Published for SISSA by Springer

Received: *May 10, 2021*
Accepted: *June 15, 2021*
Published: *July 15, 2021*

Supersymmetry, T-duality and heterotic α'-corrections

Eric Lescano,[a] **Carmen A. Núñez**[a,b] **and Jesús A. Rodríguez**[b]

[a] *Instituto de Astronomía y Física del Espacio (IAFE-CONICET-UBA), Ciudad Universitaria, Pabellón IAFE, 1428 Buenos Aires, Argentina*

[b] *Departamento de Física, FCEyN, Universidad de Buenos Aires (UBA), Ciudad Universitaria, Pabellón 1, 1428 Buenos Aires, Argentina*

E-mail: elescano@iafe.uba.ar, carmen@iafe.uba.ar, jarodriguez@df.uba.ar

Abstract: Higher-derivative interactions and transformation rules of the fields in the effective field theories of the massless string states are strongly constrained by space-time symmetries and dualities. Here we use an exact formulation of ten dimensional $\mathcal{N} = 1$ supergravity coupled to Yang-Mills with manifest T-duality symmetry to construct the first order α'-corrections of the heterotic string effective action. The theory contains a supersymmetric and T-duality covariant generalization of the Green-Schwarz mechanism that determines the modifications to the leading order supersymmetry transformation rules of the fields. We compute the resulting field-dependent deformations of the coefficients in the supersymmetry algebra and construct the invariant action, with up to and including four-derivative terms of all the massless bosonic and fermionic fields of the heterotic string spectrum.

Keywords: String Duality, Superstrings and Heterotic Strings, Supersymmetry and Duality

ArXiv ePrint: 2104.09545

https://doi.org/10.1007/JHEP07(2021)092

JHEP07(2021)092

Contents

1 Introduction **1**

2 The leading order theory **3**
2.1 Review of $\mathcal{N} = 1$ supersymmetric double field theory 4
2.2 Parameterization and choice of section 8

3 The first order α'-corrections **10**
3.1 The generalized Bergshoeff-de Roo identification 10
3.2 Induced transformation rules on $\mathrm{O}(10, 10)$ multiplets 13
3.3 Including the heterotic gauge sector 15
3.4 First order corrections to $\mathcal{N} = 1$ supersymmetric DFT 16

4 Transformation rules of the supergravity fields **17**

5 Heterotic string effective action to $\mathcal{O}(\alpha')$ **23**

6 Outlook and final remarks **25**

A Conventions and definitions **27**
A.1 Some useful gamma function identities 28
A.2 Leading order components of the generalized fluxes 28
A.3 The leading order action and equations of motion 29

B Algebra of transformations of $\mathrm{O}(10, 10 + n_g)$ fields **30**
B.1 Leading order algebra 30
B.2 First order algebra 33

C Supersymmetry of heterotic string effective action **36**
C.1 Supersymmetry algebra 37
C.2 Invariance of the action 38

1 Introduction

At low energy, or small curvature, heterotic string theory reduces to ten dimensional $\mathcal{N} = 1$ supergravity coupled to super Yang-Mills [1]. Successive terms in the α'-expansion may be expressed as higher-derivative interactions that are strongly constrained by the symmetries of string theory. There are several reasons to study the higher-order terms in the effective field theories of the massless string modes. They are needed to evaluate the stringy effects on solutions to the supergravity equations of motion [2–4], they play a central role in the tests of duality conjectures [5, 6], in the microstate counting of black hole entropy [7–9] and in moduli stabilization [10].The swampland program [11] has revealed that the effective field theories of low energy physics and cosmology are limited by their couplings to quantum gravity [12–14], and together with the string lamppost principle [15], reinforces the interest in the restrictions imposed by string theory on the higher-derivative corrections to General Relativity.

The first few orders of the heterotic string α'-expansion are known explicitly. The interactions of the bosonic fields up to $\mathcal{O}(\alpha'^3)$ were originally determined from the computation of scattering amplitudes of the massless string states at tree [1, 16–18] and one loop [19–22] levels in the string coupling and from conformal anomaly cancellations [23]. The contributions of the fermionic fields have been computed using supersymmetry and superspace methods [24–37]. Supersymmetry completely fixes the leading order terms [24] and it often provides an elegant underlying explanation of the higher-derivative corrections. But it holds iteratively in powers of α' and the transformation rules of the fields demand order by order modifications that are further restricted by other string symmetries and dualities.

In particular, the effective field theories for the massless string fields exhibit a global $\mathrm{O}(n, n; \mathbb{R})$ symmetry when the fields are independent of n spatial coordinates. This continuous T-duality symmetry holds to all orders in α' [38] (see also [39–47]) and it has been explicitly displayed recently for the quadratic and some of the quartic interactions of the bosonic fields in [48, 49]. This feature motivated the construction of field theories with T-duality covariant structures, such as double field theory (DFT) [50–56] and generalized geometry [57, 58], which provide reformulations of the string (super)gravities in which the global duality invariance is made manifest.

In the duality covariant frameworks, the standard local symmetries are generalized to larger groups: diffeomorphism invariance is extended to also include the gauge transformations of the two-form and the tangent space is enhanced with an extended Lorentz symmetry. Interestingly, the duality covariant gauge transformations completely determine the lowest order field interactions in string (super)gravities even before dimensional reduction (for reviews see [59–64] and references therein). Moreover, extensions of the duality group [65, 66] as well as enhancings of the gauge structure of DFT [67, 68] allowed to reproduce the four-derivative interactions of the massless bosonic heterotic string fields.

Supersymmetry can be naturally incorporated in the duality covariant formulations [69–76]. A supersymmetric and manifestly $\mathrm{O}(10, 10 + n_g)$ covariant DFT reformulation of ten dimensional $\mathcal{N} = 1$ supergravity coupled to n_g abelian vector multiplets was introduced in [70–73]. Although it is formally constructed on a $20 + n_g$ dimensional space-time, the apparent inconsistency of supergravity beyond eleven dimensions is avoided through a strong constraint that admits solutions removing the field dependence on $10 + n_g$ coordinates, and fermions transform as spinors under the $\mathrm{O}(9, 1)_{\mathrm{L}}$ factor of the local $\mathrm{O}(9, 1)_{\mathrm{L}} \times \mathrm{O}(1, 9 + n_g)_{\mathrm{R}}$ double Lorentz symmetry.

More recently, an exact supersymmetric and manifestly duality covariant mechanism was introduced in [76], in which the global symmetry of the theory is taken to be $\mathrm{O}(D, D + k)$, k being the dimension of the $\mathrm{O}(1, D + k - 1)$ Lorentz group. To pre-

JHEP07(2021)092

serve duality covariance, the $\mathrm{O}(D, D+k)$ multiplets are parameterized with elements of $\mathrm{O}(D, D)$. Additionally identifying the $\mathrm{O}(D, D)$ vector with the generalized spin connection of $\mathrm{O}(D, D+k)$, the construction produces an exact supersymmetric and duality covariant generalization of the Green-Schwarz transformation, which requires an infinite tower of $\mathrm{O}(D, D)$ covariant higher-derivative terms in the gauge invariant action.

With the motivation to further understand the structure of the heterotic string α'-expansion, in this paper we perform a perturbative expansion of the formal exact construction of [76] and obtain the first order corrections to $\mathcal{N} = 1$ supersymmetric DFT. Further parameterizing the duality multiplets in terms of supergravity and super Yang-Mills multiplets, we show that the supersymmetric duality covariant generalized Green-Schwarz transformation completely fixes the first order deformations of the transformation rules of the fields. We also construct the invariant action with up to and including four-derivative terms of all the massless bosonic and fermionic fields of the heterotic string and up to bilinear terms in fermions.

The paper is organized as follows. In section 2 we review the basic features of the $\mathcal{N} = 1$ supersymmetric DFT introduced in [71–73] and we trivially extend it to incorporate non-abelian gauge vectors. In section 3, after briefly recalling the relevant aspects of the duality covariant mechanism proposed in [76], we extract the first order corrections to the transformation rules of the $\mathrm{O}(10, 10+n_g)$ generalized fields from those of the $\mathrm{O}(10, 10+k)$ multiplets, and obtain the manifestly duality covariant and gauge invariant $\mathcal{N} = 1$ supersymmetric DFT action to $\mathcal{O}(\alpha')$. We then parameterize the $\mathrm{O}(10, 10+n_g)$ fields in terms of supergravity and super Yang-Mills multiplets in section 4 and find the relations between the duality and the local gauge covariant structures. We discuss the deformations induced from the generalized Green-Schwarz transformation on the transformation rules of the supergravity fields and compare with previous results in the literature. Finally, in section 5 we present the first order α'-corrections of the heterotic string effective action including up to bilinear terms in fermions. Conclusions are the subject of section 6. The conventions used throughout the paper and some useful gamma function identities are included in appendix A. Details of the proof of closure of the symmetry algebra on the duality multiplets are contained in appendix B. Finally, in appendix C we compute the deformed supersymmetry algebra on the supergravity multiplets and prove the supersymmetric invariance of the first order corrections in the heterotic string effective action.

2 The leading order theory

In this section we review the basic features of the DFT reformulation of $\mathcal{N} = 1$ supergravity coupled to n_g vector multiplets in ten dimensions that was introduced in [71–73], mainly to establish the notation. The frame formalism used in [77, 78] is most useful to achieve a manifestly $\mathrm{O}(10, 10+n_g)$ covariant rewriting of heterotic supergravity truncated to the Cartan subalgebra of $\mathrm{SO}(32)$ or $\mathrm{E}_8 \times \mathrm{E}_8$ for $n_g = 16$. Employing gauged DFT [79], we further include the full set of non-abelian gauge fields and recover the leading order terms of heterotic supergravity.

JHEP07(2021)092

2.1 Review of $\mathcal{N} = 1$ supersymmetric double field theory

$\mathcal{N} = 1$ supersymmetric Double Field Theory is defined on a space with coordinates $X^{\mathbb{M}}$ belonging to the fundamental representation of G$=$ O$(10, 10 + n_g|\mathbb{R})$, with $\mathbb{M} = (M, i)$, $M = 0, \ldots, 19; i = 1, \ldots, n_g$, and n_g is the dimension of the gauge group. The theory has a global G symmetry, a local double Lorentz H $=$ O$(9, 1|\mathbb{R})_{\rm L} \times$ O$(1, 9 + n_g|\mathbb{R})_{\rm R}$ symmetry, diffeomorphisms generated infinitesimally by $\xi^{\mathbb{M}}$ through a generalized Lie derivative $\hat{\mathcal{L}}_\xi$ and supersymmetry parameterized by an infinitesimal Majorana fermion ϵ transforming as a spinor of O$(9, 1)_{\rm L}$. The propagating degrees of freedom are:

- $E^{\mathbb{M}}{}_{\mathbb{A}}$: a generalized vielbein parameterizing the coset $\frac{\rm G}{\rm H} = \frac{{\rm O}(10,10+n_g)}{{\rm O}(9,1)_{\rm L}\times {\rm O}(1,9+n_g)_{\rm R}}$, with tangent space indices $\mathbb{A} = (\underline{A}, \overline{A})$ splitting into O$(9, 1)_{\rm L}$ and O$(1, 9 + n_g)_{\rm R}$ vector indices, $\underline{A} = 0, \ldots, 9$ and $\overline{A} = 0, \ldots, 9 + n_g$, respectively,

- d: an O$(10, 10 + n_g)$ scalar dilaton,

- $\Psi_{\overline{A}}$: a Majorana spinor generalized gravitino, transforming as a spinor of O$(9, 1)_{\rm L}$, as a vector of O$(1, 9 + n)_{\rm R}$, and as a scalar of O$(10, 10 + n_g)$,

- ρ: a Majorana spinor 'dilatino', transforming as a spinor of O$(9, 1)_{\rm L}$ and as a scalar of O$(10, 10 + n_g)$.

The group invariant symmetric and invertible O$(10, 10 + n_g)$ metric is

$$\eta_{\mathbb{MN}} = \begin{pmatrix} \eta^{\mu\nu} & \eta^{\mu}{}_{\nu} & \eta^{\mu}{}_{i} \\ \eta_{\mu}{}^{\nu} & \eta_{\mu\nu} & \eta_{\mu i} \\ \eta_i^{\nu} & \eta_{i\nu} & \eta_{ij} \end{pmatrix} = \begin{pmatrix} 0 & \delta^{\mu}{}_{\nu} & 0 \\ \delta_{\mu}{}^{\nu} & 0 & 0 \\ 0 & 0 & \kappa_{ij} \end{pmatrix} , \tag{2.1}$$

with $\mu, \nu = 0, \ldots, 9$, $i, j = 1, \ldots, n_g$ and κ_{ij} the Killing metric of the gauge group. There are two constant symmetric and invertible H-invariant metrics $\eta_{\mathbb{AB}}$ and $H_{\mathbb{AB}}$. The former is used to raise and lower the indices that are rotated by H and the latter is constrained to satisfy $H_{\mathbb{A}}{}^{\mathbb{C}} H_{\mathbb{C}}{}^{\mathbb{B}} = \delta^{\mathbb{B}}_{\mathbb{A}}$. The three metrics $\eta_{\mathbb{MN}}, \eta_{\mathbb{AB}}$ and $H_{\mathbb{AB}}$ are invariant under the action of $\hat{\mathcal{L}}$, G and H.

The generalized vielbein $E^{\mathbb{M}}{}_{\mathbb{A}}$ is constrained to relate the metrics $\eta_{\mathbb{AB}}$ and $\eta_{\mathbb{MN}}$ and defines a generalized metric $H_{\mathbb{MN}}$ from $H_{\mathbb{AB}}$

$$\eta_{\mathbb{AB}} = E^{\mathbb{M}}{}_{\mathbb{A}} \eta_{\mathbb{MN}} E^{\mathbb{N}}{}_{\mathbb{B}} , \quad H_{\mathbb{MN}} = E_{\mathbb{M}}{}^{\mathbb{A}} H_{\mathbb{AB}} E_{\mathbb{N}}{}^{\mathbb{B}} . \tag{2.2}$$

$H_{\mathbb{MN}}$ is also an element of O$(10, 10 + n_g)$, constrained as

$$H_{\mathbb{MP}} \eta^{\mathbb{PQ}} H_{\mathbb{QN}} = \eta_{\mathbb{MN}} , \qquad H_{\mathbb{AC}} \eta^{\mathbb{CD}} H_{\mathbb{DB}} = \eta_{\mathbb{AB}} . \tag{2.3}$$

It is convenient to define the projectors

$$P_{\mathbb{MN}} = \frac{1}{2} \left(\eta_{\mathbb{MN}} - H_{\mathbb{MN}}\right) \ \text{ and } \ \overline{P}_{\mathbb{MN}} = \frac{1}{2} \left(\eta_{\mathbb{MN}} + H_{\mathbb{MN}}\right) , \tag{2.4}$$

satisfying the usual properties

$$\overline{P}_{\mathbb{MQ}} \overline{P}^{\mathbb{Q}}{}_{\mathbb{N}} = \overline{P}_{\mathbb{MN}} , \quad P_{\mathbb{MQ}} P^{\mathbb{Q}}{}_{\mathbb{N}} = P_{\mathbb{MN}}, \quad P_{\mathbb{MQ}} \overline{P}^{\mathbb{Q}}{}_{\mathbb{N}} = \overline{P}_{\mathbb{MQ}} P^{\mathbb{Q}}{}_{\mathbb{N}} = 0 , \quad \overline{P}_{\mathbb{MN}} + P_{\mathbb{MN}} = \eta_{\mathbb{MN}} ,$$

JHEP07(2021)092

and related with the generalized vielbein in the following way

$$P_{\underline{AB}} = E_{\mathbb{M}\underline{A}}E^{\mathbb{M}}{}_{\underline{B}}\,, \quad \overline{P}_{\overline{AB}} = E_{\mathbb{M}\overline{A}}E^{\mathbb{M}}{}_{\overline{B}}\,, \quad P_{\mathbb{MN}} = E_{\mathbb{M}\underline{A}}E_{\mathbb{N}}{}^{\underline{A}}, \quad \overline{P}_{\mathbb{MN}} = E_{\mathbb{M}\overline{A}}E_{\mathbb{N}}{}^{\overline{A}}\,. \tag{2.5}$$

We use the convention that $P_{\underline{AB}}$, $\overline{P}_{\overline{AB}}$ and their inverse lower and raise projected indices.

The generalized Lie derivative acts as

$$\delta_\xi E^{\mathbb{M}}{}_{\mathbb{A}} = \hat{\mathcal{L}}_\xi E^{\mathbb{M}}{}_{\mathbb{A}} = \xi^{\mathbb{N}}\partial_{\mathbb{N}}E^{\mathbb{M}}{}_{\mathbb{A}} + (\partial^{\mathbb{M}}\xi_{\mathbb{N}} - \partial_{\mathbb{N}}\xi^{\mathbb{M}})E^{\mathbb{N}}{}_{\mathbb{A}} + f^{\mathbb{M}}{}_{\mathbb{NP}}\xi^{\mathbb{N}}E^{\mathbb{P}}{}_{\mathbb{A}}, \tag{2.6a}$$

$$\delta_\xi \Psi_{\overline{A}} = \hat{\mathcal{L}}_\xi \Psi_{\overline{A}} = \xi^{\mathbb{M}}\partial_{\mathbb{M}}\Psi_{\overline{A}} \tag{2.6b}$$

$$\delta_\xi d \;= \hat{\mathcal{L}}_\xi d \;= \xi^{\mathbb{M}}\partial_{\mathbb{M}} d - \frac{1}{2}\partial_{\mathbb{M}}\xi^{\mathbb{M}}\,, \qquad \delta_\xi \rho = \hat{\mathcal{L}}_\xi \rho = \xi^{\mathbb{M}}\partial_{\mathbb{M}}\rho\,, \tag{2.6c}$$

where the partial derivatives $\partial_{\mathbb{M}}$ belong to the fundamental representation of $\mathrm{O}(10, 10+n_g)$ and the so-called fluxes or gaugings $f_{\mathbb{MNP}}$ are a set of constants [77, 78] verifying linear and quadratic constraints

$$f_{\mathbb{MNP}} = f_{[\mathbb{MNP}]}\,, \qquad f_{[\mathbb{MN}}{}^{\mathbb{R}} f_{\mathbb{P}]\mathbb{R}}{}^{\mathbb{Q}} = 0\,. \tag{2.7}$$

Consistency of the construction requires constraints which restrict the coordinate dependence of fields and gauge parameters. The strong constraint

$$\partial_{\mathbb{M}}\partial^{\mathbb{M}}\cdots = 0\,, \qquad \partial_{\mathbb{M}}\cdots\ \partial^{\mathbb{M}}\cdots = 0\,, \qquad f_{\mathbb{MN}}{}^{\mathbb{P}}\partial_{\mathbb{P}}\cdots = 0\,, \tag{2.8}$$

where $\cdots$ refers to products of fields, will be assumed throughout. This constraint locally removes the field dependence on $10 + n_g$ coordinates, so that fermions can be effectively defined in a 10-dimensional tangent space.[1]

The local $\mathrm{O}(9,1)_{\mathrm{L}} \times O(1, 9 + n_g)_{\mathrm{R}}$ double Lorentz symmetry is parameterized by an infinitesimal parameter $\Gamma_{\mathbb{AB}}$ satisfying

$$\Gamma_{\mathbb{AB}} = -\Gamma_{\mathbb{BA}}\,, \tag{2.9}$$

in order to preserve the invariance of $\eta_{\mathbb{AB}}$ and $H_{\mathbb{AB}}$. The two projections of a generic vector $V^{\mathbb{A}} = V^{\underline{A}} + V^{\overline{A}}$ transform as

$$\delta_\Gamma V^{\underline{A}} = V^{\underline{B}}\Gamma_{\underline{B}}{}^{\underline{A}}\,, \qquad \delta_\Gamma V^{\overline{A}} = V^{\overline{B}}\Gamma_{\overline{B}}{}^{\overline{A}}\,, \tag{2.10}$$

where the $\Gamma_{\underline{A}}{}^{\underline{B}}$ and $\Gamma_{\overline{A}}{}^{\overline{B}}$ components generate the $\mathrm{O}(9,1)_{\mathrm{L}}$ and $O(1, 9+n_g)_{\mathrm{R}}$ transformations leaving $P_{\underline{AB}}$ and $\overline{P}_{\overline{AB}}$ invariant, respectively, and $\delta_\Lambda H_{\mathbb{AB}} = 0$ implies $\Gamma_{\underline{A}\overline{B}} = 0$.

The fields transform under double Lorentz variations as

$$\delta_\Gamma E^{\mathbb{M}}{}_{\mathbb{A}} = E^{\mathbb{M}}{}_{\mathbb{B}}\Gamma^{\mathbb{B}}{}_{\mathbb{A}}\,, \qquad \delta_\Gamma \Psi_{\overline{A}} = \Psi_{\overline{B}}\Gamma^{\overline{B}}{}_{\overline{A}} + \frac{1}{4}\Gamma_{\underline{BC}}\gamma^{\underline{BC}}\Psi_{\overline{A}}\,, \qquad \delta_\Gamma \rho = \frac{1}{4}\Gamma_{\underline{BC}}\gamma^{\underline{BC}}\rho\,, \tag{2.11}$$

where the $\mathrm{O}(9,1)_{\mathrm{L}}$ gamma matrices can be chosen to be conventional gamma matrices in ten dimensions, satisfying

$$\left\{\gamma_{\underline{A}}, \gamma_{\underline{B}}\right\} = -2P_{\underline{AB}}\,. \tag{2.12}$$

Some useful identities for the product of gamma matrices are listed in appendix A.1.

[1] A supersymmetric DFT without the strong constraint was obtained through a generalized Scherk-Schwarz reduction in [74].

JHEP07(2021)092

The Lorentz and space-time covariant derivatives act on generic vectors as

$$\nabla_{\mathbb{A}} V_{\mathbb{B}} = E_{\mathbb{A}} V_{\mathbb{B}} + \omega_{\mathbb{A}\mathbb{B}}{}^{\mathbb{C}} V_{\mathbb{C}} \,, \qquad \nabla_{\mathbb{M}} V_{\mathbb{A}} = \partial_{\mathbb{M}} V_{\mathbb{A}} + \omega_{\mathbb{M}\mathbb{A}}{}^{\mathbb{B}} V_{\mathbb{B}} \,, \tag{2.13}$$

with $E_{\mathbb{A}} \equiv \sqrt{2} E_{\mathbb{A}}{}^{\mathbb{M}} \partial_{\mathbb{M}}$, implying $\omega_{[\mathbb{A}\mathbb{B}\mathbb{C}]} = \sqrt{2}\omega_{\mathbb{M}[\mathbb{A}\mathbb{B}} E^{\mathbb{M}}{}_{\mathbb{C}]}$.

Only the totally antisymmetric and trace parts of $\omega_{\mathbb{A}\mathbb{B}\mathbb{C}}$ can be determined in terms of $E^{\mathbb{M}}{}_{\mathbb{A}}$ and d, namely

$$\omega_{[\mathbb{A}\mathbb{B}\mathbb{C}]} = -E_{[\mathbb{A}} E^{\mathbb{N}}{}_{\mathbb{B}} E_{\mathbb{N}\mathbb{C}]} - \frac{\sqrt{2}}{3} f_{\mathbb{M}\mathbb{N}\mathbb{P}} E^{\mathbb{M}}{}_{\mathbb{A}} E^{\mathbb{N}}{}_{\mathbb{B}} E^{\mathbb{P}}{}_{\mathbb{C}} \equiv -\frac{1}{3} F_{\mathbb{A}\mathbb{B}\mathbb{C}} \,, \tag{2.14}$$

$$\omega_{\mathbb{B}\mathbb{A}}{}^{\mathbb{B}} = -\sqrt{2} e^{2d} \partial_{\mathbb{M}} \left(E^{\mathbb{M}}{}_{\mathbb{A}} e^{-2d} \right) \equiv -F_{\mathbb{A}} \,, \tag{2.15}$$

the latter arising from partial integration with the dilaton density

$$\int e^{-2d} V \nabla_{\mathbb{A}} V^{\mathbb{A}} = -\int e^{-2d} V^{\mathbb{A}} \nabla_{\mathbb{A}} V \,, \tag{2.16}$$

for arbitrary V and $V^{\mathbb{A}}$. Only the combinations with the same projection on the last two indices are non-vanishing.

The covariant derivatives of the (adjoint) gravitino and dilatino are

$$\nabla_{\mathbb{A}} \Psi_{\overline{B}} = E_{\mathbb{A}} \Psi_{\overline{B}} + \omega_{\mathbb{A}\overline{B}}{}^{\overline{C}} \Psi_{\overline{C}} - \frac{1}{4} \omega_{\mathbb{A}\underline{BC}} \gamma^{\underline{BC}} \Psi_{\overline{B}} \,, \tag{2.17a}$$

$$\nabla_{\mathbb{A}} \overline{\Psi}_{\overline{B}} = E_{\mathbb{A}} \overline{\Psi}_{\overline{B}} + \omega_{\mathbb{A}\overline{B}}{}^{\overline{C}} \overline{\Psi}_{\overline{C}} + \frac{1}{4} \omega_{\mathbb{A}\underline{BC}} \overline{\Psi}_{\overline{B}} \gamma^{\underline{BC}} \,, \tag{2.17b}$$

$$\nabla_{\mathbb{A}} \rho = E_{\mathbb{A}} \rho - \frac{1}{4} \omega_{\mathbb{A}\underline{BC}} \gamma^{\underline{BC}} \rho \,, \quad \nabla_{\mathbb{A}} \bar{\rho} = E_{\mathbb{A}} \bar{\rho} + \frac{1}{4} \omega_{\mathbb{A}\underline{BC}} \, \bar{\rho} \gamma^{\underline{BC}} \,. \tag{2.17c}$$

The supersymmetry transformation rules are parameterized by an infinitesimal Majorana fermion ϵ transforming as a spinor of $\mathrm{O}(1,9)_{\mathrm{L}}$

$$\delta_\epsilon E^{\mathbb{M}}{}_{\underline{A}} = -\frac{1}{2} \bar{\epsilon} \gamma_{\underline{A}} \Psi_{\overline{B}} E^{\mathbb{M}\overline{B}} \,, \qquad \delta_\epsilon E^{\mathbb{M}}{}_{\overline{A}} = \frac{1}{2} \bar{\epsilon} \gamma_{\underline{B}} \Psi_{\overline{A}} E^{\mathbb{M}\underline{B}} \,, \qquad \delta_\epsilon d = -\frac{1}{4} \bar{\epsilon} \rho \,, \tag{2.18a}$$

$$\delta_\epsilon \Psi_{\overline{A}} = \nabla_{\overline{A}} \epsilon \,, \qquad \delta_\epsilon \rho = -\gamma^{\underline{A}} \nabla_{\underline{A}} \epsilon \,. \tag{2.18b}$$

Putting all together, the generalized fields obey the transformation rules

$$\delta E^{\mathbb{M}}{}_{\underline{A}} = \hat{\mathcal{L}}_\xi E^{\mathbb{M}}{}_{\underline{A}} + E^{\mathbb{M}}{}_{\underline{B}} \Gamma^{\underline{B}}{}_{\underline{A}} - \frac{1}{2} \bar{\epsilon} \gamma_{\underline{A}} \Psi_{\overline{B}} E^{\mathbb{M}\overline{B}} \,, \tag{2.19a}$$

$$\delta E^{\mathbb{M}}{}_{\overline{A}} = \hat{\mathcal{L}}_\xi E^{\mathbb{M}}{}_{\overline{A}} + E^{\mathbb{M}}{}_{\overline{B}} \Gamma^{\overline{B}}{}_{\overline{A}} + \frac{1}{2} \bar{\epsilon} \gamma_{\underline{B}} \Psi_{\overline{A}} E^{\mathbb{M}\underline{B}} \,, \tag{2.19b}$$

$$\delta d = \xi^{\mathbb{P}} \partial_{\mathbb{P}} d - \frac{1}{2} \partial_{\mathbb{P}} \xi^{\mathbb{P}} - \frac{1}{4} \bar{\epsilon} \rho \,, \tag{2.19c}$$

$$\delta \Psi_{\overline{A}} = \xi^{\mathbb{M}} \partial_{\mathbb{M}} \Psi_{\overline{A}} + \Gamma^{\overline{B}}{}_{\overline{A}} \Psi_{\overline{B}} + \frac{1}{4} \Gamma_{\underline{BC}} \gamma^{\underline{BC}} \Psi_{\overline{A}} + \nabla_{\overline{A}} \epsilon \,, \tag{2.19d}$$

$$\delta \rho = \xi^{\mathbb{M}} \partial_{\mathbb{M}} \rho + \frac{1}{4} \Gamma_{\underline{BC}} \gamma^{\underline{BC}} \rho - \gamma^{\underline{A}} \nabla_{\underline{A}} \epsilon \,. \tag{2.19e}$$

In appendix B.1 we review the algebra of these transformations, and show that it closes up to terms with two fermions, with the following parameters

$$\xi_{12}^{\mathbb{M}} = [\xi_1, \xi_2]^{\mathbb{M}}_{(C_f)} - \frac{1}{\sqrt{2}} E^{\mathbb{M}}{}_{\underline{A}} \bar{\epsilon}_1 \gamma^{\underline{A}} \epsilon_2, \tag{2.20a}$$

$$\Gamma_{12\mathbb{A}\mathbb{B}} = 2\xi^{\mathbb{P}}_{[1} \partial_{\mathbb{P}} \Gamma_{2]\mathbb{A}\mathbb{B}} - 2\Gamma_{[1\mathbb{A}}{}^{\mathbb{C}} \Gamma_{2]\mathbb{C}\mathbb{B}} + E_{[\mathbb{A}} \left(\bar{\epsilon}_1 \gamma_{\mathbb{B}]} \epsilon_2\right) - \frac{1}{2} \left(\bar{\epsilon}_1 \gamma^{\underline{C}} \epsilon_2\right) F_{\mathbb{A}\mathbb{B}\underline{C}}, \tag{2.20b}$$

$$\epsilon_{12} = -\frac{1}{2} \Gamma_{[1\underline{BC}} \gamma^{\underline{BC}} \epsilon_{2]} + 2\xi^{\mathbb{P}}_{[1} \partial_{\mathbb{P}} \epsilon_{2]} \,, \tag{2.20c}$$

where the C_f-bracket is defined as

$$[\xi_1, \xi_2]^{\mathbb{M}}_{(C)} = 2\xi^{\mathbb{P}}_{[1} \partial_{\mathbb{P}} \xi^{\mathbb{M}}_{2]} - \xi^{\mathbb{N}}_{[1} \partial^{\mathbb{M}} \xi_{2]\mathbb{N}} + f_{\mathbb{P}\mathbb{Q}}{}^{\mathbb{M}} \xi_1^{\mathbb{P}} \xi_2^{\mathbb{Q}} \,. \tag{2.21}$$

The transformation rules (2.19) leave the following action invariant, up to bilinear terms in fermions,

$$\mathbb{S}_{\mathcal{N}=1 \text{ DFT}} = \int d^{20+n_g} X \, e^{-2d} \left(\mathbb{L}_{\rm B} + \mathbb{L}_{\rm F}\right) \,, \tag{2.22}$$

where $\mathbb{L}_{\rm B}$ is the generalized Ricci scalar, which can be written as

$$\mathbb{L}_{\rm B} \equiv \mathbb{R} = \frac{1}{8} F_{\mathbb{A}\mathbb{B}\mathbb{C}} F_{\mathbb{D}\mathbb{E}\mathbb{F}} \left(H^{\mathbb{A}\mathbb{D}} \eta^{\mathbb{B}\mathbb{E}} \eta^{\mathbb{C}\mathbb{F}} - \frac{1}{3} H^{\mathbb{A}\mathbb{D}} H^{\mathbb{B}\mathbb{E}} H^{\mathbb{C}\mathbb{F}}\right) - H^{\mathbb{A}\mathbb{B}} \left(\frac{1}{2} F_{\mathbb{A}} F_{\mathbb{B}} + E_{\mathbb{A}} F_{\mathbb{B}}\right) \,,$$

up to terms that vanish under the strong constraint, and the fermionic Lagrangian is

$$\mathbb{L}_{\rm F} = \bar{\Psi}^{\overline{A}} \gamma^{\underline{B}} \nabla_{\underline{B}} \Psi_{\overline{A}} - \bar{\rho} \gamma^{\underline{A}} \nabla_{\underline{A}} \rho + 2 \bar{\Psi}^{\overline{A}} \nabla_{\overline{A}} \rho \,. \tag{2.23}$$

Using the Bianchi identity

$$\frac{1}{6} F_{\mathbb{A}\mathbb{B}\mathbb{C}} F^{\mathbb{A}\mathbb{B}\mathbb{C}} = 2E_{\mathbb{A}} F^{\mathbb{A}} + F_{\mathbb{A}} F^{\mathbb{A}} \,, \tag{2.24}$$

it is useful to rewrite

$$\mathbb{R} = 2E_{\underline{A}} F^{\underline{A}} + F_{\underline{A}} F^{\underline{A}} - \frac{1}{6} F_{\underline{ABC}} F^{\underline{ABC}} - \frac{1}{2} F_{\overline{A}\underline{BC}} F^{\overline{A}\underline{BC}}. \tag{2.25}$$

The supersymmetry variation of the bosonic piece of the action gives

$$e^{2d} \delta_\epsilon [e^{-2d} \mathbb{R}(E, d)] = \frac{1}{2} \bar{\epsilon} \rho \mathbb{R} + 2\Delta E_{\underline{A}\overline{B}} \mathbb{R}^{\overline{B}\underline{A}} = \frac{1}{2} \bar{\epsilon} \rho \mathbb{R} - \bar{\epsilon} \gamma^{\underline{A}} \Psi^{\overline{B}} \mathbb{R}_{\overline{B}\underline{A}} \,, \tag{2.26}$$

where we have used

$$\delta_\epsilon F_{\mathbb{A}\mathbb{B}\mathbb{C}} = -3 \left(E_{[\mathbb{A}} \Delta E_{\mathbb{B}\mathbb{C}]} + \Delta E_{[\mathbb{A}}{}^{\mathbb{D}} F_{\mathbb{B}\mathbb{C}]\mathbb{D}}\right) \tag{2.27}$$

with

$$\Delta E_{\mathbb{A}\mathbb{B}} \equiv E^{\mathbb{M}}{}_{\mathbb{A}} \delta_\epsilon E_{\mathbb{M}\mathbb{B}} = -\Delta E_{\mathbb{B}\mathbb{A}} = \begin{cases} \Delta E_{\underline{AB}} = \Delta E_{\overline{AB}} = 0 \\ \Delta E_{\underline{A}\overline{B}} = -\Delta E_{\overline{B}\underline{A}} = \frac{1}{2} \bar{\epsilon} \gamma_{\underline{A}} \Psi_{\overline{B}} \end{cases} \tag{2.28}$$

and

$$\delta_\epsilon \mathbb{R} = -\bar{\epsilon} \gamma_{\underline{A}} \Psi^{\overline{B}} \left[E_{\overline{B}} F^{\underline{A}} - E_{\underline{C}} F_{\overline{B}}{}^{\underline{AC}} + F_{\underline{C}\underline{A}\overline{B}} F^{\overline{A}\underline{AC}} - F_{\underline{D}} F_{\overline{B}}{}^{\underline{AD}}\right] = -\bar{\epsilon} \gamma^{\underline{A}} \Psi^{\overline{B}} \mathbb{R}_{\overline{B}\underline{A}} \,.$$

The supersymmetry transformation rules define the following Lichnerowicz principle

$$\left(\gamma^{\underline{A}}\nabla_{\underline{A}}\gamma^{\underline{B}}\nabla_{\underline{B}} - \nabla^{\overline{A}}\nabla_{\overline{A}}\right)\epsilon = -\frac{1}{4}\mathbb{R}\epsilon\,, \tag{2.29}$$

$$\left[\nabla_{\overline{A}}, \gamma^{\underline{B}}\nabla_{\underline{B}}\right]\epsilon = \frac{1}{2}\gamma^{\underline{B}}\mathbb{R}_{\overline{A}\underline{B}}\epsilon\,, \tag{2.30}$$

and then, the supersymmetric variation of the fermionic piece of the action

$$e^{2d}\delta_\epsilon\left(e^{-2d}\mathbb{L}_{\rm F}\right) = -2\Delta E_{\underline{B}\overline{A}}\mathbb{R}^{\overline{A}\underline{B}} - \frac{1}{2}\bar{\epsilon}\rho\mathbb{R} = \bar{\epsilon}\gamma_{\underline{B}}\Psi_{\overline{A}}\mathbb{R}^{\overline{A}\underline{B}} - \frac{1}{2}\bar{\epsilon}\rho\mathbb{R}\,, \tag{2.31}$$

exactly cancels (2.26).

2.2 Parameterization and choice of section

To make contact with ten dimensional $\mathcal{N} = 1$ supergravity coupled to n_g vector multiplets, we split the G and H indices as $\mathbb{M} = ({}_\mu, {}^\mu, i)$ and $\mathbb{A} = (\underline{A}, \overline{A})$, respectively with $\underline{A} = \underline{a}$, $\overline{A} = (\overline{a}, \overline{i})$, ${}_\mu, {}^\mu, \underline{a}, \overline{a} = 0, \dots, 9$, $i, \overline{i} = 1, \dots, n_g$, and parameterize the generalized fields as follows:

Generalized frame

$$E^{\mathbb{M}}{}_{\mathbb{A}} = \begin{pmatrix} E_{\mu\underline{a}} & E^{\mu}{}_{\underline{a}} & E^{i}{}_{\underline{a}} \\ E_{\mu\overline{a}} & E^{\mu}{}_{\overline{a}} & E^{i}{}_{\overline{a}} \\ E_{\mu\overline{i}} & E^{\mu}{}_{\overline{i}} & E^{i}{}_{\overline{i}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -e_{\mu\underline{a}} - C_{\rho\mu}e^{\rho}{}_{\underline{a}} & e^{\mu}{}_{\underline{a}} & -A_\rho{}^i e^\rho{}_{\underline{a}}\,, \\ e_{\mu\overline{a}} - C_{\rho\mu}e^{\rho}{}_{\overline{a}} & e^{\mu}{}_{\overline{a}} & -A_\rho{}^i e^\rho{}_{\overline{a}} \\ \sqrt{2}A_{\mu i}e^{i}{}_{\overline{i}} & 0 & \sqrt{2}e^{i}{}_{\overline{i}} \end{pmatrix}\,, \tag{2.32}$$

where $e_{\underline{a}}$ and $e_{\overline{a}}$ are two vielbein for the same ten dimensional metric. To guarantee that the number of DFT and supergravity degrees of freedom agree, we gauge fix $e^{\mu}{}_{\underline{a}} = e^{\mu}{}_{\overline{a}}, e_{\mu\underline{a}} = e_{\mu\overline{a}}$, and identify $e^{\mu}{}_{\overline{a}}, e_{\mu\overline{a}}$ with the supergravity vielbein $e^{\mu}{}_{a}, e_{\mu a}$, $a, b = 0, \dots, 9$, respectively, i.e. $g_{\mu\nu} = e_\mu{}^a g_{ab} e_\nu{}^b$, with g_{ab} the Minkowski metric. $C_{\mu\nu} = b_{\mu\nu} + \frac{1}{2}A^i_\mu A_{\nu i}$, with A^i_μ being the gauge connection. For consistency, we also need to impose

$$P_{\underline{ab}} = -g_{ab}\delta^a_{\underline{a}}\delta^b_{\underline{b}}, \qquad \overline{P}_{\overline{ab}} = g_{ab}\delta^a_{\overline{a}}\delta^b_{\overline{b}}\,, \qquad \overline{P}_{\overline{ij}} = e^{i}{}_{\overline{i}}\eta_{ij}e^{j}{}_{\overline{j}}\,, \tag{2.33}$$

with $e^{i}{}_{\overline{i}}$ the (inverse) vielbein for the Killing metric of the SO(32) or $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group, $\eta_{ij} = e_i{}^{\overline{i}}\eta_{\overline{ij}}e_j{}^{\overline{j}}$, as required for modular invariance of the heterotic string.

Generalized dilaton and dilatino

$$d = \phi - \frac{1}{2}\log\sqrt{-g} \qquad \text{and} \qquad \rho = 2\lambda + \gamma^\mu\psi_\mu\,, \tag{2.34}$$

where ϕ, ψ_μ and λ are the standard dilaton, gravitino and dilatino fields, respectively.

Generalized gravitino

$$\Psi_{\mathbb{A}} = (0, e^{\mu}{}_a\psi_\mu, \frac{1}{\sqrt{2}}e^{i}{}_{\overline{i}}\chi_i)\,, \tag{2.35}$$

χ_i being the standard gaugino field.

The non-abelian gauge sector is trivially incorporated through the gaugings that deform the generalized Lie derivative (2.6a) as

$$f_{\mathbb{MN}}{}^{\mathbb{P}} = \begin{cases} f_{ij}{}^k & \text{for } \mathbb{M}, \mathbb{N}, \mathbb{P} = i, j, k \\ 0 & \text{otherwise.} \end{cases} \tag{2.36}$$

The γ-functions $\gamma^{\underline{a}} = \gamma^a\delta_a^{\underline{a}}$ verify the Clifford algebra $\{\gamma^a, \gamma^b\} = 2g^{ab}$.

The gauge fixing $e^\mu{}_{\underline{a}} = e^\mu{}_{\overline{a}}$ implies $\delta e^\mu{}_{\underline{a}} = \delta e^\mu{}_{\overline{a}}$, and (2.11) lead to

$$\Gamma_{\underline{ab}} = \left(-\Lambda_{ab} + \bar\epsilon\gamma_{[a}\psi_{b]}\right)\delta^a_{\underline{a}}\delta^b_{\underline{b}}\,, \tag{2.37}$$

where Λ_{ab} denotes the generator of $\mathrm{O}(1,9)$ transformations that parameterizes $\Gamma_{\overline{ab}}$.

The additional gauge fixings $\delta E^i{}_{\bar i} = 0$ and $\delta E^\mu{}_{\bar i} = 0$ lead respectively to

$$\Gamma_{\overline{ij}} = \Lambda_{ij}\delta^i_{\bar i}\delta^j_{\bar j} = f_{ijk}\xi^k\delta^i_{\bar i}\delta^j_{\bar j} \qquad \text{and} \qquad \Gamma_{\overline{ai}} = \Lambda_{ai}\ \delta^a_{\bar a}\ \delta^i_{\bar i} = \frac{1}{2\sqrt{2}}\bar\epsilon\gamma_a\chi_i\ \delta^a_{\bar a}\ \delta^i_{\bar i}\,, \tag{2.38}$$

where we have parameterized $\xi^{\mathbb{M}} = (\xi_\mu, \xi^\mu, \xi^i)$ and Λ_{ai}, Λ_{ij} are introduced for convenience, as we will discuss in section 4.

Solving the strong constraint in the supergravity frame, parameterizing (2.18) and using the non-vanishing determined components of the generalized spin connection listed in appendix A.2, we recover the leading order supersymmetry transformation rules of the coupled ten dimensional $\mathcal{N} = 1$ supergravity and Yang-Mills fields, namely

$$\delta_\epsilon e_\mu{}^a = \frac{1}{2}\bar\epsilon\gamma^a\psi_\mu\,, \qquad \delta_\epsilon\phi = -\frac{1}{2}\bar\epsilon\lambda = -\frac{1}{4}\bar\epsilon\rho + \frac{1}{4}\bar\epsilon\gamma^\mu\psi_\mu\,, \tag{2.39a}$$

$$\delta_\epsilon b_{\mu\nu} = \bar\epsilon\gamma_{[\mu}\psi_{\nu]} + \frac{1}{2}\bar\epsilon\gamma_{[\mu}\chi^i A_{\nu]i}\,, \qquad \delta\rho = \gamma^\mu D_\mu\epsilon - \frac{1}{24}H_{abc}\gamma^{abc}\epsilon - \gamma^\mu\partial_\mu\epsilon \tag{2.39b}$$

$$\delta_\epsilon\psi_\mu = \partial_\mu\epsilon - \frac{1}{4}w^{(+)}_{\mu ab}\gamma^{ab}\epsilon\,, \qquad \delta_\epsilon\lambda = -\frac{1}{2}\gamma^\mu\partial_\mu\phi\epsilon + \frac{1}{24}H_{abc}\gamma^{abc}\epsilon\,, \tag{2.39c}$$

$$\delta_\epsilon A^i_\mu = \frac{1}{2}\bar\epsilon\gamma_\mu\chi^i\,, \qquad \delta_\epsilon\chi^i = -\frac{1}{4}F^i_{\mu\nu}\gamma^{\mu\nu}\epsilon\,, \tag{2.39d}$$

where $w^{(+)}_{\mu ab} = w_{\mu ab} + \frac{1}{2}H_{\mu ab}$ is the spin connection with torsion given by the field strength of the b-field

$$H_{abc} = e^\mu{}_{[a}e^\nu{}_b e^\rho{}_{c]}H_{\mu\nu\rho} = 3e^\mu{}_a e^\nu{}_b e^\rho{}_c\left(\partial_{[\mu}b_{\nu\rho]} - C^{(g)}_{\mu\nu\rho}\right)\,, \tag{2.40}$$

with $C^{(g)}_{\mu\nu\rho}$ the Yang-Mills Chern-Simons form

$$C^{(g)}_{\mu\nu\rho} = A^i_{[\mu}\partial_\nu A_{\rho]i} - \frac{1}{3}f_{ijk}A^i_\mu A^j_\nu A^k_\rho\,. \tag{2.41}$$

The Lorentz transformations of the supergravity and super Yang-Mills multiplets obtained from (2.11) are

$$\delta_\Lambda e_\mu{}^a = e_\mu{}^b\Lambda_b{}^a\,, \qquad \delta_\Lambda\psi_a = \psi_b\Lambda^b{}_a - \frac{1}{4}\gamma^{bc}\Lambda_{bc}\psi_a\,, \qquad \delta_\Lambda\chi = -\frac{1}{4}\Lambda_{bc}\gamma^{bc}\chi\,, \tag{2.42}$$

and the gauge transformations derived from (2.6) are

$$\delta_\xi A^i_\mu = \partial_\mu\xi^i + f^i{}_{jk}\xi^j A^k_\mu\,, \qquad \delta_\xi\chi^i = f^i{}_{jk}\xi^j\chi^k\,, \qquad \delta_\xi b_{\mu\nu} = 2\partial_{[\mu}\xi_{\nu]} - \partial_{[\mu}\xi^i A_{\nu]i}\,, \tag{2.43}$$

where the second term in the gauge transformation of the b-field is the gauge sector of the Green-Schwarz transformation required for anomaly cancellation.

JHEP07(2021)092

Parameterizing the DFT action (2.22), using the fluxes listed in appendix A.2, we get

$$
\begin{aligned}
S = \int d^{10}x \; e \; e^{-2\phi} \Big[& R(w(e)) - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{4} F^i_{\mu\nu} F_i^{\mu\nu} \\
& - \overline{\psi}^\mu \not{D} \psi_\mu + \overline{\rho} \not{D} \rho + 2 \overline{\psi}^\mu D_\mu \rho - \frac{1}{2} \overline{\chi}^i \not{D} \chi_i + \; \overline{\chi}_i \left(\gamma^\mu \psi^\nu - \frac{1}{4} \gamma^{\mu\nu} \rho \right) F^i_{\mu\nu} \\
& + \frac{1}{24} H_{\rho\sigma\tau} \left(\overline{\psi}^\mu \gamma^{\rho\sigma\tau} \psi_\mu + 12 \overline{\psi}^\rho \gamma^\sigma \psi^\tau - \overline{\rho} \gamma^{\rho\sigma\tau} \rho - 6 \overline{\psi}^\nu \gamma^{\rho\tau} \rho + \frac{1}{2} \overline{\chi}^i \gamma^{\rho\sigma\tau} \chi_i \right) \Big] . \qquad (2.44)
\end{aligned}
$$

We use standard notation defined in appendix A. Both the action and the transformation rules match the corresponding ones in [27], with the field redefinitions specified in appendix A.3, where (2.44) is rewritten in terms of the standard supergravity dilatino λ instead of ρ.

3 The first order α'-corrections

In this section we construct the first order corrections to $\mathcal{N} = 1$ supersymmetric DFT, performing a perturbative expansion of the exact formalism developed in [76].

The duality structure of the first order α'-corrections to heterotic supergravity was originally considered in [65, 66]. Exploiting a symmetry between the gauge and torsionful spin connections that exists in ten dimensional heterotic supergravity [26, 27], the duality group was extended to $\mathrm{O}(10, 10+n_g+n_l)$, with $n_g(n_l)$ the dimension of the heterotic gauge (Lorentz) group. In this construction, the gaugings in the generalized Lie derivative (2.6a) preserve a residual $\mathrm{O}(10, 10)$ global symmetry. Including one-form fields in the $\mathrm{GL}(10)$ parameterization of the generalized vielbein, the formalism reproduces the first order corrections to the interactions of the bosonic fields in the heterotic effective field theory. This construction was supersymmetrized in [75].

The lack of manifest duality covariance and the difficulties to incorporate higher orders of the α'-expansion in these formulations motivated the search of alternative frameworks. A deformation of the gauge structure of DFT was proposed in [67], introducing a generalized Green-Schwarz transformation that modifies the leading order double Lorentz variations (2.11) with two derivative corrections. The deformations fix the four derivative terms of bosonic fields in all T-duality symmetric gravitational theories, including in particular the bosonic and heterotic string effective actions [68].

The two formalisms described above were merged in the so-called generalized Bergshoeff-de Roo identification introduced in [76]. In the first part of this section we briefly review this exact supersymmetric and manifestly duality covariant formulation. Then we perform a perturbative expansion and extract the first order corrections to the transformation rules of the $\mathrm{O}(10, 10 + n_g)$ multiplets (2.19). Finally, we construct the gauge invariant action containing three and four derivatives of the duality multiplets up to bilinear terms in fermions.

3.1 The generalized Bergshoeff-de Roo identification

The theory has a global $\mathrm{O}(10, 10+k)$ symmetry, where k is the dimension of the $\mathrm{O}(1, 9+k)$ group. This differs from the construction of the previous section, where the duality group is $\mathrm{O}(10,10+n_g)$ and n_g denotes the dimension of the SO(32) or $\mathrm{E}_8\times\mathrm{E}_8$ heterotic gauge group. In the construction of [76] instead the gauge sector encodes the higher derivatives.

The vielbein $\mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}}$ is an element of $\mathrm{O}(10,10+k)$, parameterized in terms of O(10,10) fields as[2]

$$\begin{aligned} \mathcal{E}_M{}^{\overline{a}} &= E_M{}^{\overline{a}}\,, & \mathcal{E}_M{}^{\underline{a}} &= (\triangle^{\frac{1}{2}})_M{}^P\, E_P{}^{\underline{a}}\,, & \mathcal{E}_M{}^{\overline{\alpha}} &= -\mathrm{A}_M{}^{\beta}\, e_\beta{}^{\overline{\alpha}}\,, \\ \mathcal{E}_\alpha{}^{\overline{a}} &= 0\,, & \mathcal{E}_\alpha{}^{\underline{a}} &= E_M{}^{\underline{a}}\,\mathrm{A}^M{}_\alpha\,, & \mathcal{E}_\alpha{}^{\overline{\alpha}} &= (\Box^{\frac{1}{2}})_\alpha{}^\beta\, e_\beta{}^{\overline{\alpha}}\,. \end{aligned} \tag{3.1}$$

We use calligraphic symbols to distinguish the $\mathrm{O}(D,D+k)$ objects. The indices $\mathcal{M}=(M,\alpha)=({}^{\mu},{}_{\mu},\alpha)$ and $\mathcal{A}=(\underline{\mathcal{A}},\overline{\mathcal{A}})$ take values $M=0,\dots,19$, $\underline{\mathcal{A}}\equiv\underline{a}=0,\dots,9$; $\overline{\mathcal{A}}=(\overline{a},\overline{\alpha}), \overline{a}=0,\dots,9$ and $\alpha,\overline{\alpha}=1,\dots,k$. $\mathrm{A}_M{}^\alpha$ is a constrained O(10,10) vector field satisfying $\mathrm{A}_M{}^\alpha=P_M{}^N\mathrm{A}_N{}^\alpha$ (the projection is fixed by the choice of $\mathrm{O}(10,10+k)$ duality group, as opposed to $\mathrm{O}(10+k,10)$ which would give an equivalent $\mathbb{Z}_2$ transformed theory), and

$$\Box_\alpha{}^\beta = \kappa_\alpha{}^\beta - \mathrm{A}_{M\alpha}\mathrm{A}^{M\beta}\,, \tag{3.2}$$

$$\triangle_M{}^N = \eta_M{}^N - \mathrm{A}_{M\alpha}\mathrm{A}^{N\alpha}\,. \tag{3.3}$$

The gauge freedom is used to set $\mathcal{E}_\alpha{}^{\bar{a}}$ to zero and the bijective map $e_\alpha{}^{\overline{\beta}}$ relates the Cartan-Killing metrics of $\mathrm{O}(k)$, $\kappa_{\alpha\beta}$ and $\kappa_{\overline{\alpha\beta}}$, as

$$e_\alpha{}^{\overline{\alpha}}\kappa_{\overline{\alpha\beta}}e_\beta{}^{\overline{\beta}} = \kappa_{\alpha\beta}\,. \tag{3.4}$$

The parameterization (3.1) preserves the constraint

$$\mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}}\eta_{\mathcal{AB}}\mathcal{E}_{\mathcal{N}}{}^{\mathcal{B}} = \eta_{\mathcal{MN}}\,, \tag{3.5}$$

where $\eta_{\mathcal{MN}}$ and $\eta_{\mathcal{AB}}$ are the invariant metrics of $\mathrm{O}(10,10+k)$ and $\mathrm{O}(9,1)_{\mathrm{L}}\times\mathrm{O}(1,9+k)_{\mathrm{R}}$,

$$\eta_{\mathcal{MN}} = \begin{pmatrix} 0 & \delta_\mu{}^\nu & 0 \\ \delta^\mu{}_\nu & 0 & 0 \\ 0 & 0 & \kappa_{\alpha\beta} \end{pmatrix}\,, \qquad \eta_{\mathcal{AB}} = \begin{pmatrix} -g_{\underline{ab}} & 0 & 0 \\ 0 & g_{\overline{ab}} & 0 \\ 0 & 0 & \kappa_{\overline{\alpha\beta}} \end{pmatrix}\,. \tag{3.6}$$

The generalized $\mathrm{O}(10,10+k)$ gravitino splits as $\Psi_{\mathcal{A}}=(0,\Psi_{\bar{a}},\Psi_{\overline{\alpha}})$, where $\Psi_{\overline{a}}$ is a generalized O(10,10) gravitino and $\Psi_{\overline{\alpha}}$ is a gaugino of the $\mathrm{O}(1,9+k)_{\mathrm{R}}$ gauge group, that will later be identified with a function of the O(10,10) generalized fields. The gamma matrices are $\gamma^{\mathcal{A}}=(\gamma^{\underline{a}},0,0)$, with $\gamma^{\underline{a}}$ the $\mathrm{O}(9,1)_{\mathrm{L}}$ gamma matrices verifying (2.12).

The transformation rules of the $\mathrm{O}(10,10+k)$ fields have the same functional form as (2.19), namely

$$\begin{aligned} \delta\mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}} &= \xi^{\mathcal{P}}\partial_{\mathcal{P}}\mathcal{E}_{\mathcal{M}}{}^{\mathcal{A}} + (\partial_{\mathcal{M}}\xi^{\mathcal{P}} - \partial^{\mathcal{P}}\xi_{\mathcal{M}})\mathcal{E}_{\mathcal{P}}{}^{\mathcal{A}} + g f_{\mathcal{MN}}{}^{\mathcal{P}}\xi^{\mathcal{N}}\mathcal{E}_{\mathcal{P}}{}^{\mathcal{A}} \\ &\quad + \mathcal{E}_{\mathcal{M}}{}^{\mathcal{B}}\mathcal{T}_{\mathcal{B}}{}^{\mathcal{A}} - \bar{\epsilon}\gamma^{[\mathcal{A}}\Psi^{\mathcal{B}]}\mathcal{E}_{\mathcal{MB}}\,, \end{aligned} \tag{3.7a}$$

$$\delta d = \xi^{\mathcal{P}}\partial_{\mathcal{P}}d - \frac{1}{2}\partial_{\mathcal{P}}\xi^{\mathcal{P}} - \frac{1}{4}\bar{\epsilon}\rho \tag{3.7b}$$

$$\delta\Psi_{\overline{\mathcal{A}}} = \xi^{\mathcal{M}}\partial_{\mathcal{M}}\Psi_{\overline{\mathcal{A}}} + \mathcal{T}^{\overline{\mathcal{B}}}{}_{\overline{\mathcal{A}}}\Psi_{\overline{\mathcal{B}}} + \frac{1}{4}\mathcal{T}_{\underline{\mathcal{BC}}}\gamma^{\underline{\mathcal{BC}}}\Psi_{\overline{\mathcal{A}}} + \nabla_{\overline{\mathcal{A}}}\epsilon \tag{3.7c}$$

$$\delta\rho = \xi^{\mathcal{M}}\partial_{\mathcal{M}}\rho + \frac{1}{4}\mathcal{T}_{\underline{\mathcal{AB}}}\gamma^{\underline{\mathcal{AB}}}\rho - \gamma^{\underline{\mathcal{A}}}\nabla_{\underline{\mathcal{A}}}\epsilon\,, \tag{3.7d}$$

[2]Note that this differs from (2.32) and from previous constructions, e.g. [65, 66], where the generalized vielbein is parameterized with GL(10) multiplets.

where $g^{-2} \sim \alpha'$ is a dimensionful constant, $\mathcal{T}_{\mathcal{A}\mathcal{B}}$ parameterizes the local double Lorentz $\mathrm{O}(9,1)_{\mathrm{L}} \times \mathrm{O}(1,9+k)_{\mathrm{R}}$ tangent space symmetry,

$$\nabla_{\mathcal{A}}\epsilon = \mathcal{E}_{\mathcal{A}}\epsilon - \frac{1}{4}\omega_{\mathcal{A}\mathcal{B}\mathcal{C}}\gamma^{\mathcal{B}\mathcal{C}}\epsilon\,, \tag{3.8}$$

with $\mathcal{E}_{\mathcal{A}} = \sqrt{2}\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}\partial_{\mathcal{M}}$, and the identifications

$$\mathcal{F}_{\mathcal{A}\mathcal{B}\mathcal{C}} = 3\mathcal{E}_{[\mathcal{A}}\mathcal{E}^{\mathcal{N}}{}_{\mathcal{B}}\mathcal{E}_{\mathcal{N}\mathcal{C}]} + g\sqrt{2}f_{\mathcal{M}\mathcal{N}\mathcal{P}}\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}\mathcal{E}^{\mathcal{N}}{}_{\mathcal{B}}\mathcal{E}^{\mathcal{P}}{}_{\mathcal{C}} = -3\omega_{[\mathcal{A}\mathcal{B}\mathcal{C}]}\,, \tag{3.9}$$

$$\mathcal{F}_{\mathcal{A}} = \sqrt{2}e^{2d}\partial_{\mathcal{M}}\left(\mathcal{E}^{\mathcal{M}}{}_{\mathcal{A}}e^{-2d}\right) = -\omega_{\mathcal{B}\mathcal{A}}{}^{\mathcal{B}}\,, \tag{3.10}$$

$$f_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}} = \begin{cases} f_{\alpha\beta}{}^{\gamma} & \text{for } \mathcal{M},\mathcal{N},\mathcal{P} = \alpha,\beta,\gamma \\ 0 & \text{otherwise} \end{cases}\,. \tag{3.11}$$

Equivalent constraints to (2.7) and (2.8) must be imposed, i.e.

$$\partial_{\mathcal{M}}\partial^{\mathcal{M}}\cdots = 0\,, \qquad \partial_{\mathcal{M}}\cdots\partial^{\mathcal{M}}\cdots = 0\,, \qquad f^{\mathcal{M}}{}_{\mathcal{N}\mathcal{P}}\partial_{\mathcal{M}}\cdots = 0\,, \tag{3.12a}$$

$$f_{\mathcal{M}\mathcal{N}\mathcal{P}} = f_{[\mathcal{M}\mathcal{N}\mathcal{P}]}\,, \qquad f_{[\mathcal{M}\mathcal{N}}{}^{\mathcal{R}}f_{\mathcal{P}]\mathcal{R}}{}^{\mathcal{Q}} = 0\,. \tag{3.12b}$$

The gauge fixing $\delta\mathcal{E}_{\alpha}{}^{\overline{a}} = 0$ implies

$$\mathcal{T}_{\overline{\alpha}}{}^{\overline{b}} = \left(\partial^{P}\xi_{\alpha}\mathcal{E}_{P}{}^{\overline{b}} - \frac{1}{2}\bar{\epsilon}\gamma^{\underline{c}}\Psi^{\overline{b}}\mathcal{E}_{\alpha\underline{c}}\right)(\Box^{-\frac{1}{2}})^{\alpha}{}_{\beta}e^{\beta}{}_{\overline{\alpha}}\,, \tag{3.13}$$

and $\delta e_{\alpha}{}^{\overline{\alpha}} = 0$ determines

$$\mathcal{T}_{\overline{\alpha}\overline{\beta}} = \left(\delta(\Box^{\frac{1}{2}})_{\alpha\beta}e^{\beta}{}_{[\overline{\beta}} - \xi^{P}\partial_{P}\mathcal{E}_{\alpha[\overline{\beta}} + \partial_{P}\xi_{\alpha}\mathcal{E}^{P}{}_{[\overline{\beta}} - gf_{\alpha\beta}{}^{\gamma}\xi^{\beta}\mathcal{E}_{\gamma[\overline{\beta}} - \frac{1}{2}\mathcal{E}_{\alpha\underline{b}}\bar{\epsilon}\gamma^{\underline{b}}\Psi_{[\overline{\beta}}\right)e^{\delta}{}_{\overline{\alpha}]}(\Box^{-\frac{1}{2}})^{\alpha}{}_{\delta}\,. \tag{3.14}$$

The gauge generators $(t^{\alpha})_{\overline{\mathcal{A}}}{}^{\overline{\mathcal{B}}}$ implement the map

$$V_{\overline{\mathcal{A}}}{}^{\overline{\mathcal{B}}} = -gV_{\alpha}\,(t^{\alpha})_{\overline{\mathcal{A}}}{}^{\overline{\mathcal{B}}}\,, \tag{3.15}$$

allowing to write

$$-g\xi_{\alpha}(t^{\alpha})_{\overline{\mathcal{A}\mathcal{B}}} \equiv \mathcal{T}_{\overline{\mathcal{A}\mathcal{B}}}\,, \qquad -g\mathcal{E}_{\alpha}{}^{\underline{a}}\,(t^{\alpha})_{\overline{\mathcal{C}\mathcal{D}}} \equiv \frac{1}{\sqrt{2}}\mathrm{A}^{\underline{a}}{}_{\overline{\mathcal{C}\mathcal{D}}}\,. \tag{3.16}$$

They satisfy $[t_{\alpha}\,,\,t_{\beta}] = f_{\alpha\beta}{}^{\gamma}\,t_{\gamma}$ and $Tr(t^{\alpha}t_{\beta}) = X_R\delta^{\alpha}_{\beta}$, where X_R is the Dynkin index of the representation.

Parameterizing $\delta\mathcal{E}_{M}{}^{\underline{a}}$ one gets

$$\delta\mathrm{A}_{\underline{a}\overline{\mathcal{C}\mathcal{D}}} = \xi^{P}\partial_{P}\mathrm{A}_{\underline{a}\overline{\mathcal{C}\mathcal{D}}} - \mathcal{E}_{\underline{a}}\mathcal{T}_{\overline{\mathcal{C}\mathcal{D}}} - 2\mathrm{A}_{\underline{a}[\overline{\mathcal{C}}}{}^{\overline{\mathcal{B}}}\mathcal{T}_{\overline{\mathcal{D}}]\overline{\mathcal{B}}} - \mathrm{A}^{\underline{b}}{}_{\overline{\mathcal{C}\mathcal{D}}}\mathcal{T}_{\underline{ab}} + \bar{\epsilon}\gamma_{\underline{a}}\Psi_{\overline{\mathcal{C}\mathcal{D}}}\,, \tag{3.17}$$

where

$$\Psi_{\overline{\mathcal{C}\mathcal{D}}} \equiv \frac{g}{\sqrt{2}}\Psi_{\overline{\beta}}\mathcal{E}_{\alpha}{}^{\overline{\beta}}(t^{\alpha})_{\overline{\mathcal{C}\mathcal{D}}}\,. \tag{3.18}$$

In order to eliminate these extra degrees of freedom, it is convenient to define

$$\mathcal{F}^{*}_{\underline{a}\overline{\mathcal{C}\mathcal{D}}} = \mathcal{F}_{\underline{a}\overline{\mathcal{C}\mathcal{D}}} - \frac{1}{2}\overline{\Psi}_{\overline{\mathcal{C}}}\gamma_{\underline{a}}\Psi_{\overline{\mathcal{D}}}\,, \tag{3.19}$$

which allows to establish the generalized Bergshoeff-de Roo identification between the generalized gauge and spin connections

$$\mathrm{A}_{\underline{M}\overline{\mathcal{CD}}} = \mathcal{F}^{*}_{\underline{M}\overline{\mathcal{CD}}}\,, \tag{3.20}$$

and to determine $\Psi_{\overline{\mathcal{CD}}}$ as the generalized gravitino curvature

$$\Psi_{\overline{\mathcal{CD}}} = \nabla_{[\overline{\mathcal{C}}}\Psi_{\overline{\mathcal{D}}]} + \frac{1}{2}\omega^{\overline{\mathcal{B}}}{}_{\overline{\mathcal{CD}}}\Psi_{\overline{\mathcal{B}}}\,, \tag{3.21}$$

since both sides of (3.20) and (3.21) transform in the same way. The main steps of the demonstration can be found in [76].

We now proceed to extract the first order α'-corrections to the transformation rules of the $\mathrm{O}(10, 10 + n_g)$ generalized fields.

3.2 Induced transformation rules on O(10, 10) multiplets

The covariant transformation rules (3.7) induce higher derivative deformations on the transformations (2.19) of the $\mathrm{O}(10, 10 + n_g)$ fields. In this section, we work out the first order modifications, expanding the coefficients $(\Box^{\frac{1}{2}})_{\alpha}{}^{\beta}$ and $(\triangle^{\frac{1}{2}})_{M}{}^{N}$ in the parameterization of the $\mathrm{O}(10, 10 + k)$ multiplets.

To simplify the presentation, we turn off the gauge sector of the $\mathrm{O}(10, 10 + n_g)$ multiplets, i.e. we take $n_g = 0$, and obtain the induced transformation rules of the $\mathrm{O}(10, 10)$ fields. The gauge sector will be trivially included in the next subsection.

It is convenient to first express the components of the generalized $\mathrm{O}(10, 10 + k)$ fluxes (3.9) and (3.10) in terms of the $\mathrm{O}(10, 10)$ fluxes (2.14) and (2.15). Keeping only the first order terms in the expansion of the coefficients $(\Box^{\frac{1}{2}})_{\alpha}{}^{\beta}$ and $(\triangle^{\frac{1}{2}})_{M}{}^{N}$, namely

$$(\Box^{\frac{1}{2}})_{\alpha}{}^{\beta} \;\cong\; \kappa_{\alpha}{}^{\beta} - \frac{1}{2}\mathrm{A}_{M\alpha}\mathrm{A}^{M\beta}\,, \qquad (\triangle^{\frac{1}{2}})_{MN} \cong \eta_{MN} - \frac{1}{2}\mathrm{A}_{M\alpha}\mathrm{A}_{N\beta}\kappa^{\alpha\beta}\,, \tag{3.22}$$

we get the first order deformations

$$\mathcal{F}_{\underline{abc}} = F_{\underline{abc}} + F^{(3)}_{\underline{abc}} \cong F_{\underline{abc}} - \frac{3b}{4}\left(E_{[\underline{a}}F^{*\overline{cd}}_{\underline{b}} - \frac{1}{2}F_{\underline{d}[\underline{ab}}F^{*\underline{d}\overline{cd}} - \frac{2}{3}F^{*\overline{c}}{}_{\overline{e}[\underline{a}}F^{*\overline{ed}}_{\underline{b}}\right)F^{*}_{\underline{c}]\overline{cd}}\,, \tag{3.23a}$$

$$\mathcal{F}_{\overline{a}\underline{bc}} = F_{\overline{a}\underline{bc}} + F^{(3)}_{\overline{a}\underline{bc}} \cong F_{\overline{a}\underline{bc}} - \frac{b}{4}\left(E_{\overline{a}}F^{*\overline{cd}}_{[\underline{b}} + F^{*\underline{e}\overline{cd}}F_{\overline{a}\underline{e}[\underline{b}}\right)F^{*}_{\underline{c}]\overline{cd}}\,, \tag{3.23b}$$

$$\mathcal{F}_{\underline{a}\overline{bc}} = F_{\underline{a}\overline{bc}} + F^{(3)}_{\underline{a}\overline{bc}} \cong F_{\underline{a}\overline{bc}} + \frac{b}{8}F^{*}_{\underline{d}\overline{ef}}F^{*\overline{ef}}{}_{\underline{a}}F^{\underline{d}}{}_{\overline{bc}}\,, \tag{3.23c}$$

$$\mathcal{F}_{\overline{abc}} = F_{\overline{abc}}\,, \tag{3.23d}$$

$$\mathcal{F}_{\overline{ab}\underline{cd}} \cong F^{(2)}_{\overline{ab}\underline{cd}} = -2E_{[\underline{c}}F^{*}_{\underline{d}]\overline{ab}} + 2F^{*\overline{e}}_{\overline{a}}{}_{[\underline{c}}F^{*}_{\underline{d}]\overline{eb}} + F_{\underline{cde}}F^{*\underline{e}}{}_{\overline{ab}}\,, \tag{3.23e}$$

$$\mathcal{F}_{\underline{a}\overline{bcd}} \cong F^{(2)}_{\underline{a}\overline{bcd}} = \frac{1}{\sqrt{2}}E_{\overline{b}}F^{*}_{\underline{a}\overline{cd}} - \frac{1}{\sqrt{2}}F_{\underline{ad}\overline{b}}F^{*\underline{d}}{}_{\overline{cd}}\,, \tag{3.23f}$$

$$\mathcal{F}_{\underline{a}} = F_{\underline{a}} + F^{(3)}_{\underline{a}} \cong F_{\underline{a}} + \frac{b}{8}\left[F^{*\underline{b}}{}_{\overline{cd}}F^{*\overline{cd}}_{\underline{a}}F_{\underline{b}} + E_{\underline{b}}\left(F^{*\underline{b}}{}_{\overline{cd}}F^{*\overline{cd}}_{\underline{a}}\right)\right]\,, \tag{3.23g}$$

$$\mathcal{F}_{\overline{a}} = F_{\overline{a}}\,, \tag{3.23h}$$

where we used

$$F^*_{\underline{M}\overline{bc}} \equiv P_M{}^N F^*_{N\overline{bc}} = \frac{1}{\sqrt{2}} E_M{}^{\underline{a}} F^*_{\underline{a}\overline{bc}} = \frac{1}{\sqrt{2}} E_M{}^{\underline{a}} \left(F_{\underline{a}\overline{bc}} - \frac{1}{2}\overline{\Psi}_{\overline{b}}\gamma_{\underline{a}}\Psi_{\overline{c}} \right) , \tag{3.24}$$

$b = \frac{2}{(1-X_R)g^2}$, the superscripts $^{(2)}$ and $^{(3)}$ refer to the number of derivatives, and we defined

$$\mathcal{F}_{\overline{\alpha}\underline{cd}} = \frac{1}{\sqrt{2}gX_R} \mathcal{F}_{\overline{\mathcal{AB}}\underline{cd}}(t_\alpha)^{\overline{\mathcal{AB}}} e^\alpha{}_{\overline{\alpha}} . \tag{3.25}$$

The transformation rules (3.7) take the following form:

Vielbein. The identification $\mathcal{E}_M{}^{\overline{a}} = E_M{}^{\overline{a}}$ implies $\delta\mathcal{E}_M{}^{\overline{a}} = \delta E_M{}^{\overline{a}}$, and from (3.7a) we get

$$\delta E_M{}^{\overline{a}} = \hat{\mathcal{L}}_\xi E_M{}^{\overline{a}} + E_M{}^{\overline{b}}\mathfrak{T}_{\overline{b}}{}^{\overline{a}} + \mathcal{E}_M{}^{\overline{\beta}}\mathfrak{T}_{\overline{\beta}}{}^{\overline{a}} + \frac{1}{2}\overline{\epsilon}\gamma^{\underline{b}}\Psi^{\overline{a}}\mathcal{E}_{M\underline{b}} . \tag{3.26}$$

Using the gauge fixing (3.13) and the following relation

$$\mathrm{A}_M{}^\beta f(\Box)_\beta{}^\alpha = \mathrm{A}_N{}^\alpha f(\triangle)_M{}^N , \tag{3.27}$$

which holds for any function f, one gets

$$\delta E_M{}^{\overline{a}} = \hat{\mathcal{L}}_\xi E_M{}^{\overline{a}} + E_M{}^{\overline{b}}\mathfrak{T}_{\overline{b}}{}^{\overline{a}} - \mathrm{A}_M{}^\beta \partial^P \xi_\alpha E_P{}^{\overline{a}} (\Box^{-\frac{1}{2}})^\alpha{}_\beta + \frac{1}{2}\overline{\epsilon}\gamma^{\underline{b}}\Psi^{\overline{a}}(\triangle^{-\frac{1}{2}})_M{}^N E_{N\underline{b}} . \tag{3.28}$$

The second term in the r.h.s. of this expression allows to identify $\mathfrak{T}_{\overline{ab}}$ with the $\Gamma_{\overline{ab}}$ component of the Lorentz parameter (2.9). The third term contains the deformation

$$\delta^{(1)}_\Gamma E_M{}^{\overline{a}} = \frac{b}{2}\, F^*_{\underline{M}\overline{bc}} E_N{}^{\overline{a}} \partial^N \Gamma^{\overline{bc}} , \tag{3.29}$$

which is the leading order of the $\mathrm{O}(10,10)$ covariant generalization of the Green-Schwarz transformation [67]. And finally, the last term in (3.28) contains the first order correction to the supersymmetry transformation rule (2.18a), namely

$$\delta^{(1)}_\epsilon E_M{}^{\overline{a}} = -\frac{b}{8}\overline{\epsilon}\gamma^{\underline{b}}\Psi^{\overline{a}} F^*_{\underline{M}\overline{bc}} F^*_N{}^{\overline{bc}} E^N{}_{\underline{b}} . \tag{3.30}$$

Following a similar reasoning, one can see that the other projection transforms as

$$\delta^{(1)} E_M{}^{\underline{a}} = \frac{b}{2}\, F^{*N\overline{cd}} E_N{}^{\underline{a}} \left(-\partial_{\overline{M}}\Gamma_{\overline{cd}} + \frac{1}{4\sqrt{2}}\overline{\epsilon}\gamma^{\underline{b}}\Psi_{\overline{b}} F_{\underline{b}\overline{cd}} E_M{}^{\overline{b}} \right) , \tag{3.31}$$

where we have identified

$$\mathfrak{T}_{\underline{ab}} = \Gamma_{\underline{ab}} + \frac{b}{4} F^{*\overline{cd}}_{[\underline{b}} E_{\underline{a}]}\Gamma_{\overline{cd}} - \frac{b}{4}\overline{\epsilon}\gamma_{[\underline{a}}\Psi^{\overline{cd}} F^*_{\underline{b}]\overline{cd}} . \tag{3.32}$$

Gravitino. From (3.7c) we get the first order corrections to the transformation rules of the $\mathrm{O}(10,10)$ gravitino (2.19d), up to bilinear terms in fermions,

$$\delta^{(1)}\Psi_{\overline{a}} = \frac{b}{16}\, E_{\underline{b}}\Gamma_{\overline{cd}} F_{\underline{c}}{}^{\overline{cd}}\gamma^{\underline{bc}}\Psi_{\overline{a}} + \frac{b}{2}\, \Psi^{\overline{cd}} E_{\overline{a}}\Gamma_{\overline{cd}} + \frac{1}{4} F^{(3)}_{\overline{a}\underline{bc}}\gamma^{\underline{bc}}\epsilon, \tag{3.33}$$

where we have kept the leading order terms in the $\mathrm{O}(10,10{+}k)$ gaugino identification (3.21). Note that there are two corrections to the Lorentz transformations. The first term in the right hand side can be interpreted as a generalized Green-Schwarz transformation and the second one depends on the gravitino curvature, that we now define.

JHEP07(2021)092

Gravitino curvature. To leading order in (3.21), the induced $O(10,10)$ gravitino curvature is,

$$\Psi_{\overline{ab}} = \nabla_{[\overline{a}}\Psi_{\overline{b}]} + \frac{1}{2}\omega_{\overline{cab}}\Psi^{\overline{c}} . \tag{3.34}$$

From (3.7c), we find that it obeys the transformation rule

$$\delta\Psi_{\overline{ab}} = \xi^M\partial_M\Psi_{\overline{ab}} + 2\Psi_{\overline{c}[\overline{b}}\Gamma^{\overline{c}}{}_{\overline{a}]} + \frac{1}{4}\Gamma_{\underline{cd}}\gamma^{\underline{cd}}\Psi_{\overline{ab}} + \frac{1}{2}E_{\overline{c}}\Gamma_{\overline{ab}}\Psi^{\overline{c}} + \frac{1}{2}F^{*\underline{c}}{}_{\overline{ab}}E_{\underline{c}}\epsilon + \frac{1}{8}\mathcal{F}^{(2)}_{\underline{cd}\overline{ab}}\gamma^{\underline{cd}}\epsilon . \tag{3.35}$$

Dilatino. The first order corrections to the transformation rules of the generalized dilatino (2.19e) that are obtained from (3.7d) are

$$\delta^{(1)}\rho = \frac{b}{16}E_{\underline{b}}\Gamma_{\overline{cd}}F^{*\overline{cd}}_{\underline{c}}\gamma^{\underline{bc}}\rho - \frac{b}{8}\gamma^{\underline{a}}F^{*}_{\underline{a}\overline{bc}}F^{*\underline{d}\overline{bc}}E_{\underline{d}}\epsilon - \frac{1}{12}F^{(3)}_{\underline{abc}}\gamma^{\underline{abc}}\epsilon - \frac{1}{2}F^{(3)}_{\underline{c}}\gamma^{\underline{c}}\epsilon . \tag{3.36}$$

Note that the transformation rules of the dilaton (2.19c) as well as the diffeomorphisms on all the fields are not corrected.

3.3 Including the heterotic gauge sector

It is now trivial to include the gauge sector of the $O(10,10+n_g)$ formulation. We simply extend the duality group $O(10,10) \to O(10,10+n_g)$, the right Lorentz group $O(1,9)_R \to O(1,9+n_g)_R$ and the indices $M \to \mathbb{M} = (M,i), \bar{a} \to \bar{A} = (\bar{a},\bar{i})$, accordingly. Now the generalized fluxes and gravitino curvature contain the contributions of the gauge sector, and in particular the structure constants.

A straightforward extension of the indices in equations (3.30)–(3.36) gives the following transformation rules of the $O(10,10+n_g)$ generalized fields, up to first order,

$$\begin{aligned}
\delta E_{\mathbb{M}}{}^{\underline{a}} &= \hat{\mathcal{L}}_\xi E_{\mathbb{M}}{}^{\underline{a}} + E_{\mathbb{M}\underline{b}}\Gamma^{\underline{ba}} - \frac{1}{2}\bar{\epsilon}\gamma^{\underline{a}}\Psi^{\overline{B}}E_{\mathbb{M}\overline{B}} \\
&\quad - \frac{b}{2}E_N{}^{\underline{a}}F^{*N}{}_{\overline{CD}}\left(\partial_{\underline{\mathbb{M}}}\Gamma^{\overline{CD}} - \frac{1}{4\sqrt{2}}\bar{\epsilon}\gamma^{\underline{b}}\Psi_{\overline{B}}F_{\underline{b}}{}^{\overline{CD}}E_{\mathbb{M}}{}^{\overline{B}}\right) , \qquad (3.37a)\\
\delta E_{\mathbb{M}}{}^{\overline{A}} &= \hat{\mathcal{L}}_\xi E_{\mathbb{M}}{}^{\overline{A}} + E_{\mathbb{M}\overline{B}}\Gamma^{\overline{BA}} + \frac{1}{2}\bar{\epsilon}\gamma^{\underline{b}}\Psi^{\overline{A}}E_{\mathbb{M}\underline{b}} \\
&\quad + \frac{b}{2}\, F^{*}_{\underline{\mathbb{M}}}{}^{\overline{CD}}\left(E_N{}^{\overline{A}}\partial^N\Gamma_{\overline{CD}} - \frac{1}{4}\bar{\epsilon}\gamma^{\underline{b}}\Psi^{\overline{A}}F_{N\overline{CD}}E^N{}_{\underline{b}}\right) , \qquad (3.37b)\\
\delta d &= \xi^M\partial_M d - \frac{1}{2}\partial_M\xi^M - \frac{1}{4}\bar{\epsilon}\rho , \qquad (3.37c)\\
\delta\Psi_{\overline{A}} &= \hat{\mathcal{L}}_\xi\Psi_{\overline{A}} + \Psi_{\overline{B}}\Gamma^{\overline{B}}{}_{\overline{A}} + \frac{1}{4}\Gamma_{\underline{bc}}\gamma^{\underline{bc}}\Psi_{\overline{A}} + \nabla_{\overline{A}}\epsilon \\
&\quad + \frac{b}{16}E_{\underline{b}}\Gamma_{\overline{CD}}F_{\underline{c}}{}^{\overline{CD}}\gamma^{\underline{bc}}\Psi_{\overline{A}} + \frac{b}{2}\,\Psi^{\overline{DC}}E_{\overline{A}}\Gamma_{\overline{CD}} + \frac{1}{4}F^{(3)}_{\overline{A}\underline{bc}}\gamma^{\underline{bc}}\epsilon , \qquad (3.37d)\\
\delta\rho &= \hat{\mathcal{L}}_\xi\rho + \frac{1}{4}\Gamma_{\underline{bc}}\gamma^{\underline{bc}}\rho - \gamma^{\underline{a}}\nabla_{\underline{a}}\epsilon + \frac{b}{16}\,E_{\underline{b}}\Gamma_{\overline{CD}}F_{\underline{c}}{}^{\overline{CD}}\gamma^{\underline{bc}}\rho \\
&\quad - \frac{b}{8}\gamma^{\underline{a}}F_{\underline{a}\overline{BC}}F^{\underline{d}\overline{BC}}E_{\underline{d}}\epsilon - \frac{1}{12}F^{(3)}_{\underline{abc}}\gamma^{\underline{abc}}\epsilon - \frac{1}{2}F^{(3)}_{\underline{a}}\gamma^{\underline{a}}\epsilon . \qquad (3.37e)
\end{aligned}$$

JHEP07(2021)092

In appendix B.2 we show that the algebra of these transformation rules closes, up to terms with two fermions, with the following field-dependent parameters

$$\xi^{\mathbb{M}}_{12} = [\xi_1,\xi_2]^{\mathbb{M}}_{C_f} - \frac{1}{\sqrt{2}} E^{\mathbb{M}}{}_{\underline{a}}\bar{\epsilon}_1\gamma^{\underline{a}}\epsilon_2 + b\Gamma_{[1}^{\overline{CD}}\partial^{\mathbb{M}}\Gamma_{2]\overline{CD}} + \frac{b}{8}F^{\mathbb{M}}{}_{\overline{CD}}F^{*\overline{CD}}_{\underline{b}}\bar{\epsilon}_1\gamma^{\underline{b}}\epsilon_2\,, \tag{3.38a}$$

$$\begin{aligned}\Gamma_{12\overline{AB}} =& 2\xi^P_{[1}\partial_P\Gamma_{2]\overline{AB}} - 2\Gamma_{[1\overline{A}}{}^{\overline{C}}\Gamma_{2]\overline{CB}} + \frac{b}{2}E_{\overline{B}}\Gamma_{[1}^{\overline{CD}}E_{\overline{A}}\Gamma_{2]\overline{CD}}\\ &+ b\bar{\epsilon}_{[1}\gamma^{\underline{b}}\Psi_{[\overline{A}}E^M{}_{\overline{B}]}\partial_M\Gamma_{2]}^{\overline{CD}}F^*_{\underline{b}\overline{CD}} + \frac{b}{4}\bar{\epsilon}_1\gamma^{\underline{b}}\epsilon_2 F^*_{\underline{b}\overline{AB}}\,,\end{aligned} \tag{3.38b}$$

$$\begin{aligned}\Gamma_{12\underline{ab}} =& 2\xi^P_{[1}\partial_P\Gamma_{2]\underline{ab}} - 2\Gamma_{[1\underline{a}}{}^{\underline{c}}\Gamma_{2]\underline{cb}} + \frac{b}{2}E_{\underline{b}}\Gamma_{[1}^{\overline{CD}}E_{\underline{a}}\Gamma_{2]\overline{CD}}\\ &+ b\bar{\epsilon}_{[1}\gamma_{[\underline{a}}\Psi^{\overline{B}}F^{*\overline{CD}}_{\underline{b}]}E^M{}_{\overline{B}}\partial_M\Gamma_{2]\overline{CD}}\,,\end{aligned} \tag{3.38c}$$

$$\epsilon_{12} = -\frac{1}{2}\Gamma_{[1\underline{bc}}\gamma^{\underline{bc}}\epsilon_{2]} + 2\xi^P_{[1}\partial_P\epsilon_{2]} - \frac{b}{4}\gamma^{\underline{bc}}\epsilon_{[1}E^M{}_{\underline{b}}\partial_M\Gamma_{2]\overline{CD}}F^{*\overline{CD}}_{\underline{c}}\,. \tag{3.38d}$$

3.4 First order corrections to $\mathcal{N}=1$ supersymmetric DFT

The invariant action under the transformation rules (3.7) is clearly of the same functional form as (2.22) but it depends on the $\mathrm{O}(10,10+k)$ multiplets, namely

$$S_{\mathcal{N}=1\ \mathrm{DFT}} = \int d^{20+k}X\, e^{-2d}\left(\mathcal{R}(\mathcal{E},d) + \bar{\Psi}^{\overline{\mathcal{A}}}\gamma^{\underline{b}}\nabla_{\underline{b}}\Psi_{\overline{\mathcal{A}}} - \bar{\rho}\gamma^{\underline{a}}\nabla_{\underline{a}}\rho + 2\bar{\Psi}^{\overline{\mathcal{A}}}\nabla_{\overline{\mathcal{A}}}\rho\right)\,. \tag{3.39}$$

Hence it contains higher derivatives of the $\mathrm{O}(10,10+n_g)$ multiplets.

The transformation rules (3.7) define the following Lichnerowicz principle,

$$\left(\gamma^{\underline{\mathcal{A}}}\nabla_{\underline{\mathcal{A}}}\gamma^{\underline{\mathcal{B}}}\nabla_{\underline{\mathcal{B}}} - \nabla^{\overline{\mathcal{A}}}\nabla_{\overline{\mathcal{A}}}\right)\epsilon = -\frac{1}{4}\mathcal{R}\epsilon\,, \tag{3.40}$$

$$\left[\nabla_{\overline{\mathcal{A}}},\gamma^{\underline{\mathcal{B}}}\nabla_{\underline{\mathcal{B}}}\right]\epsilon = \frac{1}{2}\gamma^{\underline{\mathcal{B}}}\mathcal{R}_{\overline{\mathcal{A}}\underline{\mathcal{B}}}\epsilon\,, \tag{3.41}$$

and then the $\mathrm{O}(10,10+k)$ generalized Ricci scalar

$$\mathcal{R} = 2\mathcal{E}_{\underline{\mathcal{A}}}\mathcal{F}^{\underline{\mathcal{A}}} + \mathcal{F}_{\underline{\mathcal{A}}}\mathcal{F}^{\underline{\mathcal{A}}} - \frac{1}{6}\mathcal{F}_{\underline{\mathcal{ABC}}}\mathcal{F}^{\underline{\mathcal{ABC}}} - \frac{1}{2}\mathcal{F}_{\overline{\mathcal{A}}\underline{\mathcal{BC}}}\mathcal{F}^{\overline{\mathcal{A}}\underline{\mathcal{BC}}} \tag{3.42}$$

determines the corrections to the generalized Dirac operator.

In terms of the $\mathrm{O}(10,10+n_g)$ generalized fluxes, the $\mathrm{O}(10,10+k)$ generalized Ricci scalar is, up to first order,

$$\begin{aligned}\mathcal{R} = \mathbb{R} + b\mathbb{R}^{(1)} =& \mathbb{R} - F^{(3)}_{\overline{A}\underline{bc}}F^{\overline{A}\underline{bc}} - \frac{1}{3}F^{(3)}_{\underline{abc}}F^{\underline{abc}} + 2F^{(3)}_{\underline{d}}F^{\underline{d}} + 2E_{\underline{a}}F^{(3)\underline{a}}\\ &+ \frac{b}{4}E_{\underline{d}}F^{\underline{a}}F^{*\underline{d}}{}_{\overline{BC}}F^{*\overline{BC}}_{\underline{a}} + \frac{b}{8}F^{(2)\overline{AB}}{}_{\underline{cd}}F^{(2)\underline{cd}}_{\overline{AB}}\,,\end{aligned} \tag{3.43}$$

where $\mathbb{R}$ was defined in (2.25). Replacing the expressions (3.23) with the overlined indices extended to include the gauge sector (i.e. $\bar{c},\bar{d},\ldots \to \overline{C},\overline{D},\ldots$), $\mathbb{R}^{(1)}$ may be written as

$$\begin{aligned}
\mathbb{R}^{(1)} = \frac{1}{4}\Big[&(E_{\underline{a}}E_{\underline{b}}F^{*\underline{b}}{}_{\overline{CD}})F^{*\underline{a}\overline{CD}} + (E_{\underline{a}}E_{\underline{b}}F^{*\underline{a}}{}_{\overline{CD}})F^{*\underline{b}\overline{CD}} + 2(E_{\underline{a}}F^{*\overline{CD}}_{\underline{b}})F^{*\underline{a}}{}_{\overline{CD}}F^{\underline{b}} \\
&+ (E_{\underline{a}}F^{*\underline{a}\overline{CD}})(E_{\underline{b}}F^{*\underline{b}}{}_{\overline{CD}}) + (E_{\underline{a}}F^{*\overline{CD}}_{\underline{b}})(E^{\underline{a}}F^{*\underline{b}}{}_{\overline{CD}}) + 2(E_{\underline{a}}F_{\underline{b}})F^{*\underline{b}}{}_{\overline{CD}}F^{*\underline{a}\overline{CD}} \\
&+ (E_{\overline{A}}F^{*}_{\underline{b}\overline{CD}})F^{*\overline{CD}}_{\underline{c}}F^{*\overline{A}\underline{bc}} - (E_{\underline{a}}F^{*}_{\underline{b}\overline{CD}})F^{*\overline{CD}}_{\underline{c}}F^{\underline{abc}} + 2(E_{\underline{a}}F^{*\underline{a}}{}_{\overline{CD}})F^{*\overline{CD}}_{\underline{b}}F^{\underline{b}} \\
&- 4(E_{\underline{a}}F^{*\overline{CD}}_{\underline{b}})F^{*\underline{a}}{}_{\overline{CE}}F^{*\underline{b}\overline{E}}{}_{\overline{D}} + \frac{4}{3}F^{*\overline{E}}{}_{\underline{a}\overline{C}}F^{*}_{\underline{b}\overline{ED}}F^{*\overline{CD}}_{\underline{c}}F^{\underline{abc}} + F^{*\underline{b}}{}_{\overline{CD}}F^{*\overline{CD}}_{\underline{a}}F_{\underline{b}}F^{\underline{a}} \\
&+F^{*\overline{CE}}_{\underline{a}}F^{*}_{\underline{b}\overline{ED}}F^{*\underline{a}}{}_{\overline{CG}}F^{*\underline{b}\overline{GD}} - F^{*\overline{CE}}_{\underline{b}}F^{*}_{\underline{a}\overline{ED}}F^{*\underline{a}}{}_{\overline{CG}}F^{*\underline{b}\overline{GD}} - F_{\overline{A}\underline{bd}}F^{*\underline{d}}{}_{\overline{CD}}F^{*\overline{CD}}_{\underline{c}}F^{\overline{A}\underline{bc}}\Big] \,.
\end{aligned} \tag{3.44}$$

Note that it depends on the generalized gravitino through $F^{*}_{\underline{a}\overline{BC}}$.

Similarly, we may define

$$\bar{\Psi}^{\overline{\mathcal{A}}}\gamma^{\underline{b}}\nabla_{\underline{b}}\Psi_{\overline{\mathcal{A}}} - \bar{\rho}\gamma^{\underline{a}}\nabla_{\underline{a}}\rho + 2\bar{\Psi}^{\overline{\mathcal{A}}}\nabla_{\overline{\mathcal{A}}}\rho \equiv \mathbb{L}_{\mathrm{F}} + \mathbb{L}_{\mathrm{F}}^{(1)}\,, \tag{3.45}$$

where $\mathbb{L}_{\mathrm{F}}$ was introduced in (2.23) and the first order corrections are given by

$$\begin{aligned}
\mathbb{L}_{\mathrm{F}}^{(1)} = \frac{1}{2}\Big[&\frac{1}{4}\bar{\Psi}^{\overline{A}}\gamma^{\underline{b}}E_{\underline{c}}\Psi_{\overline{A}}F^{\underline{c}\overline{CD}}F_{\underline{b}\overline{CD}} - \frac{1}{8}\bar{\Psi}^{\overline{A}}\gamma^{\underline{bcd}}\Psi_{\overline{A}}(E_{\underline{b}}F_{\underline{c}\overline{CD}})F_{\underline{d}}{}^{\overline{CD}} \\
&+\frac{1}{16}\bar{\Psi}^{\overline{A}}\gamma^{\underline{bcd}}\Psi_{\overline{A}}F_{\underline{abc}}F^{\underline{a}}{}_{\overline{CD}}F_{\underline{d}}{}^{\overline{CD}} + \frac{1}{12}\bar{\Psi}^{\overline{A}}\gamma^{\underline{bcd}}\Psi_{\overline{A}}F_{\underline{b}\overline{C}}{}^{\overline{E}}F_{\underline{c}\overline{ED}}F_{\underline{d}}{}^{\overline{CD}} \\
&-\frac{1}{4}\bar{\Psi}^{\overline{A}}\gamma^{\underline{b}}\Psi^{\overline{C}}F_{\underline{b}}{}^{\overline{EF}}F_{\underline{d}\overline{AC}}F^{\underline{d}}{}_{\overline{EF}} + 2\bar{\Psi}^{\overline{A}}\gamma^{\underline{b}}\Psi_{\overline{CD}}(E_{\overline{A}}F_{\underline{b}}{}^{\overline{CD}}) - 2\bar{\Psi}^{\overline{A}}\gamma^{\underline{b}}\Psi_{\overline{CD}}F_{\overline{A}\underline{bc}}F^{\underline{c}\overline{CD}} \\
&-2\bar{\Psi}^{\overline{CD}}\gamma^{\underline{b}}E_{\underline{b}}\Psi_{\overline{CD}} - \frac{1}{6}\bar{\Psi}^{\overline{CD}}\gamma^{\underline{bcd}}\Psi_{\overline{CD}}F_{\underline{bcd}} - 4\bar{\Psi}_{\overline{CE}}\gamma^{\underline{b}}\Psi^{\overline{E}}{}_{\overline{D}}F_{\underline{b}}{}^{\overline{CD}} - \frac{1}{4}\bar{\rho}\gamma^{\underline{a}}E_{\underline{b}}\rho F^{\underline{b}\overline{CD}}F_{\underline{a}\overline{CD}} \\
&+\frac{1}{8}\bar{\rho}\gamma^{\underline{abc}}\rho E_{\underline{a}}F_{\underline{b}\overline{CD}}F_{\underline{c}}{}^{\overline{CD}} - \frac{1}{16}\bar{\rho}\gamma^{\underline{abc}}\rho F_{\underline{abd}}F^{\underline{d}}{}_{\overline{CD}}F_{\underline{c}}{}^{\overline{CD}} - \frac{1}{12}\bar{\rho}\gamma^{\underline{abc}}\rho F_{\underline{a}\overline{C}}{}^{\overline{E}}F_{\underline{b}\overline{ED}}F_{\underline{c}}{}^{\overline{CD}} \\
&-\frac{1}{4}\bar{\Psi}^{\overline{A}}\gamma^{\underline{bc}}\rho(E_{\overline{A}}F_{\underline{b}}{}^{\overline{CD}})F_{\underline{c}\overline{CD}} + \frac{1}{4}\bar{\Psi}^{\overline{A}}\gamma^{\underline{bc}}\rho F_{\overline{A}\underline{bd}}F^{\underline{d}}{}_{\overline{CD}}F_{\underline{c}}{}^{\overline{CD}} - 2\bar{\Psi}^{\overline{CD}}F^{\underline{a}}{}_{\overline{CD}}E_{\underline{a}}\rho \\
&+\bar{\Psi}^{\overline{CD}}\gamma^{\underline{ab}}\rho(E_{\underline{a}}F_{\underline{b}\overline{CD}}) - \bar{\Psi}^{\overline{CD}}\gamma^{\underline{ab}}\rho F_{\underline{a}\overline{C}}{}^{\overline{E}}F_{\underline{b}\overline{ED}} - \frac{1}{2}\bar{\Psi}^{\overline{CD}}\gamma^{\underline{ab}}\rho F_{\underline{abc}}F^{\underline{c}}{}_{\overline{CD}}\Big]\,.
\end{aligned} \tag{3.46}$$

In conclusion, the manifestly duality covariant first order corrections to the action of $\mathcal{N}=1$ supersymmetric DFT (2.22) in terms of $\mathrm{O}(10,10+n_g)$ multiplets are given by the addition of $\mathbb{R}^{(1)}$ and $\mathbb{L}_{\mathrm{F}}^{(1)}$, up to bilinear terms in fermions. We have explicitly verified that the action

$$S_{\mathcal{N}=1\ \mathrm{DFT}} = \int d^{20+n_g}X e^{-2d}\left(\mathbb{R} + \mathbb{R}^{(1)} + \mathbb{L}_F + \mathbb{L}_F^{(1)}\right)\,, \tag{3.47}$$

is invariant under the transformation rules (3.37), up to terms with four derivatives and two fermions. The structure constants preserve a global $\mathrm{O}(10,10;\mathbb{R})$ symmetry.

4 Transformation rules of the supergravity fields

To make contact with the heterotic string low energy effective field theory, in this section we parameterize the $\mathrm{O}(10,10+n_g)$ duality multiplets in terms of supergravity and super

JHEP07(2021)092

Yang-Mills multiplets, we analyze the deformations of the symmetry transformation rules and compare with previous proposals in the literature.

The deformed transformation rules of the duality multiplets (3.37) induce higher derivative corrections on the transformation rules of the supergravity and super Yang-Mills fields that parameterize the generalized fields (2.32), (2.34) and (2.35). We then expect an α'-expansion of the parameterizations, that we now denote $\widetilde{e}_\mu{}^a, \widetilde{b}_{\mu\nu}, \widetilde{\phi}, \widetilde{A}^i_\mu, \widetilde{\psi}_\mu, \widetilde{\lambda}, \widetilde{\chi}_i$, in terms of the gauge and Lorentz covariant fields, e.g. $\widetilde{e}_\mu{}^a = e_\mu{}^a + \mathcal{O}(\alpha')\,, \widetilde{b}_{\mu\nu} = b_{\mu\nu} + \mathcal{O}(\alpha')\,, \widetilde{\psi}_\mu = \psi_\mu + \mathcal{O}(\alpha')\,$, etc.

To find the relations between both sets of fields, it is convenient to first work out the parameterizations of the generalized fluxes and curvatures and their transformation rules. From the first order terms in the action (3.47), we see that only the leading order expressions are necessary. We denote the parameterization of $F^*_{\underline{a}\overline{CD}}$ as

$$\hat{\Omega}_{aCD} = \left(\hat{w}^{(-)}_{acd}, \hat{F}^i_{ac}, \hat{A}_a{}^{ij}\right)\,, \tag{4.1}$$

where the hats distinguish objects that contain fermions and the collective indices of the tangent space $C = (c, i)$ include the gauge indices. In terms of supergravity and super Yang-Mills fields, the components are

$$\hat{w}^{(-)}_{abc} \equiv \left(w^{(-)}_{\mu bc} - \frac{1}{2}\overline{\psi}_b\gamma_\mu\psi_c\right) e^\mu{}_a\,, \tag{4.2}$$

with $w^{(\pm)}_{abc} = w_{abc} \pm \frac{1}{2}H_{abc}$,

$$\hat{F}_{ab}{}^i \equiv -\frac{1}{\sqrt{2}}\left(F^i_{\mu\nu} - \frac{1}{2}\overline{\psi}_{[\mu}\gamma_{\nu]}\chi^i\right) e^\mu{}_a e^\nu{}_b\,, \tag{4.3}$$

and

$$\hat{A}_a{}^{ij} \equiv -\left(A^k_\mu\, f_k{}^{ij} + \frac{1}{4}\overline{\chi}^i\gamma_\mu\chi^j\right) e^\mu{}_a\,. \tag{4.4}$$

The generalized gravitino curvature $\Psi_{\overline{AB}}$ is parameterized as

$$\widetilde{\Psi}_{AB} = \Psi_{AB} - \frac{1}{2}\hat{\Omega}_{cAB}\psi^c \equiv \psi_{AB} - \frac{1}{2\sqrt{2}}\hat{\Omega}_{iAB}\chi^i - \frac{1}{2}\hat{\Omega}_{cAB}\psi^c\,, \tag{4.5}$$

with

$$\psi_{ab} \equiv e^\mu{}_{[a}e^\nu{}_{b]}D^{(+)}_\mu\psi_\nu\,, \tag{4.6a}$$

$$\psi_{ai} = \frac{1}{2\sqrt{2}}\left(\partial_c\chi_i - \frac{1}{4}\hat{w}^{(+)}_{abc}\gamma^{bc}\chi_i - \frac{1}{2\sqrt{2}}\hat{F}_{bci}\gamma^{bc}\psi_a\right)\,, \tag{4.6b}$$

$$\psi_{ij} = \frac{1}{4\sqrt{2}}\hat{F}_{bc[i}\gamma^{bc}\chi_{j]}\,, \tag{4.6c}$$

and

$$\hat{\Omega}_{iAB} = \left(\hat{F}_{abi}, \hat{A}_{aij}, \sqrt{2}f_{ijk}\right) \tag{4.7}$$

is the parameterization of the generalized flux component $F_{\overline{ABi}}$.

JHEP07(2021)092

Parameterizing the Lorentz and supersymmetry transformation rules of $F^*_{\underline{a}\overline{BC}}$, namely

$$\delta F^*_{\underline{a}\overline{BC}} = -E_{\underline{a}}\Gamma_{\overline{BC}} + \Gamma^{\underline{b}}{}_{\underline{a}}F^*_{\underline{b}\overline{BC}} - 2\Gamma^{\overline{D}}{}_{[\overline{B}}F^*_{\overline{C}]\overline{D}\underline{a}} + \bar{\epsilon}\gamma_{\underline{a}}\Psi_{\overline{BC}}\,, \tag{4.8}$$

we get

$$\delta\hat{\Omega}_{\mu CD} = -\partial_\mu\Lambda_{CD} + 2\hat{\Omega}_{\mu B[D}\Lambda^B{}_{C]} + \bar{\epsilon}\gamma_\mu\Psi_{CD}\,, \tag{4.9}$$

where the generalized Lorentz parameters $\Gamma_{\underline{ab}}$ and $\Gamma_{\overline{AB}}$ are parameterized as $-\widetilde{\Lambda}_{ab}+\bar{\epsilon}\gamma_{[a}\widetilde{\psi}_{b]}$ and $\widetilde{\Lambda}_{AB} = (\widetilde{\Lambda}_{ab}, \widetilde{\Lambda}_{ai}, \widetilde{\Lambda}_{ij})$, with $\widetilde{\Lambda}_{AB} = \Lambda_{AB} + \mathcal{O}(\alpha')$, and Λ_{ab} is the generator of $\mathrm{O}(1,9)$ transformations, while $\Lambda_{ai} = \frac{1}{2\sqrt{2}}\bar{\epsilon}\gamma_a\chi_i$ and $\Lambda_{ij} = f_{ijk}\xi^k$ depend on the supersymmetry and gauge parameters according to (2.38).

The transformation rule (4.9) contains, other than the standard Lorentz transformations, the supersymmetry variation of the torsionful spin connection [26, 27]

$$\delta_\epsilon\hat{w}^{(-)}_{\mu bc} = \bar{\epsilon}\gamma_\mu\psi_{bc} + \frac{3}{4}\bar{\epsilon}\gamma_{[\rho}\chi_i\hat{F}^i_{\mu\nu]}e^\nu{}_b e^\rho{}_c\,, \tag{4.10}$$

the supersymmetry and gauge transformations of the Yang-Mills field strength,

$$\delta_\epsilon\hat{F}_{\mu ci} = \frac{1}{2}\left[D_\mu\left(\bar{\epsilon}\gamma_c\chi_i\right) - \bar{\epsilon}\gamma_\mu D_c\chi_i + \frac{1}{4}\bar{\epsilon}\gamma_\mu\left(\frac{1}{2}H_{c\nu\rho}\gamma^{\nu\rho}\chi_i - \hat{F}_{\nu\rho i}\gamma^{\nu\rho}\psi_c\right)\right] \tag{4.11}$$

and $\delta_\xi\hat{F}_{\mu ci} = f_{ijk}\xi^j\hat{F}_{\mu c}{}^k$, as well as the leading order gauge and supersymmetry transformations of the Yang-Mills connection, (2.39d) and (2.43) respectively.

Similarly, from the transformation rule of the generalized gravitino curvature

$$\delta\Psi_{\overline{AB}} = 2\Psi_{\overline{C}[\overline{B}}\Gamma^{\overline{C}}{}_{\overline{A}]} + \frac{1}{4}\Gamma_{\underline{cd}}\gamma^{\underline{cd}}\Psi_{\overline{AB}} + \frac{1}{2}E_{\overline{C}}\Gamma_{\overline{AB}}\Psi^{\overline{C}} + \frac{1}{2}F^{*\underline{c}}{}_{\overline{AB}}E_{\underline{c}}\epsilon + \frac{1}{8}\mathcal{F}^{(2)}_{\underline{cd}\overline{AB}}\gamma^{\underline{cd}}\epsilon \tag{4.12}$$

we obtain

$$\delta\Psi_{CD} = 2\Psi_{B[D}\Lambda^B{}_{C]} + \frac{1}{8}\hat{\mathfrak{R}}_{\mu\nu CD}\gamma^{\mu\nu}\epsilon\,, \tag{4.13}$$

where we have defined

$$\hat{\mathfrak{R}}_{\mu\nu CD} = -2\partial_{[\mu}\hat{\Omega}_{\nu]CD} + 2\hat{\Omega}_{[\mu|C|}{}^E\hat{\Omega}_{\nu]ED}\,, \tag{4.14}$$

which has components

$$\hat{\mathfrak{R}}_{\mu\nu cd} = \hat{R}^{(-)}_{\mu\nu cd} - \hat{F}_{\mu\tau}{}^i\hat{F}_{\nu\lambda i}e^\tau{}_{[c}e^\lambda{}_{d]}\,, \tag{4.15}$$

$$\hat{\mathfrak{R}}_{\mu\nu c}{}^i = \sqrt{2}\left(D^{(-)}_{[\mu}\hat{F}^i_{\nu]c} + \frac{1}{4}\overline{\chi}^i\gamma_{[\mu}\chi^j\hat{F}_{\nu]cj}\right)\,, \tag{4.16}$$

$$\hat{\mathfrak{R}}_{\mu\nu}{}^{ij} = F^k_{\mu\nu}f^{ij}{}_k + \hat{F}^{i\lambda}{}_{[\mu}\hat{F}^j_{\nu]\lambda} + \frac{1}{2}D_{[\mu}\left(\overline{\chi}^i\gamma_{\nu]}\chi^j\right)\,. \tag{4.17}$$

In particular, (4.13) contains the supersymmetry transformation rule of the supergravity gravitino curvature

$$\delta_\epsilon\psi_{ab} = \frac{1}{8}\left(\hat{R}^{(-)}_{\mu\nu ab} + \frac{3}{2}\hat{T}_{\mu\nu ab}\right)\gamma^{\mu\nu}\epsilon\,, \tag{4.18}$$

where $\hat{R}^{(-)}_{\mu\nu ab}$ is the two-form curvature computed from the torsionful spin connection $\hat{w}^{(-)}_{\mu ab}$ and $\hat{T}_{\mu\nu ab} = \hat{F}^i_{[\mu\nu}\hat{F}_{ab]i}$, in agreement with [26, 27].

Now we turn to the parameterization of the elementary fields. We start from the deformed transformation rules of the components $E_M{}^{\overline{a}}$ and $E_M{}^{\underline{a}}$ given in (3.37a) and (3.37b). Of course, different definitions lead to supergravity multiplets that obey different transformation rules. An interesting one is the following

$$\tilde{e}_\mu{}^a = e_\mu{}^a - \frac{b}{8}\left(\hat{w}^{(-)}_{bcd}\hat{w}^{(-)acd} + 2\hat{T}_b{}^a + \hat{A}_{bij}\hat{A}^{aij}\right) e_\mu{}^b\,, \tag{4.19}$$

$$\tilde{\phi} = \phi - \frac{b}{16}\left(\hat{w}^{(-)acd}\hat{w}^{(-)}_{acd} + 2\hat{T} + \hat{A}^{aij}\hat{A}_{aij}\right)\,, \tag{4.20}$$

where $\hat{T}_{ab} = \hat{F}_{aci}\hat{F}_b{}^{ci}$ and $\hat{T} = \hat{F}^i_{ac}\hat{F}^{ac}_i$. The quadratic terms in spin and gauge connections are known to be necessary in order to remove the non-standard Lorentz transformations of the supergravity vielbein $e_\mu{}^a$ and dilaton ϕ fields [67, 68]. Together with the gauge covariant $\hat{T}$ terms, these parameterizations determine $e_\mu{}^a$ and ϕ fields that obey the leading order supersymmetry and Lorentz transformation rules (2.39a) and (2.42). To get this result, the gauge fixings $\tilde{e}^\mu{}_{\overline{a}} = \tilde{e}^\mu{}_{\underline{a}} \equiv \tilde{e}^\mu{}_a$, $\delta E^i{}_{\bar{i}} = 0$ and $\delta E^\mu{}_{\bar{i}} = 0$ are used to absorb several terms into the Lorentz parameters. As a consequence, the following parameterization is needed for the duality covariant gravitino

$$\tilde{\psi}_a = \psi_a - \frac{b}{2}\hat{\Omega}_{aCD}\Psi^{CD} + \frac{b}{8}\hat{\Omega}_a{}^{CD}\hat{\Omega}_{bCD}\psi^b\,. \tag{4.21}$$

Interestingly, these parameterizations induce a deformation of the gravitino supersymmetry variation (2.39c) that can be absorbed into the torsion of the spin connection through the following modification of the two-form curvature

$$\begin{aligned}\widetilde{H}_{\mu\nu\rho} = 3\Big[&\partial_{[\mu}\tilde{b}_{\nu\rho]} - \zeta C^{(g)}_{\mu\nu\rho} + \frac{b}{2}\hat{C}^{(L)}_{\mu\nu\rho} + \frac{b}{2}\hat{F}_{[\mu}{}^{ci}D^{(-)}_\nu\hat{F}_{\rho]ci} + \frac{b}{8}A^k_{[\mu}\partial_\nu\left(\overline{\chi}^i\gamma_{\rho]}\chi^j\right)f_{ijk}\\ &+\frac{b}{8}\overline{\chi}^i\gamma_{[\mu}\chi^j\left(\partial_\nu A_{\rho]}{}^k - A^l_\nu A_{\rho]}{}^m f^k{}_{lm}\right)f_{ijk} - \frac{b}{8}\overline{\chi}^i\gamma_{[\mu}\chi^j\hat{F}_\nu{}^{ci}\hat{F}^j_{\rho]c}\Big]\,.\end{aligned} \tag{4.22}$$

The Yang-Mills Chern-Simons form $C^{(g)}_{\mu\nu\rho}$ was defined in (2.41), the coefficient

$$\zeta = 1 + \frac{1}{2}b\varrho\,, \qquad \varrho\kappa_{ij} = f_i{}^{kl}f_{jlk}\,, \tag{4.23}$$

and $\hat{C}^{(L)}_{\mu\nu\rho}$ denotes the Lorentz Chern-Simons form of the torsionful spin connection $\hat{w}^{(-)}_{\mu ab}$,

$$\hat{C}^{(L)}_{\mu\nu\rho} = \hat{w}^{(-)cd}_{[\mu}\partial_\nu\hat{w}^{(-)}_{\rho]cd} + \frac{2}{3}\hat{w}^{(-)bc}_{[\mu}\hat{w}^{(-)}_{\nu cd}\hat{w}^{(-)d}_{\rho]}{}_b\,. \tag{4.24}$$

The gaugino bilinear terms in (4.22) may be absorbed into the first order deformation of the Yang-Mills Chern-Simons form replacing $A^i_\mu \to \hat{A}_\mu{}^{jk}$, but this is not convenient for reasons that will become clear shortly.

The modified three-form $\widetilde{H}_{\mu\nu\rho}$ (4.22) may be rewritten as the compact expression

$$\widetilde{H}_{\mu\nu\rho} = 3\left[\partial_{[\mu}\tilde{b}_{\nu\rho]} - C^{(g)}_{\mu\nu\rho} + \frac{b}{2}\hat{\mathcal{C}}_{\mu\nu\rho}\right]\,, \tag{4.25}$$

where

$$\hat{\mathcal{C}}_{\mu\nu\rho} = \partial_{[\mu}\hat{\Omega}_\nu{}^{CD}\hat{\Omega}_{\rho]CD} + \frac{2}{3}\hat{\Omega}_{[\mu|CD|}\hat{\Omega}_\nu{}^{DE}\hat{\Omega}_{\rho]E}{}^C\,. \tag{4.26}$$

Likewise, a parameterization of the dilatino analogous to (4.21) also induces the replacement of the lowest order $H_{\mu\nu\rho}$ by $\widetilde{H}_{\mu\nu\rho}$ in the supersymmetry transformation rule (2.39c), so that the combination $\widetilde{\rho} = 2\widetilde{\lambda} + \gamma^a\widetilde{\psi}_a$ and its supersymmetry transformation rule are not deformed, i.e. $\widetilde{\rho} = \rho$ and $\delta_\epsilon\rho = \delta_\epsilon^{(0)}\rho$.

From $\delta E_\mu{}^{\bar{i}}$ and $\delta\Psi_{\bar{i}}$ in (3.37), one can see that the gauge and gaugino transformation rules are not deformed and hence it is not necessary to redefine these fields.

Finally, from the transformation rules of the components $E_{\mu\bar{a}}$ or $E_{\mu\underline{a}}$, and using the parameterizations defined above, we get

$$\delta^{(1)}\tilde{b}_{\mu\nu} = -\frac{b}{2}\left(\partial_{[\mu}\Lambda^{CD}\hat{\Omega}_{\nu]CD} + \bar{\epsilon}\gamma_{[\mu}\Psi^{CD}\hat{\Omega}_{\nu]CD}\right) . \tag{4.27}$$

This compact expression contains information about the gauge, Lorentz and supersymmetry transformations of the $\widetilde{b}-$field, which we now analyze separately.

Expanding the first term in (4.27) one gets

$$-\frac{b}{2}\partial_{[\mu}\Lambda^{CD}\hat{\Omega}_{\nu]CD} = -\frac{b}{2}\left(\partial_{[\mu}\Lambda^{cd}\hat{w}^{(-)}_{\nu]cd} + \partial_{[\mu}\xi^k\hat{A}_{\nu]}{}^{ij}f_{ijk} - \frac{1}{2}\partial_{[\mu}\left(\bar{\epsilon}\gamma^c\chi^i\right)\hat{F}_{\nu]ci}\right) . \tag{4.28}$$

The first term in the r.h.s. is the Lorentz sector of the Green-Schwarz transformation [80], which requires the Lorentz Chern-Simons form (4.24) in $\widetilde{H}_{\mu\nu\rho}$. It cannot be eliminated through redefinitions of the b-field [67]. The bilinear fermionic terms in $\hat{w}^{(-)}_{\nu cd}$ may be canceled redefining $\widetilde{b}_{\mu\nu} = b_{\mu\nu} - \frac{b}{2}w_{[\mu}{}^{cd}\overline{\psi}_c\gamma_{\nu]}\psi_d$, but we choose not to do this because (4.24) is defined with the corresponding fermionic contribution and then $\widetilde{H}_{\mu\nu\rho}$ is Lorentz invariant.

The bosonic piece of the second term in (4.28), i.e. $\frac{b}{2}\partial_{[\mu}\xi^k A_{\nu]}{}^l f_l{}^{ij}f_{ijk}$, is the first order correction to the Yang-Mills Green-Schwarz transformation in (2.43), reflecting the ϱ deformation of the Killing metric in (4.23). This transformation cannot be eliminated through redefinitions of the b-field either. Instead, it is convenient to cancel the fermionic terms in $\hat{A}_{\mu ij}$ redefining

$$\tilde{b}_{\mu\nu} = b_{\mu\nu} + \frac{b}{8}A^k_{[\mu}\overline{\chi}^i\gamma_{\nu]}\chi^j f_{ijk} , \tag{4.29}$$

in order to compare with standard results. With this redefinition (4.22) becomes

$$\widetilde{H}_{\mu\nu\rho} = \overline{H}_{\mu\nu\rho} + \frac{3b}{2}\left(D^{(-)}_{[\mu}\hat{F}_\nu{}^{ci}\hat{F}_{\rho]ci} - \frac{1}{4}\overline{\chi}^i\gamma_{[\mu}\chi^j\hat{F}_\nu{}^{ci}\hat{F}^j_{\rho]c} + \frac{1}{4}\overline{\chi}^i\gamma_{[\mu}\chi^j F_{\nu\rho]}{}^k f_{ijk}\right) , \tag{4.30}$$

where

$$\overline{H}_{\mu\nu\rho} = 3\left(\partial_{[\mu}b_{\nu\rho]} - \zeta C^{(g)}_{\mu\nu\rho} + \frac{b}{2}\hat{C}^{(L)}_{\mu\nu\rho}\right) . \tag{4.31}$$

Finally the third term in (4.28) together with the second term in (4.27) contain the first order deformations of the supersymmetry transformation of $b_{\mu\nu}$, i.e.

$$\delta^{(1)}_\epsilon b_{\mu\nu} = \frac{b}{2}\left(\hat{w}^{(-)cd}_{[\mu}\delta_\epsilon\hat{w}^{(-)}_{\nu]cd} - \varrho A^i_{[\mu}\delta_\epsilon A_{\nu]i} + \hat{F}_{[\mu}{}^{ci}\delta_\epsilon\hat{F}_{\nu]ci} + D^{(-)}_{[\mu}\left(\bar{\epsilon}\gamma^b\chi^i\right)\hat{F}_{\nu]bi}\right) . \tag{4.32}$$

The first term in (4.32) was originally introduced in [25] to restore manifest Lorentz covariance to the supersymmetry variation of the b-field curvature. It was later reobtained in [26] as a consequence of the assumption that the Yang-Mills and torsionful spin connections should appear symmetrically in ten dimensional $\mathcal{N} = 1$ supergravity coupled to

JHEP07(2021)092

super Yang-Mills. The second term in (4.32) reflects the ϱ deformation of the Killing metric (4.23) in the zeroth order supersymmetry transformation (2.39b). These two terms are the obvious analogs of the Lorentz and Yang-Mills Green-Schwarz transformations

$$\delta_\Lambda b_{\mu\nu} = -\frac{b}{2}\partial_{[\mu}\Lambda^{cd}\hat{w}^{(-)}_{\nu]cd}\,, \qquad \delta_\xi b_{\mu\nu} = -\zeta\partial_{[\mu}\xi^k A^k_{\nu]}\,, \tag{4.33}$$

as already noticed in [25]. Here, these transformations follow directly from the manifestly duality covariant formulation of the theory.

Interestingly, the second term in (4.27) can be obtained from the leading order transformation of the 2-form in (2.39b) with the identifications $A^i_\mu \leftrightarrow \hat{\Omega}_\mu{}^{CD}, \chi^i \leftrightarrow \Psi^{CD}$, i.e. a generalization of the symmetry $A^i_\mu \leftrightarrow \hat{w}^{(-)cd}_\mu, \chi^i \leftrightarrow \psi^{cd}$ that was used in [26, 27] to obtain the Riemann squared superinvariant. The generalized identification plays a crucial role in the proof of supersymmetric invariance of the first order action, as we discuss in the next section and show in appendix C.

Summing up, the definitions (4.19)–(4.21) and (4.29) lead to supergravity and super Yang-Mills fields that obey the leading order transformation rules, except for the first order deformations in (4.32) and the replacement $H_{\mu\nu\rho} \to \widetilde{H}_{\mu\nu\rho}$ in the supersymmetry transformations of the gravitino and dilatino, i.e.

$$\delta_\epsilon\psi_\mu = \partial_\mu\epsilon - \frac{1}{4}\widetilde{w}^{(+)}_{\mu ab}\gamma^{ab}\epsilon\,, \qquad \delta_\epsilon\lambda = -\frac{1}{2}\gamma^\mu\partial_\mu\phi\epsilon + \frac{1}{24}\widetilde{H}_{abc}\gamma^{abc}\epsilon\,, \tag{4.34}$$

with $\widetilde{w}^{(+)}_{\mu ab} = w_{\mu ab} + \frac{1}{2}\widetilde{H}_{\mu\nu\rho}e^\nu{}_a e^\rho{}_b$. We show in appendix C.1 that these deformed transformation rules obey a closed algebra including up to three-derivative terms and bilinears in fermions.

Clearly, the transformation laws depend on the choice of parameterization. For instance, we could define

$$\tilde{e}'_\mu{}^a = e_\mu{}^a - \frac{b}{8}\left(\hat{w}^{(-)cd}_b\hat{w}^{(-)a}_{cd} + \hat{A}_{bij}\hat{A}^{aij}\right)e_\mu{}^b\,, \tag{4.35}$$

$$\tilde{\phi}' = \phi - \frac{b}{16}\left(\hat{w}^{(-)acd}\hat{w}^{(-)}_{acd} + \hat{A}^{aij}\hat{A}_{aij}\right)\,, \tag{4.36}$$

and similar ones for their superpartners, which are related to the previous parameterizations through gauge and Lorentz covariant field redefinitions. This parameterization is known to reproduce the four-derivative terms in the bosonic sector of the heterotic string effective action when $b = \alpha'$ [68]. Moreover, the fields defined in this way obey the same classical dynamics as the previous (4.19) and (4.20) because the corresponding effective actions will differ by terms proportional to the leading order equations of motion. However, the definitions (4.35)–(4.36) induce complicated first order corrections in the supersymmetry transformation rules of the supergravity fields. Hence, we prefer to keep the fields that obey transformation laws with the smallest amount of deformations.

Before turning to the construction of the invariant action under the modified transformations, we analyze the deformations that were proposed in references [26, 27]. In particular, we wonder if there is a parameterization of the duality covariant vielbein in

JHEP07(2021)092

terms of a gauge covariant one that transforms as proposed in [26] or [27], i.e.

$$\delta^{(1)}\mathbf{e}_\mu{}^a = -\frac{3\alpha'}{32}\bar\epsilon\gamma^{\sigma\tau}\gamma_\mu\psi^\nu T_{\lambda\nu\sigma\tau}e^{\lambda a} \quad \text{or} \quad \delta^{(1)}\mathbf{e}_\mu{}^a = \frac{3\alpha'}{16}\bar\epsilon\gamma_{[\lambda}\chi^i F_{\nu\rho]i}H_\mu{}^{\nu\rho}e^{\lambda a} \tag{4.37}$$

respectively, written here in our conventions. Note that we only examine the gauge dependent terms since the gravitational sectors coincide up to the order we are considering. Specifically, we search for a quantity $E_\mu{}^a$ such that

$$\mathbf{e}_\mu{}^a = e_\mu{}^a + E_\mu{}^a \qquad \text{and} \qquad \delta^{(1)}\mathbf{e}_\mu{}^a = \delta^{(0)}E_\mu{}^a\,. \tag{4.38}$$

The most general expressions that can reproduce either one of (4.37) can be schematically written as

$$E_\mu{}^a = a_1^m\left(\bar\psi_{..}\gamma^{\cdots}\psi_{.}e\right)_\mu{}^a + a_2^m\left(\bar\psi_{.}\gamma^{\cdots}\chi Fe\right)_\mu{}^a \tag{4.39}$$

or as

$$\begin{aligned}E_\mu{}^a = b_1^m H_{bcd}H^{acd}e_\mu{}^b + b_2^m(\bar\psi_{.}\gamma^{\cdots}\psi_{.}He)_\mu{}^a + b_3^m(\bar\rho\gamma^{\cdots}\psi_{.}He)_\mu{}^a + b_4^m(\bar\chi\gamma^{\cdots}\psi_{.}Fe)_\mu{}^a \\ + b_5^m(\bar\chi\gamma^{\cdots}\chi Fe)_\mu{}^a + b_6^m(\bar\rho\gamma^{\cdots}\chi Fe)_\mu{}^a + b_7^m(\bar\chi\gamma^{\cdots}\chi He)_\mu{}^a\,,\end{aligned} \tag{4.40}$$

where the terms between parenthesis refer to all possible contractions of indices and numbers of γ-matrices, numerated by the supraindex m, while $\psi_{.}$ and $\psi_{..}$ denote the gravitino and gravitino curvature, respectively. We found that neither of (4.37) can be reproduced.

Indeed, the supersymmetric generalized Green-Schwarz transformation (3.37), parameterized with the fields that reproduce the bosonic terms of the heterotic effective action, strongly constrains the possible deformations of the theory. In particular, it does not admit the proposals (4.37). This does not imply that the latter are in conflict with string theory. In order to establish the invariance of the action that implements those supersymmetries under $\mathrm{O}(n,n)$ transformations, it should be dimensionally reduced to $10-n$ dimensions. We stress that the deformations (4.32) and (4.34) were obtained from the transformation rules of the $\mathrm{O}(10,10+k)$ multiplets, whose algebra closes exactly. Hence the theory avoids an iterative procedure which only guarantees consistency up to a given order. Moreover, supersymmetry is manifest to all orders and dimensional reductions will preserve the expected T-duality invariance of the theory.

5 Heterotic string effective action to $\mathcal{O}(\alpha')$

In this section we parameterize the $\mathrm{O}(10,10)$ invariant $\mathcal{N}=1$ supersymmetric action (3.47) in terms of the supergravity and super Yang-Mills fields that transform under local supersymmetry according to (2.39a), (2.39d), (4.32) and (4.34). We obtain all the terms of the heterotic string effective action, up to and including four derivatives of the fields and bilinear terms in fermions.

It is a straightforward though heavy exercise to parameterize the action (3.47). Interestingly, using Bianchi identities and integrations by parts, the action of the theory to $\mathcal{O}(\alpha')$ may be written in the following compact form:

$$S = \int d^{10}x\ e\ e^{-2\phi}\mathcal{L}\,, \tag{5.1}$$

with

$$\begin{aligned}\mathcal{L} = & R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}\widetilde{H}_{\mu\nu\rho}\widetilde{H}^{\mu\nu\rho} - \frac{1}{4}F^i_{\mu\nu}F^{\mu\nu}_i + \frac{\alpha'}{8}\hat{\mathcal{R}}_{\mu\nu AB}\hat{\mathcal{R}}^{\mu\nu AB} \\ & - \overline{\psi}^\mu\gamma^\nu D_\nu\psi_\mu + \overline{\rho}\gamma^\mu D_\mu\rho + 2\overline{\psi}^\mu D_\mu\rho - \frac{1}{2}\overline{\chi}^i\gamma^\mu D_\mu\chi_i + \overline{\chi}_i\left(\gamma^\mu\psi^\nu - \frac{1}{4}\gamma^{\mu\nu}\rho\right)F^i_{\mu\nu} \\ & + \frac{1}{24}\widetilde{H}_{\rho\sigma\tau}\left(\overline{\psi}^\mu\gamma^{\rho\sigma\tau}\psi_\mu + 12\overline{\psi}^\rho\gamma^\sigma\psi^\tau - \overline{\rho}\gamma^{\rho\sigma\tau}\rho - 6\overline{\psi}^\rho\gamma^{\sigma\tau}\rho + \frac{1}{2}\overline{\chi}^i\gamma^{\rho\sigma\tau}\chi_i\right) \\ & + \alpha'\left[\overline{\Psi}^{AB}\gamma^\mu\mathcal{D}_\mu(w,\hat{\Omega})\Psi_{AB} - \frac{1}{24}\overline{\Psi}^{AB}\not{H}\Psi_{AB} - \overline{\Psi}^{AB}\left(\gamma^\mu\psi^\nu - \frac{1}{4}\gamma^{\mu\nu}\rho\right)\hat{\mathcal{R}}_{\mu\nu AB}\right],\end{aligned}$$

where we have taken $b = \alpha'$ and defined $\not{H} = \gamma^{\mu\nu\rho}H_{\mu\nu\rho}$ and

$$\mathcal{D}_\mu(w,\hat{\Omega})\Psi_{AB} = \partial_\mu\Psi_{AB} + 2\hat{\Omega}_{\mu[A}{}^C\Psi_{B]C} - \frac{1}{4}w_{\mu cd}\gamma^{cd}\Psi_{AB}\,. \tag{5.2}$$

As expected, the bosonic fields reproduce the expression obtained from the scattering amplitudes of the heterotic string massless fields up to first order in α' and field redefinitions [16–18], i.e.

$$\begin{aligned}\mathcal{S}|_{\text{bos}} = \int d^{10}x e e^{-2\phi}\Big[& R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}\overline{H}^{\mu\nu\rho}\overline{H}_{\mu\nu\rho} - \frac{1}{4}\zeta F^i_{\mu\nu}F^{\mu\nu}_i \\ & + \frac{\alpha'}{8}\left(R^{(-)ab}_{\mu\nu}R^{(-)\mu\nu}{}_{ab} - \frac{1}{2}T_{\mu\nu}T^{\mu\nu} - \frac{3}{2}T_{\mu\nu\rho\sigma}T^{\mu\nu\rho\sigma}\right) + \frac{\alpha'}{4}\ e.o.m.\Big],\end{aligned} \tag{5.3}$$

where we have included only the terms involving purely bosonic fields (recall that the hatted expressions contain fermions) and *e.o.m.* refers to the leading order equations of motion $\Delta g_{\mu\nu}, \Delta\phi$, $\Delta A_{\nu i}$ and $\Delta b_{\rho\nu}$ that are given in appendix A.3, namely

$$e.o.m. = \frac{1}{2}\Delta e_{\mu a}T^{\mu a} - \left(\frac{1}{4}\Delta\phi T_{\mu\nu} + \Delta(Ab)^i_\nu\Delta A_{i\mu} + A^i_\lambda A_{i\rho}\Delta b^\lambda{}_\mu\Delta b^\rho{}_\nu\right)g^{\mu\nu}\,, \tag{5.4}$$

with $\Delta(Ab)^i_\nu = \left(\Delta A^i_\nu - 2A^i_\lambda\Delta b^\lambda{}_\nu\right)$. The first order correction to the Killing metric included in the coefficient ζ and all the terms in *e.o.m.* may be eliminated through gauge covariant field redefinitions. However, as we argued in the previous section, the redefined fields would obey more complicated supersymmetry transformation rules. Reversing the argument, we can think that by adding terms proportional to the equations of motion in the action, the deformations of the supersymmetry transformation rules can be minimized.

The apparent simplicity of the first order corrections that involve bilinears in fermions in (5.1) is due to the definitions (4.1), (4.5) and (4.14). The terms that are independent of the super Yang-Mills fields (i.e. those in which all the collective indices $A, B, \ldots$ take the values $a, b, \ldots$) exactly agree with equation (2.11) of [27]. The latter was obtained replacing $A^i_\mu \to \hat{w}^{(-)cd}_\mu$ and $\chi^i \to \psi^{cd}$ in the leading order Lagrangian (2.44). Actually, one can recover the Lagrangian $\mathcal{L}(R^2)$ of [27] replacing

$$\Psi_{AB} \to \psi_{ab}\,, \qquad \hat{\mathcal{R}}_{\mu\nu AB} \to R^{(-)}_{\mu\nu ab}\,, \qquad \widetilde{H}_{\mu\nu\rho} \to \overline{H}_{\mu\nu\rho}$$

in (5.1). However the structures with collective tangent space indices $A, B, \ldots$ contain super Yang-Mills fields in addition to the supergravity fields. Note that $\widetilde{H}_{\mu\nu\rho}$ involves the generalization of the Lorentz Chern-Simons form (4.24) defined in (4.26). As expected, the terms in which the collective indices take the values $i, j, \ldots$ do not agree with the corresponding expressions $\mathcal{L}(RF^2)+\mathcal{L}(F^4)$ in [27], since the supersymmetry transformation rules of the fields differ by Yang-Mills field-dependent terms.

The supersymmetric invariance of the action (5.1) is shown in appendix C. It simply results from the observation that both the action and the transformation rules of the fields have the same structure as the corresponding ones in [27], albeit with collective indices, except for the terms contained in the parameter $\Lambda_{ci} = \frac{1}{2\sqrt{2}}\bar{\epsilon}\gamma_c\chi_i$, which cancel in the variation of the action.

6 Outlook and final remarks

In this paper we have obtained the first order corrections to $\mathcal{N}=1$ supersymmetric DFT performing a perturbative expansion of the exact supersymmetric and duality covariant framework introduced in [76]. The action has the same functional form as the leading order one constructed in [71–73], but it is expressed in terms of $\mathrm{O}(10,10+k)$ multiplets, where k is the dimension of the $\mathrm{O}(1,9+k)$ group. Decomposing the $\mathrm{O}(10,10+k)$ duality group in terms of $\mathrm{O}(10,10+n_g)$ multiplets, the theory contains higher derivative terms to all orders. We kept all the terms with up to and including four derivatives of the fields and bilinears in fermions.

The transformation rules of the $\mathrm{O}(10,10+k)$ multiplets obey a closed algebra and induce higher-derivative deformations on those of the $\mathrm{O}(10,10+n_g)$ fields. In particular, they produce a supersymmetric generalization of the duality covariant Green-Schwarz transformation that was found in [67]. We showed that the algebra of deformations closes up to first order and constructed the invariant action with up to and including four derivatives of the $\mathrm{O}(10,10+n_g)$ multiplets and bilinears in fermions.

To make contact with the heterotic string low energy effective field theory, we parameterized the duality covariant multiplets in terms of supergravity and super Yang-Mills fields. The inclusion of higher-derivative terms requires unconventional non-covariant field redefinitions in the parameterizations of the duality covariant structures. The definitions that reproduce the four-derivative interactions of the bosonic fields of the heterotic string effective action were found in [67, 68]. Here, we worked with a set of fields related to the latter through gauge covariant redefinitions. Except for the two-form, the fields defined in section 4 obey the leading order transformation rules with a modification of the two-form curvature in the supersymmetry variations. The Lorentz and non-abelian gauge transformations of the two-form are deformed by the standard Green-Schwarz mechanism, as expected, and its supersymmetry transformations are deformed by Green-Schwarz-like terms plus some extra Yang-Mills dependent higher-derivative terms.

The deformed transformations obey a closed algebra, which guarantees the existence of an invariant action. We constructed such action in section 5, by parameterizing the manifestly duality covariant expression (3.47) in terms of the fields that obey supersymmetry transformation rules with the minimal set of deformations. As expected, the interactions of the bosonic fields agree with the results obtained from the heterotic string scattering amplitudes [16–18], up to terms proportional to the leading order equations of motion. To our knowledge, the three-derivative low energy interactions involving fermions have not been constructed directly from string theory. The action and transformation rules that we have obtained follow from an exact supersymmetric and duality covariant formalism. Hence the theory avoids an iterative procedure which only guarantees consistency up to a given order. Moreover, supersymmetry is manifest to all orders and dimensional reductions will preserve the expected T-duality symmetry of the theory.

Supersymmetric extensions of the Yang-Mills and Lorentz Chern-Simons forms have been constructed using the Noether method. In particular, a supersymmetric $\mathcal{L}(R) + \mathcal{L}(R^2)$ invariant was obtained in [26, 27] from the leading order action (2.44), using the symmetry between the gauge and torsionful spin connections. The three-derivative terms that are independent of the Yang-Mills fields in the action (5.1) coincide with those results. But not surprisingly, the Yang-Mills field-dependent terms disagree with the corresponding expressions of the $\mathcal{L}(RF^2) + \mathcal{L}(F^4)$ invariants proposed in those references, since the deformations of the transformation rules differ by Yang-Mills field-dependent terms. The supersymmetric and T-duality covariant generalized Green-Schwarz transformation strongly restricts the modifications to the leading order supersymmetry transformation rules, and in particular, it does not allow the proposals of [26, 27]. As argued in section 4 this does not imply that the latter are in conflict with string theory. In order to establish if they are compatible with the required T-duality symmetry, the corresponding invariant action should be dimensionally reduced.

The effort employed in the construction of the higher-derivative fermionic sector of the heterotic string effective field theory is justified for various reasons. First of all, an intriguing consequence of the duality covariant formalism is the natural appearance of the generalized collective tangent space indices $C, D, \ldots$, which allows to include the higher-derivative Yang-Mills field-dependent terms into *gravitational* structures such as $\hat{\mathcal{R}}_{\mu\nu CD}, \hat{\Omega}_{\mu CD}$ or Ψ_{CD}. In particular, it leads to relatively mild modifications of the leading order supersymmetry transformation rules of the fields, which permits the use of the leading order Killing spinor equations to obtain classical solutions containing higher-derivative corrections [2]. These features not only simplify the construction of new supersymmetric solutions but also allow to easily extend the known solutions for the gravitational sector to the Yang-Mills sector.

The fermionic contributions to the action are also relevant for applications to four-dimensional physics. Both the superpotential and D-terms can be more easily computed from the fermionic couplings [10] and the higher derivative corrections to these terms as well as to the Yukawa couplings could also have interesting consequences for string phenomenology and moduli fixing.

An obvious natural extension of our work would be to determine further interactions beyond the first order. The quartic interactions of the Yang-Mills fields that we have reproduced are mirrored by corresponding quartic Riemann curvature terms [16–18]. Consequently, we expect that the higher orders of perturbation will reproduce these higher-derivative corrections. It would be interesting to see if the generalized structures with capital indices persist to higher orders. If they do, the formulation would contain information about higher than four-point functions in the string scattering amplitudes.

JHEP07(2021)092

Nevertheless, there is another quartic Riemann curvature structure that has no analog in the Yang-Mills sector [16–18]. At tree level, these terms are proportional to the transcendental coefficient $\zeta(3)$. The analysis of the higher-derivative terms is technically more challenging but also more interesting, since further duality covariant structures, or even a more drastic change of scheme, seem to be necessary as advocated in [81].

Performing a generalized Scherk-Schwarz compactification of the sub-leading corrections to $\mathcal{N} = 1$ supersymmetric DFT would be another promising line of research, as this would produce higher-derivative corrections to lower dimensional gauged supergravities [68, 82]. We hope to return to these and related questions in the future.

Acknowledgments

We sincerely thank W. Baron and D. Marqués for kindly sharing their project [76] and for important insight and comments. C.N. would also like to acknowledge correspondence with N. Berkovits, J. Gates and H. Gomez and thank the A. S. ICTP for hospitality during the initial stages of this work. This research was partially supported by PIP-CONICET-11220150100559CO, UBACyT and ANPCyT- PICT-2016-1358.

A Conventions and definitions

In this appendix we introduce the conventions and definitions used throughout the paper. Space-time and tangent space Lorentz indices are denoted $\mu, \nu, \ldots$ and $a, b, \ldots$, respectively.

The covariant derivative acting on a gauge tensor ${G_\mu}^{ci}$ and on a spinor ϵ is, respectively,

$$D^{(\pm)}_\mu G^i_{\nu c} = \partial_\mu G^i_{\nu c} - \Gamma^\rho_{\mu\nu} G^i_{\rho c} - w^{(\pm)d}_{\mu c} G^i_{\nu d} - A^j_\mu G^k_{\nu c} {f^i}_{jk}, \tag{A.1}$$

$$D^{(\pm)}_\mu \epsilon = \partial_\mu \epsilon - \frac{1}{4} w^{\pm}_{\mu ab} \gamma^{ab} \epsilon\,, \tag{A.2}$$

with

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right)\,, \tag{A.3}$$

and the torsionful spin connection

$$w^{(\pm)}_{abc} \equiv \left(w_{abc} \pm \frac{1}{2} H_{abc}\right)\,, \tag{A.4}$$

where

$$w_{\mu bc} = {e_\mu}^a \left(-{e^\mu}_{[a} {e^\nu}_{b]} \partial_\mu e_{\nu c} + {e^\mu}_{[a} {e^\nu}_{c]} \partial_\mu e_{\nu b} + {e^\mu}_{[b} {e^\nu}_{c]} \partial_\mu e_{\nu a}\right)\,. \tag{A.5}$$

The identity $D_\mu {e_\nu}^a = \partial_\mu {e_\nu}^a - \Gamma^\rho_{\mu\nu} {e_\rho}^a - {w_\mu}^a{}_b {e_\nu}^b = 0$ implies

$${w_{\mu a}}^b = -{e^\nu}_a \partial_\mu {e_\nu}^b + \Gamma^\sigma_{\mu\nu} {e_\sigma}^b {e^\nu}_a\,. \tag{A.6}$$

The commutator of covariant derivatives acting on gauge tensors and spinors is

$$\left[D^{(\pm)}_\mu, D^{(\pm)}_\nu\right] F_{\rho ci} = -{R^\sigma}_{\rho\mu\nu} F_{\sigma ci} + R^{(\pm)d}_{\mu\nu c} F_{\rho di} - {F_{\mu\nu}}^j {F_{\rho c}}^k f_{ijk} \tag{A.7}$$

$$\left[D^{(\pm)}_\mu, D^{(\pm)}_\nu\right] \epsilon = \frac{1}{4} {R^{(\pm)}}_{\mu\nu ab} \gamma^{ab} \epsilon\,, \tag{A.8}$$

where the Riemann tensor is defined as

$$
\begin{aligned}
R^{\rho}{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\kappa}\Gamma^{\kappa}_{\nu\sigma} - \Gamma^{\rho}_{\nu\kappa}\Gamma^{\kappa}_{\mu\sigma} \\
&= e^{\rho a} e_{\sigma}{}^{b} R_{\mu\nu ab} = e^{\rho a} e_{\sigma}{}^{b}\left(-2\partial_{[\mu} w_{\nu]ab} + w_{\mu a}{}^{c} w_{\nu cb} - w_{\nu a}{}^{c} w_{\mu cb}\right) ,
\end{aligned} \tag{A.9}
$$

and the Yang-Mills field strength is

$$
F^{i}_{\mu\nu} = 2\partial_{[\mu} A^{i}_{\nu]} - f^{i}{}_{jk} A^{j}_{\mu} A^{k}_{\nu} \,. \tag{A.10}
$$

The Ricci tensor and scalar are

$$
R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}\,, \qquad R = g^{\mu\nu} R_{\mu\nu} = R_{\mu\nu}{}^{ab} e^{\mu}{}_{a} e^{\nu}{}_{b}\,. \tag{A.11}
$$

A.1 Some useful gamma function identities

To distinguish $\mathrm{O}(1,9)_{\mathrm{R}}$ and $\mathrm{O}(9,1)_{\mathrm{L}}$ tangent space indices in DFT we use $\bar{a}, \bar{b}, \ldots$ and $\underline{a}, \underline{b}, \ldots$, respectively. The Clifford algebra $\{\gamma_{\underline{a}}, \gamma_{\underline{b}}\} = -2P_{\underline{ab}}$ determines the following identities for the $\mathrm{O}(9,1)_{\mathrm{L}}$ gamma matrices

$$
\gamma_{\underline{a}}\gamma_{\underline{b}} = \gamma_{\underline{ab}} - P_{\underline{ab}}\,, \tag{A.12a}
$$

$$
\gamma_{\underline{ab}}\gamma_{\underline{c}} = \gamma_{\underline{abc}} - 2\gamma_{[\underline{a}} P_{\underline{b}]\underline{c}}\,, \tag{A.12b}
$$

$$
\gamma_{\underline{a}}\gamma_{\underline{bc}} = \gamma_{\underline{abc}} - 2P_{\underline{a}[\underline{b}}\gamma_{\underline{c}]}\,, \tag{A.12c}
$$

$$
\gamma_{\underline{ab}}\gamma^{\underline{cd}} = \gamma_{\underline{ab}}{}^{\underline{cd}} - 4\gamma_{[\underline{a}}{}^{[\underline{d}} P_{\underline{b}]}{}^{\underline{c}]} + 2P_{[\underline{b}}{}^{[\underline{c}} P_{\underline{a}]}{}^{\underline{d}]}\,, \tag{A.12d}
$$

$$
\gamma_{\underline{ab}}\gamma^{\underline{cde}} = \gamma_{\underline{ab}}{}^{\underline{cde}} - 6\gamma_{[\underline{a}}{}^{[\underline{de}} P_{\underline{b}]}{}^{\underline{c}]} + 6\gamma^{[\underline{e}} P_{[\underline{b}}{}^{\underline{c}} P_{\underline{a}]}{}^{\underline{d}]}\,, \tag{A.12e}
$$

$$
\gamma_{\underline{abc}}\gamma^{\underline{de}} = \gamma_{\underline{abc}}{}^{\underline{de}} - 6\gamma_{[\underline{ab}}{}^{[\underline{e}} P_{\underline{c}]}{}^{\underline{d}]} + 6\gamma_{[\underline{a}} P_{\underline{c}}{}^{[\underline{d}} P_{\underline{b}]}{}^{\underline{e}]}\,, \tag{A.12f}
$$

$$
\gamma_{\underline{abc}}\gamma^{\underline{def}} = \gamma_{\underline{abc}}{}^{\underline{def}} - 9\gamma_{[\underline{ab}}{}^{[\underline{ef}} P_{\underline{c}]}{}^{\underline{d}]} + 18\gamma_{[\underline{a}}{}^{[\underline{f}} P_{\underline{c}}{}^{\underline{d}} P_{\underline{b}]}{}^{\underline{e}]} - 6P_{[\underline{c}}{}^{[\underline{d}} P_{\underline{b}}{}^{\underline{e}} P_{\underline{a}]}{}^{\underline{f}]}\,, \tag{A.12g}
$$

$$
C\gamma^{\underline{a}}C^{-1} = -(\gamma^{\underline{a}})^{t}\,, \qquad C^{-1}\gamma_{\underline{ab}}C = -(\gamma_{\underline{ab}})^{t}\,, \tag{A.12h}
$$

where $C^{-1} = C^{t} = -C$ and $\underline{a}, \underline{b} = 0, \ldots, 9$.

A.2 Leading order components of the generalized fluxes

Using the parameterizations introduced in section 2 and solving the strong constraint in the supergravity frame, the non-vanishing determined components of the generalized spin connection are, to leading order,

$$
F_{\bar{a}\underline{bc}} = -\left(w_{abc} + \frac{1}{2}H_{abc}\right) \equiv -w^{(+)}_{abc}\,, \tag{A.13a}
$$

$$
F_{\underline{a}\overline{bc}} = \left(w_{abc} - \frac{1}{2}H_{abc}\right) \equiv w^{(-)}_{abc}\,, \tag{A.13b}
$$

$$
F_{\overline{abc}} = 3\left(w_{[abc]} - \frac{1}{6}H_{abc}\right)\,, \tag{A.13c}
$$

$$
F_{\underline{abc}} = -3\left(w_{[abc]} + \frac{1}{6}H_{abc}\right)\,, \tag{A.13d}
$$

$$
F_{\bar{i}\underline{ab}} = F_{\underline{a}\overline{bi}} = F_{\overline{abi}} = -\frac{1}{\sqrt{2}} e^{\mu}{}_{a} e^{\nu}{}_{b} e_{i\bar{i}} F^{i}_{\mu\nu}\,, \tag{A.13e}
$$

$$F_{\underline{a}\overline{i}\overline{j}} = -e^{i}{}_{\overline{i}}e^{j}{}_{\overline{j}}e^{\mu}{}_{a}A_{\mu}{}^{k}f_{ijk}\,, \tag{A.13f}$$

$$F_{\overline{ijk}} = \sqrt{2}e^{i}_{\overline{i}}e^{j}_{\overline{j}}e^{k}_{\overline{k}}f_{ijk}\,, \tag{A.13g}$$

$$F_{\underline{a}} = F_{\overline{a}} = \left(\partial_\mu e^\mu_a + e^\mu_a e^\nu_b \partial_\mu e^b_\nu - 2e^{\mu}{}_{a}\partial_\mu\phi\right)\,, \tag{A.13h}$$

where

$$H_{abc} = e^\mu_{[a}e^\nu_b e^\rho_{c]}H_{\mu\nu\rho} = 3e^\mu_a e^\nu_b e^\rho_c\left(\partial_{[\mu}b_{\nu\rho]} - A^i_{[\mu}\partial_\nu A_{\rho]i} + \frac{1}{3}f_{ijk}A^i_\mu A^j_\nu A^k_\rho\right)\,, \tag{A.14}$$

and f_{ijk} are the structure constants of the SO(32) or $E_8 \times E_8$ gauge groups.

A.3 The leading order action and equations of motion

Here we rewrite the zeroth order action (2.44) in terms of the dilatino λ of the supergravity multiplet and compare with the corresponding expression in [27]. We also list the leading order equations of motion of all the massless fields derived from it.

Rewriting the generalized dilatino $\rho = 2\lambda + \gamma^\mu\psi_\mu$ in terms of λ and ψ and integrating by parts, the action (2.44) takes the form

$$\begin{aligned} S = \int d^{10}x\; e\; e^{-2\phi}\Big[&R(w(e)) - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}F^i_{\mu\nu}F^{\mu\nu}_i \\ &- \bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho + 4\bar\lambda\gamma^{\mu\nu}D_\mu\psi_\nu + 4\bar\lambda\gamma^\mu D_\mu\lambda - \frac{1}{2}\bar\chi^i \not{D}\chi_i \\ &+ 4\bar\psi_\mu\gamma^\nu\gamma^\mu\lambda\partial_\nu\phi - 2\bar\psi_\mu\gamma^\mu\psi^\nu\partial_\nu\phi - \frac{1}{4}\bar\chi_i\gamma^\mu\gamma^{\nu\rho}F^i_{\nu\rho}\left(\psi_\mu + \frac{1}{3}\gamma_\mu\lambda\right) \\ &+ \frac{1}{24}H_{\rho\sigma\tau}\left(\bar\psi^\mu\gamma_{[\mu}\gamma^{\rho\sigma\tau}\gamma_{\nu]}\psi^\nu + 4\bar\psi_\mu\gamma^{\mu\rho\sigma\tau}\lambda - 4\bar\lambda\gamma^{\rho\sigma\tau}\lambda + \frac{1}{2}\bar\chi^i\gamma^{\rho\sigma\tau}\chi_i\right)\Big]\,. \end{aligned} \tag{A.15}$$

It matches the corresponding expression in [27] with the following field redefinitions: $\phi^{-3} \to e^{-2\phi}$, $R \to -R$, $H_{\mu\nu\lambda} \to \frac{1}{3\sqrt{2}}H_{\mu\nu\lambda}$, $B_{\mu\nu} \to \frac{1}{\sqrt{2}}b_{\mu\nu}$, $\lambda \to \frac{1}{\sqrt{2}}\lambda$, $A_\mu \to \frac{1}{\sqrt{2}}A_\mu$, $\chi \to \frac{1}{\sqrt{2}}\chi$.

The leading order equations of motion of all the massless fields, written in terms of ρ, are

$$\begin{aligned} \Delta e_\mu{}^a = &\frac{1}{2}e_\mu{}^a\Delta\phi + 2R_\mu{}^a + 8D_\mu\phi D^a\phi - \frac{1}{2}H_{\mu\lambda\sigma}H^{a\lambda\sigma} - F_{\mu\lambda i}F^{a\lambda i} \\ &- 2\overline{\psi}_\mu\gamma^\lambda D_\lambda\psi^a - 2\overline{\psi}^\lambda\gamma_\mu e^{\nu a}D_\nu\psi_\lambda + 2\overline{\rho}\gamma_\mu D^a\rho + 4\overline{\psi}_\mu D^a\rho - \overline{\chi}^i\gamma_\mu D^a\chi_i \\ &+ \frac{1}{4}\overline{\psi}^\lambda\gamma_\mu{}^{\sigma\tau}\psi_\lambda H^a{}_{\sigma\tau} - \frac{1}{4}\overline{\rho}\gamma_\mu{}^{\sigma\tau}\rho H^a{}_{\sigma\tau} + \frac{1}{8}\overline{\chi}^i\gamma_\mu{}^{\sigma\tau}\chi_i H^a{}_{\sigma\tau} + \overline{\psi}^\sigma\gamma_\mu{}^\tau\rho H^a{}_{\sigma\tau} \\ &- \frac{1}{2}\overline{\psi}_\mu\gamma_{\sigma\tau}\rho H^{a\sigma\tau} + 2\overline{\psi}_\mu\gamma^\sigma\psi^\tau H^a{}_{\sigma\tau} - \overline{\psi}^\sigma\gamma_\mu\psi^\tau H^a{}_{\sigma\tau} + \frac{1}{12}\overline{\psi}_\mu\gamma^{\rho\sigma\tau}\psi^a H_{\rho\sigma\tau} \\ &+ 2\overline{\chi}_i\gamma_\mu\psi_\lambda F^{a\lambda i} - 2\overline{\chi}_i\gamma_\lambda\psi_\mu F^{a\lambda i} - \overline{\chi}_i\gamma_{\mu\lambda}\rho F^{a\lambda i}\,, \end{aligned} \tag{A.16}$$

$$\Delta\phi = -2\mathcal{L}\,, \tag{A.17}$$

$$\begin{aligned} \Delta b_{\nu\rho} = &\frac{1}{2}D^\mu H_{\mu\nu\rho} - D^\mu\phi H_{\mu\nu\rho} \\ &- \frac{1}{8}D^\mu\left(\overline{\psi}^\lambda\gamma_{\mu\nu\rho}\psi_\lambda + 12\overline{\psi}_{[\mu}\gamma_\nu\psi_{\rho]} - \overline{\rho}\gamma_{\mu\nu\rho}\rho - 6\overline{\psi}_{[\mu}\gamma_{\nu\rho]}\rho + \frac{1}{2}\overline{\chi}^i\gamma_{\mu\nu\rho}\chi_i\right) \\ &+ \frac{1}{4}\left(\overline{\psi}^\lambda\gamma_{\mu\nu\rho}\psi_\lambda + 12\overline{\psi}_{[\mu}\gamma_\nu\psi_{\rho]} - \overline{\rho}\gamma_{\mu\nu\rho}\rho - 6\overline{\psi}_{[\mu}\gamma_{\nu\rho]}\rho + \frac{1}{2}\overline{\chi}^i\gamma_{\mu\nu\rho}\chi_i\right)D^\mu\phi\,, \end{aligned} \tag{A.18}$$

$$
\begin{aligned}
\Delta A_\mu{}^i &= \frac{1}{2} H_{\mu\nu\rho} F^{\nu\rho i} + A_\rho{}^i \Delta b^\rho{}_\mu - D^\nu F_{\mu\nu}{}^i + 2F_{\mu\nu}{}^i D^\nu \phi - \frac{1}{2}\overline{\chi}^j \gamma_\mu \chi^k f^i{}_{jk} \\
&\quad - \frac{1}{8} F^{\nu\rho i} \left(\overline{\psi}^\lambda \gamma_{\mu\nu\rho} \psi_\lambda + 12 \overline{\psi}_{[\mu} \gamma_\nu \psi_{\rho]} - \overline{\rho} \gamma_{\mu\nu\rho} \rho - 6 \overline{\psi}_{[\mu} \gamma_{\nu\rho]} \rho + \frac{1}{2} \overline{\chi}^j \gamma_{\mu\nu\rho} \chi_j \right) \\
&\quad + 2 D^\nu \overline{\chi}^i \left(\gamma_{[\mu} \psi_{\nu]} - \frac{1}{4} \gamma_{\mu\nu} \rho \right) + 2 \overline{\chi}^i D^\nu \left(\gamma_{[\mu} \psi_{\nu]} - \frac{1}{4} \gamma_{\mu\nu} \rho \right) \\
&\quad - 4 \overline{\chi}^i \left(\gamma_{[\mu} \psi_{\nu]} - \frac{1}{4} \gamma_{\mu\nu} \rho \right) D^\nu \phi \,, \qquad \text{(A.19)} \\
\Delta \psi_\mu &= 2 D_\nu \overline{\psi}_\mu \gamma^\nu - 2 \overline{\psi}_\mu \gamma^\nu \partial_\nu \phi + 2 D_\mu \overline{\rho} + \frac{1}{12} \overline{\psi}_\mu \gamma^{\rho\sigma\tau} H_{\rho\sigma\tau} - \frac{1}{4} H_{\mu\nu\rho} \left(4 \overline{\psi}^\rho \gamma^\nu - \overline{\rho} \gamma^{\nu\rho} \right) , \\
&\quad - F_{\mu\nu}{}^i \overline{\chi}_i \gamma^\nu \qquad \text{(A.20)} \\
\Delta \rho &= -2 D_\mu \overline{\rho} \gamma^\mu + 2 \overline{\rho} \gamma^\mu \partial_\mu \phi - 2 D_\mu \overline{\psi}^\mu + 4 \overline{\psi}^\mu \partial_\mu \phi - \frac{1}{12} H_{\rho\sigma\tau} \left(\overline{\rho} \gamma^{\rho\sigma\tau} + 3 \overline{\psi}^\rho \gamma^{\sigma\tau} \right) , \\
&\quad - \frac{1}{4} F_{\mu\nu}{}^i \overline{\chi}_i \gamma^{\mu\nu} \qquad \text{(A.21)} \\
\Delta \chi_i &= D_\mu \overline{\chi}_i \gamma^\mu - \overline{\chi}_i \gamma^\mu \partial_\mu \phi + \overline{\chi}_i \frac{1}{24} \gamma^{\rho\sigma\tau} H_{\rho\sigma\tau} - \left(\overline{\psi}^\nu \gamma^\mu - \frac{1}{4} \overline{\rho} \gamma^{\mu\nu} \right) F_{\mu\nu i} \,. \qquad \text{(A.22)}
\end{aligned}
$$

B Algebra of transformations of $O(10, 10 + n_g)$ fields

In this appendix we show that the algebra of transformation rules closes, up to terms with two fermions. We first review the algebra of zeroth order transformations (2.19) and in B.2 we include the first order corrections. We define $[\delta_1, \delta_2] = -\delta_{12}$.

B.1 Leading order algebra

We focus on the algebra determined by the leading order transformations (2.19) and show that it closes with the parameters (2.20). We split the algebra of transformations on the generalized fields into the following commutators:

- Supersymmetry transformations of the dilaton

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] d &= \frac{1}{2} \bar{\epsilon}_{[2} \left(\gamma^{\underline{a}} \sqrt{2} E_{\underline{a}}{}^M \partial_M \epsilon_{1]} - \frac{1}{4} \gamma^{\underline{a}} \omega_{\underline{abc}} \gamma^{\underline{bc}} \epsilon_{1]} \right) \\
&= -\xi'^M_{12} \partial_M d + \frac{1}{2} \partial_M \xi'^M_{12} = -\delta_{\xi'_{12}} d \,, \qquad \text{(B.1)}
\end{aligned}
$$

where we have used $\bar{\epsilon}_1 \gamma^{\underline{a}} \epsilon_2 = -\bar{\epsilon}_2 \gamma^{\underline{a}} \epsilon_1$ and $\bar{\epsilon}_1 \gamma^{\underline{abc}} \epsilon_2 = \bar{\epsilon}_2 \gamma^{\underline{abc}} \epsilon_1$, and defined

$$
\xi'^M_{12} = -\frac{1}{\sqrt{2}} E^M{}_{\underline{c}} \left(\bar{\epsilon}_1 \gamma^{\underline{c}} \epsilon_2 \right) . \qquad \text{(B.2)}
$$

- Diffeomorphisms on the dilaton

$$
[\delta_{\xi_1}, \delta_{\xi_2}] d = -\xi''^M_{12} \partial_M d + \frac{1}{2} \partial_M \xi''^M_{12} = -\delta_{\xi''_{12}} d \,, \qquad \text{(B.3)}
$$

with

$$
\xi''^M_{12} = 2 \xi^N_{[1} \partial_N \xi^M_{2]} \,. \qquad \text{(B.4)}
$$

- Mixed supersymmetry and double Lorentz transformations on the dilaton

$$\delta_{[\Gamma,\epsilon]} d = -\frac{1}{8}\bar{\epsilon}_{[2}\Gamma_{1]\underline{bc}}\gamma^{\underline{bc}}\rho = -\delta_{\epsilon'_{12}} d\,, \tag{B.5}$$

where we have defined $\delta_{[\Gamma,\epsilon]} = [\delta_{\Gamma_1}, \delta_{\epsilon_2}] + [\delta_{\epsilon_1}, \delta_{\Gamma_2}]$ and

$$\bar{\epsilon}'_{12} = -\frac{1}{2}\Gamma_{[1\underline{ab}}\bar{\epsilon}_{2]}\gamma^{\underline{ab}}\,. \tag{B.6}$$

- Mixed diffeomorphisms and supersymmetry variations on the dilaton

$$\delta_{[\epsilon,\xi]} d = \frac{1}{2}\xi^M_{[1}\partial_M\bar{\epsilon}_{2]}\rho = -\delta_{\epsilon''_{12}} d\,, \tag{B.7}$$

with

$$\bar{\epsilon}''_{12} = 2\xi^M_{[1}\partial_M\bar{\epsilon}_{2]}\,. \tag{B.8}$$

- Supersymmetry variations of the frame

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] E_{\mathbb{M}\underline{a}} = \frac{1}{\sqrt{2}} E^N{}_{\overline{B}}\partial_N\left(\bar{\epsilon}_{[1}\gamma_{\underline{a}}\epsilon_{2]}\right) E_{\mathbb{M}}{}^{\overline{B}} - \frac{1}{2}\left(\bar{\epsilon}_{[1}\gamma_{\underline{c}}\epsilon_{2]}\right)\omega_{\overline{B}\underline{a}}{}^{\underline{c}} E_{\mathbb{M}}{}^{\overline{B}}\,. \tag{B.9}$$

Projecting with $E^{\mathbb{M}}{}_{\overline{C}}$, we get

$$E^{\mathbb{M}}{}_{\overline{C}}[\delta_{\epsilon_1}, \delta_{\epsilon_2}] E_{\mathbb{M}\underline{a}} = -E^{\mathbb{N}}{}_{\overline{C}}\delta_{\xi'_{12}} E_{\mathbb{N}\underline{a}} \tag{B.10}$$

where we have used (2.14) and $\xi'^{\mathbb{M}}_{12}$ is the generalization of (B.2), i.e.

$$\xi'^{\mathbb{M}}_{12} = -\frac{1}{\sqrt{2}} E^{\mathbb{M}}{}_{\underline{c}}\left(\bar{\epsilon}_1\gamma^{\underline{c}}\epsilon_2\right)\,. \tag{B.11}$$

Projecting with $E^{\mathbb{M}}{}_{\underline{c}}$ we find

$$E^{\mathbb{M}}{}_{\underline{c}}[\delta_{\epsilon_1}, \delta_{\epsilon_2}] E_{\mathbb{M}\underline{a}} = -E^{\mathbb{N}}{}_{\underline{c}}\delta_{\Gamma'_{12}} E_{\mathbb{N}\underline{a}} \tag{B.12}$$

with

$$\Gamma'^{\underline{ab}}_{12} = E^{[\underline{a}}\left(\bar{\epsilon}_1\gamma^{\underline{c}]}\epsilon_2\right) - \frac{1}{2}\left(\bar{\epsilon}_1\gamma^{\underline{c}}\epsilon_2\right) F^{\underline{ab}}{}_{\underline{c}}\,. \tag{B.13}$$

Following similar steps, we get

$$E^{\mathbb{M}}{}_{\underline{c}}[\delta_{\epsilon_1}, \delta_{\epsilon_2}] E_{\mathbb{M}\overline{A}} = -E^{\mathbb{M}}{}_{\underline{c}}\delta_{\xi'_{12}} E_{\mathbb{M}\overline{A}}\,, \quad E^{\mathbb{M}}{}_{\overline{B}}[\delta_{\epsilon_1}, \delta_{\epsilon_2}] E_{\mathbb{M}\overline{A}} = -E^{\mathbb{M}}{}_{\overline{C}}\delta_{\Gamma'_{12}} E_{\mathbb{M}\overline{A}}\,,$$

with

$$\Gamma'^{\overline{AB}}_{12} = -\frac{1}{2}\left(\bar{\epsilon}_1\gamma^{\underline{c}}\epsilon_2\right) F^{\overline{AB}}{}_{\underline{c}}\,. \tag{B.14}$$

- Diffeomorphisms and double Lorentz variations of the frame

$$\delta_{[\Gamma,\xi]} E^{\mathbb{M}}{}_{\mathbb{A}} = -\left(\delta_{\Gamma''_{12}} + \delta_{\xi''_{12}}\right) E^{\mathbb{M}}{}_{\mathbb{A}}\,, \tag{B.15}$$

where

$$\Gamma''_{12\mathbb{AB}} = 2\xi^M_{[1}\partial_M\Gamma_{2]\mathbb{AB}} - 2\Gamma_{[1\mathbb{A}}{}^{\mathbb{C}}\Gamma_{2]\mathbb{CB}} \tag{B.16}$$

$$\xi''^{\mathbb{M}}_{12} = 2\xi^P_{[1}\partial_P\xi^{\mathbb{M}}_{2]} - \xi^{\mathbb{N}}_{[1}\partial^{\mathbb{M}}\xi_{2]\mathbb{N}} + f_{\mathbb{P}\mathbb{Q}}{}^{\mathbb{M}}\xi^{\mathbb{P}}_1\xi^{\mathbb{Q}}_2\,. \tag{B.17}$$

Note that ξ''^M_{12} in (B.4) does not contain the second and third terms in the r.h.s. of this expression, due to the strong constraint.

- Mixed diffeomorphisms and supersymmetry variations of the frame

$$\delta_{[\epsilon,\xi]} E^{\mathbb{M}}{}_{\underline{a}} = \xi^N_{[1} \partial_N \bar{\epsilon}_{2]} \gamma_{\underline{a}} \Psi_{\overline{B}} E^{\mathbb{M}\overline{B}} = -\delta_{\epsilon''_{12}} E^{\mathbb{M}}{}_{\underline{a}} \,, \tag{B.18}$$

where ϵ''_{12} is defined in (B.8). A similar result is obtained for $E^{\mathbb{M}}{}_{\overline{A}}$.

- Mixed double Lorentz and supersymmetry variations of the frame

$$\delta_{[\Gamma,\epsilon]} E^{\mathbb{M}}{}_{\underline{a}} = \frac{1}{4} \bar{\epsilon}_{[1} \Gamma_{2]\underline{bc}} \gamma^{\underline{bc}} \gamma_{\underline{a}} \psi_{\overline{B}} E^{\mathbb{M}\overline{B}} = -\delta_{\epsilon'_{12}} E^{\mathbb{M}}{}_{\underline{a}} \,, \tag{B.19}$$

where ϵ'_{12} is defined in (B.6). A similar result is obtained for $E^{\mathbb{M}}{}_{\overline{A}}$.

- Mixed diffeomorphisms and supersymmetry transformations of the gravitino

$$\delta_{[\epsilon,\xi]} \Psi_{\overline{A}} = E_{\overline{A}} (2\xi^M_{[2} \partial_M \epsilon_{1]}) - \frac{1}{2} \omega_{\overline{A}\underline{bc}} \gamma^{\underline{bc}} \xi^M_{[2} \partial_M \epsilon_{1]} = -\nabla_{\overline{A}} \epsilon''_{12} = -\delta_{\epsilon''_{12}} \Psi_{\overline{A}} \,. \tag{B.20}$$

- Mixed supersymmetry and double Lorentz transformations of the gravitino

$$\delta_{[\Gamma,\epsilon]} \Psi_{\overline{A}} = \frac{1}{2} \nabla_{\overline{A}} \left(\Gamma_{[2\underline{bc}} \gamma^{\underline{bc}} \epsilon_{1]} \right) \equiv -\nabla_{\overline{A}} \epsilon'_{12} = -\delta_{\epsilon'_{12}} \Psi_{\overline{A}} \tag{B.21}$$

- Diffeomorphisms and double Lorentz transformations of the gravitino

$$\delta_{[\Gamma,\xi]} \Psi_{\overline{A}} = -\left(\delta_{\Gamma''_{12}} + \delta_{\xi''_{12}} \right) \Psi_{\overline{A}} \,. \tag{B.22}$$

- Mixed supersymmetry and double Lorentz transformations of the dilatino

$$\delta_{[\Gamma,\epsilon]} \rho = -\frac{1}{2} \gamma^a \nabla_{\underline{a}} (\Gamma_{[2\underline{bc}} \gamma^{\underline{bc}} \epsilon_{1]}) = \gamma^{\underline{a}} \nabla_{\underline{a}} \epsilon'_{12} = -\delta_{\epsilon'_{12}} \rho \,. \tag{B.23}$$

- Diffeomorphisms and double Lorentz transformations of the dilatino

$$\delta_{[\Gamma,\xi]} \rho = -(\delta_{\Gamma''_{12}} + \delta_{\xi''_{12}}) \rho \,. \,. \tag{B.24}$$

- Mixed diffeomorphisms and supersymmetry transformations of the dilatino

$$\delta_{[\xi,\epsilon]} \rho = \gamma^{\underline{a}} \nabla_{\underline{a}} \epsilon''_{12} = -\delta_{\epsilon''_{12}} \rho \,. \,. \tag{B.25}$$

Summarizing we have found, up to bi-linear terms in fermions,

$$E^{\mathbb{M}}{}_{\overline{C}} [\delta_1, \delta_2] E_{\mathbb{M}\underline{a}} = -E^{\mathbb{M}}{}_{\overline{C}} \left(\delta_{\xi_{12}} + \delta_{\Gamma_{12}} + \delta_{\epsilon_{12}} \right) E_{\mathbb{M}\underline{a}} \,, \tag{B.26a}$$

$$E^{\mathbb{M}}{}_{\underline{c}} [\delta_1, \delta_2] E_{\mathbb{M}\overline{A}} = -E^{\mathbb{M}}{}_{\underline{c}} \left(\delta_{\xi_{12}} + \delta_{\Gamma_{12}} + \delta_{\epsilon_{12}} \right) E_{\mathbb{M}\overline{A}} \,, \tag{B.26b}$$

$$E^{\mathbb{M}}{}_{\overline{B}} [\delta_1, \delta_2] E_{\mathbb{M}\overline{A}} = -E^{\mathbb{M}}{}_{\overline{C}} \delta_{\Gamma_{12}} E_{\mathbb{M}\overline{A}} \,, \tag{B.26c}$$

$$E^{\mathbb{M}}{}_{\underline{c}} [\delta_1, \delta_2] E_{\mathbb{M}\underline{a}} = -E^{\mathbb{N}}{}_{\underline{c}} \delta_{\Gamma_{12}} E_{\mathbb{N}\underline{a}} \,, \tag{B.26d}$$

$$[\delta_1, \delta_2] d = -\left(\delta_{\xi_{12}} + \delta_{\epsilon_{12}} \right) d \,, \tag{B.26e}$$

$$[\delta_1, \delta_2] \Psi_{\overline{A}} = -\left(\delta_{\xi''_{12}} + \delta_{\Gamma_{12}} + \delta_{\epsilon_{12}} \right) \Psi_{\overline{A}} \,, \tag{B.26f}$$

$$[\delta_1, \delta_2] \rho = -\left(\delta_{\xi''_{12}} + \delta_{\Gamma_{12}} + \delta_{\epsilon_{12}} \right) \rho \,, \tag{B.26g}$$

where $\delta_1 = \delta_{\xi_1} + \delta_{\epsilon_1} + \delta_{\Gamma_1}$ and $\xi^{\mathbb{M}}_{12} = \xi'^{\mathbb{M}}_{12} + \xi''^{\mathbb{M}}_{12}$, $\Gamma_{12\mathbb{A}B} = \Gamma'_{12\mathbb{A}B} + \Gamma''_{12\mathbb{A}B}$, $\epsilon_{12} = \epsilon'_{12} + \epsilon''_{12}$. The commutator of supersymmetry variations on the gravitino and dilatino as well as the missing terms $\delta_{\xi'_{12}} \rho$ and $\delta_{\xi'_{12}} \Psi_{\overline{A}}$ are not included as they are of higher order in fermions.

B.2 First order algebra

We now work out the algebra of first order transformations (3.37) and show that it closes with the parameters (3.38), up to terms with two fermions. Here we denote $\delta \equiv \delta^{(0)} + \delta^{(1)}$ and $[\delta_1, \delta_2] = \delta_1^{(1)}\delta_2^{(0)} + \delta_1^{(0)}\delta_2^{(1)} - (1 \leftrightarrow 2) = -\delta_{12}^{(1)}$. We split the algebra as in the previous section.

Double Lorentz transformations on the generalized frame

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] E_{\mathbb{M}}{}^{\overline{A}} = \frac{b}{2}\left[\delta_{\Lambda_1}\left(\mathcal{F}^*_{\underline{\mathbb{M}}}{}^{\overline{CD}}\right) E_N{}^{\overline{A}}\partial^N\Lambda_{2\overline{CD}} - \delta_{\Lambda_2}\left(\mathcal{F}^*_{\underline{\mathbb{M}}}{}^{\overline{CD}}\right) E_N{}^{\bar{A}}\partial^N\Lambda_{1\overline{CD}}\right]. \tag{B.27}$$

Rewriting

$$\delta_{\Lambda_1}\left(\mathcal{F}^*_{\underline{\mathbb{M}}}{}^{\overline{CD}}\right) E_N{}^{\overline{A}}\partial^N\Lambda_{2\overline{CD}} = (-\partial_{\underline{\mathbb{M}}}\Lambda_1^{\overline{CD}} + 2\mathcal{F}^*_{\underline{\mathbb{M}}}{}^{\overline{BD}}\Lambda_{1\overline{B}}{}^{\overline{C}})E_N{}^{\overline{A}}\partial^N\Lambda_{2\overline{CD}}\,, \tag{B.28}$$

with $\partial_{\mathbb{M}} = \partial_{\underline{\mathbb{M}}} + \partial_{\overline{\mathbb{M}}}$ and

$$-2E^{\mathbb{P}\overline{A}}\partial_{\mathbb{M}}\Lambda_{[1}{}^{\overline{CD}}\partial_{\mathbb{P}}\Lambda_{2]\overline{CD}} = E^{\mathbb{P}\overline{A}}\left[\partial_{\mathbb{M}}\left(-\Lambda_1{}^{\overline{CD}}\partial_{\mathbb{P}}\Lambda_{2\overline{CD}}\right) + \partial_{\mathbb{P}}\left(\Lambda_1{}^{\overline{CD}}\partial_{\mathbb{M}}\Lambda_{2\overline{CD}}\right)\right], \tag{B.29}$$

we get

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] E_{\mathbb{M}}{}^{\overline{A}} = -\left(\delta_{\Lambda_{12}^{(1)\prime}} + \delta_{\xi_{12}^{(1)\prime}}\right) E_{\mathbb{M}}{}^{\overline{A}}\,, \tag{B.30}$$

where

$$\xi^{(1)\prime}_{12M} = b\Lambda_{[1}^{\overline{CD}}\partial_M\Lambda_{2]\overline{CD}}\,, \quad \Lambda^{(1)\prime}_{12\overline{AB}} = \frac{b}{2}E_{\overline{B}}\Lambda_{[1}^{\overline{CD}}E_{\overline{A}}\Lambda_{2]\overline{CD}}\,. \tag{B.31}$$

Repeating the procedure for $E_{\mathbb{M}}{}^{\underline{a}}$, we find

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] E_{\mathbb{M}}{}^{\underline{a}} = -\left(\delta_{\Lambda_{12}^{(1)\prime}} + \delta_{\xi_{12}^{(1)\prime}}\right) E_{\mathbb{M}}{}^{\underline{a}}\,, \tag{B.32}$$

with $\xi_{12}^{(1)\prime\mathbb{M}}$ defined in (B.31) and

$$\Lambda^{(1)\prime}_{12\underline{ab}} = \frac{b}{2}E_{\underline{b}}\Lambda_{[1}^{\overline{CD}}E_{\underline{a}}\Lambda_{2]\overline{CD}}\,. \tag{B.33}$$

Mixed supersymmetry and double Lorentz transformations on the generalized frame. Using

$$\delta^{(0)}_{\epsilon_1}\mathcal{F}^*_{\underline{\mathbb{M}}}{}^{\overline{CD}} = -\bar{\epsilon}_1\gamma^b\left(\frac{1}{2}\Psi^{\overline{A}}E_{\mathbb{M}\overline{A}}\mathcal{F}_{\underline{b}}{}^{\overline{CD}} + E_{\mathbb{M}\underline{b}}\nabla^{[\overline{D}}\Psi^{\overline{C}]} + \frac{1}{2}E_{\mathbb{M}\underline{b}}\Psi^{\overline{A}}\mathcal{F}^{\overline{CD}}{}_{\overline{A}}\right), \tag{B.34}$$

we get the first order contribution to the mixed transformation rules of $E_{\mathbb{M}}{}^{\overline{A}}$

$$\begin{aligned}\delta_{[\epsilon,\Lambda]}E_{\mathbb{M}}{}^{\overline{A}} = \frac{b}{2}\Big[&-\frac{1}{2}\bar{\epsilon}_1\gamma^{\underline{b}}\Psi^{\overline{B}}E_{\mathbb{M}\overline{B}}\mathcal{F}_{\underline{b}}{}^{\overline{CD}}E_N{}^{\bar{A}}\partial^N\Lambda_{2\overline{CD}} - \frac{1}{2}\bar{\epsilon}_2\gamma^{\underline{b}}\Psi^{\overline{A}}\partial_{\overline{\mathbb{M}}}\Lambda_1^{\overline{CD}}\mathcal{F}_{\underline{b}\overline{CD}}\\ &+\frac{1}{16}\bar{\epsilon}_2\gamma^{\underline{b}}\Lambda_{1\underline{cd}}\gamma^{\underline{cd}}\Psi^{\overline{A}}\mathcal{F}_{\underline{\mathbb{M}}}{}^{\overline{CD}}\mathcal{F}_{\underline{b}\overline{CD}} - \frac{1}{4}\bar{\epsilon}_2\gamma^{\underline{b}}\Psi^{\overline{A}}\mathcal{F}_{\underline{\mathbb{M}}}{}^{\overline{CD}}\mathcal{F}_{\underline{a}\overline{CD}}\Lambda_{1\underline{b}}^{\underline{a}}\\ &+\frac{1}{4}\mathcal{F}_{\underline{\mathbb{M}}}{}^{\overline{CD}}\bar{\epsilon}_1\gamma^{\underline{b}}\Psi^{\overline{A}}E^N{}_{\underline{b}}\partial_N\Lambda_{2\overline{CD}} + \frac{1}{4}\bar{\epsilon}_2\gamma^{\underline{b}}\Psi^{\overline{A}}\partial_{\underline{\mathbb{M}}}\Lambda_1^{\overline{CD}}\mathcal{F}_{\underline{b}\overline{CD}}\\ &-\frac{1}{8}\bar{\epsilon}_2\gamma^{\underline{b}}E^N{}_{\underline{a}}\partial_N\Lambda_{1\overline{CD}}\mathcal{F}_{\underline{c}}{}^{\overline{CD}}\gamma^{\underline{ac}}\Psi^{\overline{A}}E_{\mathbb{M}\underline{b}} - (1\leftrightarrow 2)\Big]\,. \end{aligned} \tag{B.35}$$

The first two terms are a Lorentz transformation with parameter

$$\Lambda^{(1)''}_{12\overline{AB}} = b\;\bar{\epsilon}_{[1}\gamma^{\underline{b}}\Psi_{[\overline{A}}E^{N}_{\overline{B}]}\partial_N\Lambda_{2]}^{\overline{CD}}\mathcal{F}_{\underline{b}\overline{CD}}\,. \tag{B.36}$$

From the second line, only one term survives after commuting the gamma matrices, which corresponds to a first order supersymmetric variation with zeroth order parameter $\bar{\epsilon}'_{12} = -\frac{1}{2}\bar{\epsilon}_{[1}\gamma^{\underline{cd}}\Lambda_{2]\underline{cd}}$.

In the same way, from the remaining terms we find a first-order supersymmetry parameter

$$\bar{\epsilon}^{(1)'}_{12} = \frac{b}{4}\bar{\epsilon}_{[1}E^{M}{}_{\underline{a}}\partial_M\Lambda_{2]\overline{CD}}\mathcal{F}_{\underline{c}}{}^{\overline{CD}}\gamma^{\underline{ac}}\,. \tag{B.37}$$

Consider now the component $E_{\mathbb{M}}{}^{\underline{a}}$

$$\begin{aligned}\delta_{[\epsilon,\Lambda]}E_{\mathbb{M}}{}^{\underline{a}} = \frac{b}{2}\Big[&-\frac{1}{2}\bar{\epsilon}_1\gamma^{\underline{c}}\Psi^{\overline{B}}E_{\mathbb{M}\underline{c}}E^{N}{}_{\overline{B}}\partial_N\Lambda_{2\overline{CD}}\mathcal{F}^{\underline{a}\overline{CD}} - \frac{1}{2}\bar{\epsilon}_2\gamma^{\underline{a}}\Psi^{\overline{B}}\mathcal{F}_{\underline{\mathbb{M}}}{}^{\overline{CD}}E^{N}{}_{\overline{B}}\partial_N\Lambda_{1\overline{CD}}\\ &+\frac{1}{4}\bar{\epsilon}_2\gamma^{\underline{b}}\left(-\frac{1}{4}\Lambda_{1\underline{cd}}\gamma^{\underline{cd}}\Psi_{\overline{B}}\right)\mathcal{F}_{\underline{b}\overline{CD}}\mathcal{F}^{\underline{a}\overline{CD}}E_{\mathbb{M}}{}^{\overline{B}} - \frac{1}{4}\bar{\epsilon}_2\gamma^{\underline{b}}\Psi_{\overline{B}}\mathcal{F}_{\underline{b}\overline{CD}}E_{N}{}^{\underline{a}}\partial^N\Lambda_1^{\overline{CD}}E_{\mathbb{M}}{}^{\overline{B}}\\ &+\frac{1}{2}E_{\mathbb{M}}{}^{\overline{B}}\bar{\epsilon}_2\gamma^{\underline{c}}\Psi_{\overline{B}}E^{N}{}_{\underline{c}}\partial_N\Lambda_{1\overline{C}}{}^{\overline{D}}\mathcal{F}^{\underline{a}\overline{C}}{}_{\overline{D}} - \frac{1}{2}\bar{\epsilon}_2\gamma^{\underline{a}}E_{\mathbb{M}\overline{B}}\left(-\frac{1}{4}E^{N}{}_{\underline{b}}\partial_N\Lambda_{1\overline{CD}}\mathcal{F}_{\underline{c}}{}^{\overline{CD}}\gamma^{\underline{bc}}\Psi^{\overline{B}}\right)\\ &+\frac{1}{4}\bar{\epsilon}_2\gamma^{\underline{b}}\Psi_{\overline{B}}\left(-E^{N}{}_{\underline{b}}\partial_N\Lambda_{1\overline{CD}} + \mathcal{F}_{\underline{c}\overline{CD}}\Lambda_{1\underline{b}}^{\underline{c}}\right)\mathcal{F}^{\underline{a}\overline{CD}}E_{\mathbb{M}}{}^{\overline{B}} - (1\leftrightarrow 2)\Big]\,.\end{aligned} \tag{B.38}$$

The first line is a zeroth order Lorentz transformation with parameter

$$\Lambda^{(1)''}_{12\underline{ab}} = b\;\bar{\epsilon}_{[1}\gamma_{[\underline{a}}\Psi^{\overline{B}}\mathcal{F}_{\underline{b}]}{}^{\overline{CD}}E^{M}{}_{\overline{B}}\partial_M\Lambda_{2]\overline{CD}}\,. \tag{B.39}$$

Commuting the gamma matrices in the first term of the second line, the second contribution in the fourth line is canceled and we get again a supersymmetry transformation with zeroth order parameter $\bar{\epsilon}''_{12} = -\frac{1}{2}\bar{\epsilon}_{[1}\gamma^{\underline{cd}}\Lambda_{2]\underline{cd}}$. Finally, commuting the gamma matrices in the second term of the third line, various cancellations leave a supersymmetry transformation with first order parameter (B.37).

Supersymmetry variations on the generalized frame

$$\begin{aligned}E^{\mathbb{M}}{}_{\underline{c}}[\delta_{\epsilon_1},\delta_{\epsilon_2}]E_{\mathbb{M}\overline{A}} = \frac{b}{2}\Big[&-\frac{1}{2}\bar{\epsilon}_2(E^{\mathbb{N}}{}_{[\underline{d}}E^{P}{}_{\overline{A}}\partial_P\mathcal{F}_{\underline{\mathbb{N}}}{}^{\overline{CD}}\mathcal{F}_{\underline{c}]\overline{CD}} + E^{\mathbb{N}}{}_{\overline{A}}E^{P}{}_{[\underline{c}}\partial_P\mathcal{F}_{\mathbb{N}}{}^{\overline{CD}}\mathcal{F}_{\underline{d}]\overline{CD}}\\ &+\mathcal{F}^{\underline{N}}{}_{\overline{CD}}\mathcal{F}_{[\underline{c}}{}^{\overline{CD}}\partial_N E^{\mathbb{P}}{}_{\underline{d}]}E_{\mathbb{P}\overline{A}})\gamma^{\underline{d}}\epsilon_1 + \frac{1}{4}\bar{\epsilon}_2\mathcal{F}^{\underline{b}}{}_{\overline{A}\underline{d}}\mathcal{F}_{\underline{c}}{}^{\overline{CD}}\mathcal{F}_{\underline{b}\overline{CD}}\gamma^{\underline{d}}\epsilon_1\\ &+\frac{1}{4}\bar{\epsilon}_2E^{P}{}_{\overline{A}}\partial_P\mathcal{F}_{\underline{d}}{}^{\overline{CD}}\mathcal{F}_{\underline{c}}{}^{\overline{CD}}\gamma^{\underline{d}}\epsilon_1 - \frac{1}{4}E^{P}{}_{\overline{A}}\partial_P(\bar{\epsilon}_2\gamma^{\underline{b}}\epsilon_1\mathcal{F}_{\underline{b}\overline{CD}})\mathcal{F}_{\underline{c}}{}^{\overline{CD}} - (1\leftrightarrow 2)\Big]\,.\end{aligned}$$

The first and last terms of the r.h.s. combine into a Lorentz transformation with parameter

$$\Lambda^{(1)'''}_{12\overline{AB}} = \frac{b}{4}\bar{\epsilon}_1\gamma^{\underline{c}}\epsilon_2\mathcal{F}_{\underline{c}\overline{AB}}\,, \tag{B.40}$$

while the other terms form a diffeomorphism with first order parameter

$$\xi^{(1)''}_{12\mathbb{M}} = \frac{b}{8}\mathcal{F}_{\mathbb{M}\overline{CD}}\mathcal{F}_{\underline{b}}{}^{\overline{CD}}\bar{\epsilon}_1\gamma^{\underline{b}}\epsilon_2\,. \tag{B.41}$$

The same result holds for $E^{\mathbb{M}}{}_{\overline{C}}[\delta_{\epsilon_1},\delta_{\epsilon_2}]E_{\mathbb{M}\underline{a}}$, while

$$E^{\mathbb{M}}{}_{\overline{C}}[\delta_{\epsilon_1},\delta_{\epsilon_2}]E_{\mathbb{M}\overline{A}} = 0\,, \qquad E^{\mathbb{M}}{}_{\underline{c}}[\delta_{\epsilon_1},\delta_{\epsilon_2}]E_{\mathbb{M}\underline{a}} = 0. \tag{B.42}$$

JHEP07(2021)092

Mixed diffeomorphism and Lorentz variations of the generalized frame. Recalling that diffeomorphisms are not deformed, we get to first order

$$\delta_{[\Lambda,\xi]}E_{\mathbb{M}\mathbb{A}} = b\ E^{P}{}_{\mathbb{A}}\partial_{[P}(2\xi^{N}_{[1}\partial_N\Lambda_{2]\overline{CD}})\mathcal{F}^{*}_{\underline{\mathbb{M}}]}{}^{\overline{CD}}\,, \tag{B.43}$$

which is a first-order Lorentz transformation with a zeroth order parameter. We use the convention $A_{[\overline{A}}B_{\underline{b}]} = \frac{1}{2}A_{\overline{A}}B_{\underline{b}} - \frac{1}{2}A_{\overline{B}}B_{\underline{a}}$ to interchange projected indices.

Mixed diffeomorphism and supersymmetry variations on the generalized frame. This case is similar to the previous one. We start with

$$\delta_{[\epsilon,\xi]}E_{\mathbb{M}\overline{A}} = \frac{b}{2}\left(-\frac{1}{4}\xi^{P}_{2}\partial_P\bar{\epsilon}_1\right)\gamma^{\underline{b}}\Psi_{\overline{A}}\mathcal{F}_{\underline{\mathbb{M}}}{}^{\overline{CD}}\mathcal{F}_{\underline{b}\overline{CD}} - (1\leftrightarrow 2)\,, \tag{B.44}$$

which is a first order supersymmetry transformation with a zeroth order parameter. It is straightforward to see that the same result holds for $E_{\mathbb{M}\underline{a}}$.

Double Lorentz variations on the generalized gravitino

$$\begin{aligned}[\delta_{\Lambda_1},\delta_{\Lambda_2}]\,\Psi_{\overline{A}} = \frac{b}{2}\Big[&\Lambda_{2}{}^{\overline{B}}{}_{\overline{A}}\left(\delta^{(1)}_{\Lambda_1}\Psi_{\overline{B}}\right) - \frac{1}{4}\Lambda_{2\underline{bc}}\gamma^{\underline{bc}}\left(\delta^{(1)}_{\Lambda_1}\Psi_{\overline{A}}\right)\\ &-\delta^{(0)}_{\Lambda_1}\left(\frac{1}{4}\left(E^{M}{}_{\underline{b}}\partial_M\Lambda_{2\overline{CD}}\mathcal{F}_{\underline{c}}{}^{\overline{CD}}\gamma^{\underline{bc}}\Psi_{\overline{A}}\right) + \left(2\nabla^{\overline{D}}\Psi^{\overline{C}} - \omega_{\overline{E}}{}^{\overline{DC}}\Psi^{\overline{E}}\right)E^{M}{}_{\overline{A}}\partial_M\Lambda_{2\overline{CD}}\right)\\ &-\left(2\nabla^{\overline{D}}\Psi^{\overline{C}} - \omega_{\overline{E}}{}^{\overline{DC}}\Psi^{\overline{E}}\right)\delta^{(0)}_{\Lambda_1}\left(E^{M}{}_{\overline{A}}\partial_M\Lambda_{2\overline{CD}}\right)\Big] - (1\leftrightarrow 2)\,.\end{aligned} \tag{B.45}$$

After some straightforward manipulations, we finally obtain Lorentz transformations with the following parameters

$$\Lambda_{12\overline{AB}} = -2\Lambda_{[1\overline{A}}{}^{\overline{C}}\Lambda_{2]\overline{CB}}\,,\quad \Lambda^{(1)'}_{12\overline{AB}} = \frac{b}{2}E_{\overline{B}}\Lambda_{[1}{}^{\overline{CD}}E_{\overline{A}}\Lambda_{2]\overline{CD}}\ \text{ and }\ \Lambda^{(1)'}_{12\underline{ab}} = \frac{b}{2}E_{\underline{b}}\Lambda_{[1\overline{CD}}E_{\underline{a}}\Lambda_{2]}{}^{\overline{CD}}$$

Mixed Lorentz and supersymmetry transformations on the generalized gravitino

$$\begin{aligned}\delta_{[\Lambda,\epsilon]}\Psi_{\overline{A}} = \frac{b}{2}\Big[&\Lambda^{\overline{B}}_{2\overline{A}}\delta^{(1)}_{\epsilon_1}\Psi_{\overline{B}} + \frac{1}{16}\Lambda_{2\underline{ab}}\gamma^{\underline{ab}}\mathcal{F}^{(3)}_{\overline{A}\underline{cd}}\gamma^{\underline{cd}}\epsilon_1 + \delta^{(1)}_{\Lambda_1}E^{M}{}_{\overline{A}}\partial_M\epsilon_2\\ &-\frac{1}{4}E^{M}{}_{\underline{b}}\partial_M\Lambda_{2\overline{CD}}\mathcal{F}_{\underline{c}}{}^{\overline{CD}}\gamma^{\underline{bc}}\nabla_{\overline{A}}\epsilon_1 - \frac{1}{4}\delta^{(1)}_{\Lambda_1}\mathcal{F}_{\overline{A}\underline{bc}}\gamma^{\underline{bc}}\epsilon_2\\ &-2\delta^{(0)}_{\epsilon_1}(\nabla^{\overline{D}}\Psi^{\overline{C}})E^{M}{}_{\overline{A}}\partial_M\Lambda_{2\overline{CD}} + \omega_{\overline{B}}{}^{\overline{DC}}\delta^{(0)}_{\epsilon_1}\Psi^{\overline{B}}E^{M}{}_{\overline{A}}\partial_M\Lambda_{2\overline{CD}}\\ &-\frac{1}{4}\delta^{(0)}_{\Lambda_1}\mathcal{F}^{(3)}_{\overline{A}\underline{bc}}\gamma^{\underline{bc}}\epsilon_2 - (1\leftrightarrow 2)\Big]\,.\end{aligned} \tag{B.46}$$

Commuting the gamma matrices in the second term of the r.h.s, and combining it with the corresponding term in the $(1\leftrightarrow 2)$ operation, we recognize a supersymmetry transformation with zeroth order parameter $\epsilon'_{12} = -\frac{1}{2}\bar{\epsilon}_{[1}\gamma^{\underline{ab}}\Lambda_{2]\underline{ab}}$.

The first term in the second line together with the corresponding term in the $(1\leftrightarrow 2)$ operation, gives a zeroth order supersymmetry transformation with first order parameter $\epsilon^{(1)'}_{12} = \frac{b}{4}\gamma^{\underline{bc}}\epsilon_{[1}E^{M}{}_{\underline{b}}\partial_M\Lambda_{2]\overline{CD}}\mathcal{F}_{\underline{c}}{}^{\overline{CD}}$. The remaining terms cancel and then we get

$$\delta_{[\Lambda,\epsilon]}\Psi_{\overline{A}} = -\left(\delta^{(0)}_{\epsilon^{(1)'}_{12}} + \delta^{(1)}_{\epsilon'_{12}}\right)\Psi_{\overline{A}} \tag{B.47}$$

up to terms with two fermions.

Mixed diffeomorphisms and supersymmetry transformations on the generalized gravitino

$$\delta_{[\xi,\epsilon]}\Psi_{\overline{A}} = \frac{b}{4}\xi^M_{[1}\mathcal{F}^{(3)}_{\overline{A}\underline{bc}}\gamma^{\underline{bc}}\partial_M\epsilon_{2]} = -\delta^{(1)}_{\epsilon_{12}}\Psi_{\overline{A}}\,, \tag{B.48}$$

with $\epsilon_{12} = 2\xi^M_{[1}\partial_M\epsilon_{2]}$.

Double Lorentz variations on the generalized dilatino

$$\begin{aligned}[\delta_{\Lambda_1},\delta_{\Lambda_2}]\rho = & -\frac{1}{4}\Lambda_{2\underline{ab}}\gamma^{\underline{ab}}\Big(-\frac{1}{4}E^M{}_{\underline{b}}\partial_M\Lambda_{1\overline{CD}}\mathcal{F}_{\underline{d}}{}^{\overline{CD}}\gamma^{\underline{bd}}\rho\Big) - \frac{1}{4}\Lambda_1{}^{\underline{c}}{}_{\underline{b}}E^M{}_{\underline{c}}\partial_M\Lambda_{2\overline{CD}}\mathcal{F}_{\underline{d}}{}^{\overline{CD}}\gamma^{\underline{bd}}\rho \\ & -\frac{1}{4}E^M{}_{\underline{b}}\partial_M\Lambda_{2\overline{CD}}\Big(-E^N{}_{\underline{c}}\partial_N\Lambda_1{}^{\overline{CD}} + \mathcal{F}_{\underline{a}}{}^{\overline{CD}}\Lambda_1{}^{\underline{a}}{}_{\underline{c}} + 2\mathcal{F}_{\underline{c}}{}^{\overline{BD}}\Lambda_{1\overline{B}}{}^{\overline{C}}\Big)\gamma^{\underline{bc}}\rho \\ & -\frac{1}{4}E^M{}_{\underline{b}}\partial_M\Lambda_{2\overline{CD}}\mathcal{F}_{\underline{c}}{}^{\overline{CD}}\gamma^{\underline{bc}}\Big(-\frac{1}{4}\Lambda_{1\underline{ad}}\gamma^{\underline{ad}}\rho\Big) - (1\leftrightarrow 2)\,. \end{aligned} \tag{B.49}$$

In the second line (adding the $(1\leftrightarrow 2)$ operation) we recognize a Lorentz transformation with first and zeroth order parameters

$$\Lambda^{(1)\prime}_{12\underline{ab}} = \frac{b}{2}E_{\underline{a}}\Lambda_{[1\overline{CD}}E_{\underline{b}}\Lambda_{2]}{}^{\overline{CD}} \quad \text{and} \quad \Lambda_{12\overline{AB}} = -2\Lambda_{[2\overline{CB}}\Lambda_{1]\overline{A}}{}^{\overline{C}}. \tag{B.50}$$

Commuting the gamma matrices of the third line, it is straightforward to see that the remaining terms cancel.

Mixed Lorentz and supersymmetry transformations on the generalized dilatino. This computation is similar to the one associated to the gravitino. We find the following supersymmetry parameters

$$\epsilon'_{12} = -\frac{1}{2}\bar{\epsilon}_{[1}\gamma^{\underline{ab}}\Lambda_{2]\underline{ab}} \quad \text{and} \quad \epsilon^{(1)\prime}_{12} = \frac{b}{4}\gamma^{\underline{bc}}\epsilon_{[2}E^M{}_{\underline{b}}\partial_M\Lambda_{1]\overline{CD}}\mathcal{F}_{\underline{c}}{}^{\overline{CD}}\,, \tag{B.51}$$

so that finally

$$\delta_{[\epsilon,\Lambda]}\rho = -\delta^{(1)}_{\epsilon_{12}}\rho\,. \tag{B.52}$$

Mixed diffeomorphism and supersymmetry transformations on the generalized dilatino

$$\begin{aligned}\delta_{[\xi,\epsilon]}\rho = & \ \xi_2^M\partial_M\left(-\frac{1}{12}\mathcal{F}^{(3)}_{\underline{abc}}\gamma^{\underline{abc}}\epsilon_1 - \frac{1}{4}\left(\omega_{\underline{cd}}{}^{\underline{c}}\mathcal{F}^{\underline{d}}{}_{\overline{CD}}\mathcal{F}_{\underline{a}}{}^{\overline{CD}} + E^N{}_{\underline{d}}\partial_N(\mathcal{F}^{\underline{d}}{}_{\overline{CD}}\mathcal{F}_{\underline{a}}{}^{\overline{CD}})\right)\gamma^{\underline{a}}\epsilon_1\right) \\ & -(1\leftrightarrow 2) \\ = & -\delta^{(1)}_{\epsilon_{12}}\rho\,. \end{aligned} \tag{B.53}$$

In equations (3.38) of the main text we collect the parameters that appear in this algebra of first order transformation rules.

C Supersymmetry of heterotic string effective action

In the first part of this appendix we prove that the higher-derivative deformations of the transformation rules of the supergravity fields satisfy a closed algebra up to $\mathcal{O}(\alpha')$ and up to terms with two fermions. In the second part, we show that the action (5.1) is invariant under these supersymmetry transformations.

JHEP07(2021)092

C.1 Supersymmetry algebra

It is well known that the algebra of leading order transformations of supergravity and super Yang-Mills fields closes. Moreover, the replacement $H_{\mu\nu\rho} \to \widetilde{H}_{\mu\nu\rho}$ in the supersymmetry transformations of the gravitino and dilatino does not affect the leading order closure on any field except for the b-field. Hence we focus on the algebra of first order transformation rules on $b_{\mu\nu}$.

It is convenient to first look at the brackets acting on $\widetilde{b}_{\mu\nu} = b_{\mu\nu} + \frac{b}{8} A^k_{[\mu} \overline{\chi}^i \gamma_{\nu]} \chi^j f_{ijk}$. Up to first order and bilinear terms in fermions, we need the following transformation rules:

$$\delta\psi_a = \psi_b \Lambda^b{}_a - \frac{1}{4}\gamma^{bc}\Lambda_{bc}\psi_a + \partial_\mu\epsilon - \frac{1}{4}\widetilde{w}^{(+)}_{\mu ab}\gamma^{ab}\epsilon \,, \tag{C.1a}$$

$$\delta A^i_\mu = \partial_\mu \xi^i + f^i{}_{jk}\xi^j A^k_\mu + \frac{1}{2}\bar{\epsilon}\gamma_\mu \chi^i \,, \tag{C.1b}$$

$$\delta\chi = f^i{}_{jk}\xi^j\chi^k - \frac{1}{4}\Lambda_{bc}\gamma^{bc}\chi - \frac{1}{4}F^i_{\mu\nu}\gamma^{\mu\nu}\epsilon \,, \tag{C.1c}$$

$$\delta\widetilde{b}_{\mu\nu} = 2\partial_{[\mu}\xi_{\nu]} - \zeta\partial_{[\mu}\xi^i A_{\nu]i} - \frac{b}{2}\left(\partial_{[\mu}\Lambda^{CD}\hat{\Omega}_{\nu]CD} + \bar{\epsilon}\gamma_{[\mu}\Psi^{CD}\hat{\Omega}_{\nu]CD}\right) \,, \tag{C.1d}$$

$$\delta\hat{\Omega}_{\mu CD} = -\partial_\mu\Lambda_{CD} + 2\hat{\Omega}_{\mu E[D}\Lambda^E{}_{C]} + \bar{\epsilon}\gamma_\mu\Psi_{CD} = -\mathcal{D}_\mu\Lambda_{CD} + \bar{\epsilon}\gamma_\mu\Psi_{CD} \,, \tag{C.1e}$$

$$\delta\hat{\mathcal{R}}_{\mu\nu CD} = 2\hat{\mathcal{R}}_{\mu\nu E[D}\Lambda^E{}_{C]} - 2\mathcal{D}_{[\mu}\left(\bar{\epsilon}\gamma_{\nu}\Psi_{CD}\right) \,, \tag{C.1f}$$

$$\delta\Psi_{CD} = 2\Psi_{E[D}\Lambda^E{}_{C]} + \frac{1}{8}\hat{\mathcal{R}}_{\mu\nu CD}\gamma^{\mu\nu}\epsilon \,. \tag{C.1g}$$

We exclude the diffeomorphisms since it is trivial to see that all the transformation rules of $b_{\mu\nu}$ (i.e. Lorentz, supersymmetry, abelian and non-abelian gauge transformations) transform as tensors under diffeomorphisms and hence their commutators are trivial. Therefore, we compute the brackets

$$\left([\delta_1, \delta_2]\widetilde{b}_{\mu\nu}\right)^{(1)} = \left(\delta_1^{(1)}\delta_2^{(0)} - \delta_2^{(1)}\delta_1^{(0)}\right)\widetilde{b}_{\mu\nu} + \left(\delta_1^{(0)}\delta_2^{(1)} - \delta_2^{(0)}\delta_1^{(1)}\right)\widetilde{b}_{\mu\nu} \,. \tag{C.2}$$

The first term in the r.h.s. gives

$$\delta_1^{(1)}\delta_2^{(0)}\widetilde{b}_{\mu\nu} - (1 \leftrightarrow 2) = \frac{3\alpha'}{4}\bar{\epsilon}_2\gamma^\lambda\epsilon_1\hat{\mathcal{C}}_{\mu\nu\lambda} \,, \tag{C.3}$$

and the second one can be written as

$$\begin{aligned}\delta_1^{(0)}\delta_2^{(1)}\widetilde{b}_{\mu\nu} - (1 \leftrightarrow 2) = {}& \alpha'\partial_{[\mu}\left(\Lambda_2^{CD}\partial_{\nu]}\Lambda_{1CD}\right) + \alpha'\partial_{[\mu}\left(\Lambda_1^{CD}\Lambda_2^E{}_C\right)\hat{\Omega}_{\nu]ED} \\ &+ \frac{\alpha'}{4}\bar{\epsilon}_2\gamma^\lambda\epsilon_1\hat{\Omega}_{[\mu}{}^{CD}\hat{\mathcal{R}}_{\nu]\lambda CD} - \frac{\alpha'}{2}\partial_{[\mu}\left(\bar{\epsilon}_2\gamma^\sigma\epsilon_1\hat{\Omega}^i_{c\sigma}\right)\hat{\Omega}_{\nu]ci} \,. \end{aligned}\tag{C.4}$$

Adding both contributions, we get

$$\left([\delta_1, \delta_2]\widetilde{b}_{\mu\nu}\right)^{(1)} = 2\partial_{[\mu}\xi_{12\nu]} - \frac{\alpha'}{2}\partial_{[\mu}\Lambda_{12}^{CD}\hat{\Omega}_{\nu]CD} - \frac{\alpha'}{2}\partial_{[\mu}\left(\bar{\epsilon}_2\gamma^\sigma\epsilon_1\hat{\Omega}^i_{c\sigma}\right)\hat{\Omega}_{\nu]ci} \,, \tag{C.5}$$

with

$$\xi_{12\nu} = \frac{\alpha'}{2}\left[\Lambda_2^{CD}\partial_\nu\Lambda_{1CD} + \frac{1}{4}\bar{\epsilon}_2\gamma^\lambda\epsilon_1\hat{\Omega}_\nu{}^{CD}\hat{\Omega}_{\lambda CD}\right] \,. \tag{C.6}$$

JHEP07(2021)092

and

$$\Lambda_{12}^{CD} = 2\Lambda_1^{CE}\Lambda_{2E}{}^D + \frac{1}{2}\bar{\epsilon}_2\gamma^\lambda\epsilon_1\hat{\Omega}_\lambda{}^{CD} . \tag{C.7}$$

To see the algebra of transformations on $b_{\mu\nu}$, note that

$$([\delta_1,\delta_2])^{(1)}\, b_{\mu\nu} = \left([\delta_1,\delta_2]\widetilde{b}_{\mu\nu}\right)^{(1)} - \frac{\alpha'}{8}\,([\delta_1,\delta_2])^{(0)}\left(A_{[\mu}{}^k\overline{\chi}^i\gamma_{\nu]}\chi^j f_{ijk}\right) , \tag{C.8}$$

and it is easy to see that the second term in the r.h.s. vanishes. Rewriting (C.5) in terms of supergravity and super Yang-Mills fields, the brackets that mix supersymmetry with Lorentz and abelian gauge transformations vanish, while the supersymmetry algebra gives

$$([\delta_{\epsilon_1},\delta_{\epsilon_2}])^{(1)}\, b_{\mu\nu} = \partial_{[\mu}(\xi_{12})_{\nu]} - \alpha'\partial_{[\mu}\Lambda_{12}^{cd}\hat{w}_{\nu]cd} - \frac{\alpha'}{2}\varrho\partial_{[\mu}\xi_{12}^i A_{\nu]i}\, , \tag{C.9}$$

with

$$\begin{aligned}(\xi_{12})_\nu &= \frac{\alpha'}{4}\bar{\epsilon}_2\gamma^\lambda\epsilon_1\hat{\Omega}_\nu{}^{CD}\hat{\Omega}_{\lambda CD}\, ,\\ \Lambda_{12}^{cd} &= \frac{1}{4}\bar{\epsilon}_2\gamma^\lambda\epsilon_1\hat{w}_\lambda^{(-)cd}\, ,\\ \xi_{12}^i &= -\frac{1}{2}\bar{\epsilon}_2\gamma^\lambda\epsilon_1 A_\lambda^i\, .\end{aligned} \tag{C.10}$$

C.2 Invariance of the action

Here we prove the supersymmetric invariance of the action

$$S = \int d^{10}x\ e\ e^{-2\phi}\mathcal{L}\, , \tag{C.11}$$

with

$$\begin{aligned}\mathcal{L} = {}& R+4\partial_\mu\phi\partial^\mu\phi-\frac{1}{12}\widetilde{H}_{\mu\nu\rho}\widetilde{H}^{\mu\nu\rho}-\frac{1}{4}F^i_{\mu\nu}F_i^{\mu\nu}+\frac{\alpha'}{8}\hat{\mathfrak{R}}_{\mu\nu CD}\hat{\mathfrak{R}}^{\mu\nu CD}\\ &-\overline{\psi}^\mu\gamma^\nu D_\nu\psi_\mu+\overline{\rho}\gamma^\mu D_\mu\rho+2\overline{\psi}^\mu D_\mu\rho-\frac{1}{2}\overline{\chi}^i\gamma^\mu D_\mu\chi_i+\overline{\chi}_i\left(\gamma^\mu\psi^\nu-\frac{1}{4}\gamma^{\mu\nu}\rho\right)F^i_{\mu\nu}\\ &+\frac{1}{24}\widetilde{H}_{\rho\sigma\tau}\left(\overline{\psi}^\mu\gamma^{\rho\sigma\tau}\psi_\mu+12\overline{\psi}^\rho\gamma^\sigma\psi^\tau-\overline{\rho}\gamma^{\rho\sigma\tau}\rho-6\overline{\psi}^\rho\gamma^{\sigma\tau}\rho+\frac{1}{2}\overline{\chi}^i\gamma^{\rho\sigma\tau}\chi_i\right)\\ &+\alpha'\left(\overline{\Psi}^{CD}\gamma^\mu D_\mu(w,\hat{\Omega})\Psi_{CD}-\frac{1}{24}\overline{\Psi}^{CD}\not{H}\overline{\Psi}_{CD}-\overline{\Psi}^{CD}\left(\gamma^\mu\psi^\nu-\frac{1}{4}\gamma^{\mu\nu}\rho\right)\hat{\mathfrak{R}}_{\mu\nu CD}\right) .\end{aligned} \tag{C.12}$$

Since the leading order action is known to be invariant [24], we analyze the $\mathcal{O}(\alpha')$ variation, namely

$$(\delta S)^{(1)} = \int d^{10}x e e^{-2\phi}\left[-e_\mu{}^a\delta^{(0)}e^\mu{}_a\mathcal{L}^{(1)} - 2\delta^{(0)}\phi\mathcal{L}^{(1)} + \delta^{(0)}\mathcal{L}^{(1)} + \delta^{(1)}\mathcal{L}^{(0)}\right] . \tag{C.13}$$

JHEP07(2021)092

Using the transformation rules (C.1) we get

$$
\begin{aligned}
(\delta S)^{(1)} = & -\frac{\alpha'}{8}\bar{\epsilon}\rho\left(H_{\mu\nu\rho}\hat{\mathcal{C}}^{\mu\nu\rho}-\frac{1}{2}\hat{\mathcal{R}}_{\mu\nu CD}\hat{\mathcal{R}}^{\mu\nu CD}\right)+\frac{3\alpha'}{8}\bar{\epsilon}\gamma^{(\mu}\psi^{\lambda)}H_{\mu\nu\rho}\hat{\mathcal{C}}_{\lambda}{}^{\nu\rho} \\
& -\frac{3\alpha'}{2}\bar{\epsilon}\gamma^{\mu}\psi^{\nu}\left(\partial_{\rho}\phi\hat{\mathcal{C}}_{\mu\nu}{}^{\rho}-\frac{1}{2}\mathcal{D}_{\rho}\hat{\mathcal{C}}_{\mu\nu}{}^{\rho}+\frac{1}{12}\hat{\mathcal{R}}_{\mu\rho CD}\hat{\mathcal{R}}_{\nu}{}^{\rho CD}\right)+\frac{3\alpha'}{8}\bar{\epsilon}\gamma_{\mu}\chi^{i}F_{\nu\rho i}\hat{\mathcal{C}}^{\mu\nu\rho} \\
& +\frac{\alpha'}{2}\delta^{(0)}\hat{\Omega}_{\mu}{}^{CD}\left(\Delta b^{\mu\nu}\hat{\Omega}_{\nu CD}+\frac{1}{2}H^{\mu\nu\rho}\hat{\mathcal{R}}_{\nu\rho CD}+2\partial_{\nu}\phi\hat{\mathcal{R}}^{\mu\nu CD}-\mathcal{D}_{\nu}\hat{\mathcal{R}}^{\mu\nu CD}\right) \\
& +\frac{\alpha'}{8}\delta^{(0)}\overline{\psi}^{\mu}\left(\gamma^{\rho\sigma\tau}\psi_{\mu}\hat{\mathcal{C}}_{\rho\sigma\tau}+12\gamma^{\sigma}\psi^{\tau}\hat{\mathcal{C}}_{\mu\sigma\tau}-3\gamma^{\sigma\tau}\rho\hat{\mathcal{C}}_{\mu\sigma\tau}+8\gamma^{\nu}\Psi^{CD}\hat{\mathcal{R}}_{\mu\nu CD}\right) \\
& -\frac{\alpha'}{8}\delta^{(0)}\overline{\rho}\left(\left(\gamma^{\rho\sigma\tau}\rho+3\gamma^{\sigma\tau}\psi^{\rho}\right)\hat{\mathcal{C}}_{\rho\sigma\tau}-2\gamma^{\mu\nu}\Psi^{CD}\hat{\mathcal{R}}_{\mu\nu CD}\right)+\frac{\alpha'}{16}\delta^{(0)}\overline{\chi}^{i}\gamma^{\rho\sigma\tau}\chi_{i}\hat{\mathcal{C}}_{\rho\sigma\tau} \\
& +2\alpha'\delta^{(0)}\overline{\Psi}^{CD}\left(\not{D}(w,\hat{\Omega})\Psi_{CD}-\left(\not{\partial}\phi+\frac{1}{24}\not{H}\right)\Psi_{CD}+\frac{1}{2}\left(\gamma^{\mu}\psi^{\nu}-\frac{1}{4}\gamma^{\mu\nu}\rho\right)\hat{\mathcal{R}}_{\mu\nu CD}\right) \\
& +\delta^{(1)}\tilde{b}_{\mu\nu}\Delta b^{\mu\nu}-2\delta^{(1)}\overline{\psi}^{\mu}\Delta\psi_{\mu}+2\delta^{(1)}\overline{\rho}\Delta\rho\,.
\end{aligned}
\tag{C.14}
$$

The variations (C.1) depend on the supersymmetry parameter explicitly and through $\Lambda_{ci} = \frac{1}{2\sqrt{2}}\bar{\epsilon}\gamma_c\chi_i$. The explicit dependence has the same structure as the corresponding transformations in [26, 27], replacing the collective indices $C, D, \ldots$ by $c, d, \ldots$. Since the corresponding actions also have the same structure, we can assure that those terms cancel in (C.14). The Λ_{ci}-dependent terms are contained in $\delta^{(0)}\hat{\Omega}_{\mu CD}, \delta^{(1)}b_{\mu\nu}$ and $\delta^{(0)}\Psi_{CD}$. We can disregard the latter as they are higher than bilinear in fermions. The former two may be written as

$$
\begin{aligned}
(\delta S)^{(1)} = & \frac{\alpha'}{2}D_{\rho}\left[\partial_{\mu}\Lambda^{CD}\hat{\Omega}_{\nu CD}-\hat{\Omega}_{\mu}{}^{ED}\hat{\Omega}_{\nu}{}^{C}{}_{D}\Lambda_{EC}\right]H^{\mu\nu\rho} \\
& +\frac{\alpha'}{2}\mathcal{D}_{\nu}\mathcal{D}_{\mu}\Lambda^{CD}\hat{\mathcal{R}}^{\mu\nu}{}_{CD}-\frac{\alpha'}{4}\mathcal{D}_{\mu}\Lambda^{CD}H^{\mu\nu\rho}\hat{\mathcal{R}}_{\nu\rho CD}\,,
\end{aligned}
$$

which can be easily shown to vanish after performing some integrations by parts.

References

[1] D.J. Gross, J.A. Harvey, E.J. Martinec and R. Rohm, *Heterotic String Theory. 2. The Interacting Heterotic String*, *Nucl. Phys. B* **267** (1986) 75 [INSPIRE].

[2] A. Fontanella and T. Ortín, *On the supersymmetric solutions of the Heterotic Superstring effective action*, *JHEP* **06** (2020) 106 [arXiv:1910.08496] [INSPIRE].

[3] P.A. Cano, S. Chimento, R. Linares, T. Ortín and P.F. Ramírez, *α′ corrections of Reissner-Nordström black holes*, *JHEP* **02** (2020) 031 [arXiv:1910.14324] [INSPIRE].

[4] P.A. Cano, P. Meessen, T. Ortín and P.F. Ramírez, *α′-corrected black holes in String Theory*, *JHEP* **05** (2018) 110 [arXiv:1803.01919] [INSPIRE].

[5] K. Foerger and S. Stieberger, *Higher derivative couplings and heterotic type-I duality in eight-dimensions*, *Nucl. Phys. B* **559** (1999) 277 [hep-th/9901020] [INSPIRE].

[6] J.T. Liu and R. Minasian, *Higher-derivative couplings in string theory: dualities and the B-field*, *Nucl. Phys. B* **874** (2013) 413 [arXiv:1304.3137] [INSPIRE].

[7] Z. Elgood and T. Ortín, *T duality and Wald entropy formula in the Heterotic Superstring effective action at first-order in α'*, *JHEP* **10** (2020) 097 [*Erratum ibid.* **06** (2021) 105] [arXiv:2005.11272] [INSPIRE].

[8] T. Ortín, *O(n, n) invariance and Wald entropy formula in the Heterotic Superstring effective action at first order in* α', *JHEP* **01** (2021) 187 [arXiv:2005.14618] [INSPIRE].

[9] A. Castro, J.L. Davis, P. Kraus and F. Larsen, *String Theory Effects on Five-Dimensional Black Hole Physics*, *Int. J. Mod. Phys. A* **23** (2008) 613 [arXiv:0801.1863] [INSPIRE].

[10] D. Ciupke, J. Louis and A. Westphal, *Higher-Derivative Supergravity and Moduli Stabilization*, *JHEP* **10** (2015) 094 [arXiv:1505.03092] [INSPIRE].

[11] H. Ooguri and C. Vafa, *On the Geometry of the String Landscape and the Swampland*, *Nucl. Phys. B* **766** (2007) 21 [hep-th/0605264] [INSPIRE].

[12] U.H. Danielsson and T. Van Riet, *What if string theory has no de Sitter vacua?*, *Int. J. Mod. Phys. D* **27** (2018) 1830007 [arXiv:1804.01120] [INSPIRE].

[13] G. Obied, H. Ooguri, L. Spodyneiko and C. Vafa, *de Sitter Space and the Swampland*, arXiv:1806.08362 [INSPIRE].

[14] P. Agrawal, G. Obied, P.J. Steinhardt and C. Vafa, *On the Cosmological Implications of the String Swampland*, *Phys. Lett. B* **784** (2018) 271 [arXiv:1806.09718] [INSPIRE].

[15] H.-C. Kim, H.-C. Tarazi and C. Vafa, *Four-dimensional* $\mathcal{N} = 4$ *SYM theory and the swampland*, *Phys. Rev. D* **102** (2020) 026003 [arXiv:1912.06144] [INSPIRE].

[16] Y. Cai and C.A. Núñez, *Heterotic String Covariant Amplitudes and Low-energy Effective Action*, *Nucl. Phys. B* **287** (1987) 279 [INSPIRE].

[17] D.J. Gross and J.H. Sloan, *The Quartic Effective Action for the Heterotic String*, *Nucl. Phys. B* **291** (1987) 41 [INSPIRE].

[18] R.R. Metsaev and A.A. Tseytlin, *Order alpha-prime (Two Loop) Equivalence of the String Equations of Motion and the Sigma Model Weyl Invariance Conditions: Dependence on the Dilaton and the Antisymmetric Tensor*, *Nucl. Phys. B* **293** (1987) 385 [INSPIRE].

[19] N. Sakai and Y. Tanii, *One Loop Amplitudes and Effective Action in Superstring Theories*, *Nucl. Phys. B* **287** (1987) 457 [INSPIRE].

[20] M. Abe, H. Kubota and N. Sakai, *Loop Corrections to the Heterotic String Effective Lagrangian*, *Phys. Lett. B* **200** (1988) 461 [*Addendum ibid.* **203** (1988) 474] [INSPIRE].

[21] M. Abe, H. Kubota and N. Sakai, *Loop Corrections to the* $E_8 \times E_8$ *Heterotic String Effective Lagrangian*, *Nucl. Phys. B* **306** (1988) 405 [INSPIRE].

[22] J.R. Ellis, P. Jetzer and L. Mizrachi, *One Loop String Corrections to the Effective Field Theory*, *Nucl. Phys. B* **303** (1988) 1 [INSPIRE].

[23] M.T. Grisaru, A.E.M. van de Ven and D. Zanon, *Four Loop Divergences for the* $N = 1$ *Supersymmetric Nonlinear Sigma Model in Two-Dimensions*, *Nucl. Phys. B* **277** (1986) 409 [INSPIRE].

[24] G.F. Chapline and N.S. Manton, *Unification of Yang-Mills Theory and Supergravity in Ten-Dimensions*, *Phys. Lett. B* **120** (1983) 105 [INSPIRE].

[25] L.J. Romans and N.P. Warner, *Some Supersymmetric Counterparts of the Lorentz Chern-Simons Term*, Nucl. Phys. B **273** (1986) 320 [INSPIRE].

[26] E. Bergshoeff and M. de Roo, *Supersymmetric Chern-Simons Terms in Ten-dimensions*, Phys. Lett. B **218** (1989) 210 [INSPIRE].

[27] E.A. Bergshoeff and M. de Roo, *The Quartic Effective Action of the Heterotic String and Supersymmetry*, Nucl. Phys. B **328** (1989) 439 [INSPIRE].

[28] M. de Roo, H. Suelmann and A. Wiedemann, *The supersymmetric effective action of the heterotic string in ten-dimensions*, Nucl. Phys. B **405** (1993) 326 [hep-th/9210099] [INSPIRE].

[29] K. Peeters, P. Vanhove and A. Westerberg, *Supersymmetric higher derivative actions in ten-dimensions and eleven-dimensions, the associated superalgebras and their formulation in superspace*, Class. Quant. Grav. **18** (2001) 843 [hep-th/0010167] [INSPIRE].

[30] S.J. Gates Jr. and H. Nishino, *New $D = 10$, $N = 1$ Superspace Supergravity and Local Symmetries of Superstrings*, Phys. Lett. B **173** (1986) 46 [INSPIRE].

[31] S.J. Gates Jr. and H. Nishino, *Manifestly Supersymmetric $O(\alpha')$ Superstring Corrections in New $D = 10$, $N = 1$ Supergravity Yang-Mills Theory*, Phys. Lett. B **173** (1986) 52 [INSPIRE].

[32] L. Bonora, M. Bregola, K. Lechner, P. Pasti and M. Tonin, *Anomaly Free Supergravity and SuperYang-Mills Theories in Ten-dimensions*, Nucl. Phys. B **296** (1988) 877 [INSPIRE].

[33] S. Bellucci and S.J. Gates Jr., *$D = 10$, $N = 1$ Superspace Supergravity and the Lorentz Chern-Simons Form*, Phys. Lett. B **208** (1988) 456 [INSPIRE].

[34] S. Bellucci, D.A. Depireux and S.J. Gates Jr., *Consistent and Universal Inclusion of the Lorentz Chern-Simons Form in $D = 10$, $N = 1$ Supergravity Theories*, Phys. Lett. B **238** (1990) 315 [INSPIRE].

[35] R. Kallosh, *Strings and Superspace*, Phys. Scripta T **15** (1987) 118.

[36] B.E.W. Nilsson and A.K. Tollsten, *Supersymmetrization of Zeta (3) (R $\mu\nu\rho\sigma$)**4 in Superstring Theories*, Phys. Lett. B **181** (1986) 63 [INSPIRE].

[37] P.S. Howe, *Heterotic supergeometry revisited*, in *Gravity, Supersymmetry and Branes: A Meeting in Celebration of Kellogg Stelle's 60th Birthday*, (2008) [arXiv:0805.2893] [INSPIRE].

[38] A. Sen, *O(d) x O(d) symmetry of the space of cosmological solutions in string theory, scale factor duality and two-dimensional black holes*, Phys. Lett. B **271** (1991) 295 [INSPIRE].

[39] J. Maharana and J.H. Schwarz, *Noncompact symmetries in string theory*, Nucl. Phys. B **390** (1993) 3 [hep-th/9207016] [INSPIRE].

[40] K.A. Meissner, *Symmetries of higher order string gravity actions*, Phys. Lett. B **392** (1997) 298 [hep-th/9610131] [INSPIRE].

[41] N. Kaloper and K.A. Meissner, *Duality beyond the first loop*, Phys. Rev. D **56** (1997) 7940 [hep-th/9705193] [INSPIRE].

[42] N. Kaloper and K.A. Meissner, *Tailoring T duality beyond the first loop*, in *Conference on Black Holes: Theory and Mathematical Aspects*, (1997) [hep-th/9708169] [INSPIRE].

[43] H. Godazgar and M. Godazgar, *Duality completion of higher derivative corrections*, JHEP **09** (2013) 140 [arXiv:1306.4918] [INSPIRE].

JHEP07(2021)092

[44] O. Hohm, A. Sen and B. Zwiebach, *Heterotic Effective Action and Duality Symmetries Revisited*, *JHEP* **02** (2015) 079 [arXiv:1411.5696] [INSPIRE].

[45] M.R. Garousi, *Duality constraints on effective actions*, *Phys. Rept.* **702** (2017) 1 [arXiv:1702.00191] [INSPIRE].

[46] M.R. Garousi, *Four-derivative couplings via the T-duality invariance constraint*, *Phys. Rev. D* **99** (2019) 126005 [arXiv:1904.11282] [INSPIRE].

[47] H. Razaghian and M.R. Garousi, *R^4 terms in supergravities via T-duality constraint*, *Phys. Rev. D* **97** (2018) 106013 [arXiv:1801.06834] [INSPIRE].

[48] C. Eloy, O. Hohm and H. Samtleben, *Duality Invariance and Higher Derivatives*, *Phys. Rev. D* **101** (2020) 126018 [arXiv:2004.13140] [INSPIRE].

[49] T. Codina, O. Hohm and D. Marqués, *String Dualities at Order α'^3*, *Phys. Rev. Lett.* **126** (2021) 171602 [arXiv:2012.15677] [INSPIRE].

[50] W. Siegel, *Two vierbein formalism for string inspired axionic gravity*, *Phys. Rev. D* **47** (1993) 5453 [hep-th/9302036] [INSPIRE].

[51] W. Siegel, *Superspace duality in low-energy superstrings*, *Phys. Rev. D* **48** (1993) 2826 [hep-th/9305073] [INSPIRE].

[52] W. Siegel, *Manifest duality in low-energy superstrings*, in *International Conference on Strings 93*, (1993) [hep-th/9308133] [INSPIRE].

[53] C. Hull and B. Zwiebach, *Double Field Theory*, *JHEP* **09** (2009) 099 [arXiv:0904.4664] [INSPIRE].

[54] C. Hull and B. Zwiebach, *The gauge algebra of double field theory and Courant brackets*, *JHEP* **09** (2009) 090 [arXiv:0908.1792] [INSPIRE].

[55] O. Hohm, C. Hull and B. Zwiebach, *Background independent action for double field theory*, *JHEP* **07** (2010) 016 [arXiv:1003.5027] [INSPIRE].

[56] O. Hohm, C. Hull and B. Zwiebach, *Generalized metric formulation of double field theory*, *JHEP* **08** (2010) 008 [arXiv:1006.4823] [INSPIRE].

[57] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Quart. J. Math.* **54** (2003) 281 [math/0209099] [INSPIRE].

[58] M. Gualtieri, *Generalized complex geometry*, Ph.D. thesis, Oxford U., 2003. math/0401221 [INSPIRE].

[59] N. Hitchin, *Lectures on generalized geometry*, arXiv:1008.0973 [INSPIRE].

[60] P. Koerber, *Lectures on Generalized Complex Geometry for Physicists*, *Fortsch. Phys.* **59** (2011) 169 [arXiv:1006.1536] [INSPIRE].

[61] G. Aldazabal, D. Marqués and C. Núñez, *Double Field Theory: A Pedagogical Review*, *Class. Quant. Grav.* **30** (2013) 163001 [arXiv:1305.1907] [INSPIRE].

[62] O. Hohm, D. Lüst and B. Zwiebach, *The Spacetime of Double Field Theory: Review, Remarks, and Outlook*, *Fortsch. Phys.* **61** (2013) 926 [arXiv:1309.2977] [INSPIRE].

[63] D.S. Berman and D.C. Thompson, *Duality Symmetric String and M-theory*, *Phys. Rept.* **566** (2014) 1 [arXiv:1306.2643] [INSPIRE].

[64] D.S. Berman and C.D.A. Blair, *The Geometry, Branes and Applications of Exceptional Field Theory*, *Int. J. Mod. Phys. A* **35** (2020) 2030014 [arXiv:2006.09777] [INSPIRE].

JHEP07(2021)092

[65] O.A. Bedoya, D. Marqués and C. Núñez, *Heterotic α'-corrections in Double Field Theory*, *JHEP* **12** (2014) 074 [arXiv:1407.0365] [INSPIRE].

[66] A. Coimbra, R. Minasian, H. Triendl and D. Waldram, *Generalised geometry for string corrections*, *JHEP* **11** (2014) 160 [arXiv:1407.7542] [INSPIRE].

[67] D. Marqués and C.A. Núñez, *T-duality and α'-corrections*, *JHEP* **10** (2015) 084 [arXiv:1507.00652] [INSPIRE].

[68] W.H. Baron, J.J. Fernandez-Melgarejo, D. Marqués and C. Núñez, *The odd story of α'-corrections*, *JHEP* **04** (2017) 078 [arXiv:1702.05489] [INSPIRE].

[69] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as Generalised Geometry I: Type II Theories*, *JHEP* **11** (2011) 091 [arXiv:1107.1733] [INSPIRE].

[70] I. Jeon, K. Lee and J.-H. Park, *Incorporation of fermions into double field theory*, *JHEP* **11** (2011) 025 [arXiv:1109.2035] [INSPIRE].

[71] O. Hohm and S.K. Kwak, *$N = 1$ Supersymmetric Double Field Theory*, *JHEP* **03** (2012) 080 [arXiv:1111.7293] [INSPIRE].

[72] I. Jeon, K. Lee and J.-H. Park, *Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity*, *Phys. Rev. D* **85** (2012) 081501 [*Erratum ibid.* **86** (2012) 089903] [arXiv:1112.0069] [INSPIRE].

[73] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as Generalised Geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M-theory*, *JHEP* **03** (2014) 019 [arXiv:1212.1586] [INSPIRE].

[74] D.S. Berman and K. Lee, *Supersymmetry for Gauged Double Field Theory and Generalised Scherk-Schwarz Reductions*, *Nucl. Phys. B* **881** (2014) 369 [arXiv:1305.2747] [INSPIRE].

[75] K. Lee, *Quadratic α'-corrections to heterotic double field theory*, *Nucl. Phys. B* **899** (2015) 594 [arXiv:1504.00149] [INSPIRE].

[76] W.H. Baron, E. Lescano and D. Marqués, *The generalized Bergshoeff-de Roo identification*, *JHEP* **11** (2018) 160 [arXiv:1810.01427] [INSPIRE].

[77] O. Hohm and S.K. Kwak, *Double Field Theory Formulation of Heterotic Strings*, *JHEP* **06** (2011) 096 [arXiv:1103.2136] [INSPIRE].

[78] D. Geissbuhler, D. Marqués, C. Núñez and V. Penas, *Exploring Double Field Theory*, *JHEP* **06** (2013) 101 [arXiv:1304.1472] [INSPIRE].

[79] M. Graña and D. Marqués, *Gauged Double Field Theory*, *JHEP* **04** (2012) 020 [arXiv:1201.2924] [INSPIRE].

[80] O. Hohm, *Background independence in string theory*, *Int. J. Mod. Phys. D* **27** (2018) 1847026 [arXiv:1806.08704] [INSPIRE].

[81] S. Hronek and L. Wulff, *$O(D, D)$ and the string α' expansion: an obstruction*, *JHEP* **04** (2021) 013 [arXiv:2012.13410] [INSPIRE].

[82] G. Aldazabal, W. Baron, D. Marqués and C. Núñez, *The effective action of Double Field Theory*, *JHEP* **11** (2011) 052 [*Erratum ibid.* **11** (2011) 109] [arXiv:1109.0290] [INSPIRE].