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Large-scale model of the axisymmetric kinematic dynamo

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ABSTRACT

A formulation of a kinematic dynamo is presented, based on a previously derived self-consistent procedure for obtaining large-scale models for complex system of equations. The model has only a small number of parametrized variables: the small-scale magnetic diffusivity, the scale of the large-scale fields, and a factor in the explicit expression of the $\alpha_{\varphi\varphi}$ component of the α tensor. Explicit expressions of what corresponds to the other components of the α tensor and of the diffusivity tensor are derived in terms of the large-scale meridional flow and of the differential rotation law, without introducing additional parameters. A very simple simulation of a solar-like dynamo, employing the model without meridional flow shows reasonable magnetic field evolution, with a cycle duration of about 2/3 that of the Sun, shift of the magnetic field from mid-latitudes towards the equator, poleward migration of the radial field at high latitudes, and correct phase relation between radial and azimuthal components.

Key words: dynamo – magnetic fields – MHD – turbulence.

1 INTRODUCTION

Mean-field models of natural dynamos are fundamental for our comprehension of dynamo processes in stars and planets. However, due to the variety and complexity of the mechanisms involved, the modellization of the effect of fluctuations on the mean fields is a difficult, open problem. If separation of scales between mean fields and fluctuations can be assumed, in the sense that mean field magnitudes have a much larger spatial scale than active, fluctuating scales, and consequently assuming that the magnitude of fluctuations is relatively small, closed models for the effects of fluctuating scales on the mean scales can be obtained (Rüdiger & Hollerbach 2004), with as few as possible parameters. With the intention of testing alternative approaches to the modelling of mean field dynamos, we consider in this work the application of the formalism developed in Minotti (2000) and Minotti & Dasso (2001) to the dynamo problem. The formalism allows us to derive effective, large-scale equations for rather general fluid systems directly from the original dynamical equations. The resulting large-scale equations differ from the original ones by the addition of new terms, generically referred to as subgrid scale terms (SGST), which arise due to the non-linear character of the fundamental equations, and which represent the effect of the small (non-resolved) scales on the dynamics of the large-scale (resolved) flow. The method is particularly suited to capture the effect of non-resolved scales close in size to those resolved, thus relaxing the assumption of separation of scales, so that, for instance, one can first include the effect of 'microscopic' scales, well separated in size from those resolved, by usual methods, and then apply the formalism to include the effect of closer, non-resolved scales. It is important to mention that the formalism itself is derived from exact expressions that relate averages made on two different scales, so that no parametrization or modelling is involved, other than the assumption of smoothness of the averaged fields. In this way, the method adds only one new parameter, the scale on which the spatial average is made, which in itself has a direct meaning, so that the formalism allows a new approach with as few as possible parameters.

The mentioned formalism produces large-scale equations in their simplest form for spatial averages made on non-elongated volumes, so that it is not possible to directly obtain equations fully averaged in one of the spatial dimensions, as one would wish, for instance, to generate axisymmetric dynamo models. In this way, if a model of this kind is desired, the resulting large-scale equations depending on three spatial coordinates must be further elaborated.

To present the formalism in its simplest form, we consider in this work only the induction equation for the magnetic field, and derive its large-scale version containing SGST fully expressed in terms of the resolved magnetic and mass flow fields. The equation is assumed to be previously averaged over isotropic microscales in order to describe their effect on the large-scale flow in terms of an increased (relative to the molecular value) diffusivity. To this equation, the averaging procedure described in Minotti (2000) is applied, which can be referred to as a spatial mesoscale averaging, performed over cells of linear size λ. Effective, anisotropic diffusivity effects due to non-resolved fluctuations on scales close in size to the resolved ones are automatically given by the model, together with equivalent α-type effects. The resulting mesoscale equations depend on the three spherical coordinates (r, θ, and φ), with φ the angle about the rotation axis z, and constitute in themselves a closed kinematic

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3D large-scale dynamo model. In order to obtain a description in terms of axisymmetric fields, we perform a final average over the azimuthal angle φ. It is seen that the mesoscale average gives rise to an explicit 'α-effect' term in the poloidal field evolution equation, which on further elaboration is found to be approximated in the axisymmetric model in terms of the stellar rotation profile. The rest of the SGST can be identified with the different components of the α and diffusivity tensors, obtained explicitly in terms of the large-scale meridional circulation and rotation profile.

2 FORMALISM

In this section, we briefly describe the application of the mesoscale averaging (Minotti 2000) as applied to the magnetic induction equation:

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{b} - \eta \nabla \times \boldsymbol{b}), \tag{1}$$

where η is the microscale turbulent diffusivity, $\boldsymbol{b}$ is the magnetic field, and $\boldsymbol{u}$ is the velocity field.

The separation of scales into small and large scales is effected with a top-hat filter, defined by the volume average

$$C(\boldsymbol{X}, t) = \langle c(\boldsymbol{x}, t)\rangle_X = \frac{1}{\triangle V}\int c(\boldsymbol{x}, t)\,\mathrm{d}V, \tag{2}$$

where $\boldsymbol{X} = \langle \boldsymbol{x}\rangle_X$ denotes the 'centre of mass' of a cubic volume $\triangle V$ of size length λ, and $c(\boldsymbol{x}, t)$ is a generic field variable. Fluctuations of $c(\boldsymbol{x}, t)$ around its average are defined as the difference

$$\delta c(\boldsymbol{X}, \boldsymbol{x}, t) = c(\mathbf{x}, t) - C(\boldsymbol{X}, t), \tag{3}$$

an approach originated by Schumman (1975) which has the advantage of avoiding the generation of the so-called Leonard and cross-terms (Leonard 1973; Clark et al. 1979) in the volume-averaged equations because definitions (2) and (3) lead to averages that satisfy Reynolds' postulates,

$$\langle C(\boldsymbol{X})\rangle_X = C(\boldsymbol{X}), \qquad \langle \delta c(\boldsymbol{X}, \boldsymbol{x}, t) C(\boldsymbol{X})\rangle_X = 0, \tag{4}$$

and

$$\left\langle \frac{\partial c}{\partial \boldsymbol{x}} \right\rangle_X = \frac{\partial C}{\partial \boldsymbol{X}}. \tag{5}$$

Due to these properties, averaging of equation (1) is very simple and results in

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B} - \eta \nabla \times \boldsymbol{B}) + \nabla \times \boldsymbol{S}, \tag{6}$$

where capital letters denote the average of the fields represented by the corresponding lower case letters, and spatial derivatives are all with respect to $\boldsymbol{X}$. The last term corresponds to the SGST for which $\boldsymbol{S}$ is given by

$$\boldsymbol{S}(\boldsymbol{X}) = \langle \delta \boldsymbol{u}(\boldsymbol{X}, \boldsymbol{x}) \times \delta \boldsymbol{b}(\boldsymbol{X}, \boldsymbol{x})\rangle_X\,. \tag{7}$$

The expression of $\boldsymbol{S}$ can be directly obtained as described in Minotti (2000), and can be expressed in Cartesian index notation as

$$S_i(\boldsymbol{X}) = \frac{\lambda^2}{24}\varepsilon_{ijk}\frac{\partial U_j}{X_m}\frac{\partial B_k}{X_m}, \tag{8}$$

where ε_{ijk} is Levi-Civita's pseudo-tensor.

Since the equations are to be used in non-Cartesian coordinates, it is convenient to express equation (8) in terms of covariant operations, which results in

$$\boldsymbol{S} = \frac{\lambda^2}{48}\left[\nabla^2(\boldsymbol{U} \times \boldsymbol{B}) - \left(\nabla^2 \boldsymbol{U}\right) \times \boldsymbol{B} - \boldsymbol{U} \times \nabla^2 \boldsymbol{B}\right]. \tag{9}$$

Although second-order derivatives appear in this expression, all of them cancel after expansion.

3 AXISYMMETRIC MODEL

Given the mesoscale flow $\boldsymbol{U}$ and the microscale diffusivity η, equations (6) and (9), together with appropriate initial and boundary conditions, constitute a complete description of a kinematic dynamo given solely in terms of mesoscale fields. To obtain an axisymmetric model, we proceed to consider a description in terms of azimuthally averaged fields, so that the magnetic field is described by the azimuthal component $B(r, \theta, t)$ and the azimuthal component $A(r, \theta, t)$ of the vector potential of the zonal magnetic field,

$$\langle \boldsymbol{B}\rangle = \nabla \times \left(A\, \boldsymbol{e}_\varphi\right) + B\, \boldsymbol{e}_\varphi, \tag{10}$$

where $\langle ...\rangle$ denotes the azimuthal average, $\boldsymbol{e}_\varphi$ is the unit vector in the azimuthal direction, and $B = \langle B_\varphi\rangle$.

From equation (6), the equation for A can then be written as

$$\frac{\partial A}{\partial t} = \langle U_r B_\theta - U_\theta B_r\rangle + \eta\left(\nabla^2 A - \frac{A}{r^2 \sin^2\theta}\right) + \left\langle S_\varphi\right\rangle. \tag{11}$$

In order to proceed, we consider the φ dependence as weak, in such a way that a generic field C is expressed as $C = \langle C\rangle + \delta C$, with δC of small magnitude, which we formally denote as $O(\delta C) \equiv \varepsilon\, O(\langle C\rangle)$, with ε a small number compared to one. We note that this assumption is made on the averaged fields, so that the choice of the scale λ of the volume average can be guided by the condition that this assumption applies. Since $\langle \delta C\rangle = 0$, we see that the lowest order corrections due to the azimuthal dependence are of the order of ε^2. The equation for A can thus be written as

$$\frac{\partial A}{\partial t} = \langle U_r\rangle \langle B_\theta\rangle - \langle U_\theta\rangle \langle B_r\rangle + \eta\left(\nabla^2 A - \frac{A}{r^2\sin^2\theta}\right) + S_\varphi^0 + O\left(\varepsilon^2\right), \tag{12}$$

where the zeroth-order SGST is

$$S_\varphi^0 = \frac{\lambda^2}{48}\left[\nabla^2(\langle \boldsymbol{U}\rangle \times \langle \boldsymbol{B}\rangle) - \left(\nabla^2\langle \boldsymbol{U}\rangle\right) \times \langle \boldsymbol{B}\rangle - \langle \boldsymbol{U}\rangle \times \nabla^2\langle \boldsymbol{B}\rangle\right] \cdot \boldsymbol{e}_\varphi \tag{13}$$

and the order ε^2 terms are given by

$$\begin{aligned} O\left(\varepsilon^2\right) = {} & \langle \delta U_r \delta B_\theta - \delta U_\theta \delta B_r\rangle \\ & + \frac{\lambda^2}{24 r^2\sin^2\theta}\left[\left\langle \partial_\varphi U_r \partial_\varphi B_\theta\right\rangle - \left\langle \partial_\varphi U_\theta \partial_\varphi B_r\right\rangle\right] \\ & + \frac{\lambda^2}{24 r^2\sin^2\theta}\left[\left\langle U_\varphi \partial_\varphi B_z\right\rangle - \left\langle B_\varphi \partial_\varphi U_z\right\rangle\right]. \end{aligned} \tag{14}$$

To keep the simplicity of the final expressions, in the third line of equation (14) we have employed the components of velocity and magnetic field along the rotation axis z.

We further note that the first two lines in equation (14) correspond to $O(\varepsilon^2)$ corrections to the first two terms in equation (12), while the third line contains terms with no equivalent zeroth order (S_φ^0 given by equation (13) does not have terms containing U_φ or B_φ, as is shown explicitly below). In the spirit of obtaining the simplest possible model, we thus neglect the $O(\varepsilon^2)$ corrections to existing zeroth-order terms, and so keep only the terms in the third line of equation (14). In particular, the last term in equation (14) is responsible for a kind of α-effect with a clear physical interpretation, as the indicated φ variation of the meridional flow leads to a deformation of the lines of B_φ that contributes to the poloidal magnetic field generation. We can obtain a simple model of this term by noting the following. In cylindrical coordinates with z the rotation axis, as before, and with cylindrical radial coordinate $s = r\sin\theta$, the last term in equation (14) is written as

$$-\frac{\lambda^2}{24s^2}\left\langle B_\varphi \partial_\varphi U_z\right\rangle = -\frac{\lambda^2}{24s}\left[\left\langle B_\varphi \partial_z U_\varphi\right\rangle + \left\langle B_\varphi \omega_s\right\rangle\right],$$

with ω_s the s component of vorticity,

$$\omega_s = \frac{1}{s}\partial_\varphi U_z - \partial_z U_\varphi .$$

The idea is that, although their azimuthal averages are of equal magnitude (but different sign), ω_s and $\partial_z U_\varphi$ are differently correlated with B_φ. In effect, since ω_s corresponds to pure rotations around s of the local fluid element, which twist the B_φ lines and is thus opposed by the magnetic tension, the magnitudes of B_φ and ω_s are expected to be anticorrelated: $\langle\delta B_\varphi \delta\omega_s\rangle/(\langle B_\varphi\rangle\langle\omega_s\rangle) < 0$. The shear $\partial_z U_\varphi$, on the other hand, does not have the effect of bending B_φ lines so that its magnitude is not so strongly correlated with that of B_φ. In this way, we expect that

$$\left\langle B_\varphi \partial_z U_\varphi\right\rangle \approx \left\langle B_\varphi\right\rangle \partial_z \left\langle U_\varphi\right\rangle = -\left\langle B_\varphi\right\rangle\langle\omega_s\rangle,$$
$$\left|\left\langle B_\varphi \omega_s\right\rangle\right| = \left|\left\langle B_\varphi\right\rangle\langle\omega_s\rangle + \left\langle\delta B_\varphi \delta\omega_s\right\rangle\right| < \left|\left\langle B_\varphi\right\rangle\langle\omega_s\rangle\right|,$$

which allows us to propose the model

$$-\frac{\lambda^2}{24s^2}\left\langle B_\varphi \frac{\partial U_z}{\partial\varphi}\right\rangle \approx -\varkappa\frac{\lambda^2}{24s}B\frac{\partial\left\langle U_\varphi\right\rangle}{\partial z} \equiv \alpha B,$$

where $0 < \varkappa < 1$ is an adjustable coefficient that approximately accounts for the effect of the neglected term $\langle B_\varphi \omega_s\rangle$. In spherical coordinates,

$$\alpha = \varkappa\frac{\lambda^2}{24r^2}\left[\partial_\theta\left\langle U_\varphi\right\rangle - r\cot\theta\partial_r\left\langle U_\varphi\right\rangle\right]. \tag{15}$$

By writing

$$\left\langle U_\varphi\right\rangle = \Omega(r,\theta)\, r\sin\theta, \tag{16}$$

with $\Omega(r,\theta)$ the large-scale rotation profile, we finally have

$$\alpha = \varkappa\frac{\lambda^2}{24\,r}\left(\sin\theta\partial_\theta\Omega - r\cos\theta\partial_r\Omega\right). \tag{17}$$

Concerning the first term in the third line of equation (14), we note that it can be written as

$$\frac{\lambda^2}{24r^2\sin^2\theta}\left\langle U_\varphi \partial_\varphi B_z\right\rangle = -\frac{\lambda^2}{24r^2\sin^2\theta}\left\langle B_z \partial_\varphi U_\varphi\right\rangle. \tag{18}$$

Consistently with the approximation made, we can at this point give a couple of arguments to neglect this term altogether. One argument is that the variation of U_φ along its own direction does not have a direct dynamic effect on the meridional magnetic field lines, and so we expect B_z and U_φ to be weakly correlated, so that $\left\langle B_z\,\partial_\varphi U_\varphi\right\rangle \approx \langle B_z\rangle\left\langle\partial_\varphi U_\varphi\right\rangle = 0$. An alternative argument is that using the stationary mass conservation equation (anelastic approximation), one can relate $\partial_\varphi U_\varphi$ to derivatives of the meridional velocity with respect to r and θ, in such a way that equation (18), which is of second order in ε, is written as a series of terms similar to those already present in the zeroth-order expression S^0_φ. In this way, we end up with the axisymmetric equation for the evolution of A

$$\frac{\partial A}{\partial t} = U_r B_\theta - U_\theta B_r + \eta\left(\nabla^2 A - \frac{A}{r^2\sin^2\theta}\right) + S^0_\varphi + \alpha B, \tag{19}$$

with S^0_φ and α given by equations (13) and (17), respectively, and with the convention that azimuthal averages are not explicitly written, so that all magnitudes are considered azimuthally averaged, represented by the same symbol as the original, non-averaged magnitude. The explicit expression of S^0_φ is

$$\begin{aligned} S^0_\varphi = \frac{\lambda^2}{24\,r^2}&\left[-B_r U_\theta - U_r\partial_\theta B_r - U_\theta\partial_\theta B_\theta + B_r\partial_\theta U_r\right.\\ &+ \partial_\theta B_\theta\partial_\theta U_r - \partial_\theta B_r\partial_\theta U_\theta + B_\theta\left(U_r + \partial_\theta U_\theta\right)\\ &\left.+ r^2\left(\partial_r B_\theta\partial_r U_r - \partial_r B_r\partial_r U_\theta\right)\right]. \end{aligned} \tag{20}$$

We now consider the axisymmetric model equation for the averaged azimuthal component B, whose subscale term $[\nabla\times\langle\boldsymbol{S}\rangle]\cdot\boldsymbol{e}_\varphi$ is expressed in terms of the components $\langle S_r\rangle$ and $\langle S_\theta\rangle$. An analysis similar to that of $\langle S_\varphi\rangle$ can also be done for these terms. If one neglects order ε^2 terms that are corrections to zeroth-order ones, there are five different ε^2-order terms left to be modelled: $\langle B_r\,\partial_\varphi U_r\rangle$, $\langle B_\theta\,\partial_\varphi U_\theta\rangle$, $\langle B_\varphi\,\partial_\varphi U_\varphi\rangle$, $\langle B_r\,\partial_\varphi U_\theta\rangle$, and $\langle B_\theta\,\partial_\varphi U_r\rangle$. For all these expressions, we have that the azimuthal variation of the velocity field in each average does not directly lead to twisting or bending of lines of a magnetic field with the direction of the field indicated in the same average. We thus expect a weak correlation between the factors inside each average, so that for a generic one of them, $\langle B_1\,\partial_\varphi U_2\rangle \approx \langle B_1\rangle\langle\partial_\varphi U_2\rangle = 0$. In this way, only the zeroth-order terms are retained in the axisymmetric model for the B equation, which is thus expressed as

$$\frac{\partial B}{\partial t} = \left[\nabla\times(\boldsymbol{U}\times\boldsymbol{B} - \eta\nabla\times\boldsymbol{B}) + \nabla\times\boldsymbol{S}^0\right]\cdot\boldsymbol{e}_\varphi, \tag{21}$$

where $\boldsymbol{S}^0$ is given by equation (9), and all fields in equations (9) and (21) are to be interpreted as axisymmetric, averaged ones, independent of φ. Explicitly, the SGST is

$$\left(\nabla\times\boldsymbol{S}^0\right)\cdot\boldsymbol{e}_\varphi = \frac{1}{r}\partial_r\left(rS^0_\theta\right) - \frac{1}{r}\partial_\theta S^0_r, \tag{22}$$

where, using equation (16) to express U_φ in terms of Ω,

$$\begin{aligned} S^0_r = \frac{\lambda^2}{24\,r^2}&\left[B\cot\theta\left(U_r + \cot\theta U_\theta\right) - \Omega r B_\theta\cos\theta\cot\theta\right.\\ &- r\sin\theta\,\partial_\theta B_\theta\partial_\theta\Omega - r\,B_r\left(2\Omega\cos\theta + \sin\theta\partial_\theta\Omega\right)\\ &- r^2\left(\Omega\sin\theta\,\partial_r B_\theta - \partial_r\,B\,\partial_r U_\theta + r\sin\theta\partial_r B_\theta\partial_r\Omega\right)\\ &\left.+ U_r\partial_\theta B - \Omega r\cos\theta\,\partial_\theta B_\theta + \partial_\theta B\,\partial_\theta U_\theta\right], \end{aligned} \tag{23}$$

and

$$\begin{aligned} S^0_\theta = \frac{\lambda^2}{24\,r^2}&\left[-B\left(U_r + \cot\theta U_\theta\right) + \Omega r B_r\sin\theta\right.\\ &+ \Omega r\cos\theta\partial_\theta B_r + U_\theta\partial_\theta B - \partial_\theta B\partial_\theta U_r\\ &+ r^2\left(\Omega\sin\theta\partial_r B_r - \partial_r B\partial_r U_r + r\sin\theta\partial_r B_r\,\partial_r\Omega\right)\\ &\left.- r B_\theta\sin\theta\partial_\theta\Omega + r\sin\theta\partial_\theta B_r\,\partial_\theta\Omega\right]. \end{aligned} \tag{24}$$

The explicit expression of equation (22) can be more easily employed by separating it in the part corresponding to the differential rotation and that corresponding to the meridional flow. The differential rotation part can be simplified using the relation $\nabla\cdot\boldsymbol{B} = 0$ to obtain

$$\begin{aligned} \left(\nabla\times\boldsymbol{S}^0\right)^\Omega\cdot\boldsymbol{e}_\varphi = \frac{\lambda^2}{24r^2}&\left[F_r B_r + F_\theta B_\theta + F_{r\theta}\partial_r B_\theta\right.\\ &\left.+ F_{\theta r}\partial_\theta B_r + F_{\theta\theta}\partial_\theta B_\theta\right], \end{aligned} \tag{25}$$

where

$$F_r = 3\cos\theta\partial_\theta\Omega + \sin\theta\left(\partial_{\theta\theta}\Omega + r\partial_r\Omega - 2r^2\partial_{rr}\Omega\right),$$

$$F_\theta = \frac{3+\cos 2\theta}{2}\csc\theta\partial_\theta\Omega - r\sin\theta\partial_{r\theta}\Omega - r^2\cos\theta\partial_{rr}\Omega,$$

$$F_{r\theta} = r^2\sin\theta\partial_{r\theta}\Omega,$$

$$F_{\theta r} = r\cos\theta\partial_r\Omega - \sin\theta\partial_\theta\Omega + r\sin\theta\partial_{r\theta}\Omega,$$

$$F_{\theta\theta} = \cos\theta\partial_\theta\Omega + \sin\theta\partial_{\theta\theta}\Omega - r^2\sin\theta\partial_{rr}\Omega.$$

The part coming from the meridional flow is directly obtained as

$$\left(\nabla\times \boldsymbol{S}^0\right)^U\cdot \boldsymbol{e}_\varphi = -\frac{\lambda^2}{24r^2}\left[G_r\partial_r B + G_\theta\partial_\theta B + G_{r\theta}\partial_{r\theta}B + G_{rr}\partial_{rr}B + G_{\theta\theta}\partial_{\theta\theta}B\right], \tag{26}$$

where

$$G_r = U_r + U_\theta\cot\theta + r\partial_r U_r + r^2\partial_{rr}U_r + r\partial_{r\theta}U_\theta,$$

$$G_\theta = \left(U_r\cot\theta + U_\theta\csc^2\theta + \partial_{\theta\theta}U_\theta\right)/r - \partial_r U_\theta + \partial_{r\theta}U_r,$$

$$G_{r\theta} = -U_\theta + \partial_\theta U_r + r\partial_r U_\theta,$$

$$G_{rr} = r^2\partial_r U_r,$$

$$G_{\theta\theta} = \left(U_r + \partial_\theta U_\theta\right)/r.$$

In the expression of S^0_φ, equation (20), the terms multiplying B_r and B_θ can be identified as the (φ, r) and (φ, θ) components of the α tensor, the α given by equation (17) being the (φ, φ) component. Similarly, the terms multiplying the derivatives of B_r and B_θ can be identified with the components of the magnetic diffusivity tensor. Likewise, the other components of the α and diffusivity tensors can be identified in expressions (23) and (24). In this way, once the scale λ of the filter is chosen, and the microscopic (still turbulent in general) diffusivity η is prescribed, the magnetic field dynamics is completely determined by the large-scale toroidal and meridional flows. In this respect, the approach presented can be considered as a kind of 'flux transport dynamo' (Choudhouri 2015).

4 EXAMPLE OF APPLICATION TO A SOLAR-LIKE DYNAMO

We have developed a mean-field $\alpha\Omega$ dynamo simulation in order to test the new formulation of α equation (17) and of the subgrid effects given by $\boldsymbol{S}^0$. As the meridional circulation is known to a lesser extent (Guerrero & Muñoz 2004; Belucz, Dikpati & Forgacs-Dajka 2015) than the rotation profile of the Sun, we have chosen in this work to test the subgrid effects without any meridional circulation, so that only equation (25) enters the model, while the other terms related to $\boldsymbol{S}^0$ are zero.

The parameter λ can be chosen considering that it should be large enough in order to have a smooth dependence of the averaged fields on the coordinate φ. While meridional space variation appears naturally due to the rotation of the star, the φ dependence can be associated with the presence of structures whose sizes determine the scales of zonal space variation. In this way, for the Sun we can take supergranules as indication of structures to be averaged out in order to obtain a smooth enough φ dependence. Considering that supergranules sizes are smaller than, approximately, 35 000 km (Srikanth, Singh & Raju 2000), we have taken λ to encompass about two such lengths, and so have fixed λ to $0.1R_t$ (R_t is the solar radius). Finally, the value of $\varkappa$ in expression (17) was taken as 0.4 in order to better fit the cycle duration.

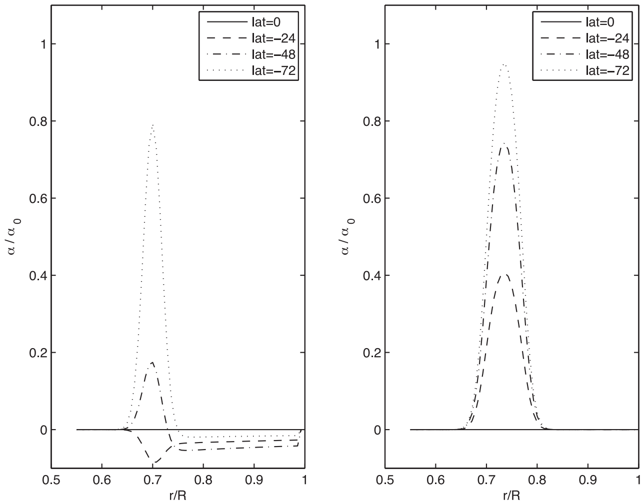

Figure 1. α radial profiles for different latitudes in the Southern hemisphere (values in the Northern hemisphere have the same magnitude, but opposite sign). Left: expression (17) for the Sun's rotation profile modelled by equation (29). Right: model given by equation (27). Both expressions are normalized to their respective maximum values.

Before considering the simulation itself, it is interesting to compare the α given by equation (17) to those usually employed in mean field dynamo simulations, like the expression given by

$$\alpha(r,\theta) = \alpha_0\cos(\theta)\frac{1}{4}\left[1 + erf\left(\frac{r-r_1}{d_1}\right)\right]\times\left[1 + erf\left(\frac{r-r_2}{d_2}\right)\right]. \tag{27}$$

By selecting the values of r_1 and r_2, the zone where the α-effect is concentrated can be imposed. Some authors using expression (27) set the maximum value of α near the surface (as a model for the Babcock–Leighton mechanism; Dikpati & Charbonneau 1999; Dikpati & Gilman 2001), while others employ an α-effect concentrated just above the tachocline (Parker 1993; Charbonneau 2010), in Cartesian geometry, and by Charbonneau & MacGregor (1997), in spherical geometry, or a combination of both (Passos et al. 2014). As a matter of fact, expression (17) with a rotation profile taken form helioseismology data is rather similar to α models acting mainly in the tachocline (Guerrero & de Gouveia Dal Pino 2008). In Fig. 1, the radial profile of two α coefficients is shown for different latitudes, one is equation (17) (α_1), with the profile $\Omega(r, \theta)$ given below in equation (29), and the other equation (27) (α_2) with $\alpha_0 = 5$, $r_1 = 0.7R_t$, $d_1 = d_2 = 0.025R_t$, $r_2 = 0.77R_t$ (Charbonneau 2010). Both profiles are quite similar, decreasing from pole to equator and with their maximum value near the tachocline. The main difference is appreciable for r/R_t above 0.8, where α_2 is zero, whereas α_1, although smaller than in the tachocline, is not zero.

In the simulation, we assume that the small-scale magnetic diffusivity in the convection zone is dominated by its turbulent contribution. Following the observed magnetic diffusivity as function of the scale (Chae, Litvinenko & Sakurai 2008), we take the diffusivity in the Sun convection zone to be $\eta_{SCZ} = 1\times10^6$ m^2 s^{-1}, which approximately corresponds to the diffusivity associated with mini-granular scales, well described by a Kolmogorov turbulence spectrum (Chae et al. 2008; Abramenko et al. 2012). In order to model the spatial transition between the non-turbulent diffusivity value η_{RZ} below the tachocline and its value in the convection zone,

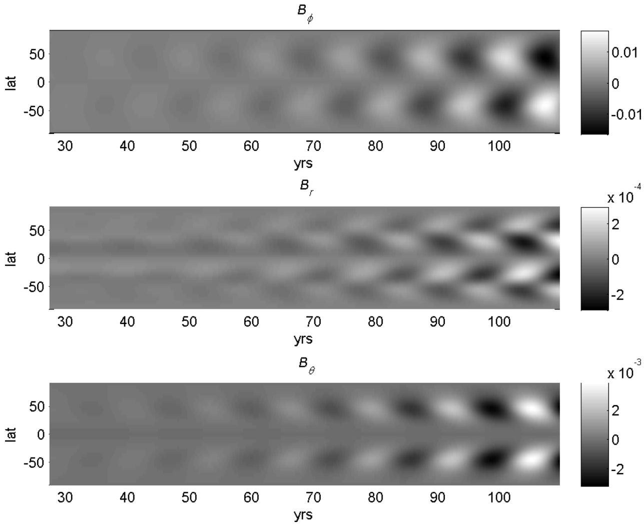

Figure 2. Magnetic field components as functions of latitude and time. Upper panel: azimuthal component near the tachocline. Middle and bottom panels: meridional components just below the surface.

we employ the functional form (Chatterjee, Nandy & Choudhouri 2004)

$$\eta\,(r) = \eta_{\rm RZ} + \frac{\eta_{\rm SCZ}}{2}\left[1 + \mathrm{erf}\left(\frac{r - r_{\rm BCZ}}{d_t}\right)\right], \tag{28}$$

where $\eta_{\rm RZ} = 2.2 \times 10^4\ \mathrm{m^2\ s^{-1}}$, $r_{\rm BCZ} = 0.7R_t$, and $d_t = 0.025R_t$.

The simulation also uses a differential rotation profile $\Omega(r, \theta)$ that fits the helioseismology data (Schou et al. 1998; Charbonneau et al. 1999), given by

$$\Omega\,(r, \theta) = \Omega_{\rm RZ} + \frac{1}{2}\left[1 + \mathrm{erf}\left(2\frac{r - r_t}{d_t}\right)\right]\left[\Omega_{\rm SCZ}\,(\theta) - \Omega_{\rm RZ}\right], \tag{29}$$

where $\Omega_{\rm SCZ}(\theta) = \Omega_{\rm EQ} + a_2\cos^2(\theta) + a_4\cos^4(\theta)$ is the surface latitudinal rotation. The value of the angular velocity of the rigidly rotating core is $\Omega_{\rm RZ} = 2\pi \times 432.8$ nHz. The other parameters are set to $r_t = 0.7R_t$, $d_t = 0.05R_t$, $\Omega_{\rm EQ} = 2\pi \times 460.7$ nHz, $a_2 = -62.69$ nHz, and $a_4 = -67.13$ nHz.

As an example of the model behaviour, we present the result of a simulation modelling 90 yr of magnetic field evolution in Fig. 2. In this figure, we can appreciate the polarity reversals in all the components of the magnetic field, although the cycle length is smaller than the observed one by approximately 7 yr. Another important observation is that despite not including meridional circulation, the field shows a shift of its maximum magnitude from middle latitudes towards the equator, as well and as a poleward migration of the radial field at high latitudes. There is also a phase lag between B_r and B_φ leading to a negative correlation between them, as observed.

The magnitude of all fields grows with time, which is understandable as in the simulations presented neither the magnetic field nor the α values were saturated. Simulations where the α value is modified depending on the value of the toroidal field (α-quenching) have also been performed, showing stable amplitudes of the magnetic field cycles as well as correct cycle durations, but since these computations involve additional modelling, not derived from the formalism used, we have not included them in this work.

5 CONCLUSIONS

We have presented a large-scale, axisymmetric, kinematic dynamo model in which only a small number of variables need to be parametrized, namely the small-scale magnetic diffusivity, the scale λ of the fields spatial average, and a factor $0 < \varkappa < 1$ in the expression of the α term (more precisely the $\alpha_{\varphi\varphi}$ component of the α tensor). Explicit expressions of what corresponds to the other components of the α tensor and of the diffusivity tensor are given in terms of the large-scale meridional flow and of the differential rotation law, without introducing additional parameters. The model was derived from a self-consistent approach applicable to general continuous systems which only requires for its application that the averaged fields be smooth, what can be checked from the solution itself. A very simple simulation of a solar-like dynamo, employing the model without meridional flow and without any saturation effects, shows reasonable magnetic field evolution, with a cycle duration of about 2/3 that of the sun, shift of the magnetic field from mid-latitudes towards the equator, poleward migration of the radial field at high latitudes, and correct phase relation between radial and azimuthal components. The explicit formulation of the dynamo-related effects in terms of the mean flows (meridional and zonal) allows in principle to explore directly their effect on dynamo action, as well as to eventually include back reaction of the magnetic field on the flow to study possible self-regulating mechanisms, saturation effects, etc.

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