papers of É. Cartan and H. Weyl; they showed that spherical harmonics arise in a natural way from the study of functions on the n-dimensional sphere $S^n = \mathrm{SO}(n + 1)/\mathrm{SO}(n)$. The first general results in this direction were obtained in 1950 by Gelfand, who considered zonal spherical functions of a Riemannian symmetric space G/K. In this case, we have a decomposition $G = KAK$. When the abelian subgroup A is one dimensional, the restrictions of zonal spherical functions to A can be identified with hypergeometric functions, providing a deep and fruitful connection between group representation theory and special functions. In particular when G is compact, this gives a one to one correspondence between all zonal spherical functions of the symmetric pair (G, K) and a sequence of orthogonal polynomials.

In light of this remarkable background, it is reasonable to look for an extension of the above results, by considering matrix-valued spherical functions on G of a general K-type. This was accomplished for the first time in the case of the complex projective plane $P_2(\mathbb{C}) = \mathrm{SU}(3)/\mathrm{U}(2)$ in [10]. This seminal work gave rise to a series of papers including [6,8,13, 27–29], where one considers matrix-valued spherical functions associated with a compact symmetric pair (G, K), arriving at sequences of matrix-valued orthogonal polynomials of one real variable satisfying an explicit three-term recursion relation, which are also eigenfunctions of a second-order matrix differential operator (bispectral property).

The theory of matrix-valued orthogonal polynomials without any consideration of differential equations was started by Krein in [23–25]. After that the theory was revived by Durán in [3], who posed the problem of finding matrix weight functions W with symmetric matrix second-order differential operators D. But the existence of such "classical pairs" (W, D) was first established in [12] and [11] as a byproduct of what was obtained in [10]. In fact, in [10] for any K-type π a matrix weight function W of size $m = \deg \pi$, a symmetric second-order differential operator D and a sequence of matrix polynomials $\{P_n\}_{n \geq 0}$ were constructed from the spherical functions of the pair $(\mathrm{SU}(3), \mathrm{U}(2))$. Such a sequence has the following properties: $\deg P_n = n + m$, $\det P_0 \not\equiv 0$, P_n and $P_{n'}$ are orthogonal with respect to W for all $n \neq n'$, $DP_n = P_n \Lambda_n$ and the sequence $\{P_n\}_{n \geq 0}$ satisfies a three-term recursion relation. Yet, the sequence $\{P_n\}_{n \geq 0}$ does not fit directly into the existing theory of matrix-valued orthogonal polynomials as given in [3]. In [12] and [11], we established such a connection defining the matrix-valued function Q_n by means of $Q_n = P_0^{-1} P_n$. It is worth to point out that whenever we are under these hypothesis one can prove that $\{Q_n\}_{n \geq 0}$ is a sequence of matrix-valued orthogonal polynomials with respect to $\widetilde{W} = P_0^* W P_0$ and that $\widetilde{D} = P_0^{-1} D P_0$ is symmetric. Related results can be also found in [11,14,26,31,32].

A different approach to find examples of classical matrix orthogonal polynomials can be found for example in [4].

The irreducible spherical functions associated with the complex projective space $P_n(\mathbb{C}) = \mathrm{SU}(n + 1)/\mathrm{U}(n)$ of a given K-type are encoded in a sequence of matrix-valued orthogonal polynomials, which are given in terms of the matrix hypergeometric function. The semi-infinite matrix corresponding to the three-term recursion relation turns out to be stochastic. This unexpected result leads to the study of the random walk with this transition probability matrix, see [7].

The present paper is an outgrowth of [38] and we are currently working on the extension of these results to the n-dimensional sphere and the n-dimensional real projective space. The starting point is to describe the irreducible spherical functions as simultaneous eigenfunctions of two commutative differential operators, one of order two and the other of order one, and

then the irreducible spherical functions of the same K-type are encoded in a sequence of matrix-valued orthogonal polynomials.

More recently in [21], the authors studied the irreducible spherical functions of the pair $(G, K) = (\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$ ($\mathrm{SU}(2)$ embedded diagonally) as projections onto K-isotypic components of irreducible representations of G. This approach is comparable with the construction of vector-valued polynomials given in [22]. Also in [20], the authors come back to the same subject but starting with the construction of the matrix orthogonal polynomials using a recursion relation and the orthogonality relations, and by ending with the differential operators. The group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is the universal covering group of $\mathrm{SO}(4)$, and the image of $\mathrm{SU}(2)$ under the covering homomorphism is $\mathrm{SO}(3)$. Thus, the pairs $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{SU}(2))$ and $(\mathrm{SO}(4), \mathrm{SO}(3))$ are very closely related. The results of this paper and those in [20] are in agreement, see Remark 10.11 at the end of this paper for details. However, the treatments are very different.

Briefly, the main results of this paper are the following. After some preliminary developed along the first sections, in Sect. 5, we are able to explicitly describe the irreducible spherical functions of the symmetric pair $(\mathrm{SO}(4), \mathrm{SO}(3))$ of a fixed K-type, by a vector-valued function $P = P(u)$, whose entries are certain Gegenbauer polynomials in a suitable variable u. This is accomplished by uncoupling a system of second-order linear differential equations using a constant matrix of Hahn polynomials, see Proposition 4.1.

In Sect. 8, it is established which are those vector-valued polynomials $P = P(u)$ that correspond to irreducible spherical functions, and it is shown how to reconstruct the spherical functions out of them.

The aim of the last two sections is to build classical sequences of matrix-valued orthogonal polynomials from our previous work. In Sect. 9, we define a sequence of polynomial matrices P_w, $w \geq 0$, whose columns are the vector-valued polynomials $P = P(u)$ corresponding to some specific irreducible spherical functions of the same K-type. In Sect. 10, we consider the sequence $\widetilde{P}_w = P_0^{-1} P_w$ and we prove that $(\widetilde{P}_w)_{w \geq 0}$ is a sequence of matrix orthogonal polynomials with respect to a weight function W explicitly given in (69). Moreover, the matrix differential operators $\widetilde{D}$ and $\widetilde{E}$ given in Theorem 9.4 satisfy $\widetilde{D}\widetilde{P}_w = \widetilde{P}_w \Lambda_w$ and $\widetilde{E}\widetilde{P}_w = \widetilde{P}_w M_w$, where the eigenvalues Λ_w and M_w are real diagonal matrices. Thus, $\widetilde{D}$ and $\widetilde{E}$ are symmetric with respect to W.

2 Preliminaries

2.1 Spherical functions

Let G be a locally compact unimodular group and let K be a compact subgroup of G. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, i.e. the dimension of any representation in the class δ, and $\chi_\delta = d(\delta)\xi_\delta$. We shall choose once and for all the Haar measure dk on K normalized by $\int_K dk = 1$.

We shall denote by V a finite dimensional vector space over the field $\mathbb{C}$ of complex numbers and by $\mathrm{End}(V)$ the space of all linear transformations of V into V. Whenever we refer to a topology on such a vector space, we shall be talking about the unique Hausdorff linear topology on it.

Definition 2.1 A spherical function Φ on G of type $\delta \in \hat{K}$ is a continuous function on G with values in $\mathrm{End}(V)$ such that

i $\Phi(e) = I$ (I = identity transformation).
ii $\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)\,dk$, for all $x, y \in G$.

The reader can find a number of general results in [34] and [5]. For our purpose, it is appropriate to recall the following facts.

Proposition 2.2 (Proposition 1.2 in [34]) *If* $\Phi : G \longrightarrow \mathrm{End}(V)$ *is a spherical function of type δ then:*

i $\Phi(k_1 g k_2) = \Phi(k_1)\Phi(g)\Phi(k_2)$, *for all* $k_1, k_2 \in K$, $g \in G$.
ii $k \mapsto \Phi(k)$ *is a representation of K such that any irreducible subrepresentation belongs to δ.*

Concerning the definition, let us point out that the spherical function Φ determines its type univocally (Proposition 2.2) and let us say that the number of times that δ occurs in the representation $k \mapsto \Phi(k)$ is called the *height* of Φ.

A spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is called *irreducible* if V has no proper subspace invariant by $\Phi(g)$ for all $g \in G$.

If G is a connected Lie group, it is not difficult to prove that any spherical function $\Phi : G \longrightarrow \mathrm{End}(V)$ is differentiable (C^∞), and moreover that it is analytic. Let $D(G)$ denote the algebra of all left invariant differential operators on G and let $D(G)^K$ denote the subalgebra of all operators in $D(G)$ which are invariant under all right translations by elements in K.

In the following proposition, (V, π) will be a finite dimensional representation of K such that any irreducible subrepresentation belongs to the same class $\delta \in \hat{K}$.

Proposition 2.3 *A function* $\Phi : G \longrightarrow \mathrm{End}(V)$ *is a spherical function of type δ if and only if*

i Φ *is analytic.*
ii $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, *for all* $k_1, k_2 \in K$, $g \in G$, *and* $\Phi(e) = I$.
iii $[D\Phi](g) = \Phi(g)[D\Phi](e)$, *for all* $D \in D(G)^K$, $g \in G$.

Proof If $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function of type δ, then Φ satisfies (iii) (see Lemma 4.2 in [34]) and Φ is analytic (see Proposition 4.3 in [34]). Conversely, if Φ satisfies (i), (ii) and (iii), then $D \mapsto [D\Phi](e)$ is a representation of $D(G)^K$ and therefore Φ satisfies the integral equation (ii) in Definition 2.1, see Proposition 4.6 in [34].   □

Moreover, we have that the eigenvalues $[D\Phi](e)$, $D \in D(G)^K$, characterize the spherical functions Φ as stated in the following proposition.

Proposition 2.4 (Remark 4.7 in [34]) *Let* $\Phi, \Psi : G \longrightarrow \mathrm{End}(V)$ *be two spherical functions on a connected Lie group G of the same type $\delta \in K$. Then $\Phi = \Psi$ if and only if $(D\Phi)(e) = (D\Psi)(e)$ for all $D \in D(G)^K$.*

Let us observe that if $\Phi : G \longrightarrow \mathrm{End}(V)$ is a spherical function, then $\Phi : D \mapsto [D\Phi](e)$ maps $D(G)^K$ into $\mathrm{End}_K(V)$ ($\mathrm{End}_K(V)$ denotes the space of all linear maps of V into V which commutes with $\pi(k)$ for all $k \in K$) defining a finite dimensional representation of the associative algebra $D(G)^K$. Moreover, the spherical function is irreducible if and only if the representation $\Phi : D(G)^K \longrightarrow \mathrm{End}_K(V)$ is irreducible. In fact, if $W < V$ is $\Phi(G)$-invariant, then clearly W is invariant as a $(D(G)^K, K)$-module. Therefore, if $\Phi : D(G)^K \longrightarrow \mathrm{End}_K(V)$ is irreducible, then the spherical function Φ is irreducible. Conversely, if $\Phi :$

$D(G)^K \longrightarrow \mathrm{End}_K(V)$ is not irreducible, then there exists a proper $(D(G)^K, K)$-invariant subspace $W < V$. Let $P : V \longrightarrow W$ be any K-projection. Let us now consider the following functions: $P\Phi P$ and ΦP. Both are analytic and $(P\Phi P)(k_1 g k_2) = \pi(k_1)(P\Phi P)(g)\pi(k_2)$ for all $g \in G$ and $k_1, k_2 \in K$. Moreover, if $D \in D(G)^K$ then $[D(P\Phi P)](e) = P[D(\Phi)](e)P = [D(\Phi)](e)P = [D(\Phi P)](e)$. Therefore, using Remark 4.7 in [34], it follows that $P\Phi P = \Phi P$. This implies that W is $\Phi(G)$-invariant. Hence, if Φ is an irreducible spherical function, then $\Phi : D(G)^K \longrightarrow \mathrm{End}_K(V)$ is an irreducible representation.

As a consequence of this we have:

Proposition 2.5 *Let G be a connected reductive linear Lie group. Then the following properties are equivalent:*

  i $D(G)^K$ *is commutative.*
  ii *Every irreducible spherical function of (G, K) is of height one.*

Lemma 2.6 *Let G be a linear Lie group. Given $D \neq 0$, $D \in D(G)$, there exists a finite dimensional representation U of G such that $[DU](e) \neq 0$.*

Proof We may assume that G is a Lie subgroup of $\mathrm{SL}(E)$ for certain real finite dimensional vector space E. The identity representation of G extends in the usual way to a representation U_s of G on $E_s = \otimes^s E$. Let U_s also denote the corresponding representation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of G. Then, as Harish-Chandra showed (see §2.3.2 of [37]), there exists $s \in \mathbb{N}$ such that $U_s(D) \neq 0$. Finally, by using the canonical isomorphism $\mathcal{U}(\mathfrak{g}) \simeq D(G)$, we obtain $[DU_s](e) = U_s(D) \neq 0$. $\qquad \square$

Proof of Proposition 2.5 i) $\Rightarrow$ ii). If Φ is an irreducible spherical function then $\Phi : D(G)^K \longrightarrow \mathrm{End}_K(V)$ is an irreducible representation. Therefore, $\mathrm{End}_K(V) \simeq \mathbb{C}$ which is equivalent to Φ being of height one.

ii) $\Rightarrow$ i). If Φ is a spherical function of height one and $D \in D(G)^K$, then $[D\Phi](e) = \lambda I$ with $\lambda \in \mathbb{C}$. Hence, if $D_1, D_2 \in D(G)^K$ we have

$$[(D_1 D_2)\Phi](e) = [D_1\Phi](e)[D_2\Phi](e) = [(D_2 D_1)\Phi](e).$$

On the other hand, we have that the irreducible spherical functions of (G, K) separate the elements of $D(G)^K$. In fact, if $D \neq 0$, $D \in D(G)^K$, by Lemma 2.6, there exists a finite dimensional representation U of G such that $[DU](e) \neq 0$. By hypothesis, we may assume that U is irreducible. Let $U = \oplus_{\delta \in \hat{K}} U_\delta$ be the decomposition of U into K-isotypic components and let P_δ be the corresponding projections of U onto U_δ. Then, there exits $\delta \in \hat{K}$ such that $[D(P_\delta U P_\delta)](e) \neq 0$. Thus, the corresponding spherical function Φ_δ is irreducible and $[D\Phi_\delta](e) \neq 0$. Therefore $D_1 D_2 = D_2 D_1$. $\qquad \square$

In this paper, the pair (G, K) is $(\mathrm{SO}(4), \mathrm{SO}(3))$. Then, it is known that $D(G)^K$ is an abelian algebra; moreover, $D(G)^K$ is isomorphic to $D(G)^G \otimes D(K)^K$ (See Theorem 10.1 in [18] or [2]), where $D(G)^G$ (respectively $D(K)^K$) denotes the subalgebra of all operators in $D(G)$ (respectively $D(K)$) which are invariant under all right translations by elements in G (respectively K).

Now, in our case, we have that $D(G)^G$ is a polynomial algebra in two algebraically independent generators. This is because the Lie algebra of G is $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$; hence, if Δ_1 and Δ_2 are the Casimirs of the corresponding $\mathfrak{so}(3)$, we have that Δ_1 and Δ_2 generate $D(G)^G$.

The first consequence of this is that all irreducible spherical functions of our pair (G, K) are of height one. The second consequence is that to find all irreducible spherical functions

of type $\delta \in \hat{K}$ is equivalent to take any irreducible representation (V, π) of K in the class δ and to determine all analytic functions $\Phi : G \longrightarrow \mathrm{End}(V)$ such that

  i $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.
 ii $[\Delta_1 \Phi](g) = \tilde{\lambda}\Phi(g)$, $[\Delta_2\Phi](g) = \tilde{\mu}\Phi(g)$ for all $g \in G$ and for some $\tilde{\lambda}, \tilde{\mu} \in \mathbb{C}$.

A particular choice of these operators Δ_1 and Δ_2 is given in (2).

Spherical functions of type δ (see Section 3 in [34]) arise in a natural way upon considering representations of G. If $g \mapsto U(g)$ is a continuous representation of G, say on a finite dimensional vector space E, then

$$P_\delta = \int_K \chi_\delta(k^{-1})U(k)\,\mathrm{d}k$$

is a projection of E onto $P_\delta E = E(\delta)$. If $P_\delta \neq 0$ the function $\Phi : G \longrightarrow \mathrm{End}(E(\delta))$ defined by

$$\Phi(g)a = P_\delta U(g)a, \quad g \in G,\; a \in E(\delta),$$

is a spherical function of type δ. In fact, if $a \in E(\delta)$, we have

$$\Phi(x)\Phi(y)a = P_\delta U(x) P_\delta U(y)a = \int_K \chi_\delta(k^{-1}) P_\delta U(x)U(k)U(y)a\,\mathrm{d}k$$

$$= \left(\int_K \chi_\delta(k^{-1})\Phi(xky)\,\mathrm{d}k\right)a.$$

If the representation $g \mapsto U(g)$ is irreducible then the associated spherical function Φ is also irreducible. Conversely, any irreducible spherical function on a compact group G arises in this way from a finite dimensional irreducible representation of G.

2.2 The groups G and K

The three-dimensional sphere S^3 can be realized as the homogeneous space G/K, where $G = \mathrm{SO}(4)$ and $K = \mathrm{SO}(3)$, where as usual we identify $\mathrm{SO}(3)$ as a subgroup of $\mathrm{SO}(4)$: for every k in K, let $k = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \in G$.

Also, we have a decomposition $G = KAK$, where A is the Lie subgroup of G of all elements of the form

$$a(\theta) = \begin{pmatrix} \cos\theta & 0 & 0 & \sin\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta & 0 & 0 & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

It is well known that there exists a double covering Lie homomorphism $\mathrm{SO}(4) \longrightarrow \mathrm{SO}(3) \times \mathrm{SO}(3)$, in particular $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. Explicitly it is obtained in the following way: Let $q : \mathrm{SO}(4) \longrightarrow \mathrm{GL}(\Lambda^2(\mathbb{R}^4))$ be the Lie homomorphism defined by

$$q(g)(e_i \wedge e_j) = g(e_i) \wedge g(e_j), \quad g \in \mathrm{SO}(4), \quad 1 \le i < j \le 4,$$

where $\{e_j\}_{j=1}^4$ is the canonical basis of $\mathbb{R}^4$. Let $\dot{q} : \mathfrak{so}(4) \longrightarrow \mathfrak{gl}(\Lambda^2(\mathbb{R}^4))$ denote the corresponding differential homomorphism.

We observe that $\Lambda^2(\mathbb{R}^4)$ is reducible as G-module. In fact, we have the following decomposition into irreducible G-modules, $\Lambda^2(\mathbb{R}^4) = V_1 \oplus V_2$, where

$$V_1 = \text{span}\{e_1 \wedge e_4 + e_2 \wedge e_3, e_1 \wedge e_3 - e_2 \wedge e_4, -e_1 \wedge e_2 - e_3 \wedge e_4\},$$
$$V_2 = \text{span}\{e_1 \wedge e_4 - e_2 \wedge e_3, e_1 \wedge e_3 + e_2 \wedge e_4, -e_1 \wedge e_2 + e_3 \wedge e_4\}.$$

Let P_1 and P_2 be the canonical projections onto the subspaces V_1 and V_2, respectively. The functions defined by

$$a(g) = P_1 q(g)_{|V_1}, \quad b(g) = P_2 q(g)_{|V_2},$$

are Lie homomorphisms from SO(4) onto SO(V_1) $\simeq$ SO(3) and SO(V_2) $\simeq$ SO(3), respectively. Therefore, in an appropriate basis, we have for each $g \in$ SO(4) and for all $X \in \mathfrak{so}(4)$

$$q(g) = \begin{pmatrix} a(g) & 0 \\ 0 & b(g) \end{pmatrix}, \quad \dot{q}(X) = \begin{pmatrix} \dot{a}(X) & 0 \\ 0 & \dot{b}(X) \end{pmatrix}. \tag{1}$$

Hence, we can consider q as a homomorphism from SO(4) onto SO(3) $\times$ SO(3) with kernel $\{I, -I\}$.

2.3 The Lie algebra structure

The Cartan involution of G (respectively of $\mathfrak{g}$) is $\Theta(g) = I_{3,1} g I_{3,1}$ (respectively $\theta(X) = I_{3,1} X I_{3,1}$), where

$$I_{3,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The subgroup K is the connected component of the fixed points of Θ in G. The corresponding Cartan decomposition of the Lie algebra of G is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $\mathfrak{k}$ being the Lie algebra of K and $\mathfrak{p}$ being the (-1)-eigenspace of θ.

A basis of $\mathfrak{g} = \mathfrak{so}(4)$ over $\mathbb{R}$ is given by

$$Y_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Y_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad Y_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad Y_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Observe that $\{Y_1, Y_2, Y_3\}$ is a basis of $\mathfrak{k}$ and $\{Y_4, Y_5, Y_6\}$ is a basis of $\mathfrak{p}$.

Consider the following vectors

$$Z_1 = \frac{1}{2}(Y_3 + Y_4), \quad Z_2 = \frac{1}{2}(Y_2 - Y_5), \quad Z_3 = \frac{1}{2}(Y_1 + Y_6),$$
$$Z_4 = \frac{1}{2}(Y_3 - Y_4), \quad Z_5 = \frac{1}{2}(Y_2 + Y_5), \quad Z_6 = \frac{1}{2}(Y_1 - Y_6).$$

It can be proved that these vectors define a basis of $\mathfrak{so}(4)$ adapted to the decomposition $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, i.e. $\{Z_4, Z_5, Z_6\}$ is a basis of the first summand and $\{Z_1, Z_2, Z_3\}$ is a basis of the second one.

The algebra $D(G)^G$ is generated by the algebraically independent elements

$$\Delta_1 = -Z_4^2 - Z_5^2 - Z_6^2, \quad \Delta_2 = -Z_1^2 - Z_2^2 - Z_3^2, \tag{2}$$

which are the Casimirs of the first and the second $\mathfrak{so}(3)$, respectively. The Casimir of K will be denoted by Δ_K, and it is given by $-Y_1^2 - Y_2^2 - Y_3^2$.

The complexification of $\mathfrak{k}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. If we define

$$e = \begin{pmatrix} 0 & i & -1 \\ -i & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & i & 1 \\ -i & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2i \\ 0 & 2i & 0 \end{pmatrix} \tag{3}$$

then we have that $\{e, f, h\}$ is an s-triple in $\mathfrak{k}_{\mathbb{C}}$, i.e.

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

We take as a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{so}(4, \mathbb{C})$ the complexification of the maximal abelian subalgebra of $\mathfrak{so}(4)$ of all matrices of the form

$$H = \begin{pmatrix} 0 & x_1 & 0 & 0 \\ -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & -x_2 & 0 \end{pmatrix}.$$

Let $\varepsilon_j \in \mathfrak{h}_{\mathbb{C}}^*$ be given by $\varepsilon_j(H) = -ix_j$ for $j = 1, 2$. Then

$$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\pm(\varepsilon_1 \pm \varepsilon_2) : \varepsilon_1, \varepsilon_2 \in \mathfrak{h}_{\mathbb{C}}^*\},$$

and we choose as positive roots those in the set $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2\}$.

We define

$$X_{\varepsilon_1+\varepsilon_2} = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & -i & -1 \\ -1 & i & 0 & 0 \\ i & 1 & 0 & 0 \end{pmatrix}, \quad X_{\varepsilon_1-\varepsilon_2} = \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & -i & 1 \\ -1 & i & 0 & 0 \\ -i & -1 & 0 & 0 \end{pmatrix},$$

$$X_{-\varepsilon_1+\varepsilon_2} = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \\ -1 & -i & 0 & 0 \\ i & -1 & 0 & 0 \end{pmatrix}, \quad X_{-\varepsilon_1-\varepsilon_2} = \begin{pmatrix} 0 & 0 & 1 & i \\ 0 & 0 & i & -1 \\ -1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \end{pmatrix}.$$

Then, for every H in $\mathfrak{h}_{\mathbb{C}}$, we get that

$$[H, X_{\pm(\varepsilon_1\pm\varepsilon_2)}] = \pm(\varepsilon_1 \pm \varepsilon_2)(H)X_{\pm(\varepsilon_1\pm\varepsilon_2)}.$$

Hence, each $X_{\pm(\varepsilon_1\pm\varepsilon_2)}$ belongs to the root-space $\mathfrak{g}_{\pm(\varepsilon_1\pm\varepsilon_2)}$.

Then, in terms of the root structure of $\mathfrak{so}(4, \mathbb{C})$, Δ_1 and Δ_2 become

$$\Delta_1 = -Z_6^2 + iZ_6 - (Z_5 + iZ_4)(Z_5 - iZ_4),$$
$$\Delta_2 = -Z_3^2 + iZ_3 - (Z_2 + iZ_1)(Z_2 - iZ_1). \tag{4}$$

We observe that $(Z_5 - iZ_4) = X_{\varepsilon_1-\varepsilon_2} \in \mathfrak{g}_{\varepsilon_1-\varepsilon_2}$ and $(Z_2 - iZ_1) = X_{\varepsilon_1+\varepsilon_2} \in \mathfrak{g}_{\varepsilon_1+\varepsilon_2}$ and $Z_3, Z_6 \in \mathfrak{h}_{\mathbb{C}}$.

2.4 Irreducible representations of G and K

Let us first consider SU(2). It is well known that the irreducible finite dimensional representations of SU(2) are, up to equivalence, $(\pi_\ell, V_\ell)_{\ell \geq 0}$, where V_ℓ is the complex vector space of all polynomial functions in two complex variables z_1 and z_2 homogeneous of degree ℓ, and π_ℓ is defined by

$$\pi_\ell \begin{pmatrix} a & b \\ c & d \end{pmatrix} P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right), \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SU(2)}.$$

Hence, since there is a Lie homomorphism of SU(2) onto SO(3) with kernel $\{\pm I\}$, the irreducible representations of SO(3) correspond to those representations π_ℓ of SU(2) with $\ell \in 2\mathbb{N}_0$. Therefore, we have $\hat{\text{SO}}(3) = \{[\pi_\ell]\}_{\ell \in 2\mathbb{N}_0}$, even more, if $\pi = \pi_\ell$ is any such irreducible representation of SO(3), it is well known (see [16, page 32]) that there exists a basis $\mathcal{B} = \{v_j\}_{j=0}^\ell$ of V_π such that the corresponding representation $\dot{\pi}$ of the complexification of $\mathfrak{so}(3)$ is given by

$$\begin{aligned} \dot{\pi}(h)v_j &= (\ell - 2j)v_j, \\ \dot{\pi}(e)v_j &= (\ell - j + 1)v_{j-1}, \quad (v_{-1} = 0), \\ \dot{\pi}(f)v_j &= (j + 1)v_{j+1}, \quad (v_{\ell+1} = 0). \end{aligned}$$

It is known (see [36, page 362]) that an irreducible representation $\tau \in \hat{\text{SO}}(4)$ has highest weight of the form $\eta = m_1\varepsilon_1 + m_2\varepsilon_2$, where m_1 and m_2 are integers such that $m_1 \geq |m_2|$. Moreover, the representation $\tau = \tau_{(m_1,m_2)}$, restricted to SO(3), contains the representation π_ℓ if and only if

$$m_1 \geq \tfrac{\ell}{2} \geq |m_2|.$$

2.5 K-Orbits in G/K

The group $G = \text{SO}(4)$ acts in a natural way in the sphere S^3. This action is transitive, and K is the isotropy subgroup of the north pole $e_4 = (0, 0, 0, 1) \in S^3$. Therefore, $S^3 \simeq G/K$. Moreover, the G-action on S^3 corresponds to the action induced by left multiplication on G/K.

In the north hemisphere of S^3

$$(S^3)^+ = \left\{ x = (x_1, x_2, x_3, x_4) \in S^3 : x_4 > 0 \right\},$$

we will consider the coordinate system $p : (S^3)^+ \longrightarrow \mathbb{R}^3$ given by the central projection of the sphere onto its tangent plane at the north pole (see Fig. 1):

$$p(x) = \left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4} \right) = (y_1, y_2, y_3). \tag{5}$$

Homogeneous coordinates were also used in the case of the complex projective plane, see [10]. The coordinate map p carries the K-orbits in $(S^3)^+$ into the K-orbits in $\mathbb{R}^3$, which are the spheres

$$S_r = \left\{ (y_1, y_2, y_3) \in \mathbb{R}^3 : ||y||^2 = |y_1|^2 + |y_2|^2 + |y_3|^2 = r^2 \right\}, \quad 0 \leq r < \infty.$$

Then, the interval $[0, \infty)$ parameterizes the set of K-orbits of $\mathbb{R}^3$.
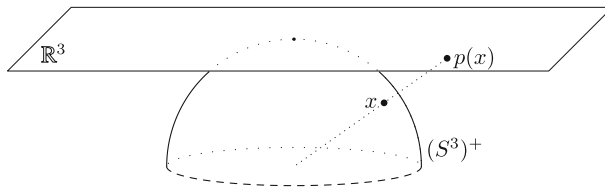
Fig. 1 The central projection p

2.6 The auxiliary function Φ_π

As in [10], to determine all irreducible spherical functions Φ of type $\pi = \pi_\ell \in \hat{K}$, an auxiliary function $\Phi_\pi : G \longrightarrow \mathrm{End}(V_\pi)$ is introduced. In this case, it is defined by

$$\Phi_\pi(g) = \pi(a(g)), \quad g \in G,$$

where a is the Lie homomorphism from SO(4) to SO(3) given in (1). It is clear that Φ_π is an irreducible representation of SO(4) and hence a spherical function of type π (see Definition 2.1).

3 The differential operators D and E

To determine all irreducible spherical functions on G of type $\pi \in \hat{K}$, it is equivalent to determine all analytic functions $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ such that

i $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.
ii $[\Delta_1 \Phi](g) = \widetilde{\lambda}\Phi(g)$, $[\Delta_2 \Phi](g) = \widetilde{\mu}\Phi(g)$ for all $g \in G$ and for some $\widetilde{\lambda}, \widetilde{\mu} \in \mathbb{C}$.

Instead of looking at an irreducible spherical function Φ of type π, we use the auxiliary function Φ_π to look at the function $H : G \longrightarrow \mathrm{End}(V_\pi)$ defined by

$$H(g) = \Phi(g)\Phi_\pi(g)^{-1}. \tag{6}$$

We observe that H is well defined on G because Φ_π is a representation of G. This function H, associated with the spherical function Φ, satisfies

i $H(e) = I$.
ii $H(gk) = H(g)$, for all $g \in G, k \in K$.
iii $H(kg) = \pi(k)H(g)\pi(k^{-1})$, for all $g \in G, k \in K$.

The fact that Φ is an eigenfunction of Δ_1 and Δ_2 makes the function H into an eigenfunction of certain differential operators D and E on G to be determined now. Let us define

$$D(H) = Y_4^2(H) + Y_5^2(H) + Y_6^2(H), \tag{7}$$

$$E(H) = \left(-Y_4(H)Y_3(\Phi_\pi) + Y_5(H)Y_2(\Phi_\pi) - Y_6(H)Y_1(\Phi_\pi)\right)\Phi_\pi^{-1}. \tag{8}$$

Proposition 3.1 *For any* $H \in C^\infty(G) \otimes \mathrm{End}(V_\pi)$ *right invariant under* K, *the function* $\Phi = H\Phi_\pi$ *satisfies* $\Delta_1\Phi = \widetilde{\lambda}\Phi$ *and* $\Delta_2\Phi = \widetilde{\mu}\Phi$ *if and only if* H *satisfies* $DH = \lambda H$ *and* $EH = \mu H$, *with*

$$\lambda = -4\widetilde{\lambda}, \quad \mu = -\tfrac{1}{4}\ell(\ell+2) + \widetilde{\mu} - \widetilde{\lambda}.$$

Proof We firstly observe that $Z_4(\Phi_\pi) = Z_5(\Phi_\pi) = Z_6(\Phi_\pi) = 0$, because Φ_π is a representation of G and $\dot{a}(Z_j) = 0$ for $j = 4, 5, 6$. In fact,

$$Z_j(\Phi_\pi)(g) = \frac{d}{dt}\big|_{t=0}\left[\Phi_\pi(g)\Phi_\pi(\exp t Z_j)\right] = \Phi_\pi(g)\dot{\pi}(\dot{a}(Z_j)) = 0.$$

On the other hand, since H is right invariant under K, we have that $Y_1(H) = Y_2(H) = Y_3(H) = 0$. Since $[Y_3, Y_4] = 0$, $[Y_2, Y_5] = 0$, and $[Y_1, Y_6] = 0$, we have that $Z_j^2(H) = \frac{1}{4}Y_j^2(H)$, for $j = 4, 5, 6$. Therefore, we obtain

$$\Delta_1(H\Phi_\pi) = -\sum_{j=4}^{6} Z_j^2(H)\,\Phi_\pi = -\frac{1}{4}\sum_{j=4}^{6} Y_j^2(H)\,\Phi_\pi = -\frac{1}{4}D(H)\Phi_\pi.$$

On the other hand, we have

$$\Delta_2(H\Phi_\pi) = -\sum_{j=1}^{3}\left(Z_j^2(H)\,\Phi_\pi + 2Z_j(H)Z_j(\Phi_\pi) + HZ_j^2(\Phi_\pi)\right),$$

We observe that $Z_1(H) = \frac{1}{2}Y_4(H)$. Since $Z_1 = Y_3 - Z_4$, we have $Z_1(\Phi_\pi) = Y_3(\Phi_\pi)$ and $Z_1^2(\Phi_\pi) = Y_3^2(\Phi_\pi)$. Similar results hold for Z_2 and Z_3. Therefore,

$$\begin{aligned}
\Delta_2(H\Phi_\pi) &= -\left(Z_1^2(H)\,\Phi_\pi + Y_4(H)Y_3(\Phi_\pi) + HY_3^2(\Phi_\pi)\right)\\
&\quad -\left(Z_2^2(H)\,\Phi_\pi - Y_5(H)Y_2(\Phi_\pi) + HY_2^2(\Phi_\pi)\right)\\
&\quad -\left(Z_3^2(H)\,\Phi_\pi + Y_6(H)Y_1(\Phi_\pi) + HY_1^2(\Phi_\pi)\right)\\
&= -\frac{1}{4}D(H)\,\Phi_\pi + E(H)\,\Phi_\pi + H\Delta_K(\Phi_\pi)\\
&= -\frac{1}{4}D(H)\,\Phi_\pi + E(H)\,\Phi_\pi + H\Phi_\pi\dot{\pi}(\Delta_K).
\end{aligned}$$

Since $\Delta_K \in D(G)^K$, Schur's Lemma tells us that $\dot{\pi}(\Delta_K) = cI$. Now we have $\Delta_1(H\Phi_\pi) = \widetilde{\lambda}H\Phi_\pi$ and $\Delta_2(H\Phi_\pi) = \widetilde{\mu}H\Phi_\pi$ if and only if $D(H) = \lambda H$ and $E(H) = \mu H$, where

$$\widetilde{\lambda} = -\frac{1}{4}\lambda \quad \text{and} \quad \widetilde{\mu} = c + \widetilde{\lambda} + \mu.$$

In order to compute the constant c, we take a highest weight vector $v \in V_\pi$, and write Y_1, Y_2, Y_3 in terms of the basis $\{e, f, g\}$ introduced in (3). It follows that

$$\begin{aligned}
\dot{\pi}(\Delta_K)v &= \dot{\pi}\left(-\left(\tfrac{-i}{2}(e+f)\right)^2 - \left(\tfrac{-1}{2}(e-f)\right)^2 - \left(\tfrac{i}{2}h\right)^2\right)v\\
&= \frac{-1}{4}\dot{\pi}\left(-2ef - 2fe - h^2\right)v = \frac{1}{4}\dot{\pi}\left(2(fe+h) + 2fe + h^2\right)v\\
&= \frac{1}{4}\left(2\ell + \ell^2\right)v = \frac{\ell(\ell+2)}{4}\,v\,.
\end{aligned}$$

Thus, $c = \ell(\ell+2)/4$ completing the proof of the proposition. $\qquad\qquad\square$

Remark 3.2 We observe that the differential operators D and E commute. In fact, from the proof of Proposition 3.1, we have that

$$D(H) = -4\Delta_1(H\Phi_\pi)\Phi_\pi^{-1},$$
$$E(H) = \Delta_2(H\Phi_\pi)\Phi_\pi^{-1} + \tfrac{1}{4}D(H) - \tfrac{\ell(\ell+2)}{2}H,$$

and Δ_1 and Δ_2 commute because they are in the center of the algebra $D(G)$.

3.1 Reduction to G/K

The quotient G/K is the sphere S^3; moreover, the canonical diffeomorphism is given by $gK \mapsto (g_{14}, g_{24}, g_{34}, g_{44}) \in S^3$.

The function H associated with the spherical function Φ is right invariant under K; then, it may be considered as a function on S^3, which we also called H. The differential operators D and E introduced in (7) and (8) define differential operators on S^3.

Lemma 3.3 *The differential operators D and E on G define differential operators D and E acting on $C^\infty(S^3) \otimes \mathrm{End}(V_\pi)$.*

Proof The only thing we need to prove is that D and E preserve the subspace $C^\infty(G)^K \otimes \mathrm{End}(V_\pi)$.

Given an irreducible spherical function Φ of type π and the function Φ_π introduced in Sect. 2.6, let $H(g) = \Phi(g)\,\Phi_\pi^{-1}(g)$. Consider any $D \in D(G)^K$ and the right translation $r_k(g) = gk$. Then

$$
\begin{aligned}
r_k^*(DH)(g) &= r_k^*(\Delta(H\Phi_\pi))(g)\pi(k)^{-1}\Phi_\pi^{-1}(g) \\
&= r_k^*(\Delta)(r_k^*(H\Phi_\pi))(g)\pi(k)^{-1}\Phi_\pi^{-1}(g) = \Delta(H\Phi_\pi)(g)\Phi_\pi^{-1}(g) = DH(g),
\end{aligned}
$$

showing that DH is right K-invariant. $\qquad\square$

Now we give the expressions of the operators D and E in the coordinate system $p : (S^3)^+ \longrightarrow \mathbb{R}^3$ introduced in (5) and given by

$$
p(x) = \left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}\right) = (y_1, y_2, y_3).
$$

We need the following propositions which require simple but lengthy computations. The outlines of the proofs appear in the "Appendix".

Proposition 3.4 *For any $H \in C^\infty(\mathbb{R}^3) \otimes \mathrm{End}(V_\pi)$ we have*

$$
\begin{aligned}
D(H)(y) = (1 + \|y\|^2) &\Big((y_1^2 + 1)H_{y_1 y_1} + (y_2^2 + 1)H_{y_2 y_2} + (y_3^2 + 1)H_{y_3 y_3} \\
&+ 2(y_1 y_2 H_{y_1 y_2} + y_2 y_3 H_{y_2 y_3} + y_1 y_3 H_{y_1 y_3}) + 2(y_1 H_{y_1} + y_2 H_{y_2} + y_3 H_{y_3})\Big).
\end{aligned}
$$

Proposition 3.5 *For any $H \in C^\infty(\mathbb{R}^3) \otimes \mathrm{End}(V_\pi)$ we have*

$$
\begin{aligned}
E(H)(y) = H_{y_1}\dot{\pi} &\begin{pmatrix} 0 & -y_2 - y_1 y_3 & -y_3 + y_1 y_2 \\ y_2 + y_1 y_3 & 0 & -1 - y_1^2 \\ y_3 - y_1 y_2 & 1 + y_1^2 & 0 \end{pmatrix} \\
+ H_{y_2}\dot{\pi} &\begin{pmatrix} 0 & -y_2 y_3 + y_1 & 1 + y_2^2 \\ y_2 y_3 - y_1 & 0 & -y_3 - y_1 y_2 \\ -1 - y_2^2 & y_3 + y_1 y_2 & 0 \end{pmatrix} \\
+ H_{y_3}\dot{\pi} &\begin{pmatrix} 0 & -1 - y_3^2 & y_1 + y_2 y_3 \\ 1 + y_3^2 & 0 & y_2 - y_1 y_3 \\ -y_1 - y_2 y_3 & -y_2 + y_1 y_3 & 0 \end{pmatrix}.
\end{aligned}
$$

3.2 Reduction to one variable

We are interested in considering the differential operators D and E given in Propositions 3.4 and 3.5 applied to functions $H \in C^\infty(\mathbb{R}^3) \otimes \text{End}(V_\pi)$ such that

$$H(ky) = \pi(k)H(y)\pi(k)^{-1}, \quad \text{for all } k \in K, y \in \mathbb{R}^3.$$

Hence, the function $H = H(y)$ is determined by its restriction to a section of the K-orbits in $\mathbb{R}^3$. We recall that the K-orbits in $\mathbb{R}^3$ are the spheres

$$S_r = \left\{(y_1, y_2, y_3) \in \mathbb{R}^3 : ||y||^2 = |y_1|^2 + |y_2|^2 + |y_3|^2 = r^2\right\}, \quad 0 \le r < \infty.$$

In each orbit S_r, we choose the point $(r, 0, 0) \in \mathbb{R}^3$ as a representative.

This allows us to find ordinary differential operators $\widetilde{D}$ and $\widetilde{E}$ defined on the interval $(0, \infty)$ such that

$$(D H)(r, 0, 0) = (\widetilde{D}\widetilde{H})(r), \quad (E H)(r, 0, 0) = (\widetilde{E}\widetilde{H})(r),$$

where $\widetilde{H}(r) = H(r, 0, 0)$.

Remark 3.6 We observe that the differential operators $\widetilde{D}$ and $\widetilde{E}$ commute because they are the restrictions of the commuting differential operators D and E.

In order to give the explicit expressions of the differential operators $\widetilde{D}$ and $\widetilde{E}$, and starting from Propositions 3.4 and 3.5, we need to compute a number of second-order partial derivatives of the function $H : \mathbb{R}^3 \longrightarrow \text{End}(V_\pi)$ at the points $(r, 0, 0)$, with $r > 0$. Given $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ in a neighborhood of $(r, 0, 0)$, $r > 0$, we need a smooth function onto $K = SO(3)$ that carries the point y to the meridian $\{(r, 0, 0) : r > 0\}$. A good choice is the following function

$$A(y) = \frac{1}{\|y\|} \begin{pmatrix} y_1 & -y_2 & -y_3 \\ y_2 & \frac{-y_2^2}{\|y\|+y_1} + \|y\| & \frac{-y_2 y_3}{\|y\|+y_1} \\ y_3 & \frac{-y_2 y_3}{\|y\|+y_1} & \frac{-y_3^2}{\|y\|+y_1} + \|y\| \end{pmatrix}. \tag{9}$$

Then

$$y = A(y)(\|y\|, 0, 0)^t.$$

It is easy to verify that $A(y)$ is a matrix in $SO(3)$ and it is well defined in $\mathbb{R}^3 - \{(y_1, 0, 0) \in \mathbb{R}^3 : y_1 \le 0\}$.

The proofs of the following propositions are similar to those in the case of the complex projective plane considered in [10], see Propositions 13.2 and 13.3 in that paper. Let us consider the following elements in $\mathfrak{k}$

$$A_1 = E_{21} - E_{12}, \quad A_2 = E_{31} - E_{13}, \quad A_3 = E_{32} - E_{23}. \tag{10}$$

Proposition 3.7 *For $r > 0$ we have*

$$\frac{\partial H}{\partial y_1}(r, 0, 0) = \frac{d\widetilde{H}}{dr}(r),$$

$$\frac{\partial H}{\partial y_2}(r, 0, 0) = \frac{1}{r}\left[\dot{\pi}(A_1), \widetilde{H}(r)\right], \quad \frac{\partial H}{\partial y_3}(r, 0, 0) = \frac{1}{r}\left[\dot{\pi}(A_2), \widetilde{H}(r)\right].$$

Proposition 3.8 *For $r > 0$ we have*

$$\frac{\partial^2 H}{\partial y_1^2}(r, 0, 0) = \frac{d^2 \widetilde{H}}{dr^2}(r),$$

$$\frac{\partial^2 H}{\partial y_2^2}(r, 0, 0) = \frac{1}{r^2}\left(r\frac{d\widetilde{H}}{dr} + \dot{\pi}(A_1)^2\,\widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}(A_1)^2 - 2\dot{\pi}(A_1)\,\widetilde{H}(r)\dot{\pi}(A_1) \right),$$

$$\frac{\partial^2 H}{\partial y_3^2}(r, 0, 0) = \frac{1}{r^2}\left(r\frac{d\widetilde{H}}{dr} + \dot{\pi}(A_2)^2\,\widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}(A_2)^2 - 2\dot{\pi}(A_2)\,\widetilde{H}(r)\dot{\pi}(A_2) \right).$$

Now we can obtain the explicit expressions of the differential operators $\tilde{D}$ and $\tilde{E}$.

Theorem 3.9 *For $r > 0$ we have*

$$\widetilde{D}(\widetilde{H})(r) = \left(1 + r^2\right)^2 \frac{d^2 \widetilde{H}}{dr^2} + 2\frac{\left(1 + r^2\right)^2}{r} \frac{d\widetilde{H}}{dr}$$

$$+ \frac{\left(1 + r^2\right)}{r^2}\left(\dot{\pi}(A_1)^2\,\widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}(A_1)^2 - 2\dot{\pi}(A_1)\widetilde{H}(r)\dot{\pi}(A_1)\right)$$

$$+ \frac{\left(1 + r^2\right)}{r^2}\left(\dot{\pi}(A_2)^2\,\widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}(A_2)^2 - 2\dot{\pi}(A_2)\widetilde{H}(r)\dot{\pi}(A_2)\right).$$

Proof Since $\widetilde{D}(\widetilde{H})(r) = D(H)(r, 0, 0)$, from Proposition 3.4 we have

$$\widetilde{D}(\widetilde{H})(r) = \left(1 + r^2\right)\left(\left(1 + r^2\right)H_{y_1 y_1} + H_{y_2 y_2} + H_{y_3 y_3} + 2r H_{y_1}\right).$$

Using Propositions 3.7 and 3.8, we get

$$\widetilde{D}(\widetilde{H})(r) = \left(1 + r^2\right)\left[\left(1 + r^2\right)\frac{d^2 \widetilde{H}}{dr^2}(r) + 2r\frac{d\widetilde{H}}{dr}(r) + \frac{2}{r}\frac{d\widetilde{H}}{dr} \right.$$

$$+ \frac{1}{r^2}\left(\dot{\pi}(A_1)^2\,\widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}(A_1)^2 - 2\dot{\pi}(A_1)\,\widetilde{H}(r)\dot{\pi}(A_1) \right)$$

$$\left. + \frac{1}{r^2}\left(\dot{\pi}(A_2)^2\,\widetilde{H}(r) + \widetilde{H}(r)\dot{\pi}(A_2)^2 - 2\dot{\pi}(A_2)\,\widetilde{H}(r)\dot{\pi}(A_2) \right) \right].$$

Now the theorem follows easily. $\square$

Theorem 3.10 *For $r > 0$, we have*

$$\widetilde{E}(\widetilde{H})(r) = \frac{d\widetilde{H}}{dr}(1 + r^2)\dot{\pi}(A_3) - \frac{1}{r}\left[\dot{\pi}(A_1), \widetilde{H}(r)\right]\dot{\pi}(rA_1 + A_2)$$

$$+ \frac{1}{r}\left[\dot{\pi}(A_2), \widetilde{H}(r)\right]\dot{\pi}(A_1 - rA_2).$$

Proof Since $\widetilde{E}(\widetilde{H})(r) = E(H)(r, 0, 0)$, from Proposition 3.5, we have

$$\widetilde{E}(\widetilde{H})(r) = H_{y_1}\dot{\pi}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 - r^2 \\ 0 & 1 + r^2 & 0 \end{pmatrix} + H_{y_2}\dot{\pi}\begin{pmatrix} 0 & r & 1 \\ -r & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + H_{y_3}\dot{\pi}\begin{pmatrix} 0 & -1 & r \\ 1 & 0 & 0 \\ -r & 0 & 0 \end{pmatrix}.$$

Now, from Proposition 3.7 we get

$$\widetilde{E}(H)(r) = \frac{d\widetilde{H}}{dr}(1+r^2)\dot{\pi}(A_3) - \frac{1}{r}\left[\dot{\pi}(A_1), \widetilde{H}(r)\right]\dot{\pi}(rA_1 + A_2)$$
$$+\frac{1}{r}\left[\dot{\pi}(A_2), \widetilde{H}(r)\right]\dot{\pi}(A_1 - rA_2),$$

which is the statement of the theorem. □

Theorems 3.9 and 3.10 are given in terms of linear transformations. Now we will give the corresponding statements in terms of matrices by choosing an appropriate basis of V_π. We take the $\mathfrak{sl}(2)$-triple $\{e, f, h\}$ in $\mathfrak{k}_\mathbb{C} \simeq \mathfrak{sl}(2, \mathbb{C})$ introduced in (3).

If $\pi = \pi_\ell$ is the only irreducible representation of SO(3) with highest weight $\ell/2$, we recalled in Sect. 2.4 that there exists a basis $\mathcal{B} = \{v_j\}_{j=0}^{\ell}$ of V_π such that

$$\dot{\pi}(h)v_j = (\ell - 2j)v_j,$$
$$\dot{\pi}(e)v_j = (\ell - j + 1)v_{j-1}, \quad (v_{-1} = 0),$$
$$\dot{\pi}(f)v_j = (j+1)v_{j+1}, \quad (v_{\ell+1} = 0). \tag{11}$$

Proposition 3.11 *The function $\widetilde{H}$ associated with an irreducible spherical function Φ of type $\pi \in \hat{K}$ simultaneously diagonalizes in the basis $\mathcal{B} = \{v_j\}_{j=0}^{\ell}$ of V_π.*

Proof Let us consider the subgroup $M = \{m_\theta : \theta \in \mathbb{R}\}$ of K, where

$$m_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}. \tag{12}$$

Then, M is isomorphic to SO(2) and fixes the points $(r, 0, 0)$ in $\mathbb{R}^3$. Also, since the function H satisfies $H(kg) = \pi(k)H(g)\pi(k^{-1})$ for all $k \in K$, we have that

$$\widetilde{H}(r) = H(r, 0, 0) = H(m_\theta(r, 0, 0)^t) = \pi(m_\theta)H(r, 0, 0)\pi(m_\theta^{-1})$$
$$= \pi(m_\theta)\widetilde{H}(r)\pi(m_\theta^{-1}).$$

Hence, $\widetilde{H}(r)$ and $\pi(m_\theta)$ commute for every r in $\mathbb{R}$ and every m_θ in M.

On the other hand, notice that $m_\theta = \exp(\theta\frac{i}{2}h)$ and then $\pi(m_\theta) = \exp(\dot{\pi}(\theta\frac{i}{2}h))$, but from (11), we know that $\dot{\pi}(h)$ diagonalizes and that its eigenvalues have multiplicity one. Therefore, the function $\widetilde{H}(r)$ simultaneously diagonalizes in the basis $\mathcal{B} = \{v_j\}_{j=0}^{\ell}$ of V_π. □

Now we introduce the coordinate functions $\widetilde{h}_j(r)$ by means of

$$\widetilde{H}(r)v_j = \widetilde{h}_j(r)v_j, \tag{13}$$

and we identify $\widetilde{H}$ with the column vector

$$\widetilde{H}(r) = (\widetilde{h}_0(r), \ldots, \widetilde{h}_\ell(r))^t. \tag{14}$$

Corollary 3.12 *The functions $\widetilde{H}(r)$, $0 < r < \infty$, satisfy $(\widetilde{D}\widetilde{H})(r) = \lambda\,\widetilde{H}(r)$ if and only if*

$$(1+r^2)^2\widetilde{h}_j'' + 2\frac{(1+r^2)^2}{r}\widetilde{h}_j' + \frac{1+r^2}{r^2}(j+1)(\ell-j)(\widetilde{h}_{j+1} - \widetilde{h}_j)$$
$$+ \frac{1+r^2}{r^2}j(\ell-j+1)(\widetilde{h}_{j-1} - \widetilde{h}_j) = \lambda\widetilde{h}_j,$$

for all $j = 0, \ldots, \ell$.

Proof Using the basis $\mathcal{B} = \{v_j\}_{j=0}^{\ell}$ of V_π [see (11)] and writing the matrices A_1 and A_2 in terms of the $\mathfrak{sl}(2)$-triple $\{e, f, h\}$, see (3),

$$A_1 = E_{21} - E_{12} = \frac{i}{2}(e + f), \quad A_2 = E_{31} - E_{13} = \frac{1}{2}(e - f),$$

we have that Theorem 3.9 says that $(\widetilde{D}\widetilde{H})(r) = \lambda\,\widetilde{H}(r)$ if and only if

$$\lambda\,\widetilde{H}(r)v_j = (1 + r^2)^2\,\widetilde{H}''(r)\,v_j + 2\frac{(1 + r^2)^2}{r}\widetilde{H}'(r)\,v_j$$
$$- \frac{(1 + r^2)}{4r^2}\left(\dot{\pi}(e + f)^2\,\widetilde{H}(r)\right.$$
$$+ \widetilde{H}(r)\dot{\pi}(e + f)^2 - 2\dot{\pi}(e + f)\widetilde{H}(r)\dot{\pi}(e + f)\big)\,v_j$$
$$+ \frac{(1 + r^2)}{4r^2}\left(\dot{\pi}(e - f)^2\,\widetilde{H}(r)\right.$$
$$+ \widetilde{H}(r)\dot{\pi}(e - f)^2 - 2\dot{\pi}(e - f)\widetilde{H}(r)\dot{\pi}(e - f)\big)\,v_j,$$

for $0 \le j \le \ell$.

As $[e, f] = h$, we have that this is equivalent to

$$\lambda\,\widetilde{H}(r)v_j = (1 + r^2)^2\,\widetilde{H}''(r)\,v_j + 2\frac{(1 + r^2)^2}{r}\widetilde{H}'(r)\,v_j$$
$$- \frac{(1 + r^2)}{2r^2}\left[(\dot{\pi}(h) + 2\dot{\pi}(f)\dot{\pi}(e))\widetilde{H}(r)\,v_j + \widetilde{H}(r)(\dot{\pi}(h) + 2\dot{\pi}(f)\dot{\pi}(e))\,v_j\right.$$
$$\left. - 2(\dot{\pi}(e)\widetilde{H}(r)\dot{\pi}(f) + \dot{\pi}(f)\widetilde{H}(r)\dot{\pi}(e))\,v_j\right],$$

for $0 \le j \le \ell$. Now, using (11), we obtain that $(\widetilde{D}\widetilde{H})(r) = \lambda\,\widetilde{H}(r)$ if and only if

$$\lambda\,\widetilde{h}_j(r)\,v_j = (1 + r^2)^2\widetilde{h}_j''(r)\,v_j + 2\frac{(1 + r^2)^2}{r}\widetilde{h}_j'(r)\,v_j$$
$$- \frac{(1 + r^2)}{2r^2}\left[((\ell - 2j) + 2j(\ell - j + 1))\widetilde{h}_j(r)\,v_j + \widetilde{h}_j(r)((\ell - 2j)\right.$$
$$\left. + 2j(\ell - j + 1))\,v_j - 2((\ell - j)\widetilde{h}_{j+1}(r)(j + 1) + j\widetilde{h}_{j-1}(r)(\ell - j + 1))\,v_j\right],$$

for $0 \le j \le \ell$.

It can be easily checked that this is the required result. $\square$

Corollary 3.13 *The functions* $\widetilde{H}(r)$, $0 < r < \infty$, *satisfy* $(\widetilde{E}\widetilde{H})(r) = \mu\widetilde{H}(r)$ *if and only if*

$$-i(\ell - 2j)\frac{1 + r^2}{2}\widetilde{h}_j' + \frac{i}{2r}\left((j + 1)(\ell - j)(\widetilde{h}_{j+1} - \widetilde{h}_j) - j(\ell - j + 1)(\widetilde{h}_{j-1} - \widetilde{h}_j)\right)$$
$$+ \frac{1}{2}\left((j + 1)(\ell - j)(\widetilde{h}_{j+1} - \widetilde{h}_j) + j(\ell - j + 1)(\widetilde{h}_{j-1} - \widetilde{h}_j)\right) = \mu\widetilde{h}_j,$$

for all $j = 0, \ldots, \ell$.

Proof We proceed in a way similar to the proof of Corollary 3.12. Using the $\mathfrak{sl}(2)$-triple $\{e, f, h\}$ and the matrices A_1, A_2, and A_3 [see (10)], from Theorem 3.10, we have that $(\widetilde{E}\widetilde{H})(r) = \mu\widetilde{H}(r)(r)$ if and only if

$$\mu \widetilde{H}(r) \, v_j = (1 + r^2) H'(r) \dot{\pi} \, (A_3) \, v_j - \frac{1}{r} \left[\dot{\pi} \, (A_1), \widetilde{H}(r) \right] \dot{\pi} \, (r A_1 + A_2) \, v_j$$

$$+ \frac{1}{r} \left[\dot{\pi} \, (A_2), \widetilde{H}(r) \right] \dot{\pi} \, (A_1 - r A_2) \, v_j,$$

for every v_j in $\mathcal{B} = \{ v_j \}_{j=0}^{\ell}$.

As in the proof of Theorem 3.12, we write A_1, A_2, and A_3 in terms of $\{ e, \ f, \ h \}$ (see (3)),

$$A_1 = \frac{i}{2}(e + f), \quad A_2 = \frac{1}{2}(e - f), \quad A_3 = -\frac{i}{2}h.$$

Hence, $(\widetilde{D} \widetilde{H})(r) = \lambda \, \widetilde{H}(r)$ if and only if

$$\mu \widetilde{H}(r) v_j = \frac{1}{4r} \left[\dot{\pi} \, (e + f), \widetilde{H}(r) \right] \dot{\pi} \, (r(e + f) - i(e - f)) \, v_j$$

$$+ \frac{1}{4r} \left[\dot{\pi} \, (e - f), \widetilde{H}(r) \right] \dot{\pi} \, (i(e + f) - r(e - f)) \, v_j - i \frac{1 + r^2}{2} H'(r) \dot{\pi} \, (h) \, v_j,$$

for $0 \leq j \leq \ell$. And that is equivalent to

$$\mu \widetilde{H}(r) \, v_j = -i \frac{1 + r^2}{2} H'(r) \dot{\pi} \, (h) \, v_j + \frac{1}{2r} (r + i) \left[\dot{\pi} \, (e), \widetilde{H}(r) \right] \dot{\pi} (f) \, v_j$$

$$+ \frac{1}{2r} (r - i) \left[\dot{\pi} \, (f), \widetilde{H}(r) \right] \dot{\pi} (e) \, v_j,$$

for $0 \leq j \leq \ell$.

Finally, we use (11) to obtain

$$\mu \widetilde{h}_j \, v_j = -i \frac{1 + r^2}{2} \widetilde{h}'_j (2\ell - j) \, v_j + \frac{1}{2r} (r + i)(\ell - j)(\widetilde{h}_{j+1} - \widetilde{h}_j)(j + 1) \, v_j$$

$$+ \frac{1}{2r} (r - i) j (\widetilde{h}_{j-1} - \widetilde{h}_j)(\ell - j + 1) \, v_j,$$

for $0 \leq j \leq \ell$. Therefore, the corollary is proved. $\qquad \square$

In matrix notation, the differential operators $\widetilde{D}$ and $\widetilde{E}$ are given by

$$\widetilde{D} \widetilde{H} = (1 + r^2)^2 \widetilde{H}'' + 2 \frac{(1 + r^2)^2}{r} \widetilde{H}' + \frac{(1 + r^2)}{r^2} (C_1 + C_0) \widetilde{H},$$

$$\widetilde{E} \widetilde{H} = -i \frac{1 + r^2}{2} A_0 \widetilde{H}' + \frac{i}{2r} (C_1 - C_0) \widetilde{H} + \frac{1}{2} (C_1 + C_0) \widetilde{H}.$$

where the matrices are given by

$$A_0 = \sum_{j=0}^{\ell} (\ell - 2j) E_{j,j},$$

$$C_0 = \sum_{j=1}^{\ell} j(\ell - j + 1)(E_{j,j-1} - E_{j,j}),$$

$$C_1 = \sum_{j=0}^{\ell-1} (j + 1)(\ell - j)(E_{j,j+1} - E_{j,j}). \tag{15}$$

When $\ell = 0$, we are in the scalar case and the matrices C_0, C_1 and A_0 are zero. It is well known that the zonal spherical functions on the sphere S^3 are given, in an appropriate variable x, in terms of Gegenbauer polynomials $C_n^\nu(x)$ with $\nu = 1$ and $n = 0, 1, 2, \ldots$ (see [1] page 302). Therefore, in some variable x, the functions $\widetilde{H}$ should satisfy a differential equation of the form

$$(1 - x^2) y'' - 3xy + n(n + 2) y = 0.$$

This suggests the following change of variable

$$u = \frac{1}{\sqrt{1+r^2}}, \quad u \in (0, 1]. \tag{16}$$

Remark It is worth noticing that if $g = ka(\theta)k'$, with $k, k' \in K$, $a(\theta) \in A$, and $gK = (x_1, x_2, x_3, x_4) \in (S^3)^+$, then

$$u = \cos(\theta),$$

because

$$u(g) = \frac{1}{\sqrt{1+r^2}} = \frac{1}{\sqrt{1+y_1^2+y_2^2+y_3^2}} = x_4 = \cos(\theta).$$

We put

$$H(u) = \widetilde{H}\left(\frac{\sqrt{1-u^2}}{u}\right) \text{ and } h_j(u) = \widetilde{H}_j\left(\frac{\sqrt{1-u^2}}{u}\right). \tag{17}$$

Under this change of variables, the differential operators $\widetilde{D}$ and $\widetilde{E}$ are converted into two new differential operators D and E. We get the following expressions for them,

$$DH = \left(1 - u^2\right)\frac{d^2 H}{du^2} - 3u\frac{dH}{du} + \frac{1}{1-u^2}(C_0 + C_1)H, \tag{18}$$

$$EH = \frac{i}{2}\sqrt{1 - u^2}A_0\frac{dH}{du} + \frac{i}{2}\frac{u}{\sqrt{1-u^2}}(C_1 - C_0)H + \frac{1}{2}(C_0 + C_1)H. \tag{19}$$

At this point, there is a slight abuse of notation, since D and E were used earlier to denote operators on $\mathbb{R}^3$.

Remark 3.14 Clearly from Remark 3.6, we have that the differential operators D and E commute.

4 Eigenfunctions of D

We are interested in determining the functions $H : (0, 1) \longrightarrow \mathbb{C}^{\ell+1}$ that are eigenfunctions of the differential operator

$$DH = \left(1 - u^2\right)\frac{d^2 H}{du^2} - 3u\frac{dH}{du} + \frac{1}{1-u^2}(C_0 + C_1)H,$$

$u \in (0, 1)$.

It is well known that such eigenfunctions are analytic functions on the interval $(0, 1)$ and that the dimension of the corresponding eigenspace is $2(\ell + 1)$.

The equation $DH = \lambda H$ is a coupled system of $\ell + 1$ second-order differential equations in the components $(h_0, \ldots, h_\ell)$ of H, because the $(\ell + 1) \times (\ell + 1)$ matrix $C_0 + C_1$ is not a diagonal matrix. But fortunately the matrix $C_0 + C_1$ is a symmetric one, thus diagonalizable. Now we quote from [9] the Proposition 5.1.

Proposition 4.1 *The matrix $C_0 + C_1$ is diagonalizable. Moreover, the eigenvalues are $-j(j + 1)$ for $0 \leq j \leq \ell$ and the corresponding eigenvectors are given by $u_j = (U_{0,j}, \ldots, U_{\ell,j})$ where*

$$U_{k,j} = {}_3F_2\left(\begin{matrix} -j, -k, j + 1 \\ 1, -\ell \end{matrix}; 1\right),$$

an instance of the Hahn orthogonal polynomials.

Therefore, if we define $\check{H}(u) = U^{-1}H(u)$, we get that $DH = \lambda H$ is equivalent to

$$\left(1 - u^2\right)\frac{d^2\check{H}}{du^2} - 3u\frac{d\check{H}}{du} - \frac{1}{1 - u^2}V_0\check{H} = \lambda\,\check{H},$$

where $V_0 = \sum_{j=0}^{\ell-1} j(j+1)E_{j,j}$.

In this way, we obtain that $DH = \lambda H$ if and only if the jth component $\check{h}_j(u)$ of $\check{H}(u)$, for $0 \le j \le \ell$, satisfies

$$\left(1 - u^2\right)\check{h}''_j(u) - 3u\check{h}'_j(u) - j(j+1)\frac{1}{\left(1 - u^2\right)}\check{h}_j(u) - \lambda\,\check{h}_j(u) = 0. \qquad (20)$$

If we write $\lambda = -n(n+2)$ with $n \in \mathbb{C}$, and $\check{h}_j(u) = (1 - u^2)^{j/2}p_j(u)$. Then, for $0 < j < \ell$, $p_j(u)$ satisfies

$$\left(1 - u^2\right)p''_j(u) - (2j+3)\,up'_j(u) + (n - j)(n + j + 2)p_j(u) = 0. \qquad (21)$$

Making a new change of variable, $s = (1-u)/2$, $s \in \left[0, \frac{1}{2}\right)$, and defining $\tilde{p}_j(s) = p_j(u)$ we have

$$s(1 - s)\tilde{p}''_j(s) + \left(j + \tfrac{3}{2} - (2j+3)\,s\right)\tilde{p}'_j(s) + (n - j)(n + j + 2)\tilde{p}_j = 0, \qquad (22)$$

for $0 < j < \ell$. This is a hypergeometric equation of parameters

$$a = -n + j, \quad b = n + j + 2, \quad c = j + \tfrac{3}{2}.$$

Hence, every solution $\tilde{p}_j(s)$ of (22) for $0 < s < \frac{1}{2}$ is a linear combination of

$$_2F_1\left(\begin{matrix}-n + j,\, n + j + 2 \\ j + 3/2\end{matrix};s\right) \quad \text{and} \quad s^{-j-1/2}\,_2F_1\left(\begin{matrix}-n - 1/2,\, n + 3/2 \\ -j + 1/2\end{matrix};s\right).$$

Therefore, for $0 \le j \le \ell$, any solution $\check{h}_j(u)$ of (20), for $0 < u < 1$, is of the form

$$\check{h}_j(u) = a_j\left(1 - u^2\right)^{j/2}\,_2F_1\left(\begin{matrix}-n + j,\, n + j + 2 \\ j + 3/2\end{matrix};\frac{1 - u}{2}\right)$$
$$+ b_j\left(1 - u^2\right)^{-(j+1)/2}\,_2F_1\left(\begin{matrix}-n - 1/2,\, n + 3/2 \\ -j + 1/2\end{matrix};\frac{1 - u}{2}\right), \qquad (23)$$

for some $a_j, b_j \in \mathbb{C}$.

Therefore, we have proved the following theorem.

Theorem 4.2 *Let $H(u)$ be an eigenfunction of D with eigenvalue $\lambda = -n(n+2)$, $n \in \mathbb{C}$. Then, H is of the form*

$$H(u) = UT(u)P(u) + US(u)Q(u)$$

where U is the matrix defined in (31),

$$T(u) = \sum_{j=0}^{\ell}\left(1 - u^2\right)^{j/2}E_{jj}, \quad S(u) = \sum_{j=0}^{\ell}\left(1 - u^2\right)^{-(j+1)/2}E_{jj},$$

$P = (p_0, \ldots, p_\ell)^t$ and $Q = (q_0, \ldots, q_\ell)^t$ are the vector-valued functions given by

$$p_j(u) = a_j \, {}_2F_1 \left(\begin{matrix} -n+j, n+j+2 \\ j+3/2 \end{matrix} ; \frac{1-u}{2} \right),$$

$$q_j(u) = b_j \, {}_2F_1 \left(\begin{matrix} -n-1/2, n+3/2 \\ -j+1/2 \end{matrix} ; \frac{1-u}{2} \right),$$

where a_j and b_j are arbitrary complex numbers for $j = 0, 1, \ldots, \ell$.

Going back to our problem of determining all irreducible spherical functions Φ, we recall that $\Phi(e) = I$; then, the associated function $H \in C^\infty(\mathbb{R}^3) \otimes \mathrm{End}(V_\pi)$ satisfies $H(0, 0, 0) = I$. In the variable $r \in \mathbb{R}$, we have that $\lim_{r \to 0^+} \widetilde{H}(r) = I$. Therefore, we are interested in those eigenfunctions of D such that

$$\lim_{u \to 1^-} H(u) = (1, 1, \ldots, 1) \in \mathbb{C}^{\ell+1}.$$

From Theorem 4.2, we observe that

$$\lim_{u \to 1^-} P(u) = (a_0, a_1, \ldots, a_\ell) \quad \text{and} \quad \lim_{u \to 1^-} Q(u) = (b_0, b_1, \ldots, b_\ell).$$

Moreover, the matrix $T(u)$ has limit when $u \to 1^-$, while $S(u)$ does not. Therefore, an eigenfunction H of D has limit when $u \to 1^-$ if and only if the limit of $Q(u)$ when $u \to 1^-$ is $(0, \ldots, 0)$. In such a case, we have that

$$\lim_{u \to 1^-} H(u) = \lim_{u \to 1^-} UT(u)P(u) = U (a_0, 0, \ldots, 0)^t = a_0(1, \ldots, 1)^t. \quad (24)$$

In this way, we have proved the following result.

Corollary 4.3 *Let $H(u)$ be an eigenfunction of D with eigenvalue $\lambda = -n(n+2)$, $n \in \mathbb{C}$, such that $\lim_{u \to 1^-} H(u)$ exists. Then, H is of the form*

$$H(u) = UT(u)P(u)$$

with U the matrix defined in (31), $T(u) = \sum_{j=0}^{\ell} (1-u^2)^{j/2} E_{jj}$, and $P = (p_0, \ldots, p_\ell)^t$ is the vector-valued function given by

$$p_j(u) = a_j \, {}_2F_1 \left(\begin{matrix} -n+j, n+j+2 \\ j+3/2 \end{matrix} ; \frac{1-u}{2} \right), \quad 0 \le j \le \ell,$$

where a_j are arbitrary complex numbers for $j = 1, 2, \ldots, \ell$. We also have that $\lim_{u \to 1^-} H(u) = a_0(1, 1, \ldots, 1)^t$. Particularly, if $H(u)$ is associated with an irreducible spherical function, then $a_0 = 1$.

5 Eigenfunctions of D and E

In this section, we shall study the simultaneous solutions of $DH(u) = \lambda H(u)$ and $EH(u) = \mu H(u)$, $0 < u < 1$.

We introduce a matrix function $P(u)$, defined from $H(u)$ by

$$H(u) = U T(u) P(u), \quad (25)$$

where U is the matrix defined in (31) and $T(u) = \sum_{j=0}^{\ell} (1-u^2)^{j/2} E_{jj}$.

The fact that H is an eigenfunction of the differential operators D and E makes P an eigenfunction of the differential operators

$$\bar{D} = (UT(u))^{-1} D (UT(u)) \quad \text{and} \quad \bar{E} = (UT(u))^{-1} E (UT(u)), \qquad (26)$$

with, respectively, the same eigenvalues λ and μ.

The explicit expressions of $\bar{D}$ and $\bar{E}$ shall be given in Theorem 5.2, but first we recall some properties of the Hahn polynomials.

For real numbers $\alpha, \beta > -1$, and for a positive integer N, the Hahn polynomials $Q_n(x) = Q_n(x; \alpha, \beta, N)$ are defined by

$$Q_n(x) = {}_3F_2 \left(\begin{matrix} -n, & -x, & n + \alpha + \beta + 1 \\ & \alpha + 1, & -N \end{matrix} ; 1 \right), \quad \text{for } n = 0, 1, \ldots, N.$$

Taking $\alpha = \beta = 0$, $N = \ell$, $x = j$, $n = k$, we obtain

$$U_{jk} = Q_k(j) = {}_3F_2 \left(\begin{matrix} -k, & -j, & k + 1 \\ & 1, & -\ell \end{matrix} ; 1 \right).$$

These Hahn polynomials are examples of orthogonal polynomials, see [19] equation (1.5.2):

$$\sum_{r=0}^{\ell} Q_j(r) Q_k(r) = \delta_{jk} \frac{(-1)^j (j+1)_{\ell+1} j!}{(2j+1)(-\ell)_j} \ell! = \delta_{jk} \frac{(\ell + j + 1)!(\ell - j)!}{(2j+1) \, \ell! \, \ell!}. \qquad (27)$$

Also these Hahn polynomials satisfy a three-term recursion relation in the variable j, see [1] equation (d) on page 346,

$$\begin{aligned} \left(j(\ell - j + 1) + (j+1)(\ell - j) - k(k+1) \right) U_{jk} \\ = j(\ell - j + 1) U_{j-1,k} + (j+1)(\ell - j) U_{j+1,k}. \end{aligned} \qquad (28)$$

Also, they satisfy a three-term recursion relation in the variable k, see [1] equation (c) on page 346,

$$(\ell - 2j) U_{jk} = \frac{k(\ell + k + 1)}{2k+1} U_{j,k-1} + \frac{(k+1)(\ell - k)}{2k+1} U_{j,k+1}. \qquad (29)$$

Karlin and McGregor in [17] also proved that the Hahn polynomials satisfy a first-order recurrence relation that combines the variables j and k [see also [30, equation (36)]]:

$$\begin{aligned} \left(k(\ell - j) - k(k + j + 1) + 2(j+1)(\ell - j) \right) U_{jk} \\ = 2(j+1)(\ell - j) U_{j+1,k} - k(k + \ell + 1) U_{j,k-1}. \end{aligned} \qquad (30)$$

We will need the following technical lemma.

Lemma 5.1 *Let* $U = (U_{jk})$ *be the matrix defined by*

$$U_{jk} = {}_3F_2 \left(\begin{matrix} -k, & -j, & k + 1 \\ & 1, & -\ell \end{matrix} ; 1 \right), \qquad (31)$$

and let A_0, C_0 *and* C_1 *be the matrices introduced in* (15). *Then,*

$$\begin{aligned} U^{-1} A_0 U &= Q_0 + Q_1, \\ U^{-1}(C_1 + C_0) U &= -V_0, \\ U^{-1}(C_1 - C_0) U &= Q_1 J - Q_0(J + 1), \end{aligned} \qquad (32)$$

where

$$V_0 = \sum_{j=0}^{\ell-1} j(j+1)E_{j,j}, \quad J = \sum_{j=0}^{\ell} jE_{jj},$$

$$Q_0 = \sum_{j=0}^{\ell-1} \frac{(j+1)(\ell+j+2)}{2j+3} E_{j,j+1}, \quad Q_1 = \sum_{j=1}^{\ell} \frac{j(\ell-j+1)}{2j-1} E_{j,j-1}.$$

Proof To prove that $U^{-1}A_0U = Q_0 + Q_1$ is equivalent to verifying that

$$A_0U = U(Q_0 + Q_1).$$

By taking a look at the jk-entry for $j, k = 0, \ldots, \ell$, we obtain that

$$(\ell - 2j)U_{jk} = U_{j,k-1}\tfrac{k(\ell+k+1)}{2k+1} + U_{j,k+1}\tfrac{(k+1)(\ell-k)}{2k+1}.$$

This is the three-term recursion relation in the variable k given in (29).

Observe that $U^{-1}(C_1 + C_0)U = -V_0$ is a direct consequence of Proposition 4.1. Also, it follows directly by considering every jk-entry of $(C_1 + C_0)U = -UV_0$ and by using the recursion relation (28).

Now we have to prove that

$$U^{-1}(C_1 - C_0)U = -Q_0(J+1) + Q_1 J.$$

By using $(C_0 + C_1)U = -UV_0$, it is equivalent to prove that

$$-2C_0U = U(-Q_0(J+1) + Q_1 J + V_0);$$

therefore, if we look at the jk-entry, what we need to verify is

$$-2(C_0)_{j,j}U_{jk} - 2(C_0)_{j,j-1}U_{j-1,k}$$
$$= -U_{j,k-1}(Q_0)_{k-1,k}(J+1)_{k,k} + U_{j,k+1}(Q_1)_{k+1,k}J_{k,k} + U_{jk}(V_0)_{k,k},$$

or, equivalently, we have to prove that

$$2j(\ell - j + 1)U_{jk} - 2j(\ell - j + 1)U_{j-1,k}$$
$$= -\tfrac{k(\ell+k+1)(k+1)}{2k+1}U_{j,k-1} + \tfrac{k(k+1)(\ell-k)}{2k+1}U_{j,k+1} + k(k+1)U_{jk}. \tag{33}$$

By using the recurrence relation (29), we can write $U_{j,k+1}$ in terms of U_{jk} and $U_{j,k-1}$. Therefore, the identity (33) becomes

$$\big(2j(\ell - j + 1) - k(\ell - 2j) - k(k+1)\big)U_{jk}$$
$$= 2j(\ell - j + 1)U_{j-1,k} - k(\ell + k + 1)U_{j,k-1},$$

Finally, we use (28) to write $U_{j-1,k}$ in terms of $U_{j+1,k}$ and U_{jk} and obtain

$$\big(k(\ell - j) - k(k + j + 1) + 2(j+1)(\ell - j)\big)U_{jk}$$
$$= 2(j+1)(\ell - j)U_{j+1,k} - k(k + \ell + 1)U_{j,k-1},$$

which is exactly the identity in (30), and this concludes the proof of the Lemma 5.1. $\qquad\square$

Theorem 5.2 *The operators $\bar{D}$ and $\bar{E}$ defined in (26) are given by*

$$\bar{D}P = (1 - u^2)P'' - uCP' - VP,$$

$$\bar{E}P = \frac{i}{2}\big((1-u^2)Q_0 + Q_1\big)P' - \frac{i2}{u}MP - \tfrac{1}{2}V_0P,$$

where

$$C = \sum_{j=0}^{\ell} (2j+3)E_{jj}, \quad V = \sum_{j=0}^{\ell} j(j+2)E_{jj},$$

$$Q_0 = \sum_{j=0}^{\ell-1} \frac{(j+1)(\ell+j+2)}{2j+3} E_{j,j+1}, \quad Q_1 = \sum_{j=1}^{\ell} \frac{j(\ell-j+1)}{2j-1} E_{j,j-1},$$

$$M = \sum_{j=0}^{\ell-1} (j+1)(\ell+j+2)E_{j,j+1}, \quad V_0 = \sum_{j=0}^{\ell-1} j(j+1)E_{jj}.$$

Proof Let $H = H(u) = UT(u)P(u)$. We start by computing $D(H)$ for the differential operator D introduced in (18).

$$DH = (1-u^2)UTP'' + (2(1-u^2)UT' - 3uUT)P'$$
$$+ \left((1-u^2)UT'' - 3uUT' + \tfrac{1}{1-u^2}(C_0+C_1)UT \right)P$$
$$= UT \Big((1-u^2)P'' + (2(1-u^2)T^{-1}T' - 3u)P'$$
$$+ \left((1-u^2)T^{-1}T'' - 3uT^{-1}T' + \frac{1}{1-u^2}T^{-1}U^{-1}(C_0+C_1)UT \right)P \Big).$$

Since T is a diagonal matrix, we easily compute

$$T^{-1}(u)T'(u) = -\frac{u}{(1-u^2)} \sum_{j=0}^{\ell} j\, E_{jj}, \quad T^{-1}T''(u) = \frac{1}{(1-u^2)^2} \sum_{j=0}^{\ell} j((j-1)u^2-1)\, E_{jj}.$$

Also, from (32) we have that $U^{-1}(C_0+C_1)U = -V_0$. Since V_0 is a diagonal matrix, it commutes with T and we get

$$(1-u^2)T^{-1}T'' \quad -3uT^{-1}T' + \frac{1}{1-u^2}T^{-1}U^{-1}(C_0+C_1)UT$$
$$= \frac{1}{(1-u^2)} \sum_{j=0}^{\ell} \left(j(j-1)u^2 - j + 3ju^2 - j(j+1) \right)E_{jj} = -V.$$

Now, for the differential operator E introduced in (18), we compute $E(H)$ with $H(u) = UT(u)P(u)$.

$$EH = \frac{i}{2}\sqrt{1-u^2}\,A_0UTP'$$
$$+ \left(\frac{i}{2}\sqrt{1-u^2}\,A_0UT' + \frac{i}{2}\frac{u}{\sqrt{1-u^2}}(C_1-C_0)UT + \frac{1}{2}(C_0+C_1)UT \right)P$$
$$= UT \left(\frac{i}{2}\sqrt{1-u^2}\,T^{-1}U^{-1}A_0UTP' + \left(\frac{i}{2}\sqrt{1-u^2}\,T^{-1}U^{-1}A_0UT' \right.\right.$$
$$+ \left.\left. \frac{i}{2}\frac{u}{\sqrt{1-u^2}}T^{-1}U^{-1}(C_1-C_0)UT + \frac{1}{2}T^{-1}U^{-1}(C_0+C_1)UT \right)P \right).$$

From Lemma 5.1 above, we have that $U^{-1}A_0U = Q_0 + Q_1$. By using $T = \sum_{j=0}^{\ell}(1-u^2)^{j/2}E_{jj}$, we get

$$\sqrt{1-u^2}\,T^{-1}U^{-1}A_0UT = \sqrt{1-u^2}\,T^{-1}(Q_0+Q_1)T = (1-u^2)Q_0 + Q_1.$$

From (32) and the fact that T is diagonal, we have that $T^{-1}U^{-1}(C_0 + C_1)UT = -V_0$. Then, it only remains to prove that

$$\sqrt{1 - u^2}T^{-1}U^{-1}A_0UT' + \frac{u}{\sqrt{1 - u^2}}T^{-1}U^{-1}(C_1 - C_0)UT = -uM. \qquad (34)$$

Since $T'(u) = \frac{-u}{1-u^2}JT(u)$, where $J = \sum_{j=0}^{\ell} j E_{jj}$, we have to prove that

$$T^{-1}\left(U^{-1}A_0UJ - U^{-1}(C_1 - C_0)U\right)T = \sqrt{1 - u^2}M. \qquad (35)$$

From Lemma 5.1 we have

$$U^{-1}A_0UJ - U^{-1}(C_1 - C_0)U = (Q_1 + Q_0)J - Q_1J + Q_0(J + 1) = Q_0(2J + 1)$$

$$= \sum_{j=0}^{\ell-1}(j + 1)(\ell + j + 2)E_{j,j+1} = M.$$

Since $T = \sum_{j=0}^{\ell}(1-u^2)^{j/2}E_{jj}$, (35) is satisfied and this completes the proof of the theorem. $\qquad \square$

The function P is an eigenfunction of the differential operator $\bar{D}$ if and only if the function $H = UT(u)P(u)$ is an eigenfunction of the differential operator D. From Theorem 4.2, we have the explicit expression of the function $P(u) = (p_0(u), \ldots, p_\ell(u))^t$,

$$p_j(u) = a_j \, {}_2F_1\left(\begin{matrix} -n + j, n + j + 2 \\ j + 3/2 \end{matrix}; \frac{1 - u}{2}\right)$$

$$+ b_j \left(1 - u^2\right)^{-(j+1/2)} {}_2F_1\left(\begin{matrix} -n - 1/2, n + 3/2 \\ -j + 1/2 \end{matrix}; \frac{1 - u}{2}\right),$$

where a_j and b_j are in $\mathbb{C}$, for $0 \le j \le \ell$.

Since we are interested in determining the irreducible spherical functions of the pair (G, K), we need to study the simultaneous eigenfunctions of D and E such that there exists a finite limit of the function H when $u \to 1^-$.

From Theorem 4.2, we have that $\lim_{u \to 1^-} H(u)$ is finite if and only if

$$\lim_{u \to 1^-} b_j \left(1 - u^2\right)^{-(j+1)/2} {}_2F_1\left(\begin{matrix} -n - 1/2, n + 3/2 \\ -j + 1/2 \end{matrix}; \frac{1 - u}{2}\right)$$

exists and is finite for all $0 \le j \le \ell$. This is true if and only if $b_j = 0$ for all $0 \le j \le \ell$. Therefore, $\lim_{u \to 1^-} H(u)$ is finite if and only if $\lim_{u \to 1^-} P(u)$ is finite.

From Corollary 4.3, we know that an eigenfunction $P = P(u)$ of $\bar{D}$ in the interval $(0, 1)$ has a finite limit as $u \to 1^-$ if and only if P is analytic at $u = 1$. Let us now consider the following vector space of functions into $\mathbb{C}^{\ell+1}$,

$$W_\lambda = \left\{P = P(u) \text{ analytic in } (0, 1] : \bar{D}P = \lambda P\right\}.$$

A function $P \in W_\lambda$ is characterized by $P(1) = (a_0, \ldots, a_\ell)$. Thus, the dimension of W_λ is $\ell + 1$ and the isomorphism $W_\lambda \simeq \mathbb{C}^{\ell+1}$ is given by

$$\nu : W_\lambda \longrightarrow \mathbb{C}^{\ell+1}, \quad P \mapsto P(1).$$

The differential operators $\bar{D}$ and $\bar{E}$ commute because the differential operators D and E commute (see Remark 3.14).

Proposition 5.3 *The linear space W_λ is stable under the differential operator $\bar{E}$ and it restricts to a linear map on W_λ. Moreover, the following is a commutative diagram*

$$
\begin{array}{ccc}
W_\lambda & \xrightarrow{\ \bar{E}\ } & W_\lambda \\
{\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \nu} \\
\mathbb{C}^{\ell+1} & \xrightarrow{\ L(\lambda)\ } & \mathbb{C}^{\ell+1}
\end{array}
\tag{36}
$$

where $L(\lambda)$ is the $(\ell+1) \times (\ell+1)$ matrix

$$
L(\lambda) = -\frac{i}{2} Q_1 C^{-1}(V+\lambda) - \frac{i}{2} M - \frac{1}{2} V_0
$$
$$
= -i \sum_{j=1}^{\ell} \frac{j(\ell-j+1)\big((j-1)(j+1)+\lambda\big)}{2(2j-1)(2j+1)} E_{j,j-1} - i \sum_{j=0}^{\ell-1} \frac{(j+1)(\ell+j+2)}{2} E_{j,j+1}
$$
$$
- \sum_{j=0}^{\ell} \frac{j(j+1)}{2} E_{jj}.
$$

Proof The differential operator $\bar{E}$ takes analytic functions into analytic functions, because its coefficients are polynomials, see Theorem 5.2. A function $P \in W_\lambda$ is analytic, then $\lim_{u \to 1^-} \bar{E}P(u)$ is finite. On the other hand, since $\bar{D}$ and $\bar{E}$ commute, the differential operator $\bar{E}$ preserves the eigenspaces of $\bar{D}$. This proves that W_λ is stable under $\bar{E}$. In particular, $\bar{E}$ restricts to a linear map $L(\lambda)$ on W_λ, to be determined now.

From Theorem 5.2, we have

$$
\nu(\bar{E}(P)) = (\bar{E}P)(1) = \tfrac{i}{2}Q_1 P'(1) - \tfrac{i}{2}MP(1) - \tfrac{1}{2}V_0 P(1).
$$

But we can obtain $P'(1)$ in terms of $P(1)$. In fact, if we evaluate $\bar{D}P = \lambda P$ at $u = 1$, we get

$$
P'(1) = -C^{-1}(V+\lambda)P(1).
$$

Notice that C is an invertible matrix. Hence,

$$
\nu(\bar{E}(P)) = -\frac{i}{2}Q_1 C^{-1}(V+\lambda)P(1) - \tfrac{i}{2}MP(1) - \tfrac{1}{2}V_0 P(1)
$$
$$
= L(\lambda)P(1) = L(\lambda)\,\nu(P).
$$

This completes the proof of the proposition.   □

Remark 5.4 If $\lambda = -n(n+2)$, with $n \in \mathbb{C}$, then we have

$$
L(\lambda) = i \sum_{j=1}^{\ell} \tfrac{j(\ell-j+1)(n-j+1)(n+j+1)}{2(2j-1)(2j+1)} E_{j,j-1} - i \sum_{j=0}^{\ell-1} \tfrac{(j+1)(\ell+j+2)}{2} E_{j,j+1}
$$
$$
- \sum_{j=0}^{\ell} \tfrac{j(j+1)}{2} E_{jj}.
\tag{37}
$$

Corollary 5.5 *All eigenvalues μ of $L(\lambda)$ have geometric multiplicity one, that is, all eigenspaces are one dimensional.*

Proof A vector $a = (a_0, a_1, \ldots, a_\ell)^t$ is an eigenvector of $L(\lambda)$ of eigenvalue μ, if and only if $\{a_j\}_{j=0}^{\ell}$ satisfies the following three-term recursion relation

$$i \frac{j(\ell - j + 1)(n - j + 1)(n + j + 1)}{2(2j - 1)(2j + 1)} a_{j-1} - \frac{j(j + 1)}{2} a_j - i \frac{(j+1)(\ell+j+2)}{2} a_{j+1} = \mu a_j, \tag{38}$$

for $j = 0, \ldots, \ell - 1$ (where we interpret $a_{-1} = 0$), and

$$i \frac{\ell(n - \ell + 1)(n + \ell + 1)}{2(2\ell - 1)(2\ell + 1)} a_{\ell-1} - \frac{\ell(\ell + 1)}{2} a_\ell = \mu a_\ell. \tag{39}$$

From these equations, we see that the vector a is determined by a_0, which proves that the geometric multiplicity of the eigenvalue μ of $L(\lambda)$ is one. $\qquad\square$

Remark 5.6 The values of μ for which the Eqs. (38) and (39) have a solution $\{a_j\}_{j=0}^{\ell}$ are exactly the eigenvalues of the matrix $L(\lambda)$.

The Eqs. (38), for $j = 0, \ldots, \ell-1$, are used to define $a_1, \ldots, a_\ell$ starting with any $a_0 \in \mathbb{C}$. The Eq. (39) is an extra condition (a "closing equation") that the coefficients a_j should satisfy in order for $a = (a_0, \ldots, a_\ell)$ to be an eigenvector of $L(\lambda)$ of eigenvalue μ.

Finally, we get the main result of this section which is the characterization of the simultaneous eigenfunctions H of the differential operators D and E in $(0, 1)$, which are continuous in $(0, 1]$. Recall that the irreducible spherical functions of the pair (G, K) give rise to such functions H.

Corollary 5.7 *Let $H(u)$ be a simultaneous eigenfunction of D and E in $(0, 1)$, continuous in $(0, 1]$, with respective eigenvalues $\lambda = -n(n + 2)$, $n \in \mathbb{C}$, and μ. Thus, H is of the form*

$$H(u) = UT(u)P(u)$$

with U the matrix given in (31), $T(u) = \sum_{j=0}^{\ell}(1 - u^2)^{j/2} E_{jj}$, and $P = (p_0, \ldots, p_\ell)^t$ is the vector-valued function given by

$$p_j(u) = a_j \, {}_2F_1\left(\begin{matrix} -n + j, n + j + 2 \\ j + 3/2 \end{matrix}; \frac{1-u}{2}\right)$$

where $\{a_j\}_{j=0}^{\ell}$ satisfies the recursion relations (38) and (39). We also have that $H(1) = a_0(1, 1, \ldots, 1)^t$. In particular, if $H(u)$ is associated with an irreducible spherical function we have that $a_0 = 1$.

Remark 5.8 The condition $H(1) = (1, \ldots, 1)^t$ implies that $P(1)$ is a vector whose first entry is equal to 1.

In S^3, the set

$$\left\{x_\theta = (\sqrt{1 - \theta^2}, 0, 0, \theta) : \theta \in [-1, 1]\right\}$$

parameterizes all the K-orbits. Notice that for $\theta > 0$, we have that $x_\theta \in (S^3)^+$, and $p(x_\theta) = (\frac{\sqrt{1-\theta^2}}{\theta}, 0, 0)$. Therefore, in terms of the variable $r \in [0, \infty)$, we have that $r = \frac{\sqrt{1-\theta^2}}{\theta}$, and then, in terms of the variable $u \in (0, 1]$, we get $u = \frac{1}{\sqrt{1+r^2}} = \theta$ Hence, given an

irreducible spherical function Φ of type $\pi \in \hat{K}$, if we consider the associated function $H : S^3 \longrightarrow \mathrm{End}(V_\pi)$ defined by

$$H(g\,(0, 0, 0, 1)^t) = \Phi(g)\Phi_\pi^{-1}(g), \quad g \in G,$$

we have that

$$H(\sqrt{1 - u^2}, 0, 0, u) = \mathrm{diag}\{H(u)\} = \mathrm{diag}\{UT(u)P(u)\}, \tag{40}$$

where $H(u)$, $u \in (0, 1]$, is the vector-valued function given in Corollary 5.7 and $\mathrm{diag}\{H(u)\}$ means the diagonal matrix-valued function whose kk-entry is equal to the kth entry of the vector-valued function $H(u)$.

6 Eigenvalues of the spherical functions

The aim of this section is to use the representation theory of G to compute the eigenvalues of an irreducible spherical function Φ corresponding to the differential operators Δ_1 and Δ_2. From these eigenvalues, we shall obtain the eigenvalues of the function H as eigenfunctions of D and E.

As we described in Sect. 2, there exists a one to one correspondence between irreducible spherical functions of (G, K) of type $\delta \in \hat{K}$ and finite dimensional irreducible representations of G that contain the K-type δ. In fact, every irreducible spherical function Φ of type $\delta \in \hat{K}$ is of the form

$$\Phi(g)v = P(\delta)\tau(g)v, \quad g \in G, \quad v \in P(\delta)V_\tau, \tag{41}$$

where (τ, V_τ) is a finite dimensional irreducible representation of G, which contains the K-type δ, and $P(\delta)$ is the projection of V_τ onto the K-isotypic component of type δ.

The irreducible finite dimensional representations τ of $G = \mathrm{SO}(4)$ are parameterized by a pair of integers (m_1, m_2) such that

$$m_1 \geq |m_2|,$$

while the irreducible finite dimensional representations π_ℓ of $K = \mathrm{SO}(3)$ are parameterized by $\ell \in 2\mathbb{N}_0$.

The representations $\tau_{(m_1,m_2)}$ restricted to $\mathrm{SO}(3)$ contain the representation π_ℓ if and only if $m_1 \geq \ell/2 \geq |m_2|$. Therefore, the equivalence classes of irreducible spherical functions of (G, K) of type π_ℓ are parameterized by the set of all pairs $(m_1, m_2) \in \mathbb{Z}^2$ such that

$$m_1 \geq \tfrac{\ell}{2} \geq |m_2|.$$

We denote by

$$\Phi_\ell^{(m_1,m_2)}, \quad \text{with} \quad m_1 \geq \tfrac{\ell}{2} \geq |m_2|,$$

the spherical function of type π_ℓ associated with the representation $\tau_{(m_1,m_2)}$ of G.

Theorem 6.1 *The spherical function* $\Phi_\ell^{(m_1,m_2)}$ *satisfies*

$$\Delta_1\Phi_\ell^{(m_1,m_2)} = \frac{1}{4}(m_1 - m_2)(m_1 - m_2 + 2)\Phi_\ell^{(m_1,m_2)},$$

$$\Delta_2\Phi_\ell^{(m_1,m_2)} = \frac{1}{4}(m_1 + m_2)(m_1 + m_2 + 2)\Phi_\ell^{(m_1,m_2)}.$$

Proof We start by observing that the eigenvalue of any irreducible spherical function Φ corresponding to a differential operator $\Delta \in D(G)^G$, given by $[\Delta\Phi](e)$, is a scalar multiple of the identity. Since Δ_1 and Δ_2 are in $D(G)^G$, we have that

$$\Delta_1\Phi_\ell^{(m_1,m_2)}](e) = \dot{\tau}_{(m_1,m_2)}(\Delta_1) \quad \text{and} \quad [\Delta_2\Phi_\ell^{(m_1,m_2)}](e) = \dot{\tau}_{(m_1,m_2)}(\Delta_2).$$

These scalars can be computed by looking at the action of Δ_1 and Δ_2 on a highest weight vector v of the representation $\tau_{(m_1,m_2)}$, whose highest weight is of the form $m_1\varepsilon_1 + m_2\varepsilon_2$.

Recall that

$$\Delta_1 = (iZ_6)^2 + iZ_6 - (Z_5 + iZ_4)(Z_5 - iZ_4),$$
$$\Delta_2 = (iZ_3)^2 + iZ_3 - (Z_2 + iZ_1)(Z_2 - iZ_1).$$

Since $(Z_5 - iZ_4)$ and $(Z_2 - iZ_1)$ are positive root vectors and $Z_6, Z_3 \in \mathfrak{h}_\mathbb{C}$, we get

$$\dot{\tau}_{(m_1,m_2)}(\Delta_1)v = \dot{\tau}_{(m_1,m_2)}(iZ_6)^2 v + \dot{\tau}_{(m_1,m_2)}(iZ_6)v = \frac{1}{4}(m_1 - m_2)(m_1 - m_2 + 2)v,$$

$$\dot{\tau}_{(m_1,m_2)}(\Delta_2)v = \dot{\tau}_{(m_1,m_2)}(iZ_3)^2 v + \dot{\tau}_{(m_1,m_2)}(iZ_3)v = \frac{1}{4}(m_1 + m_2)(m_1 + m_2 + 2)v.$$

This completes the proof of the theorem. $\square$

Now we give the eigenvalues of the function H associated with an irreducible spherical function, corresponding to the differential operators D and E.

Corollary 6.2 *The function H associated with the spherical function $\Phi_\ell^{(m_1,m_2)}$ satisfies $DH = \lambda H$ and $EH = \mu H$ with*

$$\lambda = -(m_1 - m_2)(m_1 - m_2 + 2), \quad \mu = -\frac{\ell(\ell+2)}{4} + (m_1 + 1)m_2.$$

Proof Let $\Phi = \Phi_\ell^{(m_1,m_2)}$. From Proposition 3.1, we have that $\Delta_1\Phi = \widetilde{\lambda}\Phi$ and $\Delta_2\Phi = \widetilde{\mu}\Phi$ if and only if $DH = \lambda H$ and $EH = \mu H$, where the relation between the eigenvalues of H and Φ is

$$\lambda = -4\widetilde{\lambda}, \quad \mu = -\frac{1}{4}\ell(\ell+2) + \widetilde{\mu} - \widetilde{\lambda}.$$

Now the statement follows easily from Theorem 6.1. $\square$

Corollary 6.3 *The function P associated with the spherical function $\Phi_\ell^{(m_1,m_2)}$, defined by $H(u) = UT(u)P(u)$ [see (25)], satisfies $\bar{D}P = \lambda P$ and $\bar{E}P = \mu P$ with*

$$\lambda = -(m_1 - m_2)(m_1 - m_2 + 2), \quad \mu = -\frac{\ell(\ell+2)}{4} + (m_1 + 1)m_2.$$

Remark 6.4 Notice that we have just proved that the eigenvalue λ can be written in the form

$$\lambda = -n(n+2), \quad \text{with} \quad n \in \mathbb{N}_0.$$

Proposition 6.5 *If Φ is an irreducible spherical function of $(\mathrm{SO}(4), \mathrm{SO}(3))$, then $\Phi(-e) = \pm I$. Moreover, if $\Phi = \Phi_\ell^{(m_1,m_2)}$ and $g \in \mathrm{SO}(4)$, then*

$$\Phi(-g) = \begin{cases} \Phi(g) & \text{if } m_1 + m_2 \equiv 0 \mod (2) \\ -\Phi(g) & \text{if } m_1 + m_2 \equiv 1 \mod (2). \end{cases}$$

Proof As we mentioned in Sect. 2, every irreducible spherical function of the pair $\big(\mathrm{SO}(4), \mathrm{SO}(3)\big)$ of type π is of the form $\Phi(g) = P_\pi \tau(g)$, where $\tau \in \hat{\mathrm{SO}}(4)$ contains the K-type π and P_π is the projection onto the π-isotypic component of V_τ.

Let η be the highest weight of $\tau = (m_1, m_2) \in \hat{\mathrm{SO}}(4)$, i.e. $\eta = m_1\varepsilon_1 + m_2\varepsilon_2$ (see Sect. 2.4). We have that

$$-e = \exp \begin{pmatrix} 0 & \pi & 0 & 0 \\ -\pi & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & -\pi & 0 \end{pmatrix};$$

therefore, if v is a highest weight vector in V_τ of weight η, we have

$$\tau(-e)v = e^{-\pi(m_1+m_2)i}v = \pm v.$$

Since $\tau(-e)$ commutes with $\tau(g)$ for all $g \in \mathrm{SO}(4)$, by Schur's Lemma $\tau(-e)$ is a multiple of the identity . Thus,

$$\tau(-e) = \begin{cases} I, & \text{if } m_1 + m_2 \equiv 0 \mod (2) \\ -I, & \text{if } m_1 + m_2 \equiv 1 \mod (2) \end{cases}.$$

Therefore,

$$\Phi(-g) = P_\pi \tau(-g) = P_\pi \tau(-e)\tau(g) = \tau(-e)\Phi(g).$$

Hence, the proposition is proved. $\qquad\square$

7 The function P associated with a spherical function

In the previous sections, we were interested in studying the irreducible spherical functions Φ of a K-type $\pi = \pi_\ell$. This is accomplished by associating each function Φ to a $\mathbb{C}^{\ell+1}$-valued function H, which is a simultaneous eigenfunction of the differential operators D and E in $(0, 1)$, given in (18) and (18), continuous in $(0, 1]$ and such that $H(1) = (1, \ldots, 1)$. This function H is of the form

$$H(u) = UT(u)P(u),$$

where U is the constant matrix given in (31) and $T(u) = \sum_{j=0}^{\ell}(1 - u^2)^{j/2}E_{jj}$.

In this way, we have associated each irreducible spherical function Φ to a function $P(u)$, analytic in $(0, 1]$, which is a simultaneous eigenfunction of the differential operators $\bar{D}$ and $\bar{E}$, explicitly given in Theorem 5.2 by

$$\bar{D}P = \big(1 - u^2\big)P'' - uCP' - VP,$$
$$\bar{E}P = \tfrac{i}{2}\big((1 - u^2)Q_0 + Q_1\big)P' - \tfrac{i}{2}uMP - \tfrac{1}{2}V_0 P.$$

From Corollary 4.3, we have that a vector-valued eigenfunction $P = (p_0, \ldots, p_\ell)^t$ of $\bar{D}$ with eigenvalue $\lambda = -n(n + 2)$ and such that $\lim_{u \to 1^-} P(u)$ exists is given by

$$p_j(u) = a_j \, {}_2F_1\left(\begin{matrix} -n + j, n + j + 2 \\ j + 3/2 \end{matrix}; \tfrac{1-u}{2} \right), \quad 0 \le j \le \ell,$$

where a_j are arbitrary complex numbers for $j = 1, 2, \ldots, \ell$.

Let us introduce the following vector space of functions into $\mathbb{C}^{\ell+1}$, defined for $n \in \mathbb{N}_0$ and $\mu \in \mathbb{C}$,

$$\mathcal{V}_{n,\mu} = \{P = P(u) \text{ analytic in } (0,1] : \bar{D}P = -n(n+2)P, \ \bar{E}P = \mu P\}. \quad (42)$$

We observe that $\mathcal{V}_{n,\mu} \neq 0$ if and only if μ is an eigenvalue of the matrix $L(\lambda)$ given in (37), with $\lambda = -n(n+2)$. We are interested in considering only the cases $\mathcal{V}_{n,\mu} \neq 0$.

From Corollary 5.7, we have that a function $P \in \mathcal{V}_{n,\mu}$ is of the form $P = (p_0, \ldots, p_\ell)^t$, where

$$p_j(u) = a_j \, _2F_1 \left(\begin{matrix} -n+j, n+j+2 \\ j+3/2 \end{matrix} ; \frac{1-u}{2} \right), \quad 0 \leq j \leq \ell, \quad (43)$$

and the coefficients $\{a_j\}_{j=0}^{\ell}$ satisfy the recursion relations (38).

We observe that the Eq. (39) is automatically satisfied because μ is an eigenvalue of the matrix $L(\lambda)$ given in (37).

If the function P is associated with an irreducible spherical function, then we have $a_0 = 1$, because the condition $H(1) = (1, \ldots, 1)$ implies that $P(1)$ is a vector whose first entry is 1, see (24).

Proposition 7.1 *If $P \in \mathcal{V}_{n,\mu}$ then P is a polynomial function.*

Proof Let $P(u) = (p_0(u), \ldots, p_\ell(u)) \in \mathbb{C}^{\ell+1}$. From Corollary 5.7, we have that the entries of the function P are given by

$$p_j(u) = a_j \, _2F_1 \left(\begin{matrix} -n+j, n+j+2 \\ j+3/2 \end{matrix} ; \frac{1-u}{2} \right), \quad (44)$$

where the coefficients $\{a_j\}_{j=0}^{\ell}$ satisfy the recursion relation (38), for some eigenvalue μ of $L(\lambda)$.

For $0 \leq j \leq n$, the function $p_j(u)$ is a polynomial function, while for $n < j \leq \ell$, the series defining the hypergeometric function is not finite. Hence, in this case, we have that p_j is a polynomial if and only if the coefficient a_j is zero.

From the expression of $\bar{E}$ in Theorem 5.2, for $0 \leq j \leq \ell$, we have

$$\mu p_j = \frac{i}{2} \left((1-u^2) \frac{(j+1)(\ell+j+2)}{2j+3} p'_{j+1} + \frac{j(\ell-j+1)}{2j-1} p'_{j-1} \right)$$
$$- \frac{i}{2} u(j+1)(\ell+j+2) p_{j+1} - \frac{1}{2} j(j+1) p_j, \quad (45)$$

where we interpret $p_{-1} = p_{\ell+1} = 0$.

By induction on j, suppose that p_j and p_{j-1} are polynomial functions and let $p_{j+1}(u) = \sum_{k \geq 0} b_k u^k$. Then,

$$p(u) = \frac{1}{2j+3} (1-u^2) p'_{j+1}(u) - u p_{j+1}(u)$$
$$= \frac{1}{2j+3} \sum_{k \geq 0} \left((k+1) b_{k+1} - (k+2j+2) b_{k-1} \right) u^k$$

is also a polynomial function in u, (where as usual we denote $b_{-1} = 0$). Let $m = \deg(p)$. Thus, for $k > m$ we have

$$b_{k+1} = \frac{(k+2j+1)}{k+2} b_{k-1},$$

then $|b_{k+1}| \geq |b_{k-1}|$ for $k > m$. Since $\lim_{u \to 1^-} p_{j+1}(u)$ exists, we have that $b_k = 0$ for $k > m$. Thus, p_{j+1} is a polynomial. Also, we conclude that if $n < \ell$ the jth entry of P is $p_j = 0$ for $n < j \leq \ell$. □

Corollary 7.2 *The function $P(u)$ associated with an irreducible spherical function Φ of type π_ℓ is a $\mathbb{C}^{\ell+1}$-valued polynomial function.*

Proof We only have to recall that the function P associated with the irreducible spherical function $\Phi_\ell^{(m_1,m_2)}$ of type π_ℓ belongs to $\mathcal{V}_{n,\mu}$, with $n = m_1 - m_2 \in \mathbb{N}_0$ and $\mu = -\frac{\ell(\ell+2)}{4} + (m_1 + 1)m_2$ (see Corollary 6.3). □

8 From P to Φ

8.1 Correspondence between polynomials and spherical functions

In this subsection, we will prove that, given $\pi_\ell \in \hat{K}$, there is a one to one correspondence between the vector-valued polynomial eigenfunctions $P(u)$ of $\bar{D}$ and $\bar{E}$ such that the first entry of the vector $P(1)$ is equal to 1, and the irreducible spherical functions Φ of type π_ℓ.

Theorem 8.1 *There is a one to one correspondence between the irreducible spherical functions Φ of type $\pi_\ell \in \hat{K}$, $\ell \in 2\mathbb{N}_0$, and the functions P in $\mathcal{V}_{n,\mu} \neq 0$ such that $P(1) = (1, a_1, \ldots, a_\ell)$.*

Proof Given an irreducible spherical function of type π_ℓ, we have already proved that the function P associated with it belongs to the space $\mathcal{V}_{n,\mu}$, and $P(1)$ has its first entry equal to one.

The equivalence classes of irreducible spherical functions of (G, K) of type π_ℓ are parameterized by the set of all pairs $(m_1, m_2) \in \mathbb{Z}^2$ such that

$$m_1 \geq \tfrac{\ell}{2} \geq |m_2|.$$

Every irreducible spherical function $\Phi_\ell^{(m_1,m_2)}$ corresponds to a vector-valued eigenfunction $P_\ell^{(m_1,m_2)}$ of the operators $\bar{D}$ and $\bar{E}$ whose eigenvalues, according to Corollary 6.2, are, respectively,

$$\lambda_\ell^{(m_1,m_2)} = -(m_1 - m_2)(m_1 - m_2 + 2),$$
$$\mu_\ell^{(m_1,m_2)} = -\frac{\ell(\ell+2)}{4} + (m_1 + 1)m_2.$$

Easily one can see that for different pairs (m_1, m_2), the pairs of eigenvalues $(\lambda_\ell^{(m_1,m_2)}, \mu_\ell^{(m_1,m_2)})$ are different. Thus, each eigenfunction $P_\ell^{(m_1,m_2)}$ is associated with a unique irreducible spherical function $\Phi_\ell^{(m_1,m_2)}$.

On the other hand, from (43), we know that $P \in \mathcal{V}_{n,\mu}$ if and only if $P(u) = (p_0(u), \ldots, p_\ell(u))^t$ is of the form

$$p_j(u) = a_j \, {}_2F_1 \left(\begin{matrix} -n+j, n+j+2 \\ j+3/2 \end{matrix} ; \tfrac{1-u}{2} \right), \quad \text{for all} \ \ 0 \leq j \leq \ell,$$

where $\{a_j\}_{j=0}^\ell$ satisfies (38). Thus, $a = (a_0, \ldots, a_\ell)^t$ is an eigenvector of the matrix $L(\lambda)$ with eigenvalue μ. In particular, there are no more than $\ell+1$ linear independent eigenvectors. If $P \in \mathcal{V}_{n,\mu}$, then P is a polynomial function; hence, when $n < \ell$ we have that

Fig. 2 Pairs (m_1, m_2) that contain the K-type ℓ
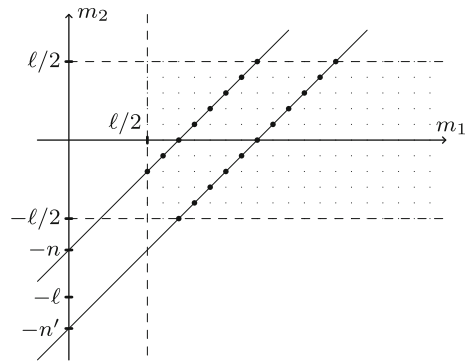
$$a_j = 0, \quad \text{for} \quad n < j \leq \ell.$$

Thus, the eigenvectors of $L(\lambda)$ live in a subspace of dimension $n + 1$ and, hence, there are at most $n + 1$ linear independent eigenvectors. From Corollary 5.5, we get that every eigenspace of $L(\lambda)$ is one dimensional. Therefore, we conclude that, up to scalars, there are no more than $\min\{\ell + 1, n + 1\}$ eigenvectors of $L(\lambda)$.

Hence, it is enough to prove that for each $\lambda = -n(n + 2)$, with $n \in \mathbb{N}_0$, there are exactly $\min\{\ell + 1, n + 1\}$ irreducible spherical functions of type $\pi_\ell \in K$.

It is easy to verify (see Fig. 2) that there are exactly $\min\{\ell+1, n+1\}$ pairs $(m_1, m_2) \in \mathbb{Z} \times \mathbb{Z}$, satisfying

$$m_1 \geq \tfrac{\ell}{2} \geq |m_2| \quad \text{and} \quad m_1 - m_2 = n. \tag{46}$$

This concludes the proof of the theorem.   □

If we take $n = m_1 - m_2$ and $k = \ell/2 - m_2$ in Corollary 6.2, we have that for an eigenfunction $P(u)$ of D and E, associated with an irreducible spherical function of type $\pi_\ell \in \hat{K}$, the respective eigenvalues are of the form

$$\lambda = -n(n + 2), \quad \mu = -\tfrac{\ell}{2}\left(\tfrac{\ell}{2} + 1\right) + \left(n - k + \tfrac{\ell}{2} + 1\right)\left(\tfrac{\ell}{2} - k\right),$$

with $0 \leq n$ and $0 \leq k \leq \min(n, \ell)$.

Now we can state the main theorem of this paper.

Theorem 8.2 *There exists a one to one correspondence between the irreducible spherical functions of type $\pi_\ell \in \hat{K}$ and the vector valued polynomial functions $P(u) = (p_0(u), \ldots, p_\ell(u))^t$ with*

$$p_j(u) = a_j \, {}_2F_1\left(\begin{matrix} -n + j, n + j + 2 \\ j + 3/2 \end{matrix}; \frac{1 - u}{2}\right),$$

where $n \in \mathbb{N}_0$, $a_0 = 1$, and $\{a_j\}_{j=0}^\ell$ satisfies the recursion relation

$$i\,\frac{j(\ell - j + 1)(n - j + 1)(n + j + 1)}{2(2j - 1)(2j + 1)} \, a_{j-1} - \frac{j(j+1)}{2} \, a_j - i\,\frac{(j+1)(\ell + j + 2)}{2} \, a_{j+1} = \mu \, a_j,$$

for $0 \leq j \leq \ell - 1$, and μ of the form

$$\mu = -\tfrac{\ell}{2}\left(\tfrac{\ell}{2} + 1\right) + \left(n + k - \tfrac{\ell}{2} + 1\right)\left(k - \tfrac{\ell}{2}\right),$$

for $k \in \mathbb{Z}$, $0 \leq k \leq \min\{n, \ell\}$.

8.2 Reconstruction of an irreducible spherical function

Fixed $\ell \in 2\mathbb{N}_0$ we know that a function $P = P(u)$ as in Theorem 8.2 is associated with a unique irreducible spherical function Φ of type $\pi_\ell \in \hat{K}$. Now we show how to explicitly construct the function Φ from such a P. Recall that P is a polynomial function.

Let us define the vector function $H(u) = UT(u)P(u) = (h_0(u), \ldots, h_\ell(u)), u \in [-1, 1]$, with U and $T(u)$ as in Corollary 5.7 and let diag$\{H(u)\}$ denote the diagonal matrix-valued function whose kk-entry is equal to the k-th entry of the vector-valued function $H(u)$.

On the other hand, if we consider the function $H : S^3 \longrightarrow \text{End}(V_\pi)$ associated with the irreducible spherical function Φ, from Corollary 5.7 and (40), we know that for $u \in (0, 1)$

$$H(\sqrt{1 - u^2}, 0, 0, u) = H(u).$$

Therefore, since both functions in the equality above are analytic in $(-1, 1)$ and continuous in $[-1, 1]$, we have that

$$H(\sqrt{1 - u^2}, 0, 0, u) = H(u),$$

for all $u \in [-1, 1]$.

Since $H(kx) = \pi_\ell(k)H(x)\pi_\ell^{-1}(k)$ for every $x \in S^3$ and $k \in K$, we have found the explicit values of the function H on the sphere S^3. Then, we can define the function $H : G \longrightarrow \text{End}(V_{\pi_\ell})$ by

$$H(g) = H(gK), \quad g \in G.$$

Finally, we have that the irreducible spherical function Φ is of the form

$$\Phi(g) = H(g)\Phi_{\pi_\ell}(g), \quad g \in G,$$

where Φ_{π_ℓ} is the auxiliary spherical function introduced in Sect. 2.6.

9 Hypergeometrization

In this section, for a fixed $\ell \in 2\mathbb{N}_0$, we shall construct a sequence of matrix-valued polynomials P_w closely related to irreducible spherical functions of type $\pi_\ell \in \hat{K}$.

Given a nonnegative integer w and $k = 0, 1, 2, \ldots, \ell$, the integers $m_1 = w + \ell/2$ and $m_2 = -k + \ell/2$ satisfy

$$w + \frac{\ell}{2} \geq \frac{\ell}{2} \geq \left| -k + \tfrac{\ell}{2} \right|.$$

Then, we can consider

$$\Phi_\ell^{(w+\ell/2, -k+\ell/2)},$$

the spherical function of type $\pi_\ell \in \hat{K}$ associated with the G-representation $\tau_{(m_1, m_2)}$.

Also let us consider the matrix-valued function $P_w = P_w(u)$, whose kth column ($k = 0, 1, 2, \ldots, \ell$) is given by the $\mathbb{C}^{\ell+1}$-valued polynomial P associated with $\Phi_\ell^{(w+\ell/2, -k+\ell/2)}$.

From Corollary 6.3, we have that the kth column of P_w is an eigenfunction of the operators $\bar{D}$ and $\bar{E}$ with eigenvalues $\lambda_{w(k)} = -(w+k)(w+k+2)$ and $\mu_w(k) = w(\tfrac{\ell}{2} - k) - k(\tfrac{\ell}{2} + 1)$, respectively.

Explicitly, we have that the jk-entry of the matrix P_w is given by

$$[P_w(u)]_{jk} = a_j^{w,k} \, {}_2F_1 \left(\begin{matrix} -w - k + j, \ w + k + j + 2 \\ j + 3/2 \end{matrix} ; (1 - u)/2 \right), \tag{47}$$

where $a_0^{w,k} = 1$ for all k and $\{a_j^{w,k}\}_{j=0}^{\ell}$ satisfies

$$i \frac{j(\ell - j + 1)(w + k - j + 1)(w + k + j + 1)}{2(2j - 1)(2j + 1)} \, a_{j-1}^{w,k} - \frac{j(j+1)}{2} \, a_j^{w,k}$$

$$- i \frac{(j+1)(\ell+j+2)}{2} \, a_{j+1}^{w,k} = \left(w(\tfrac{\ell}{2} - k) - k(\tfrac{\ell}{2} + 1) \right) a_j^{w,k}. \tag{48}$$

From Proposition 7.2, with $n = w + k$, we have that $[P_w(u)]_{jk}$ is polynomial on u. Therefore, we have the following results.

Proposition 9.1 *The matrix-valued polynomials P_w defined above satisfy*

$$\bar{D} P_w = P_w \Lambda_w \quad and \quad \bar{E} P_w = P_w M_w,$$

where $\Lambda_w = \sum_{k=0}^{\ell} \lambda_{w(k)} E_{kk}$, $M_w = \sum_{k=0}^{\ell} \mu_w(k) E_{kk}$, and

$$\lambda_{w(k)} = -(w + k)(w + k + 2) \quad and \quad \mu_w(k) = w(\tfrac{\ell}{2} - k) - k(\tfrac{\ell}{2} + 1).$$

For the particular case $w = 0$, we have the following explicit formulas for $a_j^{0,k}$.

Proposition 9.2 *We have*

$$a_j^{0,k} = \frac{(-2i)^j k! \, j!}{(k - j)!(2j)!} \quad for \ 0 \le j \le k \le \ell,$$

$$a_j^{0,k} = 0 \quad for \ 0 \le k < j \le \ell. \tag{49}$$

Proof Clearly $a_0^{0,k} = 1$; then, we only need to check that these $a_j^{0,k}$ satisfy the following three-term recursive relation:

$$i \frac{j(\ell-j+1)(k-j+1)(k+j+1)}{2(2j-1)(2j+1)} a_{j-1}^{0,k} - \frac{j(j+1)}{2} a_j^{0,k} - i \frac{(j+1)(\ell+j+2)}{2} a_{j+1}^{0,k} = -\frac{k(\ell+2)}{2} a_j^{0,k}. \tag{50}$$

Notice that if the coefficients $a_j^{0,k}$ are given by (49), for $0 \le j \le k \le \ell$, we have

$$i a_{j-1}^{0,k} = -\frac{2j-1}{k-j+1} a_j^{0,k} \quad and \quad i a_{j+1}^{0,k} = \frac{k-j}{2j+1} a_j^{0,k}.$$

Hence, for $0 \le j \le k \le \ell$ (50) is equivalent to

$$-\frac{j(\ell-j+1)(k+j+1)}{2(2j+1)} a_j^{0,k} - \frac{j(j+1)}{2} a_j^{0,k} - \frac{(j+1)(\ell+j+2)(k-j)}{2(2j+1)} a_j^{0,k} = -\frac{k(\ell+2)}{2} a_j^{0,k},$$

which can be easily checked.

If $j = k + 1$ we have

$$i \frac{(k+1)(\ell-(k+1)+1)(k-(k+1)+1)(k+(k+1)+1)}{2(2j-1)(2j+1)} a_k^{0,k} = 0,$$

which is true. And if $j \ge k + 2$ we just have $0 = 0$. Therefore, the coefficients given by (49) satisfy (48) and the proof is finished. $\qquad \square$

9.1 The hypergeometric operators

Now we introduce the matrix-valued function Ψ defined by the first "package" of spherical functions P_w with $w \geq 0$, i.e.,

$$\Psi(u) = P_0(u).$$

From (47) and (49), we observe that $\Psi(u)$ is an upper triangular matrix. Moreover, $\Psi(u) = (\Psi_{jk})_{jk}$ is the polynomial function given by

$$\Psi_{jk} = \frac{(2j+1)(-2i)^j k! j!}{(k+j+1)!} C_{k-j}^{j+1}(u), \quad \text{for } 0 \leq j \leq k \leq \ell, \tag{51}$$

where $C_{k-j}^{j+1}(u)$ is the Gegenbauer polynomial

$$C_{k-j}^{j+1}(u) = \binom{k+j+1}{k-j} \, {}_2F_1\left(\begin{matrix} -k+j, \, k+j+2 \\ j+3/2 \end{matrix}; (1-u)/2\right).$$

Since the kth column of Ψ is an eigenfunction of $\bar{D}$ and $\bar{E}$ with eigenvalues $\lambda_{0(k)} = -k(k+2)$ and $\mu_0(k) = -k(\frac{\ell}{2}+1)$, respectively, the function Ψ satisfies

$$\bar{D}\Psi = \Psi\Lambda_0 \quad \text{and} \quad \bar{E}\Psi = \Psi M_0, \tag{52}$$

where $\Lambda_0 = \sum_{k=0}^{\ell} \lambda_{0(k)} E_{kk}$ and $M_0 = \sum_{k=0}^{\ell} \mu_0(k) E_{kk}$.

Remark 9.3 The entries of the diagonal of $\Psi(u)$ are nonzero constant polynomials; thus, we have that $\Psi(u)$ is invertible.

Moreover, the inverse $\Psi(u)^{-1}$ is also an upper triangular matrix polynomial. This can be easily checked, for instance, using Cramer's rule, because the determinant of $\Psi(u)$ is a nonzero constant.

Theorem 9.4 *Let $\bar{D}$ and $\bar{E}$ be the differential operators defined in Theorem 5.2, and let Ψ the matrix-valued function whose entries are given by* (51). *Let $\widetilde{D} = \Psi^{-1}\bar{D}\Psi$ and $\widetilde{E} = \Psi^{-1}\bar{E}\Psi$, then*

$$\widetilde{D}F = (1-u^2)F'' + (-uC + S_1)F' + \Lambda_0 F,$$
$$\widetilde{E}F = (uR_2 + R_1)F' + M_0 F,$$

for any C^∞-function F on $(0, 1)$ with values in $\mathbb{C}^{\ell+1}$, where

$$C = \sum_{j=0}^{\ell}(2j+3)E_{jj}, \quad S_1 = \sum_{j=0}^{\ell-1} 2(j+1)E_{j,j+1},$$

$$R_1 = \sum_{j=0}^{\ell-1} \frac{(j+1)}{2}E_{j,j+1} - \sum_{j=0}^{\ell-1}\frac{(\ell-j)}{2}E_{j+1,j}, \quad R_2 = \sum_{j=0}^{\ell}(\tfrac{\ell}{2}-j)E_{j,j},$$

$$\Lambda_0 = \sum_{j=0}^{\ell} -j(j+2)E_{j,j}, \quad M_0 = \sum_{j=0}^{\ell} -j(\tfrac{\ell}{2}+1)E_{j,j}.$$

Proof By definition, we have

$$
\begin{aligned}
\widetilde{D}\widetilde{F} &= \left(1 - u^2\right)\widetilde{F}'' + \Psi^{-1}[2(1 - u^2)\Psi' - uC\Psi]\widetilde{F}' \\
&\quad + \Psi^{-1}\left[\left(1 - u^2\right)\Psi'' - uC\Psi' - V\Psi\right]\widetilde{F}, \\
\widetilde{E}\widetilde{F} &= \tfrac{i}{2}\Psi^{-1}[(1 - u^2)Q_0 + Q_1]\Psi\widetilde{F}' \\
&\quad + \Psi^{-1}\left[\tfrac{i}{2}\left((1 - u^2)Q_0 + Q_1\right)\Psi' - \tfrac{i}{2}uM\Psi - \tfrac{1}{2}V_0\Psi\right]\widetilde{F}.
\end{aligned}
$$

By using (52), we observe that

$$
\begin{aligned}
\left(1 - u^2\right)\Psi'' - uC\Psi' - V\Psi &= \bar{D}\Psi = \Psi\Lambda_0, \\
\tfrac{i}{2}\left((1 - u^2)Q_0 + Q_1\right)\Psi' - \tfrac{i}{2}uM\Psi - \tfrac{1}{2}V_0\Psi &= \bar{E}\Psi = \Psi M_0.
\end{aligned}
$$

To complete the proof of this theorem, we use the following properties of the Gegenbauer polynomials [for the first three see [19] page 40, and for the last one see [33], page 83, equation (4.7.27)]

$$
\frac{dC_n^\lambda}{du}(u) = 2\lambda\, C_{n-1}^{\lambda+1}(u), \tag{53}
$$

$$
2(n + \lambda)uC_n^\lambda(u) = (n + 1)C_{n+1}^\lambda(u) + (n + 2\lambda - 1)C_{n-1}^\lambda(u), \tag{54}
$$

$$
\left(1 - u^2\right)\frac{dC_n^\lambda}{du}(u) + (1 - 2\lambda)uC_n^\lambda(u) = -\frac{(n + 1)(2\lambda + n - 1)}{2(\lambda - 1)}C_{n+1}^{\lambda-1}(u), \tag{55}
$$

$$
\frac{(n + 2\lambda - 1)}{2(\lambda - 1)}C_{n+1}^{\lambda-1}(u) = C_{n+1}^\lambda(u) - uC_n^\lambda(u). \tag{56}
$$

We need to establish the following identities

$$
[2\left(1 - u^2\right)\Psi' - uC\Psi] = \Psi(-uC + S_1), \tag{57}
$$

$$
\tfrac{i}{2}[\left(1 - u^2\right)Q_0 + Q_1]\Psi = \Psi(uR_2 + R_1). \tag{58}
$$

Since (57) is a matrix identity, by looking at the jk-place, we have

$$
2\left(1 - u^2\right)\Psi'_{jk} - uC_{jj}\Psi_{jk} = -\Psi_{jk}uC_{kk} + \Psi_{j,k-1}(S_1)_{k-1,k}.
$$

Multiplying both sides by $\frac{(k+j+1)!}{(2j+1)(-2i)^j\, k!j!}$ and using (51) we have

$$
2\left(1 - u^2\right)\frac{d}{du}C_{k-j}^{j+1} - u(2j + 3)C_{k-j}^{j+1} = -u(2k + 3)C_{k-j}^{j+1} + 2(k + j + 1)C_{k-j-1}^{j+1}.
$$

and by setting $\lambda = j + 1$ and $n = k - j$, we get

$$
2\left(1 - u^2\right)\frac{dC_n^\lambda}{du} - u(2\lambda + 1)C_n^\lambda = -u(2(n + \lambda) + 1)C_n^\lambda + 2(n + 2\lambda - 1)C_{n-1}^\lambda.
$$

To see that this identity holds, we use (55) to write $\frac{dC_n^\lambda}{du}$ in terms of C_n^λ and $C_{n+1}^{\lambda-1}$, (54) to express C_{n-1}^λ in terms of C_n^λ and C_{n+1}^λ, and then we recognize the identity (56). Thus, we have proved (57).

Now we need to verify the matrix identity (58). The jk-entry is given by (see Theorem 5.2 for the definition of the matrices Q_0 and Q_1)

$$
\begin{aligned}
\tfrac{i}{2}\left(1 - u^2\right)(Q_0)_{j,j+1}\Psi_{j+1,k} + \tfrac{i}{2}(Q_1)_{j,j-1}\Psi_{j-1,k} = \\
u\Psi_{jk}(R_2)_{kk} + \Psi_{j,k+1}(R_1)_{k+1,k} + \Psi_{j,k-1}(R_1)_{k-1,k}.
\end{aligned}
$$

Again, multiplying both sides by $\frac{(k+j+1)!}{(-2i)^j k! j!}$ and setting $\lambda = j+1$ and $n = k-j$, we obtain

$$(1-u^2)\frac{\lambda^2(\ell+\lambda+1)}{n+2\lambda}C_{n-1}^{\lambda+1} - \frac{(\ell-\lambda+2)(n+2\lambda-1)}{4}C_{n+1}^{\lambda-1}$$

$$= u\left(\tfrac{\ell}{2}-(n+\lambda-1)\right)(2\lambda-1)C_n^\lambda - \frac{(\ell-n-\lambda+1)(2\lambda-1)(n+\lambda)}{2(n+2\lambda)}C_{n+1}^\lambda$$

$$+\frac{(2\lambda-1)(n+2\lambda-1)}{2}C_{n-1}^\lambda.$$

Now we firstly use (55) combined with (53) to write $C_{n-1}^{\lambda+1}$ in terms of C_n^λ and $C_{n+1}^{\lambda-1}$, and then we use (54) to express C_{n-1}^λ in terms of C_n^λ and C_{n+1}^λ. Then we get

$$\frac{(\lambda^2-\ell\lambda-3\lambda-\ell n-2n)(2\lambda-1)}{2(n+2\lambda)}\left(uC_n^\lambda(u) + \frac{(n+2\lambda-1)}{2(\lambda-1)}C_{n+1}^{\lambda-1}(u) - C_{n+1}^\lambda(u)\right) = 0,$$

which is true by (56). □

For each $w \in \mathbb{N}_0$ let us introduce the matrix-valued function

$$\widetilde{P}_w = \Psi^{-1}P_w, \tag{59}$$

where P_w is the matrix-valued polynomials introduced in (47) and Ψ the upper triangular matrix function given in (51). We recall that the function Ψ^{-1} is a polynomial function, as we observed in Remark 9.3. Therefore, $\widetilde{P}_w$ is also a polynomial function. The following result is a direct consequence of Proposition 9.1 and Theorem 9.4.

Corollary 9.5 *The matrix-valued polynomials* $\widetilde{P}_w = \Psi^{-1}P_w$ *satisfy*

$$\widetilde{D}\widetilde{P}_w = \widetilde{P}_w\Lambda_w \quad and \quad \widetilde{E}\widetilde{P}_w = \widetilde{P}_wM_w,$$

where $\Lambda_w = \sum_{k=0}^{\ell}\lambda_{w(k)}E_{kk}$, $M_w = \sum_{k=0}^{\ell}\mu_w(k)E_{kk}$, *and*

$$\lambda_{w(k)} = -(w+k)(w+k+2) \quad and \quad \mu_w(k) = w(\tfrac{\ell}{2}-k) - k(\tfrac{\ell}{2}+1).$$

9.2 The explicit expression of the coefficients $a_j^{w,k}$

In this subsection, we give the expression of the coefficients $a_j^{w,k}$, defined by the relation (48), in terms of the Racah polynomials. We recall that the Racah polynomials are defined (see for example (1.2.1) in [19]) by

$$R_k(\lambda(x); \alpha, \beta, \gamma, \delta) = {}_4F_3\left(\begin{matrix}-k, k+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}; 1\right) \tag{60}$$

for $n = 0, 1, \ldots N$, where $\lambda(x) = x(x+\gamma+\delta+1)$ and one of the numbers $\alpha+1, \beta+\delta+1$ or $\gamma+1$ are equal $-N$, with N a nonnegative integer.

If we take

$$\alpha = -(\ell+1), \quad \beta = -(w+k+1), \quad \gamma = 0, \quad \delta = 0, \quad N = \ell, \quad x = j,$$

then the Racah polynomials

$$R_k(\lambda(j)) = R_k(\lambda(j); -\ell-1, -w-k-1, 0, 0)$$

$$= {}_4F_3\left(\begin{matrix}-k, -\ell-w-1, -j, j+1 \\ -\ell, -k-w, 1\end{matrix}; 1\right)$$

satisfy the difference equation [see [19, equation (1.2.5)]]

$$\frac{(j-\ell)(j-k-w)(j+1)}{(2j+1)} R_k(\lambda(j+1)) + \frac{j(j+\ell+1)(j+k+w+1)}{(2j+1)} R_k(\lambda(j-1))$$
$$+ \big(2k(w+\ell+1) - j(j+1) - \ell(w+k)\big) R_k(\lambda(j)) = 0. \tag{61}$$

Proposition 9.6 *For $w \in \mathbb{N}_0$ and $0 \le j, k \le \ell$ let*

$$a_j^{w,k} = (-2i)^j (-w-k)_j \frac{j!}{(2j)!} \binom{\ell}{j} \binom{\ell+j+1}{j}^{-1} R_k(\lambda(j); -\ell-1, -w-k-1, 0, 0).$$

Then $a_0^{w,k} = 1$ and the sequence $\{a_j^{k,w}\}$ satisfies the recursion relation (48).

Proof It is easy to check that $a_0^{w,k} = 1$. Thus, we start by observing that

$$i \, a_{j+1}^{w,k} = (-2i)^j (-w-k)_j \frac{j!}{(2j)!} \binom{\ell}{j} \binom{\ell+j+1}{j}^{-1} \frac{(-w-k+j)(\ell-j)}{(2j+1)(\ell+j+2)} R_k(\lambda(j+1)),$$
$$i \, a_{j-1}^{w,k} = (-2i)^j (-w-k)_j \frac{j!}{(2j)!} \binom{\ell}{j} \binom{\ell+j+1}{j}^{-1} \frac{(2j-1)(\ell+j+1)}{(w+k-j+1)(\ell-j+1)} R_k(\lambda(j-1)).$$

The left-hand side of (48) becomes

$$i \frac{j(\ell-j+1)(w+k-j+1)(w+k+j+1)}{(2j-1)(2j+1)} a_{j-1}^{w,k} - j(j+1) \, a_j^{w,k} - i(j+1)(\ell+j+2) \, a_{j+1}^{w,k}$$
$$= (-2i)^j (-w-k)_j \frac{j!}{(2j)!} \binom{\ell}{j} \binom{\ell+j+1}{j}^{-1} \Big(\frac{j(w+k+j+1)(\ell+j+1)}{2j+1} R_k(\lambda(j-1))$$
$$- j(j+1) R_k(\lambda(j)) + \frac{(j+1)(\ell-j)(w+k-j)}{2j+1} R_k(\lambda(j+1)) \Big).$$

By using the difference equation (61), we have that the expression above is equal to

$$(-2i)^j (-w-k)_j \frac{j!}{(2j)!} \binom{\ell}{j} \binom{\ell+j+1}{j}^{-1} \big(\ell(w+k) - 2k(w+\ell+1) \big) R_k(\lambda(j+1)$$
$$= \big(w(\ell-2k) - k(\ell+2) \big) a_j^{w,k}.$$

This turns out to be the right-hand side of (48) and the proof is complete. □

Corollary 9.7 *For $w = 0$ we obtain*

$$a_j^{0,k} = (2i)^j (-k)_j \frac{j!}{(2j)!}.$$

Proof We start by observing that for $w = 0$ the Racah polynomial involved in the expression of $a_j^{0,k}$ can be written as a $_3F_2$ function

$$R_k(\lambda(j); -\ell-1, -k-1, 0, 0) = {}_4F_3 \left(\begin{matrix} -k, -\ell-1, -j, j+1 \\ -\ell, -k, 1 \end{matrix} ; 1 \right)$$
$$= {}_3F_2 \left(\begin{matrix} -j, j+1, -\ell-1 \\ 1, -\ell \end{matrix} ; 1 \right).$$

By Pfaff–Saalschütz identity (see for example [1], Theorem 2.2.6), we get

$${}_3F_2 \left(\begin{matrix} -j, j+1, -\ell-1 \\ 1, -\ell \end{matrix} ; 1 \right) = \frac{(-j)_j (\ell+2)_j}{(1)_j (\ell-j+1)_j} = (-1)^j \binom{\ell+j+1}{j} \binom{\ell}{j}^{-1}.$$

Now the corollary follows directly from Proposition 9.6. □

Remark 9.8 The expression of $a_j^{0,k}$ in Corollary 9.6 coincides with the result obtained in (49).

Corollary 9.9 *For $0 \leq j, k \leq \ell$ the jk-entry of the polynomial P_w is given by*

$$[P_w(u)]_{jk} = (-2i)^j (-w-k)_j \frac{j!}{(2j)!} \binom{\ell}{j} \binom{\ell+j+1}{j}^{-1} {}_4F_3 \left(\begin{array}{c} -k, -\ell-w-1, -j, j+1 \\ -\ell, -k-w, 1 \end{array}; 1 \right)$$

$$\times {}_2F_1 \left(\begin{array}{c} -w-k+j, w+k+j+2 \\ j+3/2 \end{array}; (1-u)/2 \right).$$

10 Orthogonal polynomials

The aim of this section is to build classical sequences of matrix valued orthogonal polynomials from our previous work. This means to exhibit a weight matrix W supported on the real line, a sequence $(\widetilde{P}_w)_{w \geq 0}$ of matrix polynomials such that $\deg(\widetilde{P}_w) = w$ with the leading coefficient of $\widetilde{P}_w$ nonsingular, orthogonal with respect to W, and a second-order (symmetric) differential operator $\widetilde{D}$ such that $\widetilde{D} \widetilde{P}_w = \widetilde{P}_w \Lambda_w$ where Λ_w is a real diagonal matrix. Moreover, we point out that we also have a first-order (symmetric) differential operator $\widetilde{E}$ such that $\widetilde{E} \widetilde{P}_w = \widetilde{P}_w M_w$, where M_w is a real diagonal matrix.

From $\widetilde{D}$ (see Theorem 9.4), we obtain a new differential operator D by making the change of variables $s = (1-u)/2$. Thus,

$$DF = s(1-s)F'' - \left(\frac{S_1 - C}{2} + sC \right) F' + \Lambda_0 F.$$

10.1 Polynomial solutions of $DF = \lambda F$

We are interested in studying the vector-valued polynomial solutions of the equation $DF = \lambda F$; in particular, we want to know when polynomial solutions exist. We start with

$$s(1-s)F'' + (B - sC) F' + (\Lambda_0 - \lambda) F = 0, \tag{62}$$

where

$$B = \frac{C - S_1}{2} = \sum_{j=0}^{\ell} (j + \tfrac{3}{2}) E_{jj} - \sum_{j=0}^{\ell-1} (j+1) E_{j,j+1},$$

$$C = \sum_{j=0}^{\ell} (2j+3) E_{jj}, \quad S_1 = \sum_{j=0}^{\ell-1} (j+1) E_{j,j+1}, \quad \Lambda_0 = -\sum_{j=0}^{\ell} j(j+2) E_{jj}.$$

This equation is an instance of a matrix hypergeometric differential equation studied in [35]. Since the eigenvalues of B are not in $-\mathbb{N}_0$, the function F is determined by $F_0 = F(0)$. For $|s| < 1$, it is given by

$$F(s) = {}_2H_1 \left(\begin{array}{c} C, -\Lambda_0 + \lambda \\ B \end{array}; s \right) F_0 = \sum_{j=0}^{\infty} \frac{s^j}{j!} [B; C; -\Lambda_0 + \lambda]_j F_0, \quad F_0 \in \mathbb{C}^{\ell+1},$$

where the symbol $[B; C; -\Lambda_0 + \lambda]_j$ is inductively defined by

$$[B; C; -\Lambda_0 + \lambda]_0 = 1,$$
$$[B; C; -\Lambda_0 + \lambda]_{j+1} = (B + j)^{-1} (j(C + j - 1) - \Lambda_0 + \lambda)[B; C; -\Lambda_0 + \lambda]_j,$$

for all $j \geq 0$.

Therefore, there exists a polynomial solution of (62) if and only if the coefficient $[B; C; -\Lambda_0 + \lambda]_j$ is a singular matrix for some $j \in \mathbb{N}_0$. Moreover, we have that there is a polynomial solution of degree w of (62) if and only if there exists $F_0 \in \mathbb{C}^{\ell+1}$ such that $[B; C; -\Lambda_0 + \lambda]_w F_0 \neq 0$ and

$$(w(C + w - 1) - \Lambda_0 + \lambda)F_w = 0, \quad \text{where} \quad F_w = [B; C; -\Lambda_0 + \lambda]_w F_0.$$

The matrix

$$M_w = w(C + w - 1) - \Lambda_0 + \lambda = \sum_{j=0}^{\ell}((j + w)(j + w + 2) + \lambda)E_{jj} \tag{63}$$

is diagonal. Then, it is a singular matrix if and only if λ is of the form

$$\lambda_{w(k)} = -(k + w)(k + w + 2),$$

for $0 \leq k \leq \ell$. We get the following result.

Proposition 10.1 *Given $\lambda \in \mathbb{C}$, the equation $DF = \lambda F$ has a polynomial solution if and only if λ is of the form $-n(n + 2)$ for $n \in \mathbb{N}_0$.*

Remark 10.2 Let $w \in \mathbb{N}_0$, $0 \leq k \leq \ell$. The eigenvalue $\lambda_{w(k)}$ satisfies $\lambda_{w(k)} = -n(n + 2)$ with $n \in \mathbb{N}_0$ if and only if $n = w + k$. In particular,

$$\lambda_{w(k)} = \lambda_{w'}(k') \quad \text{if} \quad \text{and} \quad \text{only if} \quad w + k = w' + k'.$$

Now we want to study in more detail the polynomial solutions of $DF = \lambda F$. Let us assume that $\lambda = -n(n + 2)$ with $n \in \mathbb{N}_0$. Let

$$F(s) = \sum_{i=0}^{w} F_i s^i$$

be a polynomial solution of degree w of the equation $DF = \lambda F$. We have that the coefficients F_i are recursively defined by

$$F_{i+1} = (B + i)^{-1}M_i F_i = [B; C; -\Lambda_0 + \lambda]_i F_0,$$

where M_i is the matrix defined in (63).

The function F is a polynomial of degree w if and only if there exists $F_0 \in \mathbb{C}^{\ell+1}$ such that

$$F_w = [B; C; -\Lambda_0 + \lambda]_w F_0 \neq 0 \quad \text{and} \quad M_w F_w = 0. \tag{64}$$

As we said, the matrix M_w is singular if and only if $\lambda = \lambda_w(k)$ for some k such that $0 \leq k \leq \ell$, and therefore, we have

$$w = n - k. \tag{65}$$

Also, we observe that $M_w F_w = 0$ if and only if F_w is in the subspace generated by e_k (the kth vector of the canonical basis of $\mathbb{C}^{\ell+1}$).

Now we want to prove that it is always possible to choose a vector $F_0 \in \mathbb{C}^{\ell+1}$ such that $[B; C; -\Lambda_0 + \lambda]_w F_0 = e_k$. Recall that

$$[B; C; -\Lambda_0 + \lambda]_w F_0 = (B + w - 1)^{-1}M_{w-1} \ldots M_1 B^{-1} M_0 F_0,$$

and that, for $0 \le i \le w$, the matrices M_{w-i} are defined by

$$M_r = \sum_{j=0}^{\ell} (\lambda_{w(k)} - \lambda_{w-i}(j)) E_{jj}.$$

In particular, the kernel of the matrix M_{w-i} is $\mathbb{C}e_{k+i}$ for $0 \le i \le \min\{w, \ell - k\}$, because $\lambda_w(k) - \lambda_{w-i}(j) = 0$ if and only if $j - i = k$ (see Remark 10.2).

Let W_k be the subspace in $\mathbb{C}^{\ell+1}$ generated by $\{e_0, e_1, \dots, e_k\}$. We observe that for every $j \in \mathbb{N}_0$, we have that W_k is invariant by $(B + j)^{-1}$ because it is an upper triangular matrix. For $j < w$, M_j is a diagonal matrix whose first $k + 1$ entries are not zero, thus the restriction of M_j to W_k is invertible. Therefore, there exists F_0 such that $[B; C; -\Lambda_0 + \lambda]_w F_0 = e_k$. Then

$$F(u) = {}_2H_1 \begin{pmatrix} C, -\Lambda_0 + \lambda \\ B \end{pmatrix}; u \end{pmatrix} F_0$$

is a vector polynomial of degree w. We observe that F_0 is unique in W_k, but not in $\mathbb{C}^{\ell+1}$. Anyhow, the kth entry of F is a polynomial of degree w, and all the other entries are of lower degrees because the leading coefficient F_w is always a multiple of e_k.

In this way, we have obtained the following results. In the first one, we fix the eigenvalue $\lambda = -n(n+2)$ with $n \in \mathbb{N}_0$, while in the second one, we fix the degree w of the polynomial F.

Proposition 10.3 *Let $n \in \mathbb{N}_0$ and $\lambda = -n(n+2)$. If P is a polynomial solution of $DF = \lambda F$ of degree w, then $n - \ell \le w \le n$.*

Conversely, for every $w \in \mathbb{N}_0$ such that $n - \ell \le w \le n$, the equation $DF = \lambda F$ has a polynomial solution of degree w. Moreover, if $w = n - k$, $0 \le k \le \ell$, the leading coefficient of any polynomial solution of $DF = \lambda F$ is a multiple of e_k.

Proof From (65), we have that there exists a polynomial solution of degree w if and only if $w = n - k$, with $0 \le k \le \ell$. In such a case, we have proved that there exists $F_0 \in \mathbb{C}^{\ell+1}$ such that (64) holds and we have that F_w is a multiple of e_k. □

Proposition 10.4 *Given $w \in \mathbb{N}_0$ there exist exactly $\ell + 1$ values of λ such that $DF = \lambda F$ has a polynomial solution of degree w, more precisely*

$$\lambda = \lambda_w(k) = -(k + w)(k + w + 2), \quad 0 \le k \le \ell.$$

For each k, the leading coefficient of any polynomial solution of $DF = \lambda_{w(k)} F$ is a multiple of e_k, the kth vector in the canonical basis of $\mathbb{C}^{\ell+1}$.

10.2 Our sequence of matrix orthogonal polynomials

The matrix polynomials

$$\widetilde{P}_w(u) = \Psi(u)^{-1} P_w(u)$$

were introduced in (59).

Proposition 10.5 *The columns $\{\widetilde{P}_w^k\}_{k=0,\dots,\ell}$ of $\widetilde{P}_w$ are polynomials of degree w. Moreover,*

$$\deg \left(\widetilde{P}_w^k \right)_k = w \quad and \quad \deg \left(\widetilde{P}_w^k \right)_j < w, \text{ for } j \ne k.$$

Proof The kth column of the matrix $\widetilde{P}_w = \Psi^{-1} P_w$ is the vector $\widetilde{P}_w^k = \Psi^{-1} P_w^k$, where P_w^k is the kth column of P_w. From Corollary 9.5, we have that $\widetilde{P}_w^k$ is a polynomial function that satisfies

$$\widetilde{D}\widetilde{P}_w^k = \lambda_{w(k)} \widetilde{P}_w^k, \quad \widetilde{E}\widetilde{P}_w^k = \mu_w(k)\widetilde{P}_w^k,$$

for $\lambda_{w(k)} = -(w+k)(w+k+2)$ and $\mu_w(k) = w(\frac{\ell}{2} - k) - k(\frac{\ell}{2} + 1)$.

If w' denotes the degree of $\widetilde{P}_w^k$, then we have that $w + k - \ell \le w' \le w + k$ (see Proposition 10.3). Hence, we write

$$\widetilde{P}_w^k = \sum_{j=0}^{w'} A_j u^j \quad \text{with } A_j \in \mathbb{C}^{\ell+1}.$$

Moreover, from Proposition 10.4, we have that the corresponding eigenvalue of D should be equal to $\lambda_{w'}(k') = -(w' + k')(w' + k' + 2)$, with $0 \le k' \le \ell$, and the leading coefficient $A_{w'}$ has all its entries equal to zero, except for the k'th one. From Remark 10.2 we obtain that

$$w - w' = k' - k.$$

On the other hand, $\widetilde{P}_w^k$ satisfies $\widetilde{E}\widetilde{P}_w^k = \mu_w(k)\widetilde{P}_w^k$, where

$$\widetilde{E}F = (uR_2 + R_1)F' + M_0 F$$

is the differential operator given in Theorem 9.4. Then, the coefficients of the polynomials $\widetilde{P}_w^k$ satisfy

$$\big(jR_2 + M_0 - \mu_w(k)\big)A_j + (j+1)R_1 A_{j+1} = 0, \quad \text{for } 0 \le j \le w',$$

denoting $A_{w'+1} = 0$. In particular, for $j = w'$ we have

$$\big(w' R_2 + M_0 - \mu_w(k)\big)A_{w'} = 0. \tag{66}$$

From Theorem 9.4, we have

$$w' R_2 + M_0 - \mu_w(k)I = \sum_{j=0}^{\ell} \big(w'(\tfrac{\ell}{2} - j) - j(\tfrac{\ell}{2} + 1) - \mu_w(k)\big)E_{jj}$$

$$= \sum_{j=0}^{\ell} \big(w'(\tfrac{\ell}{2} - j) - w(\tfrac{\ell}{2} - k) + (k - j)(\tfrac{\ell}{2} + 1)\big)E_{jj}.$$

From Eq. (66), we have that the k'th entry of the matrix $w' R_2 + M_0 - \mu_w(k)I$ must be zero, then

$$0 = w'(\tfrac{\ell}{2} - k') - w(\tfrac{\ell}{2} - k) + (k - k')(\tfrac{\ell}{2} + 1).$$

Since $w - w' = k' - k$, we have $0 = (w - w')(1 + k + w)$, which implies that $w' = w$ and $k' = k$.

Therefore, $\widetilde{P}_w^k$ is a polynomial of degree w and the only nonzero entry of the leading coefficient of $\widetilde{P}_w^k$ is the kth one. $\qquad\square$

10.3 The inner product

Given a finite dimensional irreducible representation $\pi = \pi_\ell$ of K in the vector space V_π, let $(C(G) \otimes \text{End}(V_\pi))^{K \times K}$ be the space of all continuous functions $\Phi : G \longrightarrow \text{End}(V_\pi)$ such that $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$ for all $g \in G$, $k_1, k_2 \in K$. Let us equip V_π with an inner product such that $\pi(k)$ becomes unitary for all $k \in K$. Then, we introduce an inner product in the vector space $(C(G) \otimes \text{End}(V_\pi))^{K \times K}$ by defining

$$\langle \Phi_1, \Phi_2 \rangle = \int_G \text{tr}(\Phi_1(g)\Phi_2(g)^*) \, dg \,, \tag{67}$$

where dg denotes the Haar measure of G normalized by $\int_G dg = 1$, and $\Phi_2(g)^*$ denotes the adjoint of $\Phi_2(g)$ with respect to the inner product in V_π.

By using Schur's orthogonality relations for the unitary irreducible representations of G, it follows that if Φ_1 and Φ_2 are nonequivalent irreducible spherical functions, then they are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$, i.e.

$$\langle \Phi_1, \Phi_2 \rangle = 0.$$

In particular, if Φ_1 and Φ_2 are two irreducible spherical functions of type $\pi = \pi_\ell$, we write as above (see (6)) $\Phi_1 = H_1 \Phi_\pi$ and $\Phi_2 = H_2 \Phi_\pi$ and put

$$H_1(u) = (h_0(u), \ldots, h_\ell(u))^t, \quad H_2(u) = (f_0(u), \ldots, f_\ell(u))^t,$$

as we did in Sect. 8.2.

Proposition 10.6 *If $\Phi_1, \Phi_2 \in (C(G) \otimes \text{End}(V_\pi))^{K \times K}$ then*

$$\langle \Phi_1, \Phi_2 \rangle = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - u^2} \sum_{j=0}^{\ell} h_j(u)\overline{f_j(u)} \, du = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - u^2} H_2^*(u) H_1(u) \, du.$$

Proof Let us consider the element $E_1 = E_{14} - E_{41} \in \mathfrak{g}$. Then, as $\mathfrak{so}(4)_\mathbb{C} \simeq \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, ad E_1 has 0 and $\pm i$ as eigenvalues with multiplicity 2.

Let $A = \exp \mathbb{R} E_1$ be the Lie subgroup of G of all elements of the form

$$a(t) = \exp t E_1 = \begin{pmatrix} \cos t & 0 & 0 & \sin t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin t & 0 & 0 & \cos t \end{pmatrix}, \quad t \in \mathbb{R}.$$

Now Theorem 5.10, page 190 in [15] establishes that for every $f \in C(G/K)$ and a suitable c_*

$$\int_{G/K} f(gK) \, dg_K = c_* \int_{K/M} \left(\int_{-\pi}^{\pi} \delta_*(a(t)) f(ka(t)K) \, dt \right) dk_M \,,$$

where the function $\delta_* : A \longrightarrow \mathbb{R}$ is defined by

$$\delta_*(a(t)) = \prod_{\nu \in \Sigma^+} |\sin i t \nu(E_1)|,$$

and $\mathrm{d}g_K$ and $\mathrm{d}k_M$ are, respectively, the left invariant measures on G/K and K/M normalized by $\int_{G/K} \mathrm{d}g_K = \int_{K/M} \mathrm{d}k_M = 1$. Recall that M was introduced in (12) and coincides with the centralizer of A in K. In our case, we have $\delta_*(a(t)) = \sin^2 t$.

Since the function $g \mapsto \mathrm{tr}(\Phi_1(g)\Phi_2(g)^*)$ is invariant under left and right multiplication by elements in K, we have

$$\langle \Phi_1, \Phi_2 \rangle = c_* \int_{-\pi}^{\pi} \sin^2 t \, \mathrm{tr}\left(\Phi_1(a(t))\Phi_2(a(t))^*\right) \, \mathrm{d}t. \tag{68}$$

Also, for each $t \in [-\pi, 0]$, we have that $(I - 2(E_{11} + E_{22}))a(t)(I - 2(E_{11} + E_{22})) = a(-t)$, with $I - 2(E_{11} + E_{22})$ in K. Then we have

$$\langle \Phi_1, \Phi_2 \rangle = 2c_* \int_0^{\pi} \sin^2 t \, \mathrm{tr}\left(\Phi_1(a(t))\Phi_2(a(t))^*\right) \, \mathrm{d}t.$$

By the definition of the auxiliary function $\Phi_\pi(g)$ (see Sect. 2.6), we have that $\Phi_1(a(t))\Phi_2(a(t))^* = H_1(a(t))H_2(a(t))^*$. Therefore, making the change of variables $\cos(t) = u$, we have

$$\langle \Phi_1, \Phi_2 \rangle = 2c_* \int_{-1}^1 \sqrt{1 - u^2} \sum_{j=0}^\ell h_j(u)\overline{f_j(u)}\mathrm{d}u.$$

To find the value of c_*, we consider the trivial case $\Phi_1 = \Phi_2 = I$ in (67) and (68). Therefore, we obtain

$$\ell + 1 = c_* \int_{-\pi}^{\pi} \sin^2 t \, (\ell + 1) \, \mathrm{d}t.$$

Then, we get $c_* = \pi^{-1}$ and the proposition follows. $\square$

In Theorems 5.2 and 9.4, we conjugate the differential operators D and E to hypergeometric operators $\widetilde{D}$ and $\widetilde{E}$ given by

$$\widetilde{D} = (UT(u)\Psi(u))^{-1}D(UT(u)\Psi(u)) \quad \text{and} \quad \widetilde{E} = (UT(u)\Psi(u))^{-1}E(UT(u)\Psi(u)).$$

Therefore, in terms of the functions

$$\widetilde{P}_1 = (UT(u)\Psi(u))^{-1}H_1 \quad \text{and} \quad \widetilde{P}_2 = (UT(u)\Psi(u))^{-1}H_2,$$

we have

$$\langle \widetilde{P}_1, \widetilde{P}_2 \rangle_W = \int_{-1}^1 \widetilde{P}_2(u)^* W(u)\widetilde{P}_1(u) \, \mathrm{d}u,$$

where the weight matrix $W(u)$ is given by

$$W(u) = \frac{2}{\pi}\sqrt{1 - u^2}\,\Psi^*(u)T^*(u)U^*UT(u)\Psi(u). \tag{69}$$

From (27), we notice that U^*U is a diagonal matrix by the orthogonality of the Hahn polynomials, precisely

$$U^*U = \sum_{j=0}^{\ell} \frac{(j+\ell+1)!(\ell-j)!}{(2j+1)\,\ell!\,\ell!} \, E_{jj}.$$

Since $T(u) = \sum_{j=0}^{\ell}(1-u^2)^{j/2}$, we have

$$T^*(u)U^*UT(u) = \sum_{j=0}^{\ell} \frac{(j+\ell+1)!(\ell-j)!}{(2j+1)\,\ell!\,\ell!} \, \big(1-u^2\big)^j \, E_{jj}.$$

Remark 10.7 Since $\Psi(u)$ is an upper triangular matrix, see (51), we observe that the decomposition (69) easily leads to the LDU decomposition of the weight matrix $W(u)$.

We recall that $\Psi(u)$ is polynomial in u, that U is a constant matrix, and that $T(u) = \sum_{j=0}^{\ell}(1-u^2)^{j/2}E_{jj}$. Then it follows that $W(u)$ is a continuous function on the closed interval $[-1, 1]$. Thus, W is a weight matrix on $[-1, 1]$ with finite moments of all orders.

One may be interested in the reducibility of the weight:

Definition 10.8 A $n \times n$ matrix weight function W supported on the interval $(a, b) \subset \mathbb{R}$ reduces to a smaller size if there exists a $n \times n$ matrix R such that

$$W(u) = R \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix} R^*, \quad \text{for all } u \in (a, b),$$

with W_1 and W_2 weight matrices of lower size.

Proposition 10.9 *A $n \times n$ matrix weight function W supported on the interval $(a, b) \subset \mathbb{R}$ reduces to a smaller size if the commutant*

$$\{A \in M_{n \times n} : AW(u) = W(u)A, \quad \text{for all } u \in (a, b)\}$$

is not trivial.

Proof If A is in the commutant also is A^*, assume that A is not a scalar multiple of the identity I. Then if $A + A^* = cI$, for any $c \in \mathbb{R}$, we take $B = iA + ic/2\,I$; otherwise, we take $B = A + A^*$, having then that B is not a scalar multiple of the identity in the commutant with $B = B^*$. Hence, by the spectral theorem, we have a projection $P \neq I$ which is a polynomial of B and then P and $Q = I - P$ are in the commutant. Therefore, $W(u) = (P + Q)W(u)(P + Q) = PW(u)P + QW(u)Q$, for all $u \in (a, b)$.

Let us define the matrix R as the matrix whose first columns are the vectors of an orthogonal basis of $P(\mathbb{C}^n)$ and the last columns are the vectors of an orthogonal basis of $Q(\mathbb{C}^n)$. Therefore, for all $u \in (a, b)$, we have

$$R^*W(u)R = R^*PW(u)PR + R^*QW(u)QR$$
$$= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} W(u) \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} W(u) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}.$$

Hence W reduces to a smaller size. $\qquad \square$

In [21] it is proved that the commutant of our weight is a two-dimensional vector space (see Proposition 5.5); therefore, W reduces to a smaller size. See Theorem 6.5 in [21].

Consider now the sequence of matrix polynomials $(\widetilde{P}_w(u))_{w \geq 0}$ introduced in (59). The kth column of $\widetilde{P}_w(u)$ is given by a vector $\widetilde{P}_w^k(u)$ associated with the irreducible spherical function of type π_ℓ

$$\Phi_\ell^{(w+\ell/2, -k+\ell/2)}.$$

Therefore $\widetilde{P}_w^k$ and $\widetilde{P}_{w'}^{k'}$ are orthogonal with respect to W, i.e.

$$\langle \widetilde{P}_w^k, \widetilde{P}_{w'}^{k'} \rangle_W = 0 \quad \text{if } (w, k) \neq (w', k'). \tag{70}$$

In other words, this sequence of matrix-valued polynomials squarely fits within Krein's theory, and we obtain the following theorem.

Theorem 10.10 *The matrix polynomial functions $\widetilde{P}_w$, $w \geq 0$, form a sequence of orthogonal polynomials with respect to W, which are eigenfunctions of the symmetric differential operators $\widetilde{D}$ and $\widetilde{E}$ appearing in Theorem 9.4. Moreover,*

$$\widetilde{D} \widetilde{P}_w = \widetilde{P}_w \Lambda_w \quad and \quad \widetilde{E} \widetilde{P}_w = \widetilde{P}_w M_w,$$

where $\Lambda_w = \sum_{k=0}^{\ell} \lambda_{w(k)} E_{kk}$, and $M_w = \sum_{k=0}^{\ell} \mu_w(k) E_{kk}$, with

$$\lambda_{w(k)} = -(w+k)(w+k+2) \quad and \quad \mu_w(k) = w(\tfrac{\ell}{2} - k) - k(\tfrac{\ell}{2} + 1).$$

Proof From Proposition 10.5, we obtain that each column of $\widetilde{P}_w$ is a polynomial function of degree w. Moreover, $\widetilde{P}_w$ is a polynomial whose leading coefficient is a nonsingular diagonal matrix.

Given w and w', nonnegative integers, by using (70), we have

$$\langle \widetilde{P}_w, \widetilde{P}_{w'} \rangle_W = \int_{-1}^{1} \widetilde{P}_w(u)^* W(u) \widetilde{P}_{w'}(u) \, du$$

$$= \sum_{k,k'=0}^{\ell} \int_{-1}^{1} \left(\widetilde{P}_w^k(u)^* W(u) \widetilde{P}_{w'}^{k'}(u) \, du \right) E_{k,k'}$$

$$= \sum_{k,k'=0}^{\ell} \delta_{w,w'} \delta_{k,k'} \left(\int_{-1}^{1} \widetilde{P}_w^k(u)^* W(u) \widetilde{P}_{w'}^{k'}(u) \, du \right) E_{k,k'}$$

$$= \delta_{w,w'} \sum_{k=0}^{\ell} \int_{-1}^{1} \left(\widetilde{P}_w^k(u)^* W(u) \widetilde{P}_{w'}^k(u) \, du, \right) E_{k,k},$$

which proves the orthogonality. Even more, it also shows us that $\langle \widetilde{P}_w, \widetilde{P}_w \rangle_W$ is a diagonal matrix. Now, thanks to Corollary 9.5, it only remains to prove that the operators $\widetilde{D}$ and $\widetilde{E}$ are symmetric with respect to W.

Making a few simple computations, we have that

$$\langle \widetilde{D} \widetilde{P}_w, \widetilde{P}_{w'} \rangle = \delta_{w,w'} \langle \widetilde{P}_w, \widetilde{P}_{w'} \rangle \Lambda_w = \delta_{w,w'} \Lambda_w^* \langle \widetilde{P}_w, \widetilde{P}_{w'} \rangle = \langle \widetilde{P}_w, \widetilde{D} \widetilde{P}_{w'} \rangle,$$

for every $w, w' \in \mathbb{N}_0$, because Λ_w is real and diagonal. This concludes the proof of the theorem. □

Remark 10.11 Following the suggestion of the referee, we include here some comparisons between the results in the last sections of this paper and those in [20].

Our differential operators $\widetilde{D}$ and $\widetilde{E}$ introduced in Theorem 9.4 are related to the operators $\tilde{D}$ and $\tilde{E}$ introduced in Theorem 3.1 of [20]. Unfortunately, the expression of $\tilde{E}$ in Theorem 3.1 of [20] is wrong. By starting from the differential operators D and E in Theorem 4.1 of [20] and changing the variables, we obtain the following relations

$$\widetilde{D} - 2\widetilde{E} = (\tilde{D})^t, \quad -2\widetilde{E} = \ell(\tilde{E})^t + \ell(\ell+2)I.$$

On the other hand, the operators $\mathcal{D}$ and $\mathcal{E}$, defined, respectively, in Proposition 6.1 and (6.13) of [20], are related to our operators $\bar{D}$ and $\bar{E}$ defined in (25) by

$$\bar{D} = (\mathcal{D})^t - \ell(\ell+2)I, \quad -2\bar{E} = C(\ell\mathcal{E} + \ell(\ell+2)I)^t C^{-1},$$

where $C = \sum_{j=0}^{\ell} \frac{(-2i)^j j! j!}{(2j)!} E_{jj}$.

Our weight $W(u)$ introduced in (69) is equal to the weight $W(x)$ in Theorem 2.1 of [20]. The function $L(x)$ given there is related to our function $\Psi(u)$ [see (51)] by

$$\Psi(u) = C L(u)^t.$$

Finally, the matrix polynomial function $P_w(u)$ given in (47) is related to the polynomial $\mathcal{R}_n$ in [20] by

$$P_w(u) = C \mathcal{R}_w^t (\tfrac{1-u}{2}).$$

11 Appendix

The purpose of this section is to give the proofs of Propositions 3.4 and 3.5.

Proposition 3.4 *For any $H \in C^\infty(\mathbb{R}^3) \otimes \mathrm{End}(V_\pi)$ we have*

$$D(H)(y) = (1 + \|y\|^2) \Big((y_1^2 + 1)H_{y_1 y_1} + (y_2^2 + 1)H_{y_2 y_2} + (y_3^2 + 1)H_{y_3 y_3}$$

$$+ 2(y_1 y_2 H_{y_1 y_2} + y_2 y_3 H_{y_2 y_3} + y_1 y_3 H_{y_1 y_3}) + 2(y_1 H_{y_1} + y_2 H_{y_2} + y_3 H_{y_3}) \Big).$$

Proof From (7) we have $D(H) = Y_4^2(H) + Y_5^2(H) + Y_6^2(H)$. We need to give the expressions of this operators in the coordinate system $p : (S^3)^+ \longrightarrow \mathbb{R}^3$

$$p(x) = \left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4} \right) = (y_1, y_2, y_3).$$

We have

$$Y^2(H)(g) = \frac{\mathrm{d}}{\mathrm{d}s} \frac{\mathrm{d}}{\mathrm{d}t} H \Big(\tilde{p} \big(g(\exp(s+t)Y) \big) \Big) \Big|_{s=t=0},$$

where $\tilde{p} : G \longrightarrow \mathbb{R}^3$, $\tilde{p}(g) = p(gK)$.

From Sect. 2.3 we have $Y_4 = E_{1,4} - E_{4,1}$, $Y_5 = E_{2,4} - E_{4,2}$ and $Y_6 = E_{3,4} - E_{4,3}$ then

$$\tilde{p}(g(\exp(s+t)Y_4)) = (u_1(s+t),\ v_1(s+t),\ w_1(s+t)),$$
$$\tilde{p}(g(\exp(s+t)Y_5)) = (u_2(s+t),\ v_2(s+t),\ w_2(s+t)),$$
$$\tilde{p}(g(\exp(s+t)Y_6)) = (u_3(s+t),\ v_3(s+t),\ w_3(s+t)).$$

where

$$u_j(s+t) = \frac{g_{1j}\sin(s+t) + g_{14}\cos(s+t)}{g_{4j}\sin(s+t) + g_{44}\cos(s+t)}, \quad v_j(s+t) = \frac{g_{2j}\sin(s+t) + g_{24}\cos(s+t)}{g_{4j}\sin(s+t) + g_{44}\cos(s+t)},$$

$$w_j(s+t) = \frac{g_{3j}\sin(s+t) + g_{34}\cos(s+t)}{g_{4j}\sin(s+t) + g_{44}\cos(s+t)},$$

for $j = 1, 2, 3$.

By using the chain rule, we have

$$
\begin{aligned}
D(H)(g) &= Y_4^2(H)(g) + Y_5^2(H)(g) + Y_6^2(H)(g) \\
&= \sum_{j=1}^{3}\Bigg[H_{y_1 y_1}\frac{\partial u_j}{\partial s}\frac{\partial u_j}{\partial t} + H_{y_2 y_2}\frac{\partial v_j}{\partial s}\frac{\partial v_j}{\partial t} + H_{y_3 y_3}\frac{\partial w_j}{\partial s}\frac{\partial w_j}{\partial t} \\
&\quad + H_{y_1 y_2}\left(\frac{\partial u_j}{\partial s}\frac{\partial v_j}{\partial t} + \frac{\partial u_j}{\partial t}\frac{\partial v_j}{\partial s}\right) + H_{y_1 y_3}\left(\frac{\partial u_j}{\partial s}\frac{\partial w_j}{\partial t} + \frac{\partial u_j}{\partial t}\frac{\partial w_j}{\partial s}\right) \\
&\quad + H_{y_2 y_3}\left(\frac{\partial v_j}{\partial s}\frac{\partial w_j}{\partial t} + \frac{\partial v_j}{\partial t}\frac{\partial w_j}{\partial s}\right) + H_{y_1}\frac{\partial^2 u_j}{\partial s \partial t} + H_{y_2}\frac{\partial^2 v_j}{\partial s \partial t} + H_{y_3}\frac{\partial^2 w_j}{\partial s \partial t}\Bigg].
\end{aligned}
$$

We observe that

$$
\frac{\partial u_j}{\partial s} = \frac{\partial u_j}{\partial t} = \frac{g_{1j}g_{44} - g_{14}g_{4j}}{g_{44}^2}, \quad \frac{\partial^2 u_j}{\partial s \partial t} = \frac{2g_{4j}(g_{14}g_{4j} - g_{1j}g_{44})}{g_{44}^3},
$$

$$
\frac{\partial v_j}{\partial s} = \frac{\partial v_j}{\partial t} = \frac{g_{2j}g_{44} - g_{24}g_{4j}}{g_{44}^2}, \quad \frac{\partial^2 v_j}{\partial s \partial t} = \frac{2g_{4j}(g_{24}g_{4j} - g_{2j}g_{44})}{g_{44}^3},
$$

$$
\frac{\partial w_j}{\partial s} = \frac{\partial w_j}{\partial t} = \frac{g_{3j}g_{44} - g_{34}g_{4j}}{g_{44}^2}, \quad \frac{\partial^2 w_j}{\partial s \partial t} = \frac{2g_{4j}(g_{34}g_{4j} - g_{3j}g_{44})}{g_{44}^3}.
$$

Now we observe that $y = (y_1, y_2, y_3) = \left(\frac{g_{14}}{g_{44}}, \frac{g_{24}}{g_{44}}, \frac{g_{34}}{g_{44}}\right)$ and recall that $g = (g_{jk})$ is a matrix in SO(4), therefore its rows are orthonormal vectors. In particular, we have

$$\frac{1}{g_{44}^2} = 1 + y_1^2 + y_2^2 + y_3^2 = 1 + \|y\|^2.$$

Now the proposition follows after some straightforward computations.   □

Proposition 3.5 *For any $H \in C^\infty(\mathbb{R}^3) \otimes \mathrm{End}(V_\pi)$ we have*

$$E(H)(y) = H_{y_1} \dot\pi \begin{pmatrix} 0 & -y_2 - y_1 y_3 & -y_3 + y_1 y_2 \\ y_2 + y_1 y_3 & 0 & -1 - y_1^2 \\ y_3 - y_1 y_2 & 1 + y_1^2 & 0 \end{pmatrix}$$

$$+ H_{y_2} \dot\pi \begin{pmatrix} 0 & -y_2 y_3 + y_1 & 1 + y_2^2 \\ y_2 y_3 - y_1 & 0 & -y_3 - y_1 y_2 \\ -1 - y_2^2 & y_3 + y_1 y_2 & 0 \end{pmatrix}$$

$$+ H_{y_3} \dot\pi \begin{pmatrix} 0 & -1 - y_3^2 & y_1 + y_2 y_3 \\ 1 + y_3^2 & 0 & y_2 - y_1 y_3 \\ -y_1 - y_2 y_3 & -y_2 + y_1 y_3 & 0 \end{pmatrix}.$$

Proof From (8) we obtain

$$E(H) = \Big(-Y_4(H)Y_3(\Phi_\pi) + Y_5(H)Y_2(\Phi_\pi) - Y_6(H)Y_1(\Phi_\pi) \Big) \Phi_\pi^{-1}$$

$$= \sum_{j=1}^{3} (-1)^j \, Y_{3+j}(H) \, Y_{4-j}(\Phi_\pi) \Phi_\pi^{-1}.$$

For $j = 1, 2, 3$ we have

$$Y_{3+j}(H) = \frac{d}{ds}_{|s=0} H\Big(\tilde p(g(\exp s\, Y_{3+j}))\Big) = H_{y_1} \frac{\partial u_j}{\partial s} + H_{y_2} \frac{\partial v_j}{\partial s} + H_{y_3} \frac{\partial w_j}{\partial s},$$

where u_j, v_j, and w_j are the functions introduced in the proof of Proposition 3.4.

In the other hand, we have that

$$(Y_k \Phi_\pi)(g)\Phi_\pi^{-1}(g) = \frac{d}{dt}_{|t=0} \Phi_\pi(g(\exp t Y_k))\Phi_\pi^{-1}(g) = \frac{d}{dt}_{|t=0} \Phi_\pi(g(\exp t Y_k)g^{-1})$$

$$= \frac{d}{dt}_{|t=0} \pi\big(a(g(\exp t Y_k)g^{-1})\big) = \dot\pi\big(\dot a(g Y_k g^t)\big),$$

where $\dot a$ is the function introduced in (1). It is easy to check that

$$\dot a(X) = \begin{pmatrix} 0 & -X_{12} - X_{34} & -X_{13} + X_{24} \\ X_{12} + X_{34} & 0 & -X_{23} - X_{14} \\ X_{13} - X_{24} & X_{23} + X_{14} & 0 \end{pmatrix}, \quad \text{for all } X \in \mathfrak{so}(4).$$

Therefore, at $s = t = 0$, we get

$$E(H)(g) = \sum_{j=1}^{3} \Big(H_{y_1} \frac{\partial u_j}{\partial s} + H_{y_2} \frac{\partial v_j}{\partial s} + H_{y_3} \frac{\partial w_j}{\partial s} \Big) \dot\pi\big(\dot a(g Y_3 g^t)\big)$$

$$= H_{y_1} \dot\pi\Big(\dot a\big(\textstyle\sum_{j=1}^{3}(-1)^j \frac{\partial u_j}{\partial s} g\, Y_{4-j}\, g^t\big)\Big) + H_{y_2} \dot\pi\Big(\dot a\big(\textstyle\sum_{j=1}^{3}(-1)^j \frac{\partial v_j}{\partial s} g\, Y_{4-j}\, g^t\big)\Big)$$

$$+ H_{y_3} \dot\pi\Big(\dot a\big(\textstyle\sum_{j=1}^{3}(-1)^j \frac{\partial w_j}{\partial s} g\, Y_{4-j}\, g^t\big)\Big)$$

$$= H_{y_1} \dot\pi(\dot a(B_1)) + H_{y_2} \dot\pi(\dot a(B_2)) + H_{y_3} \dot\pi(\dot a(B_3)).$$

Now we recall that $Y_1 = E_{12} - E_{21}, Y_2 = E_{13} - E_{31}, Y_3 = E_{23} - E_{32}$. It is easy to verify that

$$g(E_{ij} - E_{ji})g^t = \sum_{1 \leq k, r \leq 4} g_{ki} g_{rj}\, E_{kr}.$$

By using the expressions of $\frac{\partial u_j}{\partial s}$, $\frac{\partial v_j}{\partial s}$ and $\frac{\partial w_j}{\partial s}$ given in the proof of Proposition 3.4, we obtain that

$$B_1 = \sum_{j=1}^{3}(-1)^j \frac{\partial u_j}{\partial s} g \, Y_{4-j} \, g^t = \frac{1}{g_{44}^2} \sum_{1\leq k,r\leq 4}\Big((-g_{11}g_{44}+g_{14}g_{41})(g_{k2}g_{r3}-g_{k3}g_{r2})$$

$$+(g_{12}g_{44}-g_{14}g_{42})(g_{k1}g_{r3}-g_{k3}g_{r1})+(-g_{13}g_{44}+g_{14}g_{43})(g_{k1}g_{r2}-g_{k2}g_{r1})\Big)E_{kr}.$$

It is not difficult to verify that the krth entry of B_1 is equal to

$$(B_1)_{kr} = \frac{1}{g_{44}^2}\left(g_{44}\det\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{r1} & g_{r2} & g_{r3} \\ g_{k1} & g_{k2} & g_{k3} \end{pmatrix} - g_{14}\det\begin{pmatrix} g_{r1} & g_{r2} & g_{r3} \\ g_{k1} & g_{k2} & g_{k3} \\ g_{41} & g_{42} & g_{43} \end{pmatrix}\right).$$

Now we use the following fact: If g is a matrix in $SO(n)$ and $g(i|j)$ denotes the matrix obtained from g deleting the ith row and the jth column, then

$$g_{ij} = (-1)^{i+j}\det\big(g(i|j)\big). \tag{71}$$

Therefore

$$B_1 = \frac{1}{g_{44}^2}\begin{pmatrix} 0 & g_{14}g_{34} & -g_{14}g_{24} & 0 \\ g_{14}g_{34} & 0 & g_{44}^2+g_{14}^2 & -g_{44}g_{34} \\ -g_{14}g_{24} & g_{44}^2+g_{14}^2 & 0 & g_{44}g_{24} \\ 0 & -g_{44}g_{34} & -g_{44}g_{24} & 0 \end{pmatrix}.$$

We proceed in a similar way with $B_2 = \sum_{j=1}^{3}(-1)^j \frac{\partial v_j}{\partial s} g \, Y_{4-j} \, g^t$ and obtain

$$B_2 = \frac{1}{g_{44}^2}\begin{pmatrix} 0 & g_{24}g_{34} & -g_{44}^2-g_{24}^2 & g_{44}g_{34} \\ -g_{24}g_{34} & 0 & g_{24}g_{14} & 0 \\ g_{44}^2+g_{24}^2 & -g_{24}g_{14} & 0 & -g_{44}g_{14} \\ -g_{44}g_{34} & 0 & g_{44}g_{14} & 0 \end{pmatrix}.$$

For $B_3 = \sum_{j=1}^{3}(-1)^j \frac{\partial w_j}{\partial s} g \, Y_{4-j} \, g^t$ we get

$$B_3 = \frac{1}{g_{44}^2}\begin{pmatrix} 0 & g_{34}^2+g_{44}^2 & -g_{34}g_{24} & -g_{44}g_{24} \\ -g_{34}^2-g_{44}^2 & 0 & g_{34}g_{14} & g_{44}g_{14} \\ g_{34}g_{24} & -g_{34}g_{14} & 0 & 0 \\ g_{44}g_{24} & -g_{44}g_{14} & 0 & 0 \end{pmatrix}.$$

Therefore

$$E(H) = H_{y_1}\dot{\pi}\begin{pmatrix} 0 & -g_{24}g_{44}-g_{14}g_{34} & -g_{34}g_{44}+g_{14}g_{24} \\ g_{24}g_{44}+g_{14}g_{34} & 0 & -g_{44}^2-g_{14}^2 \\ g_{34}g_{44}-g_{24}g_{14} & g_{44}^2+g_{14}^2 & 0 \end{pmatrix}\frac{1}{g_{44}^2}$$

$$+H_{y_2}\dot{\pi}\begin{pmatrix} 0 & -g_{24}g_{34}+g_{14}g_{44} & g_{44}^2+g_{24}^2 \\ g_{24}g_{34}-g_{14}g_{44} & 0 & -g_{34}g_{44}-g_{14}g_{24} \\ -g_{44}^2-g_{24}^2 & g_{34}g_{44}+g_{14}g_{24} & 0 \end{pmatrix}\frac{1}{g_{44}^2}$$

$$+H_{y_3}\dot{\pi}\begin{pmatrix} 0 & -g_{44}^2-g_{34}^2 & g_{14}g_{44}+g_{24}g_{34} \\ g_{44}^2+g_{34}^2 & 0 & g_{24}g_{44}-g_{14}g_{34} \\ -g_{14}g_{44}-g_{24}g_{34} & -g_{24}g_{44}+g_{14}g_{34} & 0 \end{pmatrix}\frac{1}{g_{44}^2}.$$

The proposition follows by observing that $y_j = \frac{g_{j4}}{g_{44}}$, for $j = 1, 2, 3$. $\qquad\square$

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