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Similarity solutions for thawing processes with a convective boundary condition

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Abstract. *An explicit solution of similarity type for thawing in a saturated semi-infinite porous media when change of phase induces a density jump and a convective boundary condition is imposed at the fixed face, is obtained if and only if an inequality for data is verified. Relationship between this problem and the problem with temperature condition studied in [8] is analized and conditions for physical parameters under which the two problems become equivalents are obtained. Furthermore, an inequality to be satisfied for the coefficient which characterizes the free boundary of each problem is also obtained.*

Keywords: Stefan problem, free boundary problem, phase-change process, similarity solution, density jump, thawing process, convective boundary condition, Neumann solution.
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1. Introduction

In this paper, we consider the problem of thawing of a semi-infinite partially frozen porous media saturated with an incompressible liquid when change of phase induces a density jump and a convective boundary condition is imposed on the fixed face, with the aim of constructing similarity solutions (for a detailed exposition of the physical background we refer to [5, 7, 12, 14, 18]). In [8] and [10] (which generalized [19]) similarity solutions are obtained when a temperature and a heat flux condition are imposed at the fixed boundary, respectively. In this paper, we deal with the same physical situations as in [8, 10] and we study a one-dimensional model of the problem where the unknowns are the temperature $u(x, t)$ of the unfrozen zone $Q_1 = \{(x, t) : 0 < x < s(t), t > 0\}$, the temperature $v(x, t)$ of the frozen zone $Q_2 = \{(x, t) : x > s(t), t > 0\}$ and the free boundary $x = s(t)$, defined for $t > 0$, separating Q_1 and Q_2, which satisfies the following equations and boundary and initial conditions (we refer to [7] for a detailed explanation of the model):

$$u_t = d_U u_{xx} - b\rho\dot{s}(t)u_x \qquad 0 < x < s(t),\ t > 0 \quad (1)$$
$$v_t = d_F v_{xx} \qquad x > s(t),\ t > 0 \quad (2)$$
$$u(s(t),t) = v(s(t),t) = d\rho s(t)\dot{s}(t) \qquad t > 0 \quad (3)$$
$$k_F v_x(s(t),t) - k_U u_x(s(t),t) = \alpha\dot{s}(t) + \beta\rho s(t)(\dot{s}(t))^2 \qquad t > 0 \quad (4)$$
$$v(x,0) = v(+\infty,t) = -A \qquad x > 0,\ t > 0 \quad (5)$$
$$s(0) = 0 \quad (6)$$
$$k_U u_x(0,t) = \frac{h_0}{\sqrt{t}}(u(0,t) - B) \qquad t > 0 \quad (7)$$

with:

$$d_U = \alpha_U^2 = \frac{k_U}{\rho_U c_U}, \qquad d_F = \alpha_F^2 = \frac{k_F}{\rho_F c_F}, \qquad b = \frac{\epsilon\rho_W c_W}{\rho_U c_U}, \qquad d = \frac{\epsilon\gamma\mu}{K},$$

$$\rho = \frac{\rho_W - \rho_I}{\rho_W}, \qquad \alpha = \epsilon\rho_I l, \qquad \beta = \frac{\epsilon^2\rho_I(c_W - c_I)\gamma\mu}{K} = \epsilon d\rho_I(c_W - c_I) \neq 0.$$

where:
ϵ: porosity,
ρ_W and ρ_I: density of water and ice (g/cm^3),
c: specific heat at constant density ($cal/g\,^\circ C$),
k_F and k_U: conductivity of the frozen and unfrozen zones ($cal/s\,cm\,^\circ C$),
u: temperature of the unfrozen zone ($^\circ C$),
v: temperature of the frozen zone ($^\circ C$),
$u = v = 0$: melting point at atmospheric pressure,
l: latent heat at $u = 0$ (cal/g),
γ: coefficient in the Clausius-Clapeyron law ($s^2 cm^\circ C/g$),
$\mu > 0$: viscosity of the liquid ($g/s\,cm$),
$K > 0$: hydraulic permeability (cm^2),
$B > 0$: external boundary temperature at the fixed face $x = 0$ ($^\circ C$),
$B_0 > 0$: temperature at the fixed face $x = 0$ ($^\circ C$),
$-A < 0$: initial temperature ($^\circ C$),
$h_0 > 0$: coefficient which characterizes the heat transfer at the fixed face $x = 0$ ($Cal/s^{\frac{1}{2}}\,cm^2\,^\circ C$).

Remark 1.1. *The free boundary problem (1)-(7) reduces to the usual Stefan problem when*

$$\rho = 0$$

since in that case we have the cassical Stefan conditions on $x = s(t)$, i.e.:

$$u(s(t),t) = v(s(t),t) = 0\,, \quad t > 0$$

$$k_F v_x(s(t),t) - k_U u_x(s(t),t) = \alpha \dot{s}(t)$$

and therefore from now on we assume that $\rho \neq 0$.

The goal of this paper is to find the necessary and/or sufficient conditions for data (with three dimensionless parameters) in order to obtain an instantaneous phase-change process (1)-(7) with the corresponding explicit solution of the similarity type when a convective boundary condition of type (7) is imposed on the fixed face $x = 0$ [1, 4, 16, 23, 25]. We remark that the solution given in [25] is not correct for any data (in particular, for small heat transfer coefficient) for the classical two-phase Stefan problem ($\rho = 0$) which was improved in [20] obtaining the necessary and sufficient condition to get the corresponding explicit solution. Furthermore, we study the relationship between the problem (1)-(7) and the problem studied in [8], which consists of equations (1)-(6) and the following temperature boundary condition at the fixed face $x = 0$:

$$u(0,t) = B_0 \qquad t > 0, \tag{8}$$

where B_0 is the boundary temperature at the fixed face $x = 0$, with the aim of finding conditions under which the two problems become equivalent.

Recently Stefan-like problems were studied in [2, 3, 6, 9, 11, 15, 17, 21, 22, 24]. The plan is the following: first (Sect. 2) we obtain the necessary and sufficient condition in order to have a similarity solution of the free boundary problem (1)-(7) as a function of a positive parameter which must be the solution of a transcendental equation with three dimensionless parameters defined by the thermal coefficients, and initial and boundary conditions. We also find a monotonicity property between the coefficients which characterize the free boundary and the heat transfer at the fixed face $x = 0$. Then (Sect. 3) we give the necessary and/or sufficient conditions for the three real parameters involved in the trascendental equation in order to obtain an instantaneous phase-change process (1)-(7) with the corresponding similarity solution. We generalize results obtained for particular cases given in [13, 20]. Finally (Sect. 4), we analize the relationship between the problems (1)-(7) and (1)-(6) and (8), and we obtain conditions for data under which the two problems become equivalent. Furthermore, we obtain an inequality which satisfies the coefficient involved in the definition of the free boundary of the problem (1)-(6) and (8), which become the inequality obtained in [19] for the classical Stefan problem.

2. Similarity solutions

We have:

THEOREM 2.1. *The free boundary value problem (1)-(7) has the similarity solution:*

$$u(x,t) = \frac{Bg(p,\xi) + \frac{AMk_U}{2h_0\alpha_U}\xi^2 + (AM\xi^2 - B)\int_0^{\frac{x}{2\alpha_U\sqrt{t}}} exp(-r^2 + pr\xi)\,dr}{g(p,\xi) + \frac{k_U}{2h_0\alpha_U}} \tag{9}$$

$$v(x,t) = \frac{AM\xi^2 + A\ erf(\gamma_0\xi) - A(1+M\xi^2)\ erf\left(\frac{x}{2\alpha_F\sqrt{t}}\right)}{erfc(\gamma_0\xi)} \tag{10}$$

$$s(t) = 2\xi\alpha_U\sqrt{t} \tag{11}$$

if and only if the dimensionless coefficient $\xi > 0$ *satisfies the following equation:*

$$G(M,p,y) = y + Ny^3, \qquad y > 0 \tag{12}$$

involving three dimensionless parameters, N, M *and* p*, defined by:*

$$N = \frac{2\beta\rho\alpha_U^2}{\alpha} \in \mathbb{R}, \qquad M = \frac{2d\rho\alpha_U^2}{A} \in \mathbb{R}, \qquad p = 2b\rho \in \mathbb{R}, \tag{13}$$

where:

$$G(M,p,y) = \delta_1\left(1 - \frac{AM}{B}y^2\right)G_1(p,y) - \delta_2(1+My^2)G_2(y) \qquad y > 0, \tag{14}$$

$$G_1(p,y) = \frac{exp((p-1)y^2)}{K_0 + g(p,y)} \qquad p \in \mathbb{R},\ y > 0, \tag{15}$$

$$G_2(y) = \frac{exp(-\gamma_0^2 y^2)}{erfc(\gamma_0 y)} \qquad M \in \mathbb{R},\ y > 0, \tag{16}$$

$$g(p,y) = \int_0^y exp(-r^2 + pry)\,dr \qquad p \in \mathbb{R},\ y > 0, \tag{17}$$

$$\delta_1 = \frac{k_U B}{2\alpha\alpha_U^2} > 0,\ \delta_2 = \frac{k_F A}{\alpha\alpha_U\alpha_F\sqrt{\pi}} > 0,\ K_0 = \frac{k_U}{2\alpha_U h_0} > 0,\ \gamma_0 = \frac{\alpha_U}{\alpha_F} > 0. \tag{18}$$

Proof. We will follow the method introduced in [8, 10]. First of all, we note that the function u defined by:

$$u(x,t) = \Phi(\eta) \qquad \text{with } \eta = \frac{x}{2\alpha_U\sqrt{t}}$$

is a solution of equation (1) if and only if Φ satisfies the following equation:

$$\frac{1}{2}\Phi''(\eta) + \left(\eta - \frac{b\rho}{\alpha_U}\dot{s}(t)\sqrt{t}\right)\Phi'(\eta) = 0$$

Then, if we consider the function $x = s(t)$ defined as in (11), for some $\xi > 0$ to be determined, we obtain that:

$$\Phi(\eta) = C_1 + C_2 \int_0^{\eta} \exp(-r^2 + 2b\rho\xi r)\, dr \tag{19}$$

where ξ, C_1 and C_2 are constant values.

Therefore, a solution of equation (1) is given by:

$$u(x,t) = C_1 + C_2 \int_0^{x/2\alpha_U\sqrt{t}} \exp(-r^2 + 2b\rho\xi r)\, dr \tag{20}$$

where C_1, C_2 and ξ are constant values to be determined.

On the other hand, it is well known that:

$$v(x,t) = C_3 + C_4 \mathrm{erf}\left(\frac{x}{2\alpha_F\sqrt{t}}\right), \tag{21}$$

is a solution of equation (2), where C_3 and C_4 are constant values to be determined and erf is the error function defined by:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-r^2)\, dr$$

From conditions (3), (5) and (7) we have that:

$$C_1 = \frac{Bg(2b\rho,\xi) + \frac{d\rho\alpha_U k_U}{h_0}\xi^2}{g(2b\rho,\xi) + \frac{k_U}{2h_0\alpha_U}}, \qquad C_2 = \frac{2d\rho\alpha_U^2\xi^2 - B}{g(2b\rho,\xi) + \frac{k_U}{2h_0\alpha_U}},$$

$$C_3 = \frac{2d\rho\alpha_U^2\xi^2 + A\mathrm{erf}\,(\gamma_0\xi)}{\mathrm{erfc}\,(\gamma_0\xi)}, \qquad C_4 = -\frac{2d\rho\alpha_U^2\xi^2 + A}{\mathrm{erfc}\,(\gamma_0\xi)}$$

and by imposing condition (4), we obtain that the functions s, u and v defined above by (9)-(11), are a solution of the free boundary value problem (1)-(7) if and only if the dimensionless parameter ξ satisfies the equation (12), and therefore the thesis holds. □

Remark 2.2. *The similarity solution (19) is a generalized Neumann solution for the classical Stefan problem (see [1]) when a convective boundary condition at the fixed face $x = 0$ is imposed (see [20]).*

The following result give us a relationship between the coefficient ξ which characterizes the free boundary of the problem (1)-(7) and the coefficient $h_0 > 0$ which characterizes the heat transfer at the fixed face $x = 0$.

Corollary 2.3. *If the free boundary value problem (1)-(7) has a unique similarity solution of the type (9)-(11), $M > 0$, and the dimensionless coefficient $\xi > 0$ satisfies:*

$$0 < \xi < \sqrt{\frac{B}{AM}}, \tag{22}$$

then ξ is a strictly increasing function of $h_0 > 0$.

Proof. Let us assume that the free boundary value problem (1)-(7) has the similarity solution (9)-(11). Due to previous theorem we know that the dimensionless coefficient ξ satisfies equation (12). Since the LHS of equation (12) is a strictly increasing function of $h_0 > 0$, for all $y \in \left(0, \sqrt{\frac{B}{AM}}\right)$, as well as its limit when y tends to 0^+:

$$\lim_{y\to 0^+}\left(\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y)-\delta_2(1+My^2)G_2(y)\right)=\frac{\delta_1}{K_0}-\delta_2,$$

and the RHS of equation (12) is a strictly increasing function of $y > 0$, we can conclude that ξ is a strictly increasing function of $h_0 > 0$. □

3. Existence and uniqueness of similarity solutions

Due to previous theorem, in order to analize the existence and uniqueness of similarity solutions of the free boundary value problem (1)-(7), we can focus on the solvability of equation (12). With this aim, we split our analysis into four cases which correspond to four possible combinations of the signs of the dimensionless parameters M and N.

To prove the following results we will use properties of functions g, G_1 and G_2 involved in equation (12) which we proved in the Appendix.

Theorem 3.1. *If $M > 0$ and $N > 0$ then:*

1. *If:*

$$h_0 > \frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}} \tag{23}$$

   then equation (12) has at least one solution ξ, which satisfies $0 < \xi < \sqrt{\frac{B}{AM}}$.

2. *If $p \leq 1$ then equation (12) has a solution ξ, which satisfies $0 < \xi < \sqrt{\frac{B}{AM}}$, if and only if (23) holds. Moreover, when (23) holds, equation (12) has a unique solution ξ, which satisfies $0 < \xi < \sqrt{\frac{B}{AM}}$.*

Proof. Let $M > 0$ and $N > 0$.

1. We have proved in the Appendix that:

$$\lim_{y\to+\infty}\left(1-\frac{AM}{B}y^2\right)G_1(p,y)=\begin{cases}0 & p<2\\ -\infty & p\geq 2\end{cases}$$

and

$$\lim_{y\to+\infty}(1+My^2)G_2(y)=+\infty.$$

Then:

$$\lim_{y\to+\infty}\left[\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y)-\delta_2(1+My^2)G_2(y)\right]=-\infty.$$

Furthermore:

$$\lim_{y\to0^+}\left[\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y)-\delta_2(1+My^2)G_2(y)\right]=\frac{\delta_1}{K_0}-\delta_2.$$

Let us assume that $h_0>\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$. Since this inequality is equivalent to $\frac{\delta_1}{K_0}-\delta_2>0$, we have that the last limit is positive. Now taking into account that the RHS of (12) is an increasing function from 0 to $+\infty$, it follows that equation (12) has at least one positive solution.

Moreover, since:

$$\lim_{y\to0^+}\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y)=\frac{\delta_1}{K_0}\,,\quad \lim_{y\to0^+}\delta_2(1+My^2)G_2(y)=\delta_2\,,$$
$$\lim_{y\to+\infty}\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y)=-\infty\,,\ \lim_{y\to+\infty}\delta_2(1+My^2)G_2(y)=+\infty$$

and

$$\frac{\delta_1}{K_0}>\delta_2,$$

we have that the LHS of (12) has positive zeros $q_1<q_2<\cdots$. Thus we can find a solution ξ of (12) such that $\xi<q_1$. To prove that $0<\xi<\sqrt{\frac{B}{AM}}$, only remains to note that each q_k satisfies $q_k<\sqrt{\frac{B}{AM}}$.

2. Assume that $p\leq 1$ and $h_0>\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$. We know that (see [8]):

$$\frac{\partial}{\partial y}G_1(p,y)<0,\qquad y>0.$$

Then:

$$\frac{\partial}{\partial y}\left(1-\frac{AM}{B}y^2\right)G_1(p,y)=$$
$$\frac{-2AMy}{B}G_1(p,y)+\left(1-\frac{AM}{B}y^2\right)\frac{\partial}{\partial y}G_1(p,y)<0, \qquad 0<y<\sqrt{\frac{B}{AM}},$$

which implies that $\left(1-\frac{AM}{B}y^2\right)G_1(p,y)$ is strictly decreasing in the interval $\left(0,\sqrt{\frac{B}{AM}}\right)$. Furthermore, $\left(1-\frac{AM}{B}y^2\right)G_1(p,y)$ is negative in the interval $\left(\sqrt{\frac{B}{AM}},+\infty\right)$. On the other hand, we know from the Appendix that $(1+My^2)G_2(y)$ is a positive strictly incresing function of $y>0$. Therefore, the LHS of (12) is strictly decreasing in $\left(0,\sqrt{\frac{B}{AM}}\right)$ and negative in $\left(\sqrt{\frac{B}{AM}},+\infty\right)$. We also have that the limit of the LHS of (12) when y tends to 0^+ is positive because we are assuming that $h_0>\frac{A}{B}\frac{k_F}{\sqrt{\pi\alpha_F}}$. Then, since RHS of (12) is strictly increasing from 0 to $+\infty$, we have that equation (12) has a unique solution $\xi\in\left(0,\sqrt{\frac{B}{AM}}\right)$.

Now, let us assume that (12) has a solution $\xi\in\left(0,\sqrt{\frac{B}{AM}}\right)$. Suppose h_0 does not verify the inequality (23), that is $\frac{\delta_1}{K_0}-\delta_2\leq 0$. As LHS of (12) is strictly decreasing in $\left(0,\sqrt{\frac{B}{AM}}\right)$, follows that LHS of (12) is negative in $\left(0,\sqrt{\frac{B}{AM}}\right)$. Since RHS of (12) is positive in $\left(0,\sqrt{\frac{B}{AM}}\right)$, we have a contradiction. Therefore $h_0>\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$ and the thesis holds.

□

REMARK 3.2. *The inequality (23) was obtained in [20] for the particular case* $\rho=0$.

The following corollary summarizes previous results.

COROLLARY 3.3. *If* $M>0$, $N>0$, $p\leq 1$ *and* $h_0>\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$ *then the free boundary value problem (1)-(7) has a unique similarity solution of the type (9)-(11) and the dimensionless coefficient* ξ *is a strictly increasing function of the parameter* h_0 *on the interval* $\left(\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}},+\infty\right)$.

THEOREM 3.4. *If* $M>0$ *and* $N<0$ *then, if:*

$$h_0>\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}} \qquad \text{and} \qquad B<\frac{AM}{|N|},$$

then equation (12) has at least one solution ξ, which satisfies $0<\xi<\sqrt{\frac{B}{AM}}$.

Proof. Let $M>0$ and $N<0$. We have that LHS of (12) is positive in $(0,q_1)$, where $q_1<\sqrt{\frac{B}{AM}}$. On the other hand, RHS of (12) is positive in $\left(0,\sqrt{\frac{1}{|N|}}\right)$. Since $\sqrt{\frac{B}{AM}}<\sqrt{\frac{1}{|N|}}$, we can find a solution ξ of (12) which satisfies $0<\xi<\sqrt{\frac{B}{AM}}$. $\square$

Theorem 3.5. *If $M<0$ and $N>0$ then:*

1. *If:*
$$h_0>\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$$
and the LHS of (12) has positive zeros being q_1 the smaller one, then equation (12) has at least one solution ξ, which satisfies $0<\xi<q_1$.

2. *If:*
$$h_0\leq\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}\qquad \textit{and} \qquad N<\delta_2\sqrt{\pi}|M|\gamma_0,$$
then equation (12) has at least one positive solution.

Proof. Let $M<0$ and $N>0$.

1. It is analogous to the proof of the Theorem 3.1 (1).
2. Let us assume that $h_0\leq\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$ and $N<\delta_2\sqrt{\pi}|M|\gamma_0$. From the assymptotic behavior of $\left(1-\frac{AM}{B}y^2\right)G_1(p,y)$ and $(1+My^2)G_2(y)$ when y tends to $+\infty$ (see Appendix), we have that:
$$\lim_{y\to+\infty}\frac{\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y)-\delta_2(1+My^2)G_2(y)}{y+Ny^3}=$$
$$=\begin{cases} -\frac{\delta_2\sqrt{\pi}\gamma_0 M}{N} & p\leq 2\\ -\frac{AM(p-2)}{BN}-\frac{\delta_2\sqrt{\pi}\gamma_0 M}{N} & p>2\end{cases}.$$
Therefore:
$$\lim_{y\to+\infty}\frac{\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y)-\delta_2(1+My^2)G_2(y)}{y+Ny^3}>\frac{\delta_2\sqrt{\pi}\gamma_0|M|}{N}.$$
Then, it follows from the last inequality that the LHS of (12) tends to $+\infty$ faster than the RHS when y tends to $+\infty$. Now, taking into account that:
$$\lim_{y\to0^+}\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y)-\delta_2(1+My^2)G_2(y)=\frac{\delta_1}{K_0}-\delta_2<0,$$
we have that equation (12) has at least one positive solution.

□

REMARK 3.6. *When we have the condition $h_0 \leq \frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$ in the classical Stefan problem with $\rho = 0$ we only obtain a heat transfer problem without a phase-change process [20].*

THEOREM 3.7. *If $M < 0$ and $N < 0$ then:*

1. *If:*
$$h_0 > \frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$$
and the LHS of the equation (12) has positive zeros being q_1 the smaller one which satisfies:
   (a) *$q_1 < \sqrt{1/|N|}$, then equation (12) has at least two positive solutions, one of them satisfies $0 < \xi < \sqrt{1/|N|}$).*
   (b) *$q_1 = \sqrt{1/|N|}$, then equation (12) has $\xi = q_1$ as solution.*
2. *If:*
$$h_0 < \frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$$
then equation (12) has at least one positive solution ξ.

Proof. Let $M < 0$ and $N < 0$.

1. (a) Assume that $h_0 > \frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$.
We have proved in the Appendix that:
$$\lim_{y\to+\infty}\left(1-\frac{AM}{B}y^2\right)G_1(p,y) = \begin{cases} 0 & p<2 \\ +\infty & p \geq 2 \end{cases}$$
and
$$\lim_{y\to+\infty}(1+My^2)G_2(y) = -\infty.$$
Then:
$$\lim_{y\to+\infty}\left[\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y) - \delta_2(1+My^2)G_2(y)\right] = +\infty.$$
Furthermore:
$$\lim_{y\to 0^+}\left[\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y) - \delta_2(1+My^2)G_2(y)\right] = \frac{\delta_1}{K_0} - \delta_2.$$
On the other hand, the RHS of (12) is positive in $\left(0, \sqrt{\frac{1}{|N|}}\right)$ and negative in $\left(\sqrt{\frac{1}{|N|}}, +\infty\right)$. Then, since $\frac{\delta_1}{K_0} - \delta_2 > 0$ and $q_1 < \sqrt{1/|N|}$

we have that equation (12) has at least two positive solutions, one of them satisfies $0 < \xi < \sqrt{\frac{1}{|N|}}$.

(b) It is inmediate.

2. Now, let us assume that $h_0 < \frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$. We have:

$$\lim_{y\to+\infty}\left[\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y)-\delta_2(1+My^2)G_2(y)\right]=+\infty\,.$$

Furthermore:

$$\lim_{y\to0^+}\left[\delta_1\left(1-\frac{AM}{B}y^2\right)G_1(p,y)-\delta_2(1+My^2)G_2(y)\right]=\frac{\delta_1}{K_0}-\delta_2.$$

Taking into account that $\lim\limits_{y\to0^+}\left(y+Ny^3\right)=0$ and $\lim\limits_{y\to+\infty}\left(y+Ny^3\right)=-\infty$, since $\frac{\delta_1}{K_0}-\delta_2<0$ we have that equation (12) has at least one positive solution ξ.

□

REMARK 3.8. *Previous results imply that under the conditions specified in each case it is posible to find a similarity solution of the problem (1)-(7). Moreover, that solution guarantees a change of phase, that is,* $AM\xi^2 < u(0,t) < B$.

4. Relationship between the solutions of the Stefan problem with convective and temperature boundary conditions

In this section, we analyze the relationship between problem (1)-(7) and problem (1)-(6),(8) studied in [8], corresponding to a temperature condition at the fixed face $x = 0$.

In [8] it was proved that if $M > 0$, $N > 0$ and $p \le 2$, then the problem (1)-(6),(8) has a unique similarity solution of the type:

$$U(x,t)=B_0+\frac{AM\omega^2-B_0}{g(p,y)}\int_0^{\frac{x}{2\alpha_U\sqrt{t}}}\exp(-r^2+pr\omega)\,dr \tag{24}$$

$$V(x,t)=\frac{AM\omega^2\operatorname{erf}\left(\frac{x}{2\alpha_F\sqrt{t}}\right)+A\left(\operatorname{erf}(\gamma_0\omega)-\operatorname{erf}\left(\frac{x}{2\alpha_F\sqrt{t}}\right)\right)}{\operatorname{erfc}(\gamma_0\omega)} \tag{25}$$

$$S(t)=2\omega\alpha_U\sqrt{t} \tag{26}$$

where ω is the unique positive solution of the trascendental equation:

$$\widetilde{G}(M,p,y)=y+Ny^3, \tag{27}$$

with:

$$\widetilde{G}(M,p,y) = \tilde{\delta}_1 \left(1 - \frac{AM}{B_0} y^2\right) \widetilde{G_1}(p,y) - \delta_2 (1+My^2) G_2(M,y) \qquad y>0, \tag{28}$$

$$\widetilde{G}_1(p,y) = \frac{\exp\left((p-1)y^2\right)}{g(p,y)} \qquad p\in\mathbb{R},\ y>0, \tag{29}$$

$$\tilde{\delta}_1 = \frac{k_U B_0}{2\alpha\, \alpha_U^2}. \tag{30}$$

Furthermore, $0<\omega<\sqrt{\frac{B_0}{AM}}$.

Henceforth, we will only deal with situations in which existence and uniqueness of similarity solutions of type (9)-(11) for problem (1)-(8) or of type (24)-(26) for problem (1)-(6),(8), are guarantee.

Theorem 4.1. *If $M>0$, $N>0$, $p\leq 1$ and $h_0 > \frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$ then the dimensionless coefficient ξ which characterizes the free boundary of the problem (1)-(7) satisfies:*

$$0<\xi(h_0)<\omega_\infty \qquad \forall\, h_0 \in \left(\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}, +\infty\right) \tag{31}$$

where ω_∞ is the coefficient which characterizes the free boundary of the problem (1)-(6),(8) when the temperature condition is given by B.

Proof. Assume that $M>0$, $N>0$, $p\leq 1$ and $h_0 > \frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$. We know from Corollary 3.3 that the dimensionless coefficient ξ is a strictly increasing function of the coefficient h_0 on the interval $\left(\frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}, +\infty\right)$. We also know that ξ satisfies equation (12) which became:

$$\delta_1 \left(1 - \frac{AM}{B_0} y^2\right) \widetilde{G_1}(p,y) - \delta_2(1+My^2)G_2(M,y) = y + Ny^3 \tag{32}$$

when h_0 tends to $+\infty$. Only remains to note that this last equation (32) has a unique solution ω_∞ because (32) is the corresponding equation to problem (1)-(6),(8) when $B_0 = B$, which has a unique similarity solution under the hypothesis considered here. □

Theorem 4.2. *If $M>0$, $N>0$, $p\leq 1$ and $h_0 > \frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$ then the similarity solution (9)-(11) of the the problem (1)-(7), coincide with the similarity solution (24)-(26) of the problem (1)-(6),(8), when the external boundary temperature at $x=0$ is defined as:*

$$B_0 = \frac{2h_0\alpha_U B g(p,\xi) + AMk_U\xi^2}{2h_0\alpha_U g(p,\xi) + k_U} \tag{33}$$

Moreover, the dimensionless parameters ξ and ω which characterize the free boundary in each problem, are equals.

REMARK 4.3. *Note that B_0 is positive since $B_0 = \frac{2h_0\alpha_U B g(p,\xi)+AMk_U\xi^2}{2h_0\alpha_U g(p,\xi)+k_U} = u(0,t)$, $t>0$, and $u(0,t)>0$.*

Proof. Assume that $M>0$, $N>0$, $p\le 1$, $h_0 > \frac{A}{B}\frac{k_F}{\sqrt{\pi d_F}}$ and B_0 is defined as in (33). First of all, we note that function $\widetilde{G}(M,p,y)$, given in (28), can be written as:

$$\widetilde{G}(M,p,y) = \delta_1\left(\frac{B_0}{B} - \frac{AM}{B}y^2\right)\widetilde{G_1}(p,y) - \delta_2(1+My^2)G_2(y) \qquad y>0.$$

Because of the definition of B_0, given in (33), we have:

$$\frac{B_0}{B} - \frac{AM}{B}y^2 = \frac{g(p,\xi)\left(1-\frac{AM}{B}y^2\right)+\frac{AM}{B}K_0(\xi^2-y^2)}{g(p,\xi)+K_0}.$$

Then, equation (27) can be written as:

$$\begin{aligned}\delta_1\left(\frac{g(p,\xi)}{g(p,y)}\left(1-\frac{AM}{B}y^2\right)+\frac{AMK_0}{Bg(p,y)}(\xi^2-y^2)\right)G_1(p,y)\\ -\delta_2(1+My^2)G_2(y) = y+Ny^3.\end{aligned} \tag{34}$$

Now, since ξ satisfies equation (12), it follows that ξ satisfies equation (34). Therefore, ξ coincide with the coefficient ω which characterize the free boundary of the problem (1)-(8). Finally, from elementary calculations, we have that similarity solutions given in (9)-(11) and (24)-(26) are coincident. □

In the same way that the proof of the previous theorem, it can be shown the following result.

THEOREM 4.4. *If $M>0$, $N>0$ and $p\le 1$ then the similarity solution (24)-(26) of the problem (1)-(6),(8) coincide with the similarity solution (9)-(11) of the problem (1)-(7), when the coefficient which characterizes the heat transfer at the fixed face $x=0$ is defined by:*

$$h_0 = \frac{k_U(B_0 - AM\omega^2)}{2\alpha_U(B-B_0)g(p,\omega)} \tag{35}$$

and the boundary temperature B at $x=0$ is such that $B>B_0$.

Moreover, the dimensionless parameters ω and ξ which define the free boundary in each problem, are equals.

Remark 4.5. *Note that* h_0 *is a positive number since* $B_0 - AM\omega^2 > 0$ *and* $B - B_0 > 0$.

We can conclude now that in the sense established in Theorems 4.2 and 4.4, Stefan problems with convective and temperature conditions at the fixed face $x = 0$ given by (1)-(7) and (1)-(6),(8), respectively, are equivalent when inequality (23) is verified by data.

Theorem 4.6. *If* $M > 0$*,* $N > 0$ *and* $p \leq 1$ *then the coefficient* ω *which characterizes the free boundary of the problem (1)-(6),(8) satisfies the following inequality:*

$$\frac{B_0 - AM\omega^2}{g(p,\omega)} > \frac{2\alpha_U k_F A(B - B_0)}{\alpha_F k_U B\sqrt{\pi}}, \qquad \forall\, B > B_0 \tag{36}$$

Proof. Assume that $M > 0$, $N > 0$ and $p \leq 1$, and let $B > B_0$. We know from Theorem 4.4, that the problem (1)-(7) has a similarity solution of the type (9)-(11) when the external boundary temperature is given by B and the coefficient h_0 is defined as in (35). We also know that the coefficients which characterize the free boundary in problems (1)-(7) and (1)-(6), (8) are equals, that is $\xi = \omega$. Therefore, since $0 < \omega < \sqrt{\frac{B_0}{AM}}$ and $B > B_0$, we have that $0 < \xi < \sqrt{\frac{B}{AM}}$. Then, due to Theorem 3.1, part 2, inequality (23) holds. Only remains to note that inequality (23) becames inequality (36) when h_0 is defined as in (35). □

By taking limit when B tends to $+\infty$ into both sides of inequality (36), we have the following corollary.

Corollary 4.7. *If* $M > 0$*,* $N > 0$ *and* $p \leq 1$ *then the coefficient* ω *which characterizes the free boundary of the problem (1)-(6),(8) satisfies the following inequality:*

$$\frac{B_0 - AM\omega^2}{g(p,\omega)} > \frac{2\alpha_U A k_F}{\alpha_F k_U \sqrt{\pi}} \tag{37}$$

Remark 4.8. *For the classical Stefan problem with* $\rho = 0$ *the inequality (37) for the coefficient* ω*, which characterizes the free boundary* $s(t)$*, given by (26), become:*

$$erf(\omega) < \frac{B_0}{A}\frac{k_U}{k_F}\sqrt{\frac{d_F}{d_U}}$$

which was obtained in [19].

5. Appendix

PROPOSITION 5.1. *For any $M \in \mathbb{R}$ and $p \in \mathbb{R}$:*

1. *If $p < 2$ then:*
$$\lim_{y\to+\infty}\left(1-\frac{AM}{B}y^2\right)G_1(p,y)=0. \tag{38}$$

2. *If $p \geq 2$ then:*
$$\lim_{y\to+\infty}\left(1-\frac{AM}{B}y^2\right)G_1(p,y)=\begin{cases} -\infty & M>0 \\ +\infty & M<0\end{cases} \tag{39}$$

and
$$\left(1-\frac{AM}{B}y^2\right)G_1(p,y)\simeq\begin{cases} -\dfrac{2AM}{B\sqrt{\pi}}y^2 & p=2 \\ \\ -\dfrac{AM(p-2)}{B}y^3 & p>2\end{cases} \tag{40}$$

when $y \to +\infty$.

Proof. Let M and p real numbers.

1. Proof of (38) is an immediate consecuence of the following facts:
$$\lim_{y\to+\infty}G_1(p,y)=0 \qquad \text{and} \qquad \lim_{y\to+\infty}y^2G_1(p,y)=0 \tag{41}$$

Then, we will prove (41). We split the proof into three cases depending on the sign of the parameter p.

(a) $0 < p \leq 1$
Since $\lim_{y\to+\infty} g(p,y) = +\infty$ (see [8]), we have:
$$\lim_{y\to+\infty}G_1(p,y)=0$$

On the other hand,
$$\lim_{y\to+\infty}\left(y^2G_1(p,y)\right)^{-1}=$$
$$\lim_{y\to+\infty}\left(\frac{K_0}{y^2\exp((p-1)y^2)}+\frac{g(p,y)}{y^2\exp((p-1)y^2)}\right)=+\infty$$

since $\lim_{y\to+\infty}\dfrac{g(p,y)}{y^2}=+\infty$ (see [10]). Then, $\lim_{y\to+\infty} y^2G_1(p,y)=0$.

(b) $1 < p < 2$

First of all, we note that:

$$g(p,y) = \frac{\sqrt{\pi}}{2}\exp\left(\left(\frac{p}{2}y^2\right)^2\right)\left(\operatorname{erf}\left(\frac{p}{2}y\right) + \operatorname{erf}\left(\frac{2-p}{2}y\right)\right).$$

Then:

$$\lim_{y\to+\infty} G_1(p,y) =$$
$$\lim_{y\to+\infty} \frac{\exp\left(-\left(\frac{p}{2}-1\right)^2 y^2\right)}{K_0\exp\left(-\left(\frac{p}{2}y\right)^2\right) + \frac{\sqrt{\pi}}{2}}\left(\operatorname{erf}\left(\frac{p}{2}y\right) + \operatorname{erf}\left(\frac{2-p}{2}y\right)\right) = 0.$$

The proof of $\lim_{y\to+\infty} y^2 G_1(p,y) = 0$ is similar to the previous one.

(c) $p \leq 0$

It is useful to write $G_1(p,y)$ in the following way:

$$G_1(p,y) = \Big[K_0\exp((1-p)y^2) +$$
$$\frac{\sqrt{\pi}}{2}\exp\left(\left(1-\frac{p}{2}\right)^2 y^2\right)\left(\operatorname{erf}\left(\left(1-\frac{p}{2}\right)y\right) - \operatorname{erf}\left(-\frac{p}{2}y\right)\right)\Big]^{-1}.$$

Then, since:

$$\lim_{y\to+\infty}\Big[K_0\exp((1-p)y^2) +$$
$$\frac{\sqrt{\pi}}{2}\exp\left(\left(1-\frac{p}{2}\right)^2 y^2\right)\left(\operatorname{erf}\left(\left(1-\frac{p}{2}\right)y\right) - \operatorname{erf}\left(-\frac{p}{2}y\right)\right)\Big] = +\infty,$$

we have that $\lim_{y\to+\infty} G_1(p,y) = 0$.

The proof of $\lim_{y\to+\infty} y^2 G_1(p,y) = 0$ is similar to the previous one.

2. Now, we have:

$$\lim_{y\to+\infty} y^2 G_1(p,y) = 0. \tag{42}$$

In fact:

$$\lim_{y\to+\infty} G_1(p,y) = \lim_{y\to+\infty} \frac{2(p-1)y\exp((p-1)y^2)}{\frac{\partial}{\partial y}g(p,y)} \geq$$
$$\lim_{y\to+\infty} \frac{2(p-1)y\exp((p-1)y^2)}{\exp((p-1)y^2) + pyg(p,y)}$$

and

$$\lim_{y\to+\infty}\left(\frac{2(p-1)y\exp((p-1)y^2)}{\exp((p-1)y^2)+pyg(p,y)}\right)^{-1}=$$
$$\lim_{y\to+\infty}\frac{1}{2(p-1)y}+\frac{1}{2y(p-1)}\frac{yg(p,y)}{\exp((p-1)y^2)}=0.$$

For the last limit, we are using the fact that $\lim_{y\to+\infty}\frac{yg(p,y)}{\exp((p-1)y^2)}=\frac{1}{p-2}$ (see [10]). Then $\lim_{y\to+\infty}y^2G_1(p,y)=0$.

Finally, (40) follows from similar arguments.

□

The following result is proved in [8].

PROPOSITION 5.2. *We have:*

$$\lim_{y\to+\infty}(1+My^2)G_2(y)=\begin{cases}+\infty & M>0\\ -\infty & M<0\end{cases}\tag{43}$$

and

$$(1+My^2)G_2(y)\simeq\sqrt{\pi}\gamma_0 My^3 \text{ as } y\to+\infty.\tag{44}$$

Furthermore, when $M>0$, $(1+My^2)G_2(y)$ is an increasing function of $y>0$.

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