Soc Choice Welf
DOI 10.1007/s00355-011-0559-9

ORIGINAL PAPER

Hedonic games related to many-to-one matching problems

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Received: 26 February 2010 / Accepted: 26 May 2011
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Abstract We consider the existence problem of stable matchings in many-to-one matching problems. Unlike other approaches which use algorithmic techniques to give necessary and sufficient conditions, we adopt a game theoretic point of view. We first associate, with each many-to-one matching problem, a hedonic game to take advantage of recent results guaranteeing the existence of core-partitions for that class of games, to build up our conditions. The main result states that a many-to-one matching problem, with no restrictions on individual preferences, has stable* matchings if and only if a related hedonic game is pivotally balanced. In the case that the preferences in the matching problem are substitutable, the notions of stability and stability* coincide.

1 Introduction

Since the study of Banerjee et al. (2001) and Bogomolnaia and Jackson (2002), the literature on hedonic or simple coalition formation games has grown considerably, dealing mainly with existence and uniqueness issues of core-partitions. Hedonic games are simple game models where each player ranks the coalitions to which he join, while the value of a coalition for him depends solely on the identity of the other players in the coalition. Relevant to our purpose are, besides Bogomolnaia and Jackson (2002), the articles of Iehlé (2007), where a necessary and sufficient condition for the existence problem of core-partitions in hedonic games is given, and Pápai (2004)

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Published online: 11 June 2011

who provides a uniqueness result with restrictions on the family of feasible coalitions that are allowed to form. On the other hand, many-to-one matching problems, which are a natural extension of the classical college admission and marriage models of Gale and Shapley (1962), have also experimented an increasing degree of attention, specially after the publication of the book of Roth and Sotomayor (1990). Existence of stable matchings and the order structure of the set of them as well, have been the object of the main bulk of the literature. The best positive results have been obtained under restriction on individual preferences, for instance, by letting the firms' preferences being "substitutable" (see Roth and Sotomayor (1990) or our Appendix). However, Echenique and Oviedo (2004), in a general setting, with no individual preference restrictions, propose a fixed point algorithm (the T-algorithm) to compute points in the core of a many-to-one matching problem when this set is non-empty. The core coincides with the set of stable matchings in the matching problems with responsive or substitutable preferences but, in the general case, a complete characterization is given in terms of a related concept of stability*. The results in Echenique and Oviedo (2004) allow themselves to characterize the many-to-one matching problems with non-empty core as those for which the T-algorithm converge. The purpose of this note is to offer another characterization for those problems, but from a game theoretic point of view. With each many-to-one matching problem we associate a hedonic game, with a restricted family of admissible coalitions, in such a way that core-partitions are related to stable matchings. Then, we use recent necessary and sufficient conditions for the existence of core-partitions in hedonic games (see for instance Iehlé (2007)), properly reformulated to our environment, to state conditions to guarantee the existence of stable matchings in the matching problem. Dimitrov and Lazarova (2008) have also exploited the relationships between the two models of coalition formation, namely, matching models, which allow coalitions from two sides of a market, and hedonic games, where the coalitions contain members from a one-side market, to study the existence of matchings exhibiting a certain kind of stability (core stable coalitional matchings) in the many-to-many matching problems.

The article is organized as follows. In the next section, we introduce hedonic games with restrictions on the set of coalitions and the concept of core-partition as well in this setting. We also recall the notion of ordinal balancedness given by Bogomolnaia and Jackson (2002) and state their existence result based on this concept for games with restricted family of feasible coalitions. In Sect. 3 we introduce the many-to-one matching problem along with some stability notions. We also construct its associated hedonic game with restricted family of coalitions and show how the pre-solutions (matchings) in the matching problem are related to the pre-solutions (partitions) in its associated game. In Sect. 4 we state that ordinal balancedness of a related hedonic game, is a sufficient condition for the existence of stable matchings in a many-to-one problem. Another well-known sufficient condition is the substitutability of the firms' preferences. In the Appendix, we show that both conditions are independent. In the final section, we state a necessary and sufficient condition for the existence of stable* matchings in terms of the pivotal balancedness of the related hedonic game to each many-to-one matching problem, condition which turns to be a necessary and sufficient condition for the existence of stable matchings in many-to-one matching problems with substitutable preferences.

2 Hedonic games

Let $N = \{1, \ldots, n\}$ be a finite set, $\mathcal{N}$ the set of all nonempty subsets of N, and for each $i \in N$, let $\mathcal{N}(i)$ be the family of elements of $\mathcal{N}$ containing i. A *hedonic game* is a pair $(N, \hat{\succeq})$, where $\hat{\succeq} = (\hat{\succeq}_i)_{i \in N}$ is a *preference profile* with $\hat{\succeq}_i$ being a reflexive, complete and transitive binary relation on $\mathcal{N}(i)$ for each $i \in N$. The elements of N are the *players* of the game and the elements of $\mathcal{N}$ are the *coalitions*. For each $i \in N$, $\hat{\succ}_i$ will stand for the strict preference relation related to $\hat{\succeq}_i$ ($x \hat{\succ}_i y$ iff $x \hat{\succeq}_i y$ but not $y \hat{\succeq}_i x$). $\mathcal{P}(N)$ will denote the family of partitions of N. Given $\pi = \{\pi_1, \ldots, \pi_p\} \in \mathcal{P}(N)$ and $i \in N$, $\pi(i)$ will denote the unique set in π containing player i.

Given a hedonic game $(N, \hat{\succeq})$ and a partition $\pi \in \mathcal{P}(N)$, we say that $T \in \mathcal{N}$ *strongly blocks* π if for each $i \in T$, $T \hat{\succ}_i \pi(i)$. The *core* of $(N, \hat{\succeq})$, denoted by $C(N, \hat{\succeq})$, is the set of partitions strongly blocked by no coalition.

In what follows we will be restricted to consider a subfamily of coalitions of players. Let us consider $\mathcal{A} \subseteq \mathcal{N}$, a collection of non-empty subsets of N, such that $\{i\} \in \mathcal{A}$ for each $i \in N$. $\mathcal{A}$ is the family of *admissible coalitions*. A hedonic game with $\mathcal{A}$ as its family of admissible coalitions $(N, \hat{\succeq}; \mathcal{A})$ is a hedonic game $(N, \hat{\succeq})$ where the preference profile $\hat{\succeq}$ is restricted to the coalitions in $\mathcal{A}$. A partition $\pi \in \mathcal{P}(N)$ is an $\mathcal{A}$-partition if $S \in \mathcal{A}$ for each $S \in \pi$. $\mathcal{P}^{\mathcal{A}}(N)$ will stand for the collection of all $\mathcal{A}$-partitions of N.

Definition 1 A partition $\pi \in \mathcal{P}^{\mathcal{A}}(N)$ belongs to the core $C(N, \hat{\succeq}; \mathcal{A})$ of $(N, \hat{\succeq}; \mathcal{A})$ if it is strongly blocked by no coalition $T \in \mathcal{A}$.

A preference $\hat{\succeq}_i$ is strict if, besides being a reflexive, complete and transitive binary relation on $\mathcal{A}(i)$, the subfamily of $\mathcal{A}$ containing player i, it satisfies that $S \hat{\succeq}_i T$ and $S \neq T$ implies that $S \hat{\succ}_i T$. In the case that all the preferences in the preference profile $\hat{\succeq}$ of a game $(N, \hat{\succeq}; \mathcal{A})$ are strict, $\pi \in C(N, \hat{\succeq}; \mathcal{A})$ if and only if there is no coalition $T \in \mathcal{A}$ such that $T \hat{\succeq}_i \pi(i)$ for all $i \in T$ and $T \hat{\succ}_i \pi(i)$ for at least one $i \in T$. In this case we just say that T *blocks* the partition π.

A non-empty subfamily $\mathcal{B} \subseteq \mathcal{N}$ is *balanced* (see for instance, Shapley (1967)) if there exist positive weights $\lambda = (\lambda_S)_{S \in \mathcal{B}}$ such that

$$\sum_{S \in \mathcal{B}} \lambda_S . \chi^S = \chi^N.$$

Here χ^S is the n-dimensional indicator vector of a coalition $S \subseteq N$, namely, $\chi_i^S = 1$ if $i \in T$ and 0 otherwise.

Definition 2 A hedonic game $(N, \hat{\succeq})$ is ordinally balanced if for each balanced family $\mathcal{B} \subseteq \mathcal{N}$ there exists $\pi \in \mathcal{P}(N)$ such that, for each $i \in N$, $\pi(i) \hat{\succeq}_i S$ for some $S \in \mathcal{B}(i)$. A game $(N, \hat{\succeq}; \mathcal{A})$ is ordinally balanced if for each balanced family $\mathcal{B} \subseteq \mathcal{A}$ there exists $\pi \in \mathcal{P}^{\mathcal{A}}(N)$ such that, for each $i \in N$, $\pi(i) \hat{\succeq}_i S$ for some $S \in \mathcal{B}(i)$, $\mathcal{B}(i)$ being the subfamily of $\mathcal{B}$ containing player i.

The following is an existence result of core-partitions for hedonic games with coalitional restrictions which parallels the first part of Theorem 1 in Bogomolnaia and Jackson (2002) about general hedonic games, and whose proof is carried out in a similar way.

Theorem 1 *An ordinally balanced game* $(N, \hat{\succeq}; \mathcal{A})$ *with strict individual preferences always has a non-empty core.*

3 Many-to-one matching problem and its associated hedonic game

A many-to-one matching problem consists of two disjoint finite sets of agents, the sets F of "firms" and the set W of "workers". It is assumed that each firm $f \in F$ is endowed with a preference over the family of subsets of W, and that each worker w has a preference on the set $F \cup \{\phi\}$, where ϕ stands for the empty set. Individual preferences are assumed to be strict, reflexive, complete and transitive on their corresponding domains. A preference $\succeq$ is strict if $x \succeq y$ and $x \neq y$ implies that $x \succ y$, where $\succ$ is the strict preference derived from $\succeq$ (see Sect. 2). We denote a many-to-one matching problem by $(F, W; \succeq_F, \succeq_W)$, where $\succeq_F = (\succeq_f)_{f \in F}$ and $\succeq_W = (\succeq_w)_{w \in W}$ are the preference profiles of the firms and workers.

A *matching* in $(F, W; \succeq_F, \succeq_W)$ is a function $\mu : F \cup W \to F \cup \mathcal{W}$ satisfying:

(a) for each $f \in F$, if $\mu(f) \neq \phi$, then $\mu(f) \in \mathcal{W}$.
(b) For each $w \in W$, if $|\mu(w)| = 1$, then $\mu(w) \in F$ and $\mu(w) = f$ if and only if $w \in \mu(f)$.

Here $\mathcal{W}$ denotes the family of subsets of W.

In the many-to-one matching problems, a widely studied solution concept is that of the *stable set*. To define this set, we need to introduce first some new notions. The *choice operator* associated to each firm $f \in F$ is defined as

$$Ch_f : \mathcal{W} \to \mathcal{W},$$

and for any $S \in \mathcal{W}$,

$$T = Ch_f(S)$$

if and only if $T \subseteq S$ and $T \succeq_f T'$ for all $T' \subseteq S$. When the preference $\succeq_f$ is strict, for each $S \in \mathcal{W}$ there is only one set in $\mathcal{W}$ satisfying these two conditions. So the Ch_f operator is well-defined as a function. The set $Ch_f(S)$ could be, sometimes, either S itself or the empty set.

A matching μ is *blocked* by an agent $w \in W$ if $\phi \succ_w \mu(w)$, and by an agent $f \in F$ if $\mu(f) \neq Ch_f(\mu(f))$. A matching μ is *individually stable* if it is blocked by no individual agent. A matching μ is blocked by a pair $f \in F$ and $w \in W$ if

$f \neq \mu(w)$, $f \succ_w \mu(w)$, and $w \in Ch_f(\mu(f) \cup \{w\})$. A matching μ is *pairwise stable* if it is blocked by no pair of agents $f \in F$, $w \in W$.

Definition 3 The stable set of a matching problem $(F, W; \succeq_F, \succeq_W)$ is the set of matchings which are both individually and pairwise stable.

We refer the reader to Roth and Sotomayor (1990, Chap. 6) for an interpretation of these stability notions.

Given a many-to-one matching problem $(F, W; \succeq_F, \succeq_W)$, we define its associated hedonic game with a restricted family of coalition as $(N, \hat{\succeq}; \mathcal{A})$ where $N = F \cup W$, and $\mathcal{A}$ is the family including the individual coalition $\{w\}$ for all $w \in W$, along with all $S \subseteq N$ such that $|S \cap F| = 1$. In what follows, and when we were using the preferences of the workers, we are going to identify the set $S \cap F = \{f\}$ with its unique element f. The preference profile $\hat{\succeq} = (\hat{\succeq}_i)_{i \in N}$ is defined as follows. For each $i \in F$, namely, when $i = f$ for some $f \in F$, and S and T are two coalitions in $\mathcal{A}(i)$, we say that

$$S \hat{\succeq}_i T \text{ if and only if } (S \cap W) \succeq_f (T \cap W).$$

On the other hand, if $i \in W$, namely, if $i = w$ for some $w \in W$, and S and T are two different coalitions in $\mathcal{A}(i)$, $S \hat{\succeq}_i T$ if and only if

$$\begin{aligned} &(S \cap F) \succeq_w (T \cap F) \text{ if } (S \cap F) \neq (T \cap F) \text{ or} \\ &(S \cap W) \succeq_f (T \cap W) \text{ if } (S \cap F) = (T \cap F) = \{f\}. \end{aligned}$$

If we agree in putting $\{w\} \hat{\succeq}_w \{w\}$, it turns out that $\hat{\succeq}_i$ is a strict, reflexive, complete and transitive preference on $\mathcal{A}(i)$.

Remark 1 When $i = w$ for some $w \in W$, the preference $\hat{\succeq}_i$ is lexicographical in the sense that, when i is comparing two coalitions in $\mathcal{A}$ with the same firm f, she prefers the coalition which is also preferred by the firm f. This is another aspect of our study which is shared with the article of Dimitrov and Lazarova (2008). They start with a two-sided market ("students" and "researchers" of a university) where the members of one side have preferences over the coalitions of the other side of the market. Then, they build up hedonic games, where the set of players is formed by the agents of both sides of the market, and where the individual preferences are somehow lexicographical. As the authors quote, one possibility, among others included in the article, "to induce agents' preferences over universities is to assume that priority is given to groups on one of the market sides and then, in case of indifference, groups on the other market side also play a role."

Given a partition $\pi \in \mathcal{P}^{\mathcal{A}}(N)$, there is a related matching μ^{π} for the many-to-one matching problem, namely,

$$\mu^{\pi}(f) = \pi(f) \cap W,$$

and

$$\mu^{\pi}(w) = \pi(w) \cap F.$$

Conversely, given any matching μ of a many-to-one matching $(F, W; \succeq_F, \succeq_W)$ there is a related partition $\pi^{\mu} \in \mathcal{P}^{\mathcal{A}}(N)$ defined as follows. For each $f \in F$, let $\pi_f^{\mu} = \mu(f) \cup \{f\}$, and for each $w \in W$ such that $\mu(w) = \phi$, let $\pi_w^{\mu} = \{w\}$. Then $\pi^{\mu} = \{\pi_f^{\mu} : f \in F\} \cup \{\pi_w^{\mu} : w \in W \text{ and } \mu(w) = \phi\}$. It is clear that, for each $w \in W$ such that $\mu(w) \neq \phi$,

$$\pi^{\mu}(w) = \{\mu(w)\} \cup \mu(\mu(w)).$$

It is easy to see that, for all matching μ in $(F, W; \succeq_F, \succeq_W)$, $\mu^{\pi^{\mu}} = \mu$, and for all partition π in $(N, \hat{\succeq}; \mathcal{A})$, $\pi^{\mu^{\pi}} = \pi$.

In the next section, we will see how the associated hedonic game can be used to gather information about the matching problem.

4 Existence of stable matchings

In what follows, we will show that stability properties of a matching μ in a many-to-one matching problem $(F, W; \succeq_F, \succeq_W)$ are related to stability properties of the partition π^{μ} in its associated hedonic game $(N, \hat{\succeq}; \mathcal{A})$. As an interesting consequence we will state a sufficient condition for the existence of stable matchings, in a very general setting, in terms of the characteristic of the hedonic associated game. We start with the following result.

Lemma 1 *Let $(F, W; \succeq_F, \succeq_W)$ be a many-to-one matching problem, μ a matching and π^{μ} its related $\mathcal{A}$-partition in $(N, \hat{\succeq}; \mathcal{A})$. Then, if μ is individually stable in $(F, W; \succeq_F, \succeq_W)$, π^{μ} is blocked by no individual coalition in $(N, \hat{\succeq}; \mathcal{A})$ (individual rationality). Conversely, if $\pi^{\mu} \in C(N, \hat{\succeq}; \mathcal{A})$, then μ is individually stable.*

Proof Let μ be individually stable. We will show that π^{μ} is individually rational. To see this, let $\{w\}$ be an individual coalition with $w \in W$. If $\mu(w) \neq \phi$, since $\pi^{\mu}(w) \cap F = \mu(w) \succeq_w \phi$ because of the individual stability of μ, it can not hold, according to the definition of $\hat{\succeq}_w$, that $\{w\} \hat{\succ}_w \pi^{\mu}(w)$. And if $\mu(w) = \phi$, $\pi^{\mu}(w) = \{w\}$, so $\pi^{\mu}(w) \hat{\succeq}_w \{w\}$ and $\{w\}$ can not block π^{μ} in this case either. Thus, no individual coalition containing a member of W can block π^{μ}. On the other hand, let us assume that there is $i = f$ for some $f \in F$ such that $\{i\} \hat{\succ}_i \pi^{\mu}(i)$ in $(N, \hat{\succeq}; \mathcal{A})$. This would imply that $\phi \succ_f \mu(f)$. But then, $Ch_f(\mu(f)) \neq \mu(f)$, a contradiction with the individual stability of μ. Thus, π^{μ} is individually rational.

Conversely, let π^{μ} be a core partition in $(N, \hat{\succeq}; \mathcal{A})$. If for some $w \in W, \phi \succ_w \mu(w)$, then $\mu(w) \neq \phi$ and $\{w\} \hat{\succ}_w \{\mu(w)\} \cup \mu(\mu(w)) = \pi^{\mu}(w)$, which contradicts the individual rationality of π^{μ}. Thus, $\mu(w) \succeq_w \phi$ for all $w \in W$. On the other hand, if for some $f \in F, Ch_f(\mu(f)) \neq \mu(f), Ch_f(\mu(f)) \succ_f \mu(f)$, and the coalition $\{f\} \cup Ch_f(\mu(f)) \hat{\succ}_f \{f\} \cup \mu(f) = \pi^{\mu}(f)$ in $(N, \hat{\succeq}; \mathcal{A})$. Moreover, since for all $w \in \{f\} \cup Ch_f(\mu(f))$, clearly $w \in Ch_f(\mu(f)) \subseteq \mu(f)$, we have that

$\pi^{\mu}(w) = \{f\} \cup \mu(f)$. Then, since $F \cap (\{f\} \cup Ch_f(\mu(f))) = F \cap \pi^{\mu}(w)$, and because $\{f\} \cup Ch_f(\mu(f)) \hat{\succ}_f \{f\} \cup \mu(f)$, it follows from the definition of $\hat{\succeq}_w$ that $\{f\} \cup Ch_f(\mu(f)) \hat{\succ}_w \pi^{\mu}(w)$ too. Thus, $\{f\} \cup Ch_f(\mu(f))$ (strongly) blocks π^{μ}, which is a contradiction with its assumed core property, showing that μ should be individually stable in $(F, W; \succeq_F, \succeq_W)$. □

Lemma 2 *Let $\pi \in \mathcal{P}^{\mathcal{A}}(N)$ be a core-partition for the hedonic game $(N, \hat{\succeq}; \mathcal{A})$ related to a many-to-one matching problem $(F, W; \succeq_F, \succeq_W)$. Then the matching μ^{π} is stable in $(F, W; \succeq_F, \succeq_W)$.*

Proof From Lemma 1, and since $\pi^{\mu^{\pi}} = \pi$, we get that μ^{π} is individually stable. On the other hand, let us assume that $f \in F$ and $w \in W$ is a blocking pair to μ^{π}. Then $f \neq \mu^{\pi}(w)$ and $f \succ_w \mu^{\pi}(w)$. Since $\mu^{\pi}(f) \subseteq \mu^{\pi}(f) \cup \{w\}$ we conclude that $Ch_f(\mu^{\pi}(f) \cup \{w\}) \succeq_f \mu^{\pi}(f)$. Thus, the coalition $\{f\} \cup Ch_f(\mu^{\pi}(f) \cup \{w\})$, which certainly belongs to $\mathcal{A}$, satisfies that $\{f\} \cup Ch_f(\mu^{\pi}(f) \cup \{w\}) \hat{\succeq}_f \{f\} \cup \mu^{\pi}(f) = \pi(f)$. On the other hand, for w, which also belongs to $Ch_f(\mu^{\pi}(f) \cup \{w\})$, and from $f \succ_w \mu^{\pi}(w)$, we get that $\{f\} \cup Ch_f(\mu^{\pi}(f) \cup \{w\}) \hat{\succ}_w \pi(w) = \{\mu(w)\} \cup \mu(\mu(w))$. Finally, for any other $w' \in Ch_f(\mu^{\pi}(f) \cup \{w\})$, $\mu^{\pi}(w') = f$ and the relationship $\{f\} \cup Ch_f(\mu^{\pi}(f) \cup \{w\}) \hat{\succeq}_{w'} \pi^{\mu}(w') = \{f)\} \cup \mu(f)$ follows, according to the definition of $\hat{\succeq}_{w'}$, from the relationship $Ch_f(\mu^{\pi}(f) \cup \{w\}) \succeq_f \mu^{\pi}(f)$. Thus, $\{f\} \cup Ch_f(\mu^{\pi}(f) \cup \{w\})$ blocks π, which once more, contradicts the core-partition property of π in $(N, \hat{\succeq}; \mathcal{A})$. This proves that μ^{π} is also pairwise stable and therefore, stable in $(F, W; \succeq_F, \succeq_W)$. □

The proof of Lemma 3 shows indeed that the set of matchings associated to partitions in the core of $(N, \hat{\succeq}; \mathcal{A})$ is included in the set of stable matching of the associated many-to-one matching problem $(F, W; \succeq_F, \succeq_W)$. Thus, sufficient conditions guaranteeing the non-emptiness of the core of $(N, \hat{\succeq}; \mathcal{A})$ could be used to state sufficient conditions guaranteeing the existence of stable matchings for the related many-to-one matching problem.

Corollary 1 *Let $(F, W; \succeq_F, \succeq_W)$ be a many-to-one matching problem. If its associated hedonic game $(N, \hat{\succeq}; \mathcal{A})$ is ordinally balanced, then the set of stable matchings of $(F, W; \succeq_F, \succeq_W)$ is non-empty.*

Proof Since $(N, \hat{\succeq}; \mathcal{A})$ is ordinally balanced and because of Theorem 1, there exists a partition $\pi \in \mathcal{P}^{\mathcal{A}}(N)$ belonging to $C(N, \hat{\succeq}; \mathcal{A})$. Thus, according to Lemma 2, μ^{π} is a stable matching for the many-to-one matching problem. □

5 A necessary and sufficient condition for the existence of stable* matchings

An interesting issue in the study of many-to-one matching problems is to find sufficient conditions for the existence of stable matchings in a class of problems including those with substitutable preferences, and Corollary 1 is a contribution into this field. On the other hand, Echenique and Oviedo (2004) give a necessary and sufficient condition in terms of fixed points of a certain operator suitably related to a given many-to-one problem with general preference profiles. Their approach characterizes indeed, the core of matching problems, which in the case of substitutable individual preferences, coincides with the set of stable matchings. In this section, we are going to give another characterization of the core of matching problems with a game theoretic approach. To this end we will use a recent result of Iehlé (2007) giving a necessary and sufficient condition for the existence of core-partitions in hedonic games.

Given a family $\mathcal{A} \subseteq \mathcal{N}$ like that defined in Sect. 3, a family $\mathcal{I} = (\mathcal{I}(S))_{S \in \mathcal{A}}$ is called an $\mathcal{A}$-distribution if, for each nonempty coalition $S \in \mathcal{A}, \phi \neq \mathcal{I}(S) \subseteq S$. Given an $\mathcal{A}$-distribution $\mathcal{I}$, a family $\mathcal{B} \subseteq \mathcal{A}$ is $\mathcal{I}$-*balanced* if the family $(\mathcal{I}(S))_{S \in \mathcal{B}}$ is balanced.

Definition 4 Let $(N, \hat{\succeq}; \mathcal{A})$ be a hedonic game. Given an $\mathcal{A}$-distribution $\mathcal{I}$, the game is pivotally balanced with respect to $\mathcal{I}$ if for each $\mathcal{I}$-balanced family $\mathcal{B}$, there exists a partition $\pi \in \mathcal{P}^{\mathcal{A}}(N)$ such that, for each $i \in N, \pi(i) \hat{\succeq}_i S$ for some $S \in \mathcal{B}(i)$. The game is pivotally balanced if it is pivotally balanced with respect to some distribution $\mathcal{I}$.

Remark 2 The notions of distribution and pivotal balancedness were first introduced by Iehlé (2007) to study the existence of a core-partition in a hedonic game $(N, \hat{\succeq})$.

With the same proof given by Iehlé (2007, Theorem 3, equivalence between parts (i) and (ii)) we can state the following theorem.

Proposition 1 *Let $(N, \hat{\succeq}; \mathcal{A})$ be a hedonic game with $\mathcal{A}$ as its family of admissible coalitions. Then the core of the game is nonempty if and only if it is pivotally balanced.*

Definition 5 Given a many-to-one matching problem, $(F, W; \succeq_F, \succeq_W)$, the core $C(F, W; \succeq_F, \succeq_W)$ is the set of matchings for which there is no $\hat{F} \subseteq F, \hat{W} \subseteq W, \hat{F} \cup \hat{W} \neq \phi$, and a matching $\hat{\mu}$ such that, for all $f \in \hat{F}, w \in \hat{W}$ it holds that:

(a) $\hat{\mu}(f) \subseteq \hat{W}, \hat{\mu}(w) \in \hat{F}$ whenever $\hat{\mu}(w) \neq \phi$.
(b) $\hat{\mu}(f) \succeq_f \mu(f)$.
(c) $\hat{\mu}(w) \succeq_w \mu(w)$.
(d) $\hat{\mu}(s) \succ_s \mu(s)$ for at least one $s \in \hat{F} \cup \hat{W}$.

It is a well-known result that the core and the set of stable matchings coincide in matching problems with substitutable preferences for the firms (Roth and Sotomayor 1990), although, in a general setting, this result is no longer true. However, with the following notion of stability* proposed by Echenique and Oviedo (2004), the equivalence is recovered in general.

A matching μ is blocked* by $f \in F, \phi \neq B \subseteq W$ if $f \succ_w \mu(w)$ for all $w \in B$, and there exists $A \subseteq \mu(f)$ such that $A \cup B \succ_f \mu(f)$.

Definition 6 A matching μ in $(F, W; \succeq_F, \succeq_W)$ is stable* if it is individually rational (Sect. 3) and there is no pair $f \in F, \phi \neq B \subseteq W$ blocking*μ. The following result is from Echenique and Oviedo (2004, Theorem 2).

Theorem 2 *Let $(F, W; \succeq_F, \succeq_W)$ be a many-to-one matching problem. Then, the core $C(F, W; \succeq_F, \succeq_W)$ coincides with the set of stable* matchings.*

Clearly, the three sets, namely, the set of stable matchings, the set of stable matchings and the core, coincide if* $(F, W; \succeq_F, \succeq_W)$ *has substitutable preferences for the firms in* F. *In general, the set of stable* matching, and therefore the core, is a subset of the set of stable matchings. Now we are going to show that the core of* $(F, W; \succeq_F, \succeq_W)$ *also coincides with the set matchings obtained from the core-partitions of the associated hedonic game* $(N, \hat{\succeq}; \mathcal{A})$.

Lemma 3 *Let* $(F, W; \succeq_F, \succeq_W)$ *be a one-to-many matching problem and* $(N, \hat{\succeq}; \mathcal{A})$ *its associated hedonic game. Then,* $\mu \in C(F, W; \succeq_F, \succeq_W)$ *if and only if* $\pi^\mu \in C(N, \hat{\succeq}; \mathcal{A})$.

Proof Let $\mu \in C(F, W; \succeq_F, \succeq_W)$, and assume that there is a coalition $S \in \mathcal{A}$ blocking π^μ in $(N, \hat{\succeq}; \mathcal{A})$. Then, $S \cap F \neq \phi$. Otherwise $S = \{w\}$ for some $w \in W$, and $S \hat{\succ}_w \pi^\mu(w)$. Therefore, $\phi = (S \cap F) \succ_w (\pi^\mu(w) \cap F) = \mu(w)$ (note that $\mu(w)$ should be non-empty) which contradicts the individual stability of μ. Thus, $S \cap F = \{f\}$. Now let $\hat{F} = \{f\}$, $\hat{W} = S \cap W$, $\hat{\mu}$ the matching assigning $\hat{W}$ to f and f to each $w \in \hat{W}$, and letting any other agent in the matching be single. We claim that $\hat{F}$, $\hat{W}$ and $\hat{\mu}$ blocks μ, contradicting the assumption that the matching μ is in the core of $(F, W; \succeq_F, \succeq_W)$. We first note that, since S blocks π^μ,

$$\hat{\mu}(f) = (S \cap W) \succeq_f (\pi^\mu(f) \cap W) = \mu(f), \tag{1}$$

relationship which also implies that $\hat{\mu}(w) \hat{\succeq}_w \mu(w)$ for any $w \in \hat{W}$ such that $\mu(w) = \{f\} = \hat{\mu}(w)$. On the other hand, for any $w \in \hat{W}$ with $\mu(w) \neq \{f\}$, we also have that

$$\hat{\mu}(w) = (S \cap F) \succeq_w (\pi^\mu(w) \cap F) = \mu(w). \tag{2}$$

Thus, for each $s \in \hat{F} \cup \hat{W}$ we have shown that $\hat{\mu}(s) \succeq_s \mu(s)$. But once more, since S blocks π^μ, there is a $s^* \in N$ such that $S \hat{\succ}_{s^*} \pi^\mu(s^*)$. If $s^* = w$ for some $w \in \hat{W}$ with $\mu(w) \neq \{f\}$, then the strict preference should appear in (2), and the strict preference should prevail in (1) if $s^* = f$ for some $f \in F$ as well. And if $s^* = w$ for some $w \in \hat{W}$ with $\mu(w) = \{f\}$, then $S \hat{\succ}_{s^*} \pi^\mu(s^*)$ implies

$$\hat{\mu}(f) = (S \cap W) \succ_f (\pi^\mu(f) \cap W) = \mu(f).$$

Thus, we have shown that always $\hat{\mu}(s^*) \succ_{s^*} \mu(s^*)$ for at least on $s^* \in \hat{F} \cup \hat{W}$ completing the proof of our claim and showing that $\pi^\mu \in C(N, \hat{\succeq}; \mathcal{A})$.

Conversely, now let $\pi^\mu \in C(N, \hat{\succeq}; \mathcal{A})$. If μ does not belong to $C(F, W; \succeq_F, \succeq_W)$, there exist $\hat{F} \subseteq F$, $\hat{W} \subseteq W$ such that $\hat{F} \cup \hat{W} \neq \phi$, and a matching $\hat{\mu}$ such that, for all $f \in \hat{F}$, $w \in \hat{W}$, conditions (a) to (d) from Definition 5 are satisfied. From condition (d) we know that there is $s^* \in \hat{F} \cup \hat{W}$ such that $\hat{\mu}(s^*) \succ_{s^*} \mu(s^*)$. We claim that $s^* = f$ for some $f \in F$ leads to a contradiction, for in this case the coalition $S = \{f\} \cup \hat{\mu}(f)$ would block π^μ. To see this, we first note that $\hat{\mu}(f) \subseteq \hat{W}$ and certainly S belongs to $\mathcal{A}$. Moreover, since $\hat{\mu}(f) = (S \cap W) \succ_f (\pi^\mu(f) \cap W) = \mu(f)$, then $\hat{\mu}(f) \succ_f \mu(f)$ implies that $S \hat{\succ}_f \pi^\mu(f)$. The same argument also shows that $S \hat{\succ}_w \pi^\mu(w)$ for any

$w \in \hat{\mu}(f)$ such that $\{f\} = \hat{\mu}(w) = \mu(w)$ (recall that $w \in \hat{\mu}(f)$ implies that $\hat{\mu}(w) = f$). And in the event that $w \in \hat{\mu}(f)$, and $\hat{\mu}(w) \neq \mu(w)$, $\hat{\mu}(w) \succeq_w \mu(w)$ implies that $(S \cap F) = \hat{\mu}(w) \succeq_w \mu(w) = (\pi^{\mu}(w) \cap F)$ completing the proof of the claim.

On the other hand, if $s^* = \hat{w}$ for some $\hat{w} \in \hat{W}$, namely, if $\hat{\mu}(\hat{w}) \succ_{\hat{w}} \mu(\hat{w})$ for some $\hat{w} \in \hat{W}$, then $\hat{\mu}(\hat{w}) \neq \phi$, or μ would not be individually stable otherwise. Moreover, we claim that the coalition $S = \{\hat{\mu}(\hat{w})\} \cup \hat{\mu}(\hat{\mu}(\hat{w})) \in \mathcal{A}$ would block π^{μ} in this case. To see this, we first note that, since $S \cap F = \hat{\mu}(\hat{w})$, $\pi^{\mu}(\hat{w}) \cap F = \mu(\hat{w})$ and $\hat{\mu}(\hat{w}) \neq \mu(\hat{w})$, the condition $\hat{\mu}(\hat{w}) \succ_{\hat{w}} \mu(\hat{w})$ implies that $S \hat{\succ}_{\hat{w}} \pi^{\mu}(\hat{w})$. Furthermore, for $\hat{\mu}(\hat{w}) \in \hat{F}$, since $S \cap W = \hat{\mu}(\hat{\mu}(\hat{w}))$, $\pi^{\mu}(\hat{\mu}(\hat{w})) \cap W = \mu(\hat{\mu}(\hat{w}))$ and $\hat{\mu}(s)) \succeq_s \mu(s)$ for all $s \in \hat{F}$, and in particular for $\hat{\mu}(\hat{w})$, we conclude that $S \hat{\succeq}_{\hat{\mu}(\hat{w})} \pi^{\mu}(\hat{\mu}(\hat{w}))$. Finally, if $w \in \hat{\mu}(\hat{\mu}(\hat{w})) \subseteq \hat{W}$, we have that $\hat{\mu}(w) \succeq_w \mu(w)$. This condition, by a similar argument than that used for the case $s^* = \hat{w}$ for some $\hat{w} \in \hat{W}$, directly indicates that $S \hat{\succeq}_w \pi^{\mu}(w)$ in the case that $\hat{\mu}(w) \neq \mu(w)$. And if $\hat{\mu}(s) = \mu(s) = \hat{\mu}(\hat{w})$, the relationship $S \hat{\succeq}_w \pi^{\mu}(w)$ follows from $S \hat{\succeq}_{\hat{\mu}(\hat{w})} \pi^{\mu}(\hat{\mu}(\hat{w}))$ already proved. Thus, in any case, the assumption $\mu \notin C(F, W; \succeq_F, \succeq_W)$ leads to a contradiction. Therefore, $\mu \in C(F, W; \succeq_F, \succeq_W)$. □

Corollary 2 *Let $(F, W; \succeq_F, \succeq_W)$ be a one-to-many matching problem and $(N, \hat{\succeq}; \mathcal{A})$ its associated hedonic game. Then, $C(F, W; \succeq_F, \succeq_W) \neq \phi$ if and only if $C(N, \hat{\succeq}; \mathcal{A})$ $\neq \phi$.*

Proof It follows from Lemma 3 by noting that $\pi^{\mu^{\pi}} = \pi$. □

Next we state our main characterization result.

Theorem 3 *Let $(F, W; \succeq_F, \succeq_W)$ be a many-to-one matching problem and $(N, \hat{\succeq}; \mathcal{A})$ its associated hedonic game. A necessary and sufficient condition for the existence of stable* matchings is that $(N, \hat{\succeq}; \mathcal{A})$ be pivotally balanced. In the case that the preferences for the firms in F are substitutable, the latter condition is necessary and sufficient for the existence of stable matchings.*

Proof The second statement of the theorem is due to the coincidence of the set of stable* matchings and the set of stable matchings when the firms' preferences are substitutable.

To prove the first assertion, we note that, by Proposition 1, the game $(N, \hat{\succeq}; \mathcal{A})$ is pivotally balanced if and only if $C(N, \hat{\succeq}; \mathcal{A}) \neq \phi$, and by Corollary 2, that $C(N, \hat{\succeq}; \mathcal{A}) \neq \phi$ if and only if $C(F, W; \succeq_F, \succeq_W) \neq \phi$. Finally, from Theorem 2, we get that $C(F, W; \succeq_F, \succeq_W) \neq \phi$ if and only if set of stable* matchings is non empty. Thus, pivotal balancedness of $(N, \hat{\succeq}; \mathcal{A})$ implies the existence of stable* matchings in $(F, W; \succeq_F, \succeq_W)$ and conversely. □

Remark 3 The condition stated in Theorem 3 is also necessary and sufficient to guarantee the existence of a fixed point for the T-algorithm of Echenique and Oviedo (2004). It is also a necessary and sufficient condition for guaranteeing the convergence of that algorithm from at least one starting point in the framework of many-to-one matching problems.

Acknowledgments The author would like to thank CONICET and UNSL (Argentina) for their financial support and to J. Oviedo, J. Massó, A. Neme and F. Echenique for some helpful comments. The presentation of the paper has improved substantially thanks to several suggestions of one of the referees.

Appendix

Both substitutability and the ordinal balancedness of the lexicographical extension of the associated hedonic game to a many-to-one matching problem are sufficient condition for the existence of stable matching. But, as the following two examples show, they are independent.

Example 1 The following example exhibits that substitutability in a many-to-one matching problem does not imply the ordinal balancedness of the associated hedonic game.

Let $(F, W; \succeq_F, \succeq_W)$ be a many-to-one matching problem. The preference $\succeq_f$ of a firm $f \in F$ is substitutable if for each $S \subseteq W$, and every pair of different workers $w, \hat{w}$ in S it holds that $w \in Ch_f(S)$ implies that $w \in Ch_f(S \backslash \{\hat{w}\})$. We recall that the matching problem satisfies substitutability if firms' preferences are substitutable.

Let us consider the matching problem $(F, W; \succeq_F, \succeq_W)$, where $F = \{f_1, f_2\}$, $W = \{w_1, w_2, w_3\}$, and the individual preferences are given as follows.

$f_1 : \{w_1, w_2\} \succ_{f_1} \{w_1, w_3\} \succ_{f_1} \{w_2, w_3\} \succ_{f_1} \{w_3\} \succ_{f_1} \{w_2\} \succ_{f_1} \{w_1\}$.
$f_2 : \{w_3\} \succ_{f_2} \phi \succ_{f_2} \{w_1, w_2\} \succ_{f_2} \{w_1, w_3\} \succ_{f_2} \{w_2, w_3\} \succ_{f_2} \{w_2\} \succ_{f_2} \{w_1\}$.
$w_1 : f_1 \succ_{w_1} f_2$.
$w_2 : f_1 \succ_{w_2} f_2$.
$w_3 : f_1 \succ_{w_3} f_2$.

This is Example 6.6 of Roth and Sotomayor (1990) except for the fact that we have completed, somehow, the inessential part of the preference of f_2. It is easy to see that this matching problem satisfies substitutability.

The associated game $(N, \hat{\succeq}; \mathcal{A})$ has $N = \{f_1, f_2, w_1, w_2, w_3\}$ and
$\mathcal{A} = \{\{f_1\}, \{f_2\}, \{w_1\}, \{w_2\}, \{w_3\}, \{f_1, w_1\}, \{f_1, w_2\}, \{f_1, w_3\}, \{f_1, w_1, w_2\}, \{f_1, w_1, w_3\}, \{f_1, w_2, w_3\}, \{f_2, w_1\}, \{f_2, w_2\}, \{f_2, w_3\}, \{f_2, w_1, w_2\}, \{f_2, w_1, w_3\}, \{f_2, w_2, w_3\}\}$.

The individual preferences in this game are the following.

$f_1 : \{f_1, w_1, w_2\} \hat{\succ}_{f_1} \{f_1, w_1, w_3\} \hat{\succ}_{f_1} \{f_1, w_2, w_3\} \hat{\succ}_{f_1} \{f_1, w_3\} \hat{\succ}_{f_1} \{f_2, w_2\} \hat{\succ}_{f_1} \{f_2, w_1\} \hat{\succ}_{f_1} \{f_1\}$.
$f_2 : \{f_2, w_3\} \hat{\succ}_{f_2} \{f_2\} \hat{\succ}_{f_2} \{f_2, w_1, w_2\} \hat{\succ}_{f_2} \{f_2, w_1, w_3\} \hat{\succ}_{f_2} \{f_2, w_2, w_3\} \hat{\succ}_{f_2} \{f_2, w_2\} \hat{\succ}_{f_2} \{f_2, w_1\}$.
$w_1 : \{f_1, w_1, w_3\} \hat{\succ}_{w_1} \{f_1, w_1, w_2\} \hat{\succ}_{w_1} \{f_1, w_1\} \hat{\succ}_{w_1} \{f_2, w_1\} \hat{\succ}_{w_1} \{f_2, w_1, w_2\} \hat{\succ}_{w_1} \{f_1, w_1, w_3\} \hat{\succ}_{w_1} \{w_1\}$.
$w_2 : \{f_1, w_1, w_2\} \hat{\succ}_{w_2} \{f_1, w_2, w_3\} \hat{\succ}_{w_2} \{f_1, w_2\} \hat{\succ}_{w_2} \{f_2, w_2\} \hat{\succ}_{w_2} \{f_2, w_1, w_2\} \hat{\succ}_{w_2} \{f_2, w_2, w_3\} \hat{\succ}_{w_2} \{w_2\}$.
$w_3 : \{f_1, w_1, w_3\} \hat{\succ}_{w_3} \{f_1, w_2, w_3\} \hat{\succ}_{w_3} \{f_1, w_3\} \hat{\succ}_{w_3} \{f_2, w_3\} \hat{\succ}_{w_2} \{f_2, w_1, w_3\} \hat{\succ}_{w_3} \{f_2, w_2, w_3\} \hat{\succ}_{w_3} \{w_3\}$.

The family $\mathcal{B} = \{\{f_1, w_1, w_3\}, \{f_1, w_2, w_3\}, \{f_2, w_1, w_2\}, \{f_2\}\}$ is balanced but there does not exists a partition satisfying the condition that, for each $i \in N$, $\pi(i) \hat{\succeq}_i S$ for some $S \in \mathcal{B}(i)$. In fact, such a partition should contain either the coalition

$\{f_1, w_1, w_3\}$ or the coalition $\{f_1, w_2, w_3\}$ in order to satisfy the condition for player f_1. In the case that the coalition $\{f_1, w_1, w_3\}$ be in the partition, the other coalitions in it could be $\{f_2\}$ and $\{w_2\}$ or $\{f_2, w_2\}$. In any case, the condition for ordinal balancedness is not satisfied for either w_2 or f_2. A similar argument applies if $\{f_1, w_2, w_3\}$ is in the partition.

However, $\pi = \{\{f_1, w_1, w_2\}, \{f_2, w_3\}\}$ is a core-partition for the game $(N, \hat{\succeq}; \mathcal{A})$ and consequently, $(N, \hat{\succeq}; \mathcal{A})$ is pivotally balanced with respect to, for instance, the distribution $\mathcal{I}$ defined as follows (see Iehlé 2007).

$\mathcal{I}(\{f_1\}) = \{f_1\}, \mathcal{I}(\{f_2\}) = \{f_2\}, \mathcal{I}(\{w_1\}) = \{w_1\}, \mathcal{I}(\{w_2\}) = \{w_2\}, \mathcal{I}(\{w_3\}) = \{w_3\}$,

$\mathcal{I}(\{f_1, w_1\}) = \{f_1, w_1\}, \mathcal{I}(\{f_1, w_2\}) = \{f_1, w_2\}, \mathcal{I}(\{f_1, w_3\}) = \{f_1\}$,

$\mathcal{I}(\{f_1, w_1, w_2\}) = \{f_1, w_1, w_2\}, \mathcal{I}(\{f_1, w_1, w_3\}) = \{f_1\}, \mathcal{I}(\{f_1, w_2, w_3\}) = \{f_1, w_2\}$,

$\mathcal{I}(\{f_2, w_1\}) = \{f_2, w_1\}, \mathcal{I}(\{f_2, w_2\}) = \{f_2, w_2\}, \mathcal{I}(\{f_2, w_3\}) = \{f_2, w_3\}$,

$\mathcal{I}(\{f_2, w_1, w_2\}) = \{f_2, w_1, w_2\}, \mathcal{I}(\{f_2, w_1, w_3\}) = \{f_2, w_3\}, \mathcal{I}(\{f_2, w_2, w_3\}) = \{f_2, w_2, w_3\}$.

Example 2 This example, which is adapted from Example 2 of Echenique and Oviedo (2004) is a case of a many-to-one matching problem with non substitutable preference which has, however, an associated hedonic game which is ordinally balanced.

Let us consider the matching problem $(F, W; \succeq_F, \succeq_W)$, where $F = \{f_1\}$, $W = \{w_1, w_2, w_3\}$, and the individual preferences are given as follows.

$f_1 : \{w_1, w_2\} \succ_{f_1} \{w_3\} \succ_{f_1} \phi \succ_{f_1} \{w_1, w_3\} \succ_{f_1} \{w_2, w_3\} \succ_{f_1} \{w_2\} \succ_{f_1} \{w_1\} \succ_{f_1} \{w_1, w_2, w_3\}$.

$w_1 : f_1$.

$w_2 : f_1$.

$w_3 : f_1$.

$(F, W; \succeq_F, \succeq_W)$ has two stable matchings, $\mu^1(f_1) = \{w_1, w_2\}$, $\mu^1(w_3) = \phi$, and $\mu^2(f_1) = \{w_3\}$, $\mu^2(w_1) = \phi$, $\mu^2(w_2) = \phi$, but only μ^1 is a stable* matching. This shows that $(F, W; \succeq_F, \succeq_W)$ has not substitutable preferences, since otherwise the set of stable matchings should coincide with the set of stable* matchings.

Now let us consider the associated hedonic game $(N, \hat{\succeq}; \mathcal{A})$ which has $N = \{f_1, w_1, w_2, w_3\}$, $\mathcal{A} = \{\{f_1\}, \{w_1\}, \{w_2\}, \{w_3\}, \{f_1, w_1\}, \{f_1, w_2\}, \{f_1, w_3\}, \{f_1, w_1, w_2\}, \{f_1, w_1, w_3\}, \{f_1, w_2, w_3\}\}$, and individual preferences given by:

$f_1 : \{f_1, w_1, w_2\} \hat{\succ}_{f_1} \{f_1, w_3\} \hat{\succ}_{f_1} \{f_1\} \hat{\succ}_{f_1} \{f_1, w_1, w_3\} \hat{\succ}_{f_1} \{f_1, w_2, w_3\} \hat{\succ}_{f_1} \{f_2, w_2\} \hat{\succ}_{f_1} \{f_2, w_1\}$.

$w_1 : \{f_1, w_1, w_2\} \hat{\succ}_{w_1} \{f_1, w_1, w_3\} \hat{\succ}_{w_1} \{f_1, w_1\} \hat{\succ}_{w_1} \{w_1\}$.

$w_2 : \{f_1, w_1, w_2\} \hat{\succ}_{w_2} \{f_1, w_2, w_3\} \hat{\succ}_{w_2} \{f_1, w_2\} \hat{\succ}_{w_2} \{w_2\}$.

$w_3 : \{f_1, w_3\} \hat{\succ}_{w_3} \{f_1, w_1, w_3\} \hat{\succ}_{w_3} \{f_1, w_2, w_3\} \hat{\succ}_{w_3} \{w_3\}$.

Since the only $\mathcal{A}$-balanced families of coalitions in $(N, \hat{\succeq}; \mathcal{A})$ are partitions, it follows that this game is ordinally balanced.

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