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From quantum correlations in dissipative quantum walk to two-qubit systems

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HIGHLIGHTS

- From a Dissipative Quantum Walk we have studied quantum correlations.
- Two bipartitions in the lattice have been defined to solve dissipative Qubits.
- We calculate analytically the Negativity, the Concurrence and the Quantum Discord.
- Our results can be used as an indicator of the quantum to the classical transition.
- A two-qubit density matrix with rotational symmetry has been studied.

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ABSTRACT

A dissipative quantum walk (according to the semigroup approach) has been used as the starting point from which to study quantum correlations in an open system. This system is a fruitful model that allows the definition of several bipartite systems (sets of qubits). Thus the quantum correlations and the decoherence properties induced by a phonon bath can be investigated analytically using tools from quantum information. In particular we have studied the negativity, concurrence and quantum discord for different bipartitions in our dissipative system, and we have found analytical expression for these measures, using a local initial condition for the density matrix of the walker. In general quantum correlations are affected by dissipation in a complex non-monotonic way, showing at long time an expected asymptotic decrease with the increase of the dissipation. In addition, our results for the quantum correlations can be used as an indicator of the transition from the quantum to the classical regimen, as has recently been shown experimentally.

1. Introduction

A quantum walk (QW) can be thought of as an excitation moving in a lattice, and is the quantum analogous to the classical diffusion random walk model [1,2]. The former, i.e., the QW, has advantages over the classical one due to the quantum issue of the superposition of states, a fact clearly reflected, for example, in the dispersion of the probability profile σ, where it is linear in time for the QW ($\sigma \sim t$), and is a power low for classical random walk (CRW) ($\sigma \sim t^{1/2}$). Numerically this is an advantage that can be used for implementing quantum algorithms, because the excitation will spread faster than its classical counterpart [3].

Interestingly, the concept of QW, borrowed from classical statistics [1,2,4,5], has the same properties as a tight-binding free particle [5,6]. Two kinds of QW are considered in the literature: discrete-time quantum coined walk [7,8,1,4,9–16] and

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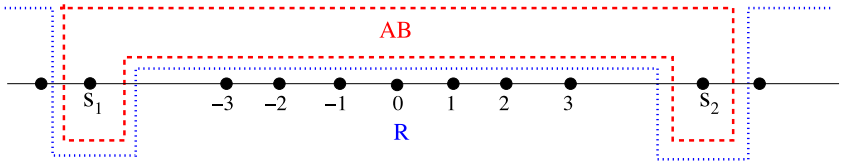

Fig. 1. Graphical representation of the first bipartition on the lattice, the sites $s_1 = -s$ and $s_2 = s$ is the subset AB (system of interest), and the remaining sites are the complement subset R. The set AB represents a two-qubit system.

continuous-time QW [17,5,6,18–21]. In the former (proposed by Aharonov et al. [1]), a two-level state, the so-called "coin", rules the unitary discrete-time evolution of a particle moving in a lattice. On the other hand, the evolution of the particle in the continuous-time QW is determined by a tight-binding like Hamiltonian [6,19]. It is not difficult to see, by simple comparison, that a mapping between the *tight-binding* Hamiltonian and the QW model can be established.

A related analysis is the study of two QWs, and this important scenario can be simulated experimentally using different mechanisms such as trapped atoms [22], ions [16,7], excitations in nuclear magnetic resonance schemes [23], waveguide structures [20], beam array [12], and loop configurations [8]. These results reflect the key role of the coherent superposition of the quantum mechanics, enabling the QW framework to be a good numerical emulation of real systems.

In all these experimental systems the decoherence mechanisms turn out to be important issues for us to understand because, for instance, the measurement of some properties in the system could introduce noise in several complex ways, which must be taken into account for clear description of the excitation movement. This fact has actually been studied theoretically and experimentally [1,4,7,8,17,12–14,20,16,21,24,22,23,10]. In this context, in the present work, we will focus on a more realistic QW model, i.e., the continuous-time Dissipative Quantum Walk (DQW) [24–26]. A DQW is a particle moving in a unidimensional lattice (using a tight binding approximation) and interacting simultaneously with a dissipative ohmic thermal bath [2,4]. In addition, let us comment here that there are many other mechanism of decoherence, see for example Refs. [27,28].

In the present paper we assume that the whole closed system, i.e., the particle in the lattice plus the thermal ohmic bath are together in a pure state. Then we trace out the degree of freedom of the thermal bath in order to arrive at a well-defined DQW. This analysis has been done by solving the reduced density matrix in the Markov approximation, i.e., the time-evolution for the reduced density matrix is described using a completely positive infinitesimal generator [29,30], assuring in this way that the reduced density matrix of the system of interest will be positive definite at any time. This approximation, in a lattice, allows us to tackle the quantum master equation analytically and solve the time-evolution of our excitation.

The goal of the present paper is to study quantum correlations in an open system, starting from a single-particle Hamiltonian and then introducing suitable bipartitions in the lattice in order to define two-qubit systems. We would like to note that related studies, for closed systems, have been carried out by calculating the quantum entanglement (concurrence) from a single-particle state [31], as well as for multi-particle states [32]. The non-locality effect has also been analyzed for a one-particle model [33,26]. We emphasize that the present analytical results, of quantum correlations, from a DQW allow us to tackle straightforwardly a problem which would demand a lot of work for a multi-particle model. This latter problem and the non-local case we will be reported elsewhere.

The competition between dissipation (mechanism of decoherence) and quantum correlations turn out to be a very important fact in understanding and improving quantum computation algorithm, since this issue could help improve the efficiency of quantum simulations by correcting the decoherence appropriately [34,35]. To study the quantum correlations in our system we make several calculations to quantify the quantum entanglement analytically (i.e., the concurrence [36] and the negativity [37]). In addition, a related measure which has no direct connection with the entanglement, called the Quantum Discord (QD) [38–40] will be studied within our framework. In order to do this task we first have to consider some bipartitions for our system, then starting with our DQW we trace out some of the information from the degree of freedom of the lattice in order to define different dissipative bipartite systems (qubit).

As we mention above, having at hand a dissipative bipartite system we will be able to study the entanglement between the parties induced by the environment. As far as we know, this is the first time that analytical expressions for these measures are reported as a function of the dissipative parameter. The two bipartitions, in the lattice, that we are going to use were inspired in non-dissipative magnetic systems [41,42], allowing us to carry on all the calculations analytically.

Using a local initial condition for the DQW we define two different bipartitions in our systems (see Figs. 1 and 2). Therefore, tracing out some of the degrees of freedom on the lattice, we keep two parties $\{A, B\}$, allowing us to define two quantum bits as our open systems, which allow us to study several quantum correlations in the system. In this way we have succeeded in defining a set of "qubits" interacting with a phonon bath (one qubit is defined by one site, where it can be occupied or empty).

Using the first bipartition (Fig. 1) we trace out sites on the rest of the lattice, keeping only two sites $\{s_1 s_2\}$ (one site is the reflection of the other) in order to define a two-level system. Therefore we obtain analytical expressions for the concurrence, and the quantum discord (QD) for this bipartition. The second bipartition (Fig. 2) is defined as left and right subsystems with respect to the origin in the lattice. Then to study the correlations between both subsystems (the left and right A, B parties) we

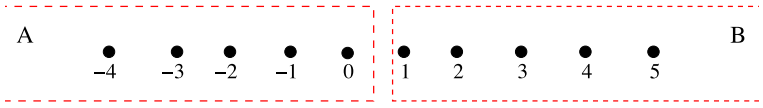

Fig. 2. Graphical representation of our second bipartition on the lattice, where the sites $s \leq 0 (>0)$ belong to the subsystem A (the subsystem B). This mathematical bipartition also represent a set of two qubits.

measure the negativity. All these measures will afford information on the influence of the initial state on the quantum correlations (the preparation of the density matrix) by tackling a soluble analytical model. In general, the present approach would also allow us to tackle the many-body problem in an open quantum framework, showing the interplay between the quantum correlations and the quantum thermal bath. A possible experimental implementation for our dissipative continuous-time QW approach can be the use of photonic devices on lattices, as in Refs. [21,20].

This article is organized as follows: In Section 2 we review the DQW and show the main features of this system. In Section 3 we introduce the bipartitions on the lattice in order to calculate quantum correlations such as concurrence, negativity and quantum discord, and also derive analytical expressions for these measures. In Section 4.1 we show the main results analyzed from our quantum open system using different bipartitions. Finally, we close our work with some general conclusions in Section 5.

2. A review of the dissipative quantum walk

In this section we review the one-particle dissipative quantum walk model [24–26]. This system is based on the study of a free particle in a one-dimensional infinity lattice interacting with a thermal bath of phonon $\mathcal{B}$. In this case, the total Hamiltonian of the system $\mathcal{S}$ plus the bath $\mathcal{B}$ can be written as:

$$H_{\mathcal{T}} = E_0 \sum_{s=-\infty}^{\infty} c_s^{\dagger} c_s - \frac{\Omega}{2} \sum_{s=-\infty}^{\infty} \left(c_{s-1}^{\dagger} c_s + c_{s+1}^{\dagger} c_s \right) + \hbar\Gamma \left(\sum_{s=-\infty}^{\infty} c_{s-1}^{\dagger} c_s \otimes B_1 + \sum_{s=-\infty}^{\infty} c_{s+1}^{\dagger} c_s \otimes B_2 \right) + \sum_k \hbar\omega_k \mathcal{B}_k^{\dagger} \mathcal{B}_k, \tag{1}$$

where $c_s^{\dagger}$ and c_s are creation and destruction operators of one particle in the site s on the lattice respectively ($|\ldots, 0, 1_s, 0, \ldots\rangle = c_s^{\dagger}|\phi\rangle$, where $|\phi\rangle$ is the empty state). The first and second terms correspond to the free *tight-binding* Hamiltonian H_S for spinless particles (fermion): the system $\mathcal{S}$. Here E_0 is the energy of site and Ω the associated next-neighbor hopping energy. The third term in Eq. (1) considers the Hamiltonian interaction between $\mathcal{S}$ and the bath $\mathcal{B}$, and describes a linear coupling between phonon operators $B_1 = \sum_k v_k \mathcal{B}_k = B_2^{\dagger}$ and shift operators $c_{s-1}^{\dagger} c_s$ of the system $\mathcal{S}$, here Γ is the coupling parameter, $\mathcal{B}_k$, $\mathcal{B}_k^{\dagger}$ are bosonic operators destroying and creating energy $\hbar\omega_k$, and v_k characterizes the spectral function of the thermal bath $g(\omega) = \sum_k |v_k|^2 \delta(\omega_k - \omega)$. In the Ohmic approximation we take $g(\omega) = \omega\, \Theta(\omega_c - \omega)$, with $\Theta(\omega)$ being the step function. Finally the fourth term is the phonon Hamiltonian written in terms of boson operators $\mathcal{B}_k^{\dagger}$ and $\mathcal{B}_k$ [28].

Considering now that there is only one particle, it is straightforward to compare Eq. (1) with the total Hamiltonian associated to a QW in interaction with the thermal phonon bath [18,24,2,26]. First we define translational operators a and $a^{\dagger}$ as a combination of $c_{s'}^{\dagger}$ and $c_{s'}$, in the following way:

$$\sum_{s=-\infty}^{\infty} c_{s-1}^{\dagger} c_s \equiv \mathcal{R} \Rightarrow a \quad \text{and} \quad \sum_{s=-\infty}^{\infty} c_{s+1}^{\dagger} c_s \equiv \mathcal{R}^{\dagger} \Rightarrow a^{\dagger}, \tag{2}$$

where c_s and $\mathcal{R}$ are acting in the Fock-space, and a in the Wannier basis. We can also check that $\mathcal{R}$ and $\mathcal{R}^{\dagger}$ commute even in the general case for many particles (note that $\mathcal{R}^{\dagger}\mathcal{R} = \mathcal{R}\mathcal{R}^{\dagger} = \sum_{s=-\infty}^{\infty} c_s^{\dagger} c_s - \sum_{s,s'=-\infty}^{\infty} c_{s+1}^{\dagger} c_{s'-1}^{\dagger} c_s c_{s'}$), but only for one particle in the lattice we get the result $\mathcal{R}\mathcal{R}^{\dagger} = \mathbf{1}$. Eq. (2) shows the expected mapping from Fock's space into the Wannier basis. Then the connection between the tight-binding Hamiltonian and the QW model can be established.

In what follows it is much more simpler to work out the elimination of the bath variables using Wannier representation and the shift operators [2]:

$$a = \sum_{s=-\infty}^{\infty} |s-1\rangle \langle s| \quad \text{and} \quad a^{\dagger} = \sum_{s=-\infty}^{\infty} |s+1\rangle \langle s|. \tag{3}$$

The Quantum Master Equation (QME) for the DQW can be obtained in the Born–Markov approximation using regular techniques, noting that $a^{\dagger}(t) = a^{\dagger}(0)$, $a(t) = a(0)$, i.e., in the Heisenberg representation they are constant in time, and

$aa^\dagger = \mathbf{1}$, then the QME can finally be written as [25,24]:

$$\dot{\rho} \equiv \frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{-\mathrm{i}}{\hbar}\left[H_{\mathrm{eff}}, \rho\right] + D\left(a\rho a^\dagger + a^\dagger \rho a - 2\rho\right), \tag{4}$$

where $H_{\mathrm{eff}} = H_S - \hbar\omega_c a^\dagger a$ is a trivial effective Hamiltonian, the diffusion constant D is given in terms of the quantum thermal bath temperature T and the coupling constant which can be written as $D \equiv \Gamma^2 k_B T/\hbar$; the additive energy $\hbar\omega_c$ is related to the Caldeira and Leggett frequency cut-off in the Ohmic approximation [43,44]. Without loss of generality we add a constant $-E_0 + \hbar\omega_c + \Omega$, and so we can write the effective Hamiltonian in the form Ref. [25]:

$$H_{\mathrm{eff}} = \Omega\left(\mathbf{1} - \frac{a + a^\dagger}{2}\right). \tag{5}$$

The particular form of the QME (4) is the result of the type of linear interaction between $\mathcal{S}$ and $\mathcal{B}$, compare for example Eq. (4) with the QME introduced in van Kampen's paper [2]. From Eq. (4) it can be seen that as $T \to 0$ (or $D \to 0$), the unitary evolution is recovered (von Neumann evolution equation for the *tight-binding* model). The opposite case is $\Omega \to 0$ (or $D \to \infty$). Interestingly, it is simple to see that if the initial condition of the density matrix is diagonal then the classical random walk profile can be reobtained in this limit. This is analyzed in detail in the Appendix A.

2.1. Moments of the DQW

It is well known that the variance of the QW is quadratic in time t, in contrast with the CRW which is lineal in t. Within our framework we can re-obtain these results in a simple way. For this purpose, we define a characteristic function as [45,46]: $\mathcal{G}(\xi) = \mathrm{Tr}[\rho(t)\mathrm{e}^{\mathrm{i}\xi\mathbf{q}}] = \sum_{s=-\infty}^{\infty}\langle s|\rho(t)|s\rangle \mathrm{e}^{\mathrm{i}\xi s}$, thus all quantum moments of the position operator $\mathbf{q}$ can be obtained as, $\langle \mathbf{q}(t)^m\rangle = \frac{1}{i^m}\frac{\mathrm{d}^m}{\mathrm{d}\xi^m}\mathcal{G}(\xi)\Big|_{\xi=0}$. Using as initial condition a localized state in the origin ($\rho(0) = |s_0\rangle\langle s_0|$, where $s_0 = 0$), we can rewrite the characteristic function in the form

$$\mathcal{G}(\xi) = \mathrm{e}^{-2Dt(1-\cos\xi)} J_0\left(2t\frac{\Omega}{\hbar}\sin\frac{\xi}{2}\right), \tag{6}$$

here $J_n(x)$ is a Bessel function of integer order with $n = 0, \pm1, \pm2, \pm3, \ldots$, and argument x [47]. This is a product of the classical $\mathcal{G}_{CRW}(\xi) = \mathrm{e}^{-2Dt(1-\cos\xi)}$ and the quantum $\mathcal{G}_{QW}(\xi) = J_0\left(2t\frac{\Omega}{\hbar}\sin\frac{\xi}{2}\right)$ characteristic functions, indicating that the total quantum time-evolution will be a space-convolution between the classical and quantum characteristics. In particular the variance of the DQW can be written as:

$$\sigma(t)^2 = \langle q(t)^2\rangle - \langle q(t)\rangle^2 = \frac{1}{2}\left(\frac{\Omega}{\hbar}t\right)^2 + 2Dt,$$

showing the competition between the ballistic behavior (controlled by the motion of Anderson's boundaries at the velocity $\frac{\Omega}{\sqrt{2}\hbar}$) and the diffusive behavior (controlled by the diffusion constant D).

From this result we are tempted to establish that asymptotically, the important energy scale will be dominated by the rate $\propto D/\frac{\Omega}{\hbar}$. Nevertheless, there are many transient correlation functions that are important at short-time where we cannot expect such simple asymptotic scaling; these facts will be analyzed in the remainder of the paper. From the latter analysis we note that a transition from quantum to classic behavior can be expected from the two important limits: the non-dissipative case (pure unitary evolution case) for $D \to 0$ where we get $\sigma(t)^2 \propto t^2$, and the classical limit (completely dissipative case $\Omega \to 0$) where we get $\sigma(t) \propto t$.

In a previous work [25], we found the exact solution for the QME Eq. (4) using as initial condition a localized state $\rho(0) = |s_0\rangle\langle s_0|$ (where $s_0 = 0$). In that case the solution $\rho(t)$ can be written in the Wannier basis in the form:

$$\langle s_1|\rho(t)|s_2\rangle = i^{s_1-s_2}\mathrm{e}^{-2Dt}\sum_{n=-\infty}^{\infty} J_{s_1+n}\left(\frac{\Omega t}{\hbar}\right) J_{s_2+n}\left(\frac{\Omega t}{\hbar}\right) I_n(2Dt), \tag{7}$$

where s_1, s_2 are Wannier indices. Here $\langle s_1|\rho(t)|s_2\rangle$ gives the time evolution of the reduced density matrix element between sites s_1 and s_2 on the lattice, which is a mixed convolution of a pure QW with a CRW contribution. A detailed analysis of how to obtain that solution can be found in our previous work [25]. In that work, we have calculated different quantum measures in order to study significant observables, such as the profile of probability, the quantum purity, the Wigner function, and the quantum entanglement between the system $\mathcal{S}$ and the quantum bath $\mathcal{B}$. In Ref. [25] we have shown that these mentioned measures are only functions of the two parameters: the bandwidth of the tight-binding model $\frac{\Omega}{\hbar}$ and the diffusion constant D, showing the competition between the quantum correlations and the dissipative decoherence. Interestingly, for instance, we have shown that in the case where the dissipative term is smaller than the bandwidth, the quantum correlations are important and dominate the behavior of the system even for a long time. In the opposite case, the decoherence becomes

important and we observe the transition from the DQW to the CRW. In the present work, we will go one step forward and show important results concerning the quantum entanglement for different bipartitions on the lattice.

The purpose of this article is to evaluate quantum correlations, analytically, for certain models of dissipative two-level systems (using different bipartitions from the dynamics of a DQW in the lattice) and investigate the comparative relations between quantum and classical correlations, as measured by negativity, concurrence and quantum discord (and its respective classical correlations). We will explicitly show the dependence of all these measures as a function of time and study its functionality in terms of the dissipation parameter.

3. Quantum correlations on the lattice

In this section we study different quantum correlations for our dissipative two-qubit systems. These quantum measures such as concurrence [36], negativity [37], and quantum discord [38], provide a different point of view from the standard measures (probability profile, dispersion, etc.). Several studies of quantum correlations for dissipative and non-dissipative systems have been reported in the literature [48,11,49], but to our knowledge a bipartition like the one we are going to present from a DQW have never been reported before. These bipartitions (set of qubits) allow us to find analytical expressions for all these measures.

3.1. Mirror concurrence

To study a mechanism of decoherence we analyze how quantum correlations are affected by dissipation; therefore, we propose here to introduce a mathematical bipartition onto the lattice in order to define a set of qubits (a qubit is a two-level quantum system [34]). In fact, in the remaining part of the paper we will be concerned with two different bipartitions (see Figs. 1 and 2). For the first bipartition we have considered two sites $\{s_1, s_2\}$ on the lattice in order to define a two-qubit system $\{A, B\}$, then the reduced density matrix of AB is obtained by tracing out the remaining sites $s \neq \{s_1, s_2\}$ on the lattice, wherewith we have defined two qubits interacting with the environment $\mathcal{B}$. It is well known that the quantum entanglement between two qubits in a mixed state is calculated using the concurrence measure proposed by Wootters [36]. To quantify this concurrence, we need to obtain the reduced density matrix for the two-qubit (in our case as a function of s_1, s_2, see inset of Fig. 1).

In order to proceed we start with the density matrix associated with the DQW on the lattice $\rho(t)$ (solution of the QME Eq. (4)). Using the Wannier basis we can write:

$$\begin{aligned}
\rho(t) &= \sum_{s,s'=-\infty}^{\infty} \rho_{s,s'}(t)|s\rangle\ \langle s'| \\
&= \sum_{s=-\infty}^{\infty} \rho_{s,s}(t)|s\rangle\ \langle s| + \sum_{\{s\neq s'\}} \rho_{s,s'}(t)|s\rangle\ \langle s'| \\
&= \rho_{s_1,s_1}(t)|s_1\rangle\ \langle s_1| + \rho_{s_2,s_2}(t)|s_2\rangle\ \langle s_2| + \rho_{s_1,s_2}(t)|s_1\rangle\ \langle s_2| + \rho_{s_2,s_1}(t)|s_2\rangle\ \langle s_1| + \sum_{s\neq\{s_1,s_2\}} \rho_{s_1,s}(t)|s_1\rangle\langle s| \\
&\quad + \sum_{s\neq\{s_1,s_2\}} \rho_{s,s_1}(t)|s\rangle\ \langle s_1| + \sum_{s\neq\{s_1,s_2\}} \rho_{s_2,s}(t)|s_2\rangle\langle s| + \sum_{s\neq\{s_1,s_2\}} \rho_{s,s_2}(t)|s\rangle\ \langle s_2| + \sum_{\{s,s'\neq s_1,s_2\}} \rho_{s,s'}(t)|s\rangle\langle s'|. \qquad (8)
\end{aligned}$$

3.1.1. From the DQW to the occupation two-qubit definition

We can redefine the Wannier basis (state of one particle using the outer product notation) in the form

$$|s_1\rangle = |1_A\rangle \otimes |0_B\rangle \otimes |0_R\rangle \equiv |1_A 0_B\rangle \otimes |0_R\rangle, \qquad (9)$$

and so

$$|s_2\rangle = |0_A 1_B\rangle \otimes |0_R\rangle, \qquad (10)$$

here $|1_J\rangle$ and $|0_R\rangle$ are occupied or empty states in the subsystem J, with $J = A, B, R$, where $s_1 < 0$ and $s_2 > 0$.

Finally, we trace out over the subsystem C and obtain the reduced density matrix for the two-qubit $\rho^{AB}(t)$ in the form:

$$\begin{array}{c} \\ |1_A1_B\rangle \\ |1_A0_B\rangle \\ |0_A1_B\rangle \\ |0_A0_B\rangle \end{array}
\begin{array}{c}
\begin{array}{cccc} |1_A1_B\rangle & |1_A0_B\rangle & |0_A1_B\rangle & |0_A0_B\rangle \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \rho_{s_1,s_1}(t) & \rho_{s_1,s_2}(t) & 0 \\
0 & \rho_{s_2,s_1}(t) & \rho_{s_2,s_2}(t) & 0 \\
0 & 0 & 0 & \sum_{s\neq\{s_1,s_2\}} \rho_{s,s}(t)
\end{pmatrix}
\end{array}. \qquad (11)$$

Note that this exact result for the reduced density matrix $\rho^{AB}(t)$ represents a genuine non-Markov process. It can be seen that the matrix element associated to the two particles states is zero (there is only one walker on the lattice). Thus in the present case we get that the reduced density matrix for two-qubit has the form Ref. [50]:

$$\rho^{AB} = \begin{pmatrix} v & 0 & 0 & 0 \\ 0 & w & z^* & 0 \\ 0 & z & x & 0 \\ 0 & 0 & 0 & y \end{pmatrix}, \tag{12}$$

with $v = 0$. In general, the concurrence is obtained as $C = \max\{0, 2(|z| - \sqrt{yv})\}$, where z^* is the complex conjugate of z, thus in our case the concurrence between systems A and B is $C(s_1, s_2, t) = 2|\rho_{s_1,s_2}(t)|$. It is worth noting that when $C = 0$, ρ^{AB} is separable, and for $C > 0$ the state is entangled. Finally, $C = 1$ means it is maximally entangled. To measure the total mirror quantum entanglement on the lattice (we sum the contribution over all pairs of mirror sites s and $-s$ on the lattice), we are going to consider the sum of the mirror concurrence [41,42] between the site $s_1 = -s$ and its mirror site $s_2 = s$, see Fig. 1. This will measure the total mirror concurrence on the lattice as a function of time. We will call this measurement C_M, which we can write in the form:

$$C_M \equiv \sum_{s=1}^{\infty} C_s = 2\sum_{s=1}^{\infty} |\rho_{s,-s}(t)|, \tag{13}$$

the explicit value of $\rho_{s,-s}(t)$ can be obtained from the solution of the QME, Eq. (4). In particular using as initial condition a localized state, the solution is given by Eq. (7). In this case C_M is given by the following exact expression:

$$\begin{aligned} C_M(\Omega, D, t) &= 2\sum_{s=1}^{\infty} |\langle s|\rho(t)|-s\rangle| \\ &= 2\mathrm{e}^{-2Dt}\sum_{s=1}^{\infty}\left|\sum_{n=-\infty}^{\infty} J_{s+n}\left(\frac{\Omega t}{\hbar}\right) J_{-s+n}\left(\frac{\Omega t}{\hbar}\right) I_n(2Dt)\right|. \end{aligned} \tag{14}$$

This expression is valid for all values of D. Thus we can study all regime, sweeping from the pure unitary evolution case (non-dissipative $D \to 0$ case) to the completely dissipative case ($\Omega \to 0$), i.e., when $\Omega/\hbar \gg D$ or $\Omega/\hbar \ll D$, respectively. In the pure unitary evolution case ($D = 0$), we obtain a simple expression for C_M (using that $I_n(0) = \delta_{n,0}$):

$$\begin{aligned} C_M(\Omega, 0, t) &= 2\sum_{s=1}^{\infty}\left|J_s\left(\frac{\Omega t}{\hbar}\right) J_{-s}\left(\frac{\Omega t}{\hbar}\right)\right| \\ &= 2\sum_{s=1}^{\infty} J_s^2\left(\frac{\Omega t}{\hbar}\right) \\ &= 1 - J_0^2\left(\frac{\Omega t}{\hbar}\right) \leq 1, \end{aligned} \tag{15}$$

where we have used that $2\sum_{n=1}^{\infty} J_n^2(x) + J_0^2(x) = 1$ and $J_{-n}(x) = (-1)^n J_n(x)$ [47]. It can be seen that for $t = 0$ the total mirror concurrence C_M is zero, and grows for $t > 0$. This fact is ruled by the temporal oscillatory behavior of $J_0^2(\frac{\Omega t}{\hbar})$. Therefore, the roots of $J_0^2(\frac{\Omega t}{\hbar})$ give the maximum value of C_M, i.e., $C_M = 1$. In the long-time regimen the C_M converges to 1 only for $D = 0$. The opposite case is when the dissipation ruled out the evolution of the system ($D \gg \Omega/\hbar$); in this case $\rho_{s,-s}(t) \to 0$ (off-diagonal elements of the DQW are zero, i.e., the system reached the classical behavior) and therefore the C_M goes to zero (the system is completely decoherent).

Note that we can obtain a lower bound for C_M in the case when $D \neq 0$. Using that $J_{-n}(x) = (-1)^n J_n(x)$, and $I_{-n}(x) = I_n(x)$ we can rewrite Eq. (14) in the form:

$$\begin{aligned} C_M(\Omega, D, t) &= \mathrm{e}^{-2Dt}\sum_{s=1}^{\infty}\left|\sum_{n=-\infty}^{\infty}(-1)^{s-n} J_{s+n}\left(\frac{\Omega t}{\hbar}\right) J_{s-n}\left(\frac{\Omega t}{\hbar}\right) I_n(2Dt)\right| \\ &= \mathrm{e}^{-2Dt}\sum_{s=1}^{\infty}\left|\sum_{n=-\infty}^{\infty}(-1)^{n} J_{s+n}\left(\frac{\Omega t}{\hbar}\right) J_{s-n}\left(\frac{\Omega t}{\hbar}\right) I_n(2Dt)\right| \\ &\geq \mathrm{e}^{-2Dt}\left|\sum_{s=1}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{n} J_{s+n}\left(\frac{\Omega t}{\hbar}\right) J_{s-n}\left(\frac{\Omega t}{\hbar}\right) I_n(2Dt)\right| \\ &\geq \mathrm{e}^{-2Dt}\left|\sum_{n=-\infty}^{\infty}(-1)^{n} I_n(2Dt)\sum_{s=1}^{\infty} J_{s+n}\left(\frac{\Omega t}{\hbar}\right) J_{s-n}\left(\frac{\Omega t}{\hbar}\right)\right|. \end{aligned} \tag{16}$$

Then using that $\sum_{s=1}^{\infty} J_{s+n}(x)J_{s-n}(x) = \frac{1}{2}[\delta_{n,0} - (-1)^n J_n^2(x)]$ [51,52] after some algebra from Eq. (16) we obtain:

$$C_M(\Omega, D, t) \geq \mathrm{e}^{-2Dt}\left| I_0(2Dt) - \sum_{n=-\infty}^{\infty} I_n(2Dt) J_n^2\left(\frac{\Omega t}{\hbar}\right)\right|.$$

3.2. Negativity

In this subsection we will analyze a second bipartition on the lattice (see Fig. 2). In order to study the total quantum correlation associated with a dissipative system, with many degrees of freedom, we introduce now our second bipartition and we will compare the negativity measure with the total mirror concurrence from the first bipartition (see Fig. 1).

Here we consider the following bipartition on the lattice: all sites with Wannier index $s \leq 0$ belong to the party A, otherwise they belong to B. In Fig. 2 the graphical representation of such a bipartition is shown. Now to measure the entanglement between A and B we use the negativity [37]. Indeed this measure can be used for a bipartite system with many degrees of freedom. The negativity is one quantitative version of the criterion for separability that can be used in a mixed bipartite system characterized by a density matrix ρ (see Ref. [53]). The explicit expression of the negativity is defined as a function of a partial transpose of ρ in one of the two subspaces (now our system is composed of parties A and B), then the partial transpose ρ^{T_A} is calculated as

$$\langle i_A; j_B|\rho^{T_A}|k_A; l_B\rangle = \langle k_A; j_B|\rho|i_A; l_B\rangle. \tag{17}$$

Here the basis $|i_A; j_B\rangle$ belongs to the Hilbert space $\mathcal{H}_{AB}$, which is a direct product of the Hilbert spaces of the subsystems A and B, i.e.,

$$|i_A, j_B\rangle = |i_A\rangle \otimes |j_B\rangle. \tag{18}$$

Then the negativity can be calculated as: $\mathcal{N}'(\rho) = \sum_i |\mu_i|$, where μ_i is a negative eigenvalue of ρ^{T_A}. To compare this measure with the concurrence associated with the first bipartition (see Fig. 1), we use a normalized negativity between two qubits (note that the parties A and B behave, in fact, as a two-qubit system) which ranges from 0 to 1 (normalized negativity [54]). Then we define a normalized negativity as:

$$\mathcal{N}(\rho) = 2\sum_i |\mu_i|. \tag{19}$$

3.2.1. From the lattice degrees of freedom to the effective two-qubit definition

First we note that for the mathematical bipartition on the lattice represented in Fig. 2 the reduced density matrix $\rho(t)$ can be written on the basis of the parties A and B. This is so because if $s > 0$ the Wannier basis can be written as:

$$|s\rangle = |\phi\rangle \otimes |1_{s_B}\rangle \equiv |\phi\ 1_{s_B}\rangle, \tag{20}$$

otherwise if $s \leq 0$ we write

$$|s\rangle = |s_A\rangle \otimes |\phi\rangle \equiv |1_{s_A}\ \phi\rangle, \tag{21}$$

where $|\phi\rangle$ is the empty state in $A(B)$, and $|1_{s_{A(B)}}\rangle$ is an occupied state in $A(B)$. Therefore, the reduced density matrix can be rewritten as follows:

$$\begin{aligned}
\rho(t) &= \sum_{s,s'=-\infty}^{\infty} \rho_{s,s'}(t)|s\rangle\ \langle s'| \\
&= \sum_{s,s'\leq 0} \rho_{s,s'}(t)|s\rangle\langle s'| + \sum_{s,s'>0} \rho_{s,s'}(t)|s\rangle\ \langle s'| + \sum_{s\leq 0,s'>0} \rho_{s,s'}(t)|s\rangle\langle s'| + \sum_{s>0,s'\leq 0} \rho_{s,s'}(t)|s\rangle\ \langle s'| \\
&= \sum_{s_A,s'_A} \rho_{s_A,s'_A}(t)|1_{s_A}\ \phi\rangle\ \langle 1_{s'_A}\phi| + \sum_{s_B,s'_B} \rho_{s_B,s'_B}(t)|\phi\ 1_{s_B}\rangle\ \langle \phi\ 1_{s'_B}| \\
&\quad + \sum_{s_A,s_B} \rho_{s_A,s_B}(t)|1_{s_A}\ \phi\rangle\langle \phi\ 1_{s_B}| + \sum_{s_A,s_B} \rho_{s_B,s_A}(t)|\phi\ 1_{s_B}\rangle\langle 1_{s_A}\phi|.
\end{aligned} \tag{22}$$

Now using Eqs. (17) and (22) we can calculate the partial transpose ρ^{T_A} as follows:

$$\begin{aligned}
\rho^{T_A}(t) &= \sum_{s_A,s'_A} \rho_{s_A,s'_A}(t)|1_{s'_A}\ \phi\rangle\ \langle 1_{s_A}\ \phi| + \sum_{s_B,s'_B} \rho_{s_B,s'_B}(t)|\phi\ 1_{s_B}\rangle\ \langle \phi\ 1_{s'_B}| \\
&\quad + \sum_{s_A,s_B} \rho_{s_A,s_B}(t)|\phi\ \phi\rangle\langle 1_{s_A}1_{s_B}| + \sum_{s_A,s_B} \rho_{s_B,s_A}(t)|1_{s_A}1_{s_B}\rangle\ \langle \phi\ \phi|,
\end{aligned}$$

where $s_A = 0, -1, -2, \ldots$ and $s_B = 1, 2, \ldots$. Then we can express ρ^{T_A} in the matrix form:

$$\begin{array}{c} \\ |\phi\ \phi\rangle \\ \vdots \\ |1_{s_A} 1_{s_B}\rangle \\ \vdots \\ \vdots \\ |1_{s'_A} \phi\rangle \\ \vdots \\ |\phi\ 1_{s_B}\rangle \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} |\phi\ \phi\rangle & \cdots & |1_{s_A} 1_{s_B}\rangle & \cdots & \cdots & |1_{s_A}\phi\rangle & \cdots & |\phi\ 1_{s'_B}\rangle \cdots \end{array} \\ \left(\begin{array}{cccc|cccc} 0 & \cdots & \rho_{s_A,s_B}(t) & \ddots & 0 & \cdots & 0 & \ddots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ \rho_{s_B,s_A}(t) & \cdots & 0 & \ddots & 0 & \cdots & 0 & \ddots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ \hline \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \ddots & \rho_{s_A,s'_A}(t) & \cdots & 0 & \ddots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \ddots & 0 & \cdots & \rho_{s_B,s'_B}(t) & \ddots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \end{array}\right) \end{array}, \tag{23}$$

this infinite dimensional matrix can be written in block as follows:

$$\rho^{T_A} = \begin{bmatrix} \mathcal{M}_1 & \mathcal{O}_1 \\ \mathcal{O}_2 & \mathcal{M}_2 \end{bmatrix},$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are null matrices, and $\mathcal{M}_1$ is a matrix that mixes states $|\phi\,\phi\rangle$ with states $|1_{s_A} 1_{s_B}\rangle$, and $\mathcal{M}_2$ mixes states $|1_{s_A}\,\phi\rangle$ with states $|1_{s'_A}\,\phi\rangle$ and $|\phi\ 1_{s_B}\rangle$ with states $|\phi\ 1_{s'_B}\rangle$. In this way ρ^{T_A} can be diagonalized by blocks. It can be seen that the only block matrix with a negative eigenvalue is $\mathcal{M}_1$. Therefore, in accordance with Eq. (19) the normalized negativity is two times the sum of the eigenvalues of ρ^{T_A}, noting that in our case we have only one negative eigenvalue $|\mu_1| = \sqrt{\sum_{s_A,s_B} |\rho_{s_A,s_B}(t)|^2}$ (this eigenvalue comes from the block matrix $\mathcal{M}_1$ and was obtained in a deductive way, see Appendix B), finally we obtain:

$$\begin{aligned} \mathcal{N}(\Omega, D, t) &= 2\sqrt{\sum_{s_A,s_B} |\rho_{s_A,s_B}(t)|^2} \\ &= 2\mathrm{e}^{-2Dt} \left\{ \sum_{s=1}^{\infty} \sum_{s'=-\infty}^{0} \sum_{n,n'=-\infty}^{\infty} \left| J_{s+n}\left(\frac{\Omega t}{\hbar}\right) J_{s+n'}\left(\frac{\Omega t}{\hbar}\right) J_{s'+n} \right.\right. \\ &\quad \left.\left. \times \left(\frac{\Omega t}{\hbar}\right) J_{s'+n'}\left(\frac{\Omega t}{\hbar}\right) I_n(2Dt)\, I_{n'}(2Dt) \right| \right\}^{1/2}, \end{aligned} \tag{24}$$

where $\rho_{s_A,s_B}(t)$ is given in Eq. (7). This expression allows us to study the negativity as a function of the two characteristic parameters of the system $\{D, \Omega/\hbar\}$.

In the non-dissipative case ($D = 0$) (i.e., pure unitary evolution case) Eq. (24) reduces to:

$$\begin{aligned} \mathcal{N}(\Omega, 0, t) &= 2\left\{ \sum_{s=1}^{\infty} \sum_{s'=-\infty}^{0} J_s^2\left(\frac{\Omega t}{\hbar}\right) J_{s'}^2\left(\frac{\Omega t}{\hbar}\right) \right\}^{1/2} \\ &= 2\left\{ \sum_{s=1}^{\infty} J_s^2\left(\frac{\Omega t}{\hbar}\right) \sum_{s'=-\infty}^{0} J_{s'}^2\left(\frac{\Omega t}{\hbar}\right) \right\}^{1/2} \\ &= 2\left\{ \sum_{s=1}^{\infty} J_s^2\left(\frac{\Omega t}{\hbar}\right) \sum_{s'=0}^{\infty} J_{s'}^2\left(\frac{\Omega t}{\hbar}\right) \right\}^{1/2} \\ &= 2\left\{ \frac{1}{2}\left[1 - J_0^2\left(\frac{\Omega t}{\hbar}\right)\right] \frac{1}{2}\left[1 + J_0^2\left(\frac{\Omega t}{\hbar}\right)\right] \right\}^{1/2} \\ &= \left[1 - J_0^4\left(\frac{\Omega t}{\hbar}\right)\right]^{1/2}, \quad \text{(QW)} \end{aligned} \tag{25}$$

where $I_n(0) = \delta_{n,0}$ was used. For the non-dissipative case and comparing (25) against (15), we note that in the long-time regimen, the "negative measure" for the second bipartition (see Fig. 2) goes to 1 faster than the total "mirror concurrence"

for the first bipartition (see Fig. 1). Indeed, the negativity includes mirror correlations between sites $\{-s, s\}$ and correlations between other sites on the lattice, leading therefore to a faster entanglement in the system.

In the classical case ($\Omega = 0$) the off-diagonal elements of $\rho(t)$ are equal to zero [25], therefore the negativity is equal to zero. Interestingly, our analytical expression of the negativity allows us to study the entanglement in this bipartition as a function of the rate of energies $r_D = 2D/\frac{\Omega}{\hbar}$. This analysis will be shown in Section 4.

3.3. Mirror quantum discord

We have also studied QD, which is another measurement of quantum correlations in a bipartite system. The different nature of QD in comparison with entanglement, for instance, from the measure of the concurrence, was noticed by several authors some years ago [38–40]. The QD is the difference between two natural quantum extensions of the classical mutual information. This quantity reveals the quantum aspect of correlations in bipartite states, including separable ones. The quantum mutual information for a bipartite system AB, ρ^{AB} being the density matrix of the bipartite system, is obtained as:

$$\mathcal{I}(\rho^{AB}) = S(\rho^A) + S(\rho^B) - S(\rho^{AB}), \tag{26}$$

where $\rho^{A(B)}$ is the reduced density matrix of the subsystem $A(B)$ (i.e. $\rho^A = \mathrm{Tr}_B[\rho^{AB}]$), and $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy of the state ρ, which measures the quantum entanglement between the system and the environment. The quantum mutual information $\mathcal{I}(\rho^{AB})$ measures the total correlations between subsystems A and B, i.e., $\mathcal{I}(\rho^{AB})$ retains the quantum and classical correlations. This means that $\mathcal{I}(\rho^{AB})$ can be written as $\mathcal{I}(\rho^{AB}) = CC(\rho^{AB}) + QD(\rho^{AB})$ where $CC(\rho^{AB})$ represents the classical correlations and $QD(\rho^{AB})$ the quantum correlations between parties A and B. This latter quantity is called QD [38]. Indeed, to measure this quantity we first need to obtain $CC(\rho^{AB})$, thus the QD can be calculated. In Ref. [38], Olliver and Zurek proposed the use of any type of von Neumann measurement, a one-dimensional set of projectors that sum up to the identity. These projection operators B_k represent a von Neumann measurement for the system B, therefore the conditional density operator ρ_k related to the measurement outcome k, can to be written in the form:

$$\rho_k = \frac{1}{p_k}(I \otimes B_k)\rho^{AB}(I \otimes B_k), \tag{27}$$

where $p_k = \mathrm{Tr}[(I \otimes B_k)\rho^{AB}(I \otimes B_k)]$ is the probability of the outcome k. Then the quantum conditional entropy is defined as:

$$S(\rho^A|\{B_k\}) \equiv \sum_k p_k S(\rho_k). \tag{28}$$

Thus a variant of quantum mutual information related to this measurement is obtained as:

$$\mathcal{I}(\rho^A|\{B_k\}) \equiv S(\rho^A) - S(\rho^A|\{B_k\}). \tag{29}$$

Noting that the classical correlations involve an optimization process, defined as:

$$CC(\rho^{AB}) \equiv \sup_{\{B_k\}} \mathcal{I}(\rho^A|\{B_k\}). \tag{30}$$

Finally the quantum correlations can be defined as:

$$QD(\rho^{AB}) = \mathcal{I}(\rho^{AB}) - CC(\rho^{AB}). \tag{31}$$

In recent years few works have been devoted to finding analytical expressions for a two-qubit system [38,39,55,56,40], because founding analytical expression for the QD is a complex calculation for arbitrary dimensions of the composite system (AB).

Here we focus on our set of dissipative qubits; in this system we can study the classical and quantum correlations analytically for the bipartition shown in Fig. 1. In this case the reduced density matrix for the system AB is given by Eq. (11), where $s_1 = -s$ and $s_2 = s$. This density matrix has the form of the matrix given in Eq. (12), then the CC and QD can be found straightforwardly. Following the steps in Ref. [56], we can express ρ^{AB} in the alternative way (noting that in our case $\rho_{-s,-s}(t) = \rho_{s,s}(t) \in \mathcal{R}_e$ and $\rho^{AB} = \rho^{BA}$, i.e., reflection symmetry from Eq. (11))

$$\rho^{AB} = \frac{1}{4}\left(I \otimes I + \sum_{i=1}^{3}(c_i\sigma_i \otimes \sigma_i) + c_4 I \otimes \sigma_3 + c_4\sigma_3 \otimes I\right), \tag{32}$$

where σ_i is the Pauli matrix, with $i = 1, 2, 3$. The relation between c_k (with $k = 1, 2, 3, 4$) and the elements of ρ^{AB} is simple to obtain [56]:

$$\begin{aligned} c_1 &= c_2 = 2z = 2\rho_{s,-s}(t), \\ c_3 &= v + y - w - x = 1 - 2\rho_{s,s}(t) - 2\rho_{-s,-s}(t) \\ &= 1 - 4\rho_{s,s}(t), \\ c_4 &= v - y = -\frac{1}{2}(1 + c_3) = -(1 - 2\rho_{s,s}(t)). \end{aligned} \tag{33}$$

We recall that z and $\rho_{s,-s}(t)$ are real numbers in our case. Note that c_3 and c_4 are real observables proportional to the probability $\rho_{s,s}(t)$, while c_1 and c_2 are real quantities but proportional to the off-diagonal element $\rho_{s,-s}(t)$. In order to obtain the classical and quantum correlations, we first calculate the quantum mutual information (Eq. (26)). This is a function of the eigenvalues of ρ^{AB}, ρ^{A} and ρ^{B} as follows Ref. [56]:

$$\mathcal{I}(s,t) = -\sum_{k=1}^{2} r_k^A \log_2 r_k^A - \sum_{k=1,2} r_k^B \log_2 r_k^B + \sum_{j=1}^{4} \lambda_j \log_2 \lambda_j, \tag{34}$$

where $r_1^{A(B)} = (1+c_4)/2$, $r_2^{A(B)} = (1-c_4)/2$ are the eigenvalues of $\rho^{A(B)}$, and λ_j is an eigenvalue of ρ^{AB}, which are given as follows:

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{4}\left(1 + c_3 \pm |2c_4|\right) \\ \lambda_{3,4} &= \frac{1}{4}\left(1 - c_3 \pm |2c_1|\right), \end{aligned} \tag{35}$$

where c_1, c_2, c_3 and c_4 are explicit functions of $\rho_{s,s}(t)$ and $\rho_{s,-s}(t)$, see Eq. (33).

In a similar way, the classical correlations can be found explicitly from Eq. (30), and then the mirror QD can be analytically obtained between sites s and $-s$, using the reduced density matrix given in Eq. (11). Following similar steps as in Ref. [56], by using Eq. (7) in (11), we can obtain an explicit expression for QD and CC as functions of the dissipative parameter $r_D \equiv \frac{2D}{\Omega/\hbar}$. We stress that the state given by Eq. (11) is not a Werner state [34], it is in fact a type of X state [40,56], where the classical correlations $CC(s)$ between the site s and its mirror $-s$ read as follows:

$$CC(s,t) = H_{\text{bin}}(p_s) - H_{\text{bin}}(p_s'), \tag{36}$$

where $H_{\text{bin}}(x) = -x\log_2(x) - (1-x)\log_2(1-x)$, with

$$\begin{aligned} p_s &= \frac{1}{2}(1+c_4) = \rho_{s,s}(t), \\ p_s' &= \frac{1}{2}\left(1 + \sqrt{c_1^2 + c_4^2}\right) = \frac{1}{2}\left(1 + \sqrt{4\rho_{s,s}^2(t) + (2\rho_{s,-s}(t) - 1)^2}\right). \end{aligned} \tag{37}$$

Finally, the mirror quantum discord $QD(s,t)$ is calculated analytically using Eqs. (36) and (34) in Eq. (31) (we have used the optimization process and numerical relations found in Ref. [56]. In addition, we have found and argue in favor of this numerical conjecture which is determined analytically, concerning Eqs. (37) and (36)). Therefore the total mirror QD is obtained as: $\sum_{s=1}^{\infty} QD(s,t)$.

4. Results from an initial localized QDW state

4.1. On the dissipative two-qubit models

In this section we show numerical results for the concurrence, the negativity and the QD as a function of the dissipative parameter interpolating from the pure unitary evolution case (zero dissipation $D = 0$) to the pure classical limit (completely dissipative case $\Omega \to 0$). In our system we have two characteristic energy scales ($\Omega/\hbar$ and $2D$), therefore we can analyze the competition between these scales. We can accomplish this task using as initial condition a localized state ($\rho(t) = |s_0\rangle\langle s_0|$, with $s_0 = 0$). Then we define the dissipative parameter $r_D = \frac{2D}{\Omega/\hbar}$ (rates of energies of the system) and the dimensionless time $t' = \frac{\Omega}{\hbar}t$. With these parameters in mind it is straightforward to plot all the interesting correlations for the bipartite system.

In Fig. 3 we plot C_M as a function of time t' and the dissipation parameter r_D (see Eq. (14)). This plot shows the decay of the concurrence as the dissipative parameter increases for fixed time t', and vice-versa, displaying oscillatory behavior for small values of r_D and t'. Similar behavior is noted for the negativity N, as it is shown in Fig. 4 (see Eq. (24)) for the same values of time and dissipation parameter as in the plot for C_M.

The measure of time evolution of quantum entanglement E, such as we have shown previously through concurrence and negativity, are compared in Fig. 5 for several values of $r_D = 0, 0.01, 0.05, 0.1, 05, 1, 2$. It can be noted that at $t' = 0$ entanglement is zero (for $r_D \geq 0$), because the initial localized state: $\rho(0) = |s_0\rangle\langle s_0|$ (with $s_0 = 0$) is a disentangled state for both partitions shown in Figs. 1 and 2. In the case $r_D = 0$ (closed quantum system, i.e., without dissipation), the quantum entanglement displays an oscillatory behavior (see Eqs. (15) and (25)); these are the exact solutions for the mirror concurrence and the negativity for the bipartitions used in Figs. 1 and 2 respectively. Both functions go asymptotically to the value one, due to the entanglement produced by the QW in the degree of freedom of the lattice as time goes on in the bipartition. Since the negativity provides all the correlations between all the sites, in contrast with the total mirror concurrence (i.e., this measure gives the total correlations only between mirror sites), the negativity goes asymptotically in time t' faster to the value one than the total mirror concurrence.

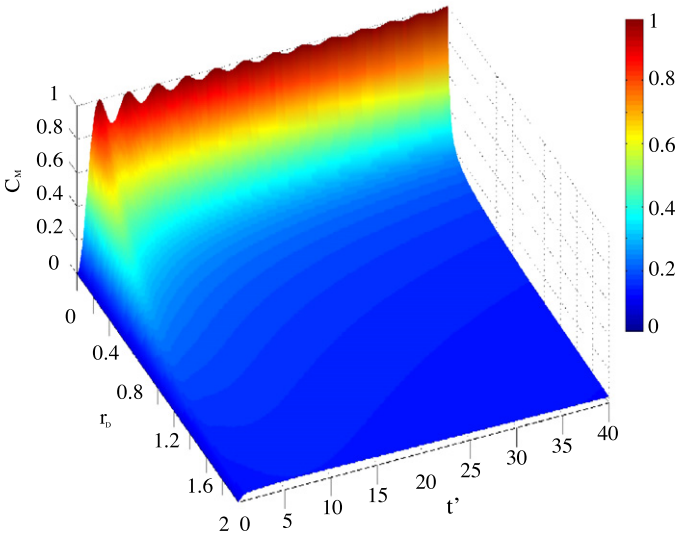

Fig. 3. The total mirror concurrence C_M (sum over all mirror sites) from a localized initial state $\rho(0)$ as function of r_D and t' (see Eq. (14)), where $r_D = \frac{2D}{\Omega/\hbar}$ and $t' = \frac{\Omega}{\hbar}t$. It can be noted that the total mirror concurrence decays for increasing t' (for a fixed value of r_D). On the other hand, the concurrence also decays for fixed t' and increasing r_D.

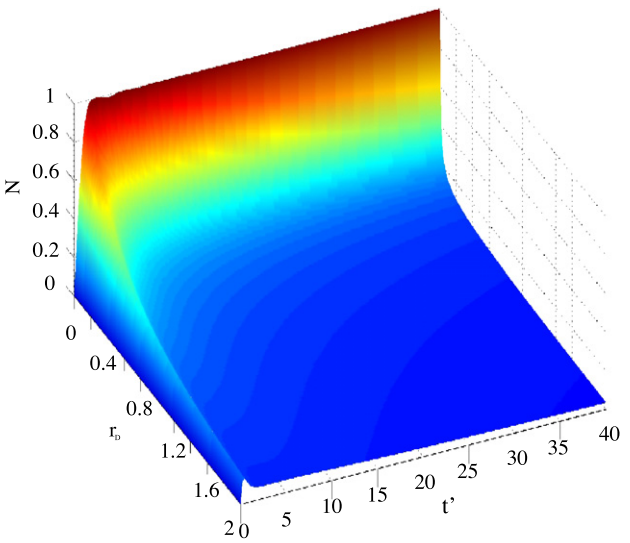

Fig. 4. The negativity N for the second bipartition (see Fig. 2) as a function of r_D and t'. The negativity for a localized state is given by Eq. (24). This measure shows similar behavior to that of the total mirror concurrence.

In contrast, for $r_D > 0$ the entanglement increases with time until it reaches a maximum value after which it decreases in time due to the dissipation in the system. Thus there is an interesting typical time $\tau(r_D)$ characterizing this maximum. This behavior can be interpreted as clear competition between the quantum correlations and decoherence mechanisms working on the bipartitions that we have used. From Eqs. (15) and (25) it is possible to find an upper bound for the characteristic time $\tau(r_D)$, in fact form the 1st zero of the 0th Bessel coefficient j_{01} we get:

$$\tau(r_D) < \tau(r_D = 0) = j_{01} \equiv 2.4048. \tag{38}$$

So the scaling time given in Eq. (38) which is valid for behavior of $C_M(t)$ and $N(t)$ can be used to characterize the quantum to classical transition in time (for fixed value of r_D).

In Fig. 6 we show the negativity decreasing with the dissipative parameter r_D (for several values of fixed times $t' = 5, 10, 20, 30, 40, 50$). As expected at the limit $\Omega \to 0$ ($r_D \to \infty$) the entanglement is zero, because the off-diagonal elements of $\rho(t)$ are all null due to the initial localized state that we have used. Otherwise, for non-localized initial states there is a remanent quantum correlation at the asymptotic limit [26].

In addition, we now show the mirror QD from our localized initial condition for $\rho(0)$. In Fig. 7 (left panel), the total QD is shown, calculated as the sum of the mirror QD between sites $-s$ and s, using the bipartition shown in Fig. 2 for different

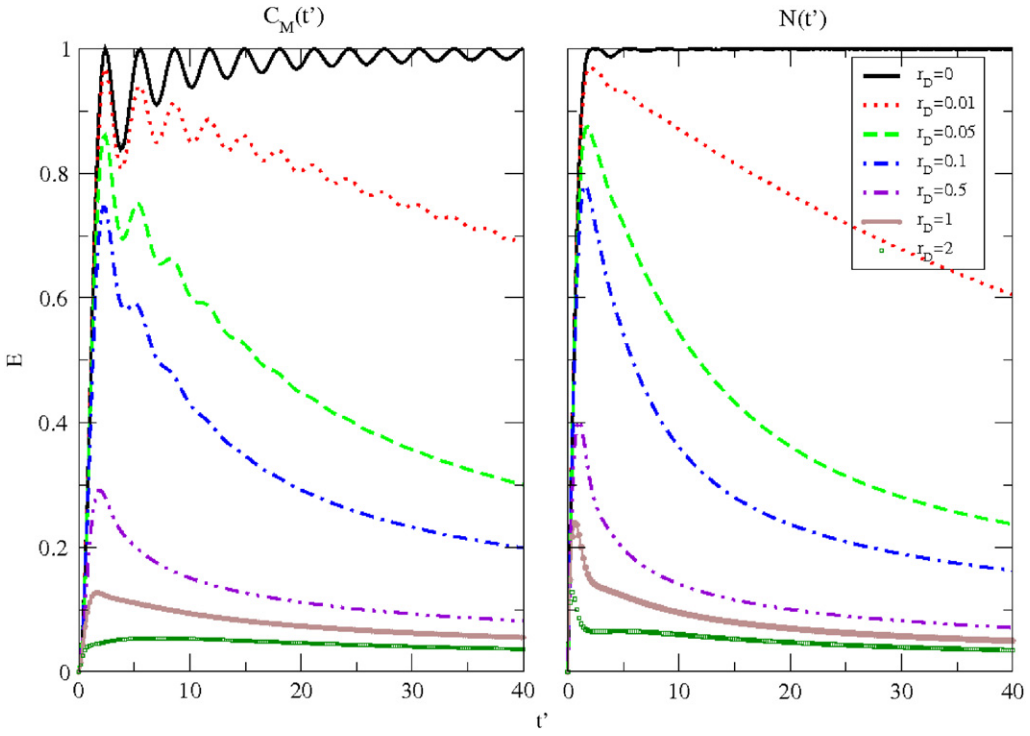

Fig. 5. The quantum entanglement E i.e., mirror concurrence and negativity (for the two bipartitions on the lattice, see Figs. 1 and 2) as a function of t', and for values of $r_D = 0, 0.01, 0.005, 0.1, 0.5, 1, 2$. The left panel shows the mirror concurrence C_M and in the right panel the negativity N can be observed.

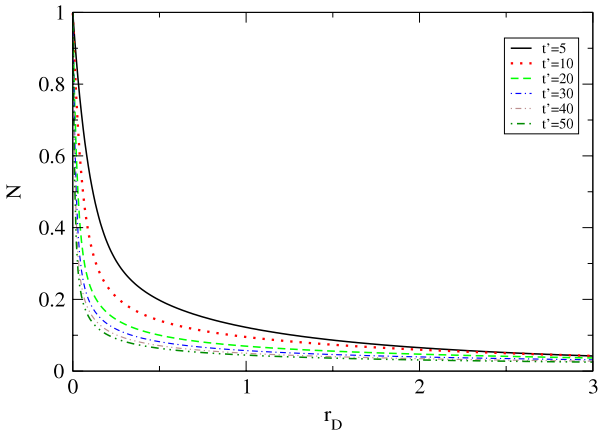

Fig. 6. The negativity as a function r_D for values of $t' = 5, 10, 20, 30, 40, 50$. These measures show a monotonous behavior, and for $r_D \gg 1$ the negativity goes asymptotically to zero.

values of dissipation $r_D = 0, 0.01, 0.05, 0.1, 05, 1, 2$. The total QD is in good agreement with the behavior of the total mirror concurrence and the negativity analysis, see Fig. 5, as both display a non-monotonic behavior and asymptotically decay at long time for $r_D > 0$. In Fig. 7 we note that these correlations (quantum and classical) increase and then decrease with time t' for several values of dissipation r_D. As expected, the behavior of these quantum measures (CC, N, QD) are converse to the increase in entropy for a DQW interacting with a quantum bath [25].

4.2. On the two-qubit density matrix with rotational symmetry around the z-axes

Finally, we show a 3D plot of the QD and CC associated with the state ρ^{AB} of our two-qubit system in a general way. Using Eq. (11) with $s_1 = -s$; $s_2 = s$ and a localized initial condition for the DQW we arrive to the symmetries pointed out in Eq. (32), it means that we can define non-null quantities as given by:

$$\Gamma_{xx} = \Gamma_{yy} = c_1 = c_2$$

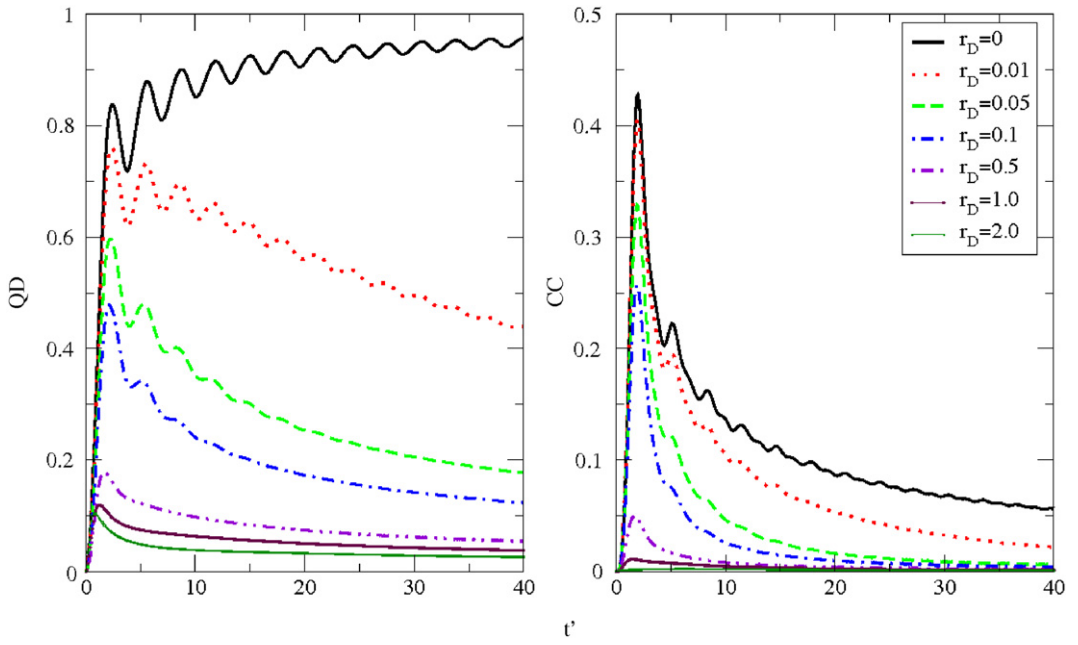

Fig. 7. The sum of the mirror quantum discord for the sites $-s$ and s as function of t', for values of $r_D = 0, 0.01, 0.005, 0.1, 0.5, 1, 2$ (left panel). In the right panel the total classical correlations (CC) are shown. Using a localized initial condition for $\rho(0)$.

$$\Gamma_{zz} = c_3$$
$$\Gamma_{0z} = c_4 = -\frac{1}{2}(1 + c_3) = -\frac{1}{2}(1 + \Gamma_{zz}), \tag{39}$$

where $\Gamma_{\beta\gamma} = \mathrm{Tr}[\rho^{AB}\sigma_\beta \otimes \sigma_\gamma]$, with $\beta = 0, x, y, z$; here σ_β is a Pauli matrix with $\beta = x, y, z$, and σ_0 is the identity matrix, where $\rho^{AB} = \sum_{(\alpha,\beta)=0,x,y,z} \Gamma_{\alpha\beta}\, \sigma_\alpha \otimes \sigma_\beta$. In this way Eq. (39) tells us that ρ^{AB} depends only on two parameters, showing the rotational symmetry around the z-axes.

In Fig. 8 we show the results for QD and CC (with the symmetry considered above). In the top panel of Fig. 8 the QD is shown; it can be noted that for $\Gamma_{zz} = -1$ ($\Gamma_{0z} = 0$) and $|\Gamma_{xx}| \leq 1$ the quantum discord reaches its maximum value (one), this type of state was analyzed in Ref. [57]. And the QD is zero in the point defined by $\Gamma_{xx} = 0$, $\Gamma_{zz} = 1$ ($\Gamma_{0z} = -1$), which means that $\rho^{AB} = \{I\otimes I + \sigma_z \otimes \sigma_z - (\sigma_z \otimes I + I \otimes \sigma_z)\}/4$, this corresponds to the triplet state with effective magnetization in the z-direction. In the bottom panel the CC can be observed, where we show that it is maximum for $\Gamma_{zz} = -1$; $\Gamma_{xx} = -1$ (singlet state, $|\psi_s^{(-)}\rangle = (|1_{-s}, 0_s\rangle - |0_{-s}, 1_s\rangle)/\sqrt{2}$) and $\Gamma_{zz} = -1$; $\Gamma_{xx} = 1$ (triplet state, $|\psi_s^{(+)}\rangle = (|1_{-s}, 0_s\rangle + |0_{-s}, 1_s\rangle)/\sqrt{2}$); these states can be used as initial condition as we did in our previous work [26]. It can also be noted that the CC is zero in the line defined by $\Gamma_{xx} = 0$ and $|\Gamma_{zz}| < 1$. This 3D phase diagram, for these correlation functions, are valid for two-qubit states with the symmetries mentioned above.

5. Conclusions

A dissipative quantum walk in interaction with a thermal phonon bath has been used to define a set of dissipative qubits. The fact that we have solved the quantum master equation associated with the dissipative quantum walk allows us to work out several correlation functions analytically.

By introducing two different bipartitions on the lattice we were able to measure several quantum correlations associated with a set of qubits (our dissipative system of interest AB). This provides alternative ways to study the effect of the dissipation on the lattice. From the first bipartition (see Fig. 1) we define two qubits by tracing out the rest of lattice sites $s' \neq \pm s$. Then exact expressions for mirror concurrence and mirror quantum discord was obtained. Using a second bipartition (see Fig. 2) i.e., considering the left and right sites with respect to the origin of the lattice, a different subsystem AB was defined, in this case with infinite degrees of freedom, then the negativity was used to measure entanglement in the system AB.

The evolution of correlations (such as concurrence, negativity, quantum discord, and classical correlations) depends on a non-trivial form of the rate of energies of the system, so we defined the rate $r_D = \frac{2D}{\Omega/\hbar}$ as the important dissipative parameter of the system. Intriguingly, all these correlation measures are in good agreement with each other in a qualitative way. Our results allow us to study analytically the non-dissipative limit $D = 0$ ($r_D \ll 1$), and the classical limit $\Omega = 0$ ($r_D \gg 1$), which have been found to be in agreement with the physical expectations made on a related quantum system [25]. This

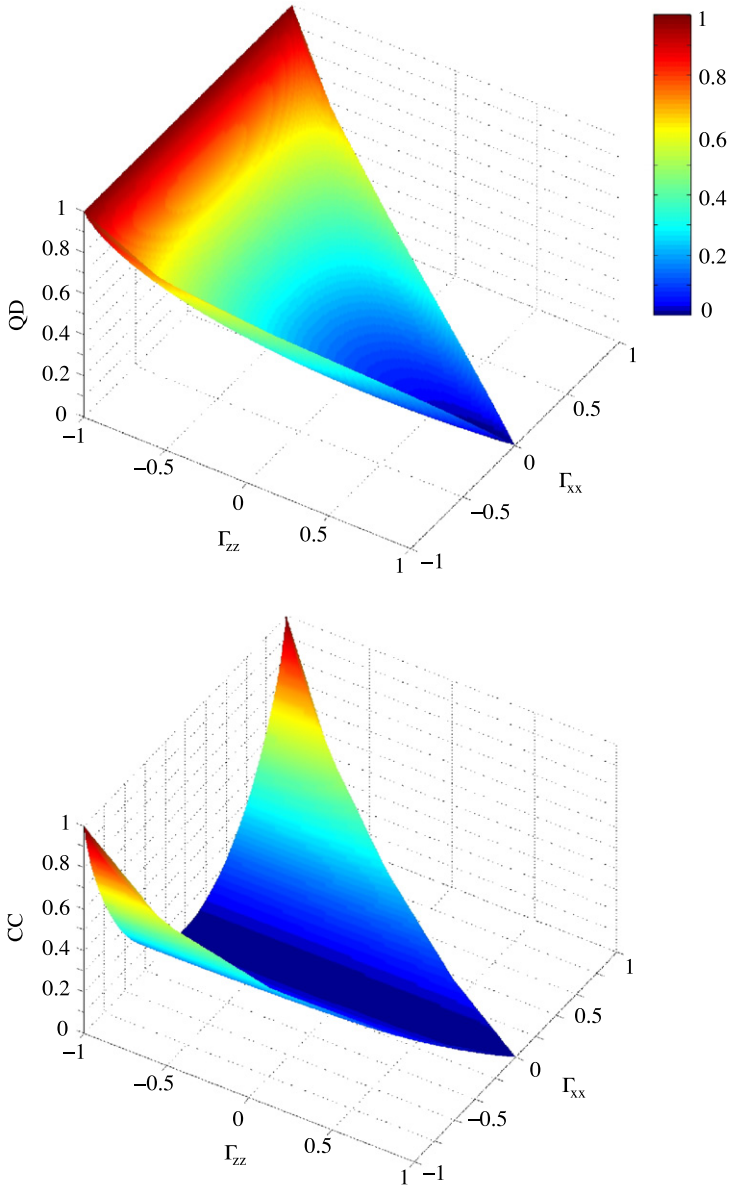

Fig. 8. Quantum discord as a function of physical observables Γ_{xx} and Γ_{zz} (top panel). The bottom panel shows the total classical correlations (CC). The physical observables are calculated as $\Gamma_{\beta\beta} = \mathrm{Tr}[\rho^{AB}\sigma_\beta \otimes \sigma_\beta]$, with $\beta = x, z$, and ρ^{AB} is expressed in a similar way to Eq. (11).

analysis allows us to understand the effects induced by the decoherence mechanism in the quantum correlations by using quantum information tools.

Our framework opens up the possibility of carrying out an analytical analysis on several quantities related to the study of the quantum correlations in dissipative systems. In particular, using the same type of dynamic (semigroup) it is possible to write down the quantum master equation for a set of two free particles interacting with a thermal phonon bath, which is of great interest in quantum information theory, and work along this line is in progress.

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Appendix A. The completely dissipative regime

From the QME (see Eq. (4)) in the case $\Omega = 0$, we obtain the following expression for the quantum semigroup:

$$\dot{\rho} = D\left(a\rho a^{\dagger} + a^{\dagger}\rho a - 2\rho\right). \tag{A.1}$$

This equation can be worked out on the Wannier basis, where we know that $a|s\rangle = |s-1\rangle$ and $a^{\dagger}|s\rangle = |s+1\rangle$. Then the time evolution of the density matrix elements follows the equation:

$$\langle s|\dot{\rho}|s'\rangle = D\left(\langle s-1|\rho|s'-1\rangle + \langle s+1|\rho|s'+1\rangle - 2\langle s|\rho|s'\rangle\right). \tag{A.2}$$

In particular, for the diagonal elements ($s' = s$), we get using the notation $\rho_{s,s'} = \langle s|\rho|s'\rangle$ and $P_s \equiv \rho_{s,s}$, that the probability of finding the walker in the position s follows the evolution:

$$\dot{P}_s = D\left(P_{s-1} + P_{s+1} - 2P_s\right). \tag{A.3}$$

For the time evolution of the off-diagonal elements $\rho_{s,s'}$ (where $s' \neq s$), from (A.1) we obtain the expression:

$$\dot{\rho}_{s,s'} = D\left(\rho_{s-1,s'-1} + \rho_{s+1,s'+1} - 2\rho_{s,s'}\right). \tag{A.4}$$

Then it can be noted that for any initial condition fulfilling $\rho_{s,s'}(0) = 0, \forall s' \neq s$, the time evolution of the off-diagonal elements of ρ is zero ($\rho_{s,s'}(t) = 0$, with $t \geq 0$), so the time evolution of ρ is ruled by Eq. (A.3), which is the evolution equation for a classical random walk [45,46] (some-times called the Pauli master equation for the diagonal elements). In the case that $\rho_{s,s'}(0) \neq 0$ for some value of $s' \neq s$, the off-diagonal elements contribute to the quantum correlations (for instance: concurrence, negativity and QD), which have been called the quantum remanence of the correlations [26].

Appendix B. Diagonalizing the matrix $\mathcal{M}_1$

To diagonalize the infinite matrix $\mathcal{M}_1$ (see the matrix ρ^{T_A} after Eq. (23)), we study the finite matrix ($N \times N$) with the same symmetry given by the following expression

$$M = \begin{pmatrix} 0 & a_1^* & a_2^* & \cdots & a_N^* \\ a_1 & 0 & 0 & \cdots & 0 \\ a_2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_N & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{B.1}$$

Using an appropriated program for diagonalizing this matrix, it is straightforwardly to see that M has only two eigenvalues different from zero. These eigenvalues are:

$$\mu = -\sqrt{\sum_{i=1}^{N} |a_i|^2}$$

$$\beta = \sqrt{\sum_{i=1}^{N} |a_i|^2}. \tag{B.2}$$

Then, the matrix M has only one negative eigenvalue. This result can be extend for the limit $N \to \infty$, namely for the matrix $\mathcal{M}_1$.

To end this appendix, we note here that the block $\mathcal{M}_2$, which is not altered by the transport partial operation T_A, has not negative eigenvalues as it is expected.

References

[1] Y. Aharonov, L. Davidovich, N. Zagury, Phys. Rev. A 48 (1993) 1687.
[2] N.G. van Kampen, J. Stat. Phys. 78 (1995) 299.
[3] A. Ambainis, Int. J. Quantum Inf. 1 (2003) 507;
N. Shenvi, J. Kempe, K. BirgittaWhaley, Phys. Rev. A 67 (2003) 052307;
A. Childs, E. Farhi, S. Gutmann, J. Quantum Inf. Process. 1 (2002) 35.
[4] J. Kempe, Contemp. Phys. 44 (2003) 307.
[5] D.E. Katsanos, S.N. Evangelou, S.J. Xiong, Phys. Rev. B 51 (1995) 895.
[6] M. Esposito, P. Gaspard, Phys. Rev. B 71 (2005) 214302.
[7] F. Zähringer, G. Kirchmair, R. Gerritsma, E. Solano, R. Blatt, C.F. Roos, Phys. Rev. Lett. 104 (2010) 100503.
[8] A. Schreiber, K.N. Cassemiro, V. Potoček, A. Gábris, I. Jex, Ch. Silberhorn, Phys. Rev. Lett. 106 (2011) 180403.
[9] N. Konno, Quantum Inf. Process. 8 (2009) 387.

[10] Balaji R. Rao, R. Srikanth, C.M. Chandrashekar, Subhashish Banerjee, Phys. Rev. A 83 (2011) 064302;
C.M. Chandrashekar, R. Srikanth, S. Banerjee, Phys. Rev. A 76 (2007) 022316;
R. Srikanth, S. Banerjee, C.M. Chandrashekar, Phys. Rev. A 81 (2010) 062123.
[11] A. Romanelli, Phys. Rev. A 76 (2007) 054306.
[12] M.A. Broome, A. Fedrizzi, B.P. Lanyon, I. Kassal, A. Aspuru-Guzik, A.G. White, Phys. Rev. Lett. 104 (2010) 153602.
[13] D. Shapira, O. Biham, A.J. Bracken, M. Hackett, Phys. Rev. A 68 (2003) 062315.
[14] W. Dür, R. Raussendorf, V.M. Kendon, H.-J. Briegel, Phys. Rev. A 66 (2002) 052319.
[15] A. Joye, M. Merkli, J. Stat. Phys. 140 (2010) 1.
[16] H. Schmitz, R. Matjeschk, Ch. Schneider, J. Glueckert, M. Enderlein, T. Huber, T. Schaetz, Phys. Rev. Lett. 103 (2009) 090504.
[17] Yue Yin, D.E. Katsanos, S.N. Evangelou, Phys. Rev. A 77 (2008) 022302.
[18] M.O. Cáceres, A.K. Chattah, J. Mol. Liq. 71 (1997) 187.
[19] Oliver Mülken, Alexander Blumen, Phys. Rev. E 71 (2005) 036128.
[20] H.B. Perets, Y. Lahini, F. Pozzi, M. Sorel, R. Morandotti, Y. Silberberg, Phys. Rev. Lett. 100 (2008) 170506.
[21] A. Peruzzo, M. Lobino, J.C.F. Matthews, N. Matsuda, A. Politi, K. Poulios, X. Zhou, Y. Lahini, N. Ismail, K. Wörhoff, Y. Bromberg, Y. Silberberg, M.G. Thompson, J.L. O'Brien, Science 329 (2010) 1500.
[22] M. Karski, et al., Science 325 (2009) 174.
[23] C.A. Ryan, M. Laforest, J.C. Boileau, R. Laflamme, Phys. Rev. A 72 (2005) 062317.
[24] M.O. Cáceres, M. Nizama, J. Phys. A: Math. Theor. 43 (2010) 455306.
[25] M. Nizama, M.O. Cáceres, J. Phys. A: Math. Theor. 45 (2012) 335303.
[26] M. Nizama, M.O. Cáceres, Physica A 392 (2013) 6155.
[27] V. Kendon, Math. Structures Comput. Sci. 17 (2006) 1169.
[28] H.P. Breuer, F. Petrucione, The Theory of Open Quantum Systems, Oxford University Press, Oxford, 2003.
[29] V. Gorini, A. Kossakowski, J. Math. Phys. 17 (1976) 821.
[30] G. Lindblad, Comm. Math. Phys. 48 (1976) 119.
[31] A. Lakshminarayan, V. Subrahmanyam, Phys. Rev. A 67 (2003) 052304.
[32] Y. Omar, N. Paunković, L. Sheridan, S. Bose, Phys. Rev. A 74 (2006) 042304.
[33] S.M. Tan, D.F. Walls, M.J. Collett, Phys. Rev. Lett. 66 (1991) 252;
L. Hardy, Phys. Rev. Lett. 73 (1994) 2279;
J. Dunningham, V. Vedral, Phys. Rev. Lett. 99 (2007) 180404;
J.J. Cooper, J.A. Dunningham, New J. Phys. 10 (2008) 113024.
[34] M. Nielsen, I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, Cambridge, 2000.
[35] J. Preskill, Phys. Today 52 (1999) 24.
[36] W.K. Wootters, Phys. Rev. Lett. 80 (1998) 2245.
[37] G. Vidal, R.F. Werner, Phys. Rev. A 65 (2002) 032314.
[38] H. Ollivier, et al., Phys. Rev. Lett. 88 (2001) 017901;
L. Henderson, et al., J. Phys. A 34 (2001) 6899.
[39] S. Luo, Phys. Rev. A 77 (2008) 042303.
[40] M. Ali, A.R.P. Rau, G. Alber, Phys. Rev. A 81 (2010) 042105.
[41] X. Qian, Y. Li, Z. Song, C.P. Sun, Phys. Rev. A 72 (2005) 062329.
[42] P. Zanardi, X. Wang, J. Phys. A: Math. Gen. 35 (2002) 7947.
[43] A.O. Caldeira, A.J. Legget, Ann. Phys. (USA) 149 (1983) 374.
[44] A.A. Budini, A.K. Chattah, M.O. Cáceres, J. Phys. A: Math. Gen. 32 (1999) 631.
[45] N.G. van Kampen, Stochastic Processes in Physics and Chemistry, second ed., North Holland, Amsterdam, 1992.
[46] M.O. Cáceres, Elementos de Estadística de no Equilibrio y sus Aplicaciones al Transporte en Medios Desordenados, Reverté S.A., Barcelona, 2003 (in Spanish).
[47] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, Dover Publications, Nueva York, 1995.
[48] A. Romanelli, Phys. Rev. A 85 (2012) 012319.
[49] A. Patel, et al., Phys. Rev. A 71 (2005) 032347.
[50] K.M.O. Connor, W.K. Wootters, Phys. Rev. A 63 (2001) 052302.
[51] E.A. Evangelidis, J. Math. Phys. 25 (1984) 2151.
[52] P.A. Martin, J. Phys. A: Math. Theor. 41 (2008) 015207.
[53] A. Peres, Phys. Rev. Lett. 77 (1996) 1413.
[54] S. Lee, D.P. Chi, S.D. Oh, J. Kim, Phys. Rev. A 68 (2003) 062304.
[55] R. Dillenschneider, Phys. Rev. B 78 (2008) 224413.
[56] M.S. Sarandy, Phys. Rev. A 80 (2009) 022108.
[57] M. Nizama, D. Frustaglia, K. Hallberg, Phys. Rev. B 86 (2012) 075413.