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Variance squeezing and information entropy squeezing via Bloch coherent states in two-level nonlinear spin models

Horacio Grinberg*,[1]

IFIBA, Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina

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ABSTRACT

The nonclassical squeezing effect emerging from a nonlinear coupling model (generalized Jaynes–Cummings model) of a two-level atom interacting resonantly with a bimodal cavity field via two-photon transitions is investigated in the rotating wave approximation. Various Bloch coherent initial states (rotated states) for the atomic system are assumed, i.e., (i) ground state, (ii) excited state, and (iii) linear superposition of both states. Initially, the atomic system and the field are in a disentangled state, where the field modes are in Glauber coherent states via Poisson distribution. The model is numerically tested against simulations of time evolution of the based Heisenberg uncertainty relation variance and Shannon information entropy squeezing factors. The quantum state purity is computed for the three possible initial states and used as a criterion to get information about the entanglement of the components of the system. Analytical expression of the total density operator matrix elements at $t>0$ shows, in fact, the present nonlinear model to be strongly entangled, where each of the definite initial Bloch coherent states is reduced to statistical mixtures. Thus, the present model does not preserve the modulus of the Bloch vector.

1. Introduction

The Jaynes–Cummings model (JCM) was originally proposed for describing the spontaneous emission in a semi-classical manner [1–4]. This model consists of a single two-level atom (qubit) and a single cavity mode of the electromagnetic field. The JCM interaction between the atom and the cavity mode is obtained by the rotating wave approximation (RWA), so that each photon creation causes an atomic de-excitation and each photon annihilation causes an atomic excitation. The JCM is an analytically tractable quantum mechanical model. Moreover, it is simple enough for expressing the basic and most important characteristics of the matter–radiation interaction.

In the atom–field interaction scenario, where the atom is initially prepared in the ground or excited state and the cavity mode in the coherent state, the level structure of the atom leads to the prediction of a wide range of experimentally verifiable coherent phenomena. Probably the most notable among them is the observation of the periodic spontaneous collapse and the revival of the Rabi oscillations during the time-evolution [5–9], a phenomenon that was experimentally demonstrated in the 1980s [10]. This phenomenon can be regarded as a direct evidence for discreteness of energy states of photons and clearly is a manifestation of the role of quantum mechanics in the coherence and fluctuation properties of radiation–matter systems. Particularly relevant is the atomic squeezing phenomenon. It reflects the nonclassical behavior for the quantum systems and is one of the most interesting phenomena in the field of quantum optics. In fact, the spin squeezing can be considered as a quantum strategy [11] which aims at redistributing the fluctuations of two orthogonal spin directions between each other. It was theoretically shown that spin squeezed states are useful quantum resources to enhance the precision of atom interferometers [12] and the connection between spin squeezing and entanglement was pointed out [13]. Thus, the JCM has a fully quantum property, which cannot be explained by semi-classical physics.

Because of its distinct advantages the extensions of the basic JCM have been extremely plentiful, e.g., the generalized N-level JCM [14,15], JCM beyond RWA [16–19], non-linear JCM [8,9,20,21], and JCM with intensity-dependent coupling [19,22–24].

In the present paper, the so far uninvestigated non-dissipative generalized JCM using Bloch coherent states [25] in the initial

* Permanent address: Department of Physics, Facultad de Ciencias Exactas y Naturales, University of Buenos Aires, Pabellón 1, Ciudad Universitaria, 1428 Buenos Aires, Argentina. Tel.: +54 11 4576 3353; fax: +54 11 4576 3357.

E-mail address: grinberg@df.uba.ar

[1] Under Contract, Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), República Argentina.

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atomic state will be explored through numerical simulations of time evolution of the based Heisenberg uncertainty relation variance (which is regarded as the standard limitation on measurements of quantum fluctuations), and Shannon information entropy squeezing factors.

Spin or angular momentum systems have often been regarded as squeezed if the uncertainty of one spin component, say $\langle \Delta S_x^2 \rangle$ or $\langle \Delta S_y^2 \rangle$, is smaller than $1/2|\langle S_z \rangle|$. This definition implies that a coherent spin state is already squeezed if it is placed in an appropriate system of coordinates, and also that spin can be squeezed by just rotating the coherent spin state. In spin systems, the squeezing occurs on the phase sphere (spherical phase space). Unlike boson squeezing, the quasiprobability distribution cannot be homogeneously or globally squeezed in one direction over the whole phase space. If a spin component is shrunk around a certain point on the sphere, it must be stretched around another point. This imposes a fundamental restriction on the reduction in quantum noise.

As an alternative to the Heisenberg uncertainty relation the quantum uncertainty can also be studied by using quantum entropy theory which can overcome the limitations of the Heisenberg uncertainty relation [26]. In this paper, we find that the entropic uncertainty relation can be used as a general criterion for the squeezing of a two-level atomic system.

The use of Bloch coherent states (rotated states) is a novel feature of the present model since it offers the possibility of considering various initial atomic states. In fact, an appropriate choice of the rotation angle θ leads to different initial states on the Bloch sphere, namely, (a) ground state ($\theta = \pi$); (b) excited state ($\theta = 0$); and (c) a linear superposition of both states ($0 < \theta < \pi$). It will be assumed that two degenerate modes (i.e., $\nu_1 = \nu_2$) of the electromagnetic field and two-photon transitions are involved in the resulting highly non-linear Hamiltonian and that the spin transition frequency ω is nearly resonant with the two (degenerate) modes, i.e., $\Delta_j = \omega - \nu_j = 0$ ($j = 1, 2$), where Δ_j is the detuning parameter for the mode j. The atom and the cavity modes are initially in thermal equilibrium at a certain finite low temperature $\beta (= 1/kT)$. Thus, we can describe an initial probability distribution of quantum states of the system with the canonical ensemble. Moreover, we assume that the time-evolution of the system is governed by a unitary operator generated with the JCM Hamiltonian. This implies that the system does not suffer from dissipation and its time-evolution is reversible.

The remainder of the paper is organized as follows. Section 2 describes the initial conditions and introduces the interaction picture representation of the model, leading to a time evolution operator from which the dynamics of the atomic system–field interaction is examined. Section 3 discusses the numerical simulations. The quantum *purity* is computed for the three possible initial states and used as a criterion to get information about the entanglement of the components of the system. The second-order statistical moments necessary to compute the Heisenberg uncertainty relation variance and the information entropy squeezing factors are computed in the rotated basis for the three above mentioned possible initial atomic states. The time-evolution of the atomic population inversion is also computed. The paper ends up with conclusions in Section 4.

2. Theoretical background

2.1. Bloch sates and initial conditions

Let us consider a bosonic system S, with Hilbert space $\mathcal{H}^{(S)}$ which is coupled with a two-level atom ($S = 1/2$), with Hilbert space $\mathcal{H}^{(B)}$. It is assumed that the complete system is in thermal equilibrium with a reservoir at temperature β^{-1}. It is important to keep in mind that the presence of the reservoir only takes the two-level atom and the bosonic modes in thermal equilibrium. Let us denote by $\mathcal{H}_S$, $\mathcal{H}_B$, and $\mathcal{H}_I$ the Hamiltonians of the bosonic field, the two-level atom, and the interaction between both systems, respectively. The Hamiltonian for the total system can be written as

$$\mathcal{H} = \mathcal{H}_S \otimes I_B + I_S \otimes \mathcal{H}_B + \mathcal{H}_I \equiv \mathcal{H}_0 + \mathcal{H}_I, \qquad (1)$$

where I_S and I_B denote the identities in the Hilbert spaces of the bosonic field and the two-level atom. Thus, under the RWA the Hamiltonian of a two-level atom interacting resonantly with a bimodal cavity via two-photon transitions becomes (a system of units in which $\hbar$=1 is used throughout the paper)

$$\mathcal{H} = \underbrace{\sum_{j=1}^{2} \nu_j a_j^\dagger a_j \otimes I_B + \omega I_S \otimes S_z}_{\mathcal{H}_0} + \underbrace{\sum_{j=1}^{2} G_j (\sigma_+ \otimes a_j^2 + \sigma_- \otimes a_j^{\dagger 2})}_{\mathcal{H}_I}, \qquad (2)$$

where G_j is the atom–field coupling constant (vacuum Rabi frequency) for the mode j and $a_j^\dagger$ (a_j) is the associated canonical creation (annihilation) bosonic operator.

The connection of spins to interferometry becomes obvious when realizing that an interferometer is essentially a two-level system with states $|a\rangle$ and $|b\rangle$. Each particle in the interferometer can be regarded as an elementary spin, $\sigma = 1/2$, corresponding to the two interferometer states. Thus, we start by describing the single-qubit computational basis representing the ground and the excited state of the atom as two-components vectors

$$|a\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |b\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad (3)$$

and therefore the Pauli atom-flip operators in Eq. (2) are given by

$$\sigma_+ = |b\rangle\langle a| \quad \sigma_- = |a\rangle\langle b|. \qquad (4)$$

The Hilbert space of the atomic operators is spanned by the Dicke states, which are simply the usual spin angular momentum states $|SM\rangle$ ($M = -S, -S+1$) obtained as the simultaneous eigenstates of the $SU(2)$ Casimir operators S^2 and $S_z = 1/2[S_+, S_-]$, where $S_\pm$ are the raising and lowering spin operators, i.e., $S_\pm = 1/2\sigma_\pm$. The Dicke states are then given by

$$|SM\rangle = \frac{1}{(M+S)!} \begin{pmatrix} 2S \\ M+S \end{pmatrix}^{-1/2} S_+^{M+S} |S-S\rangle, \qquad (5)$$

with eigenvalue M and where the ground state $|S-S\rangle$ is defined by $S_-|S-S\rangle$=0. Let us consider the rotation operator $R_{\theta,\phi}$ which produces a rotation through the coherence angle θ about an axis $\hat{n} = (\sin\phi, -\cos\phi, 0)$

$$R_{\theta,\phi} = e^{-i\theta S_n} = e^{-i\theta(S_x \sin\phi - S_y \cos\phi)} = e^{\vartheta S_+ - \vartheta^* S_-}, \qquad (6)$$

where

$$\vartheta = \frac{1}{2}\theta e^{-i\phi} \quad (0 \le \theta \le \pi; 0 \le \phi \le 2\pi). \qquad (7)$$

A coherent atomic state, or Bloch state, $|\theta, \phi\rangle$ is obtained by rotation of the ground state $|S-S\rangle$, i.e.,

$$|\theta, \phi\rangle \equiv R_{\theta,\phi}|S-S\rangle, \qquad (8)$$

which is the group definition of the atomic coherent states. The Bloch spin coherent states $|\theta, \phi\rangle$ satisfy a completeness relation given by

$$(2S+1)\int |\theta, \phi\rangle \frac{d\Omega}{4\pi} \langle \theta, \phi| = 1, \qquad (9)$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$ is the solid-angle volume element at (θ, ϕ) on $\mathcal{S}^2$ (Bloch sphere). The Bloch sphere is a well-known tool in quantum optics, where the simple qubit state is faithfully represented, up to an overall phase factor, by a point on a standard sphere with radius unity, whose coordinates are expectation values of the atomic set operators of the system. Thus, the position (θ, ϕ) on the spin-1/2 Bloch sphere describes the full quantum state. Remarkably, in this geometrical description a rotation of a spin-S state corresponds to a rigid-body rotation of the corresponding points on the sphere.

Using the disentangling theorem for angular momentum operators, the rotation $R_{\theta,\phi}$ given by Eq. (5) becomes

$$R_{\theta,\phi} = e^{-\tau^* S_-}\, e^{-\ln(1+|\tau|^2)S_z}\, e^{\tau S_+} = e^{\tau S_+}\, e^{\ln(1+|\tau|^2)S_z}\, e^{-\tau^* S_-}, \tag{10}$$

where

$$\tau = e^{-i\phi} \tan \frac{1}{2}\theta, \tag{11}$$

defines the stereographic projection from the south pole of the sphere to the plane passing through the equator, with complex coordinates τ, τ^*.

The last form of Eq. (10) which we call the normally ordered form, immediately gives the expansion of $|\theta, \phi\rangle$ in terms of Dicke states

$$|\tau\rangle \equiv |\theta, \phi\rangle = R_{\theta,\phi}|S - S\rangle = \left(\frac{1}{1+|\tau|^2}\right)^S e^{\tau S_+}|S - S\rangle, \tag{12}$$

whence, expanding the exponential and using Eq. (5)

$$\begin{aligned}\langle SM|\theta, \phi\rangle &= \begin{pmatrix} 2S \\ M+S \end{pmatrix}^{1/2} \frac{\tau^{M+S}}{[1+|\tau|^2]^S} \\ &= \begin{pmatrix} 2S \\ M+S \end{pmatrix}^{1/2} \sin^{S+M}\left(\frac{1}{2}\theta\right) \cos^{S-M}\left(\frac{1}{2}\theta\right) e^{-i(S+M)\phi}.\end{aligned} \tag{13}$$

The probability distribution over the Dicke states is binomial. Thus, the θ dependent part of the coefficients $\langle SM|\theta, \phi\rangle$ follows a binomial distribution peaked around θ.

It should be pointed out that the phase ϕ in Eq. (13) appears in two connected contexts. On the one hand, for coherent spin states, it is a variable characterizing the full quantum state of the system (together with θ). On the other hand, it has the meaning of a relative phase between the two states $|a\rangle$ and $|b\rangle$ of the two-mode system. In both cases, the phase ϕ can be interpreted as the mean angle of the spin state in the longitudinal direction (i.e., in the S_x, S_y or S_χ, S_ξ planes on the Bloch sphere, where S_χ, S_ξ are rotated operators).

In the present model the initial state vector involving two modes of the field is represented by

$$|Z\tau\rangle \equiv |Z\rangle \otimes |\tau\rangle = |z_1 z_2\rangle \otimes |\theta, \phi\rangle \tag{14}$$

i.e., the direct product of a field canonical coherent state $|Z\rangle$ and a pseudospin coherent state $|\tau\rangle \equiv |\theta, \phi\rangle$. Thus, while $|Z\rangle$ is defined in the "particle" Hilbert space, $|\tau\rangle$ is defined in the $(2S+1)$-dimensional space. The coherent states (Glauber states) $|z_j\rangle$ are given by $(j=1, 2)$

$$|z_j\rangle = \exp\left(z_j a_j^\dagger - \frac{1}{2}|z_j|^2\right)|0\rangle, \tag{15}$$

where the z_j are complex numbers and $|0\rangle$ is the bosonic oscillator vacuum state for the j-mode. These coherent states are non-orthogonal

$$\langle z_1|z_2\rangle = \exp\left(-\frac{|z_1|^2}{2} + z_1^* z_2 - \frac{|z_2|^2}{2}\right), \tag{16}$$

and satisfy the closure relation

$$\int |z_j\rangle \frac{d^2 z_j}{\pi} \langle z_j| = 1_j, \tag{17}$$

with 1_j the unit operator for the j-mode and the overlap $\langle n_1 n_2|z_1 z_2\rangle$ given by a Poisson distribution

$$\langle n_1 n_2|z_1 z_2\rangle = \langle n_1|z_1\rangle\langle n_2|z_2\rangle, \tag{18}$$

with

$$|\langle n_j|z_j\rangle|^2 = \exp(-|z_j|^2)\frac{|z_j|^{2n_j}}{n_j!}. \tag{19}$$

The dynamics of the present model is not stationary and depends on the initial conditions of the system and the cavity field. Thus, it is assumed that, initially, the field modes are in coherent states $|z_1 z_2\rangle$ and the atomic system is a superposition of states $|SM\rangle$, i.e., the atomic system and the field are initially in a disentangled state with density operator

$$\rho(0) = \rho^S(0) \otimes \rho^B(0) = |\psi(0)\rangle\,\langle\psi(0)|, \tag{20}$$

where $\rho^S(0)$ and $\rho^B(0)$ are density operators at $t=0$ of the field and the atomic system respectively, and

$$|\psi(0)\rangle = \sum_{M=-1/2}^{1/2} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} C_M(0) C_{m_1 m_2}(0)|SM\rangle \otimes |m_1 m_2\rangle, \tag{21}$$

with coefficients $C_M(0) \equiv \langle SM|\theta, \phi\rangle$, $C_{n_1 n_2}(0) \equiv \langle n_1 n_2|z_1 z_2\rangle$. It is further assumed that at $t=0$, the two modes have the same photon distribution, i.e., the field density operator is written as

$$\rho^S(0) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} C_{n_1}(0) C_{n_2}(0) C^*_{m_1}(0) C^*_{m_2}(0) \mathcal{P}^{n_1 m_1}_{n_2 m_2}, \tag{22}$$

where $\mathcal{P}^{n_1 m_1}_{n_2 m_2} = |m_1 m_2\rangle\,\langle n_1 n_2|$ is the two-mode Fock space projection operator. Thus, we write

$$C_{m_1 m_2}(0) = [\rho^S_{m_1 m_1}(0)\rho^S_{m_2 m_2}(0)]^{1/2}, \tag{23}$$

while the density operator of the two-level atomic system is given by

$$\rho^B(0) = \sum_{M=-1/2}^{1/2} \sum_{M'=-1/2}^{1/2} C_M(0) C^*_{M'}(0) \mathcal{P}^{SM}_{SM'}, \tag{24}$$

where $\mathcal{P}^{SM}_{SM'} = |SM\rangle\,\rangle SM'|$ is the projection operator in the spin angular momentum space.

It is well known that the quantum coherences which are built up during the interaction process significantly affect the dynamics of the atomic system. Thus, in order to investigate the nonclassical behavior of the present model we introduce in the next subsection the interaction picture representation of this generalized JCM.

2.2. Interaction picture representation of the generalized JCM

Assuming exact simultaneous resonance of both modes of the cavity field with the spin transition frequency, in the interaction picture and under the RWA, the coupling between the qubit and the reservoir in a two-photon transition involving two (degenerate) modes of the electromagnetic field is represented by $\mathcal{H}_I$, the second term on the RHS of Eq. (2).

A simplified model is achieved if we assume that the atomic system and the bosonic modes are in the interior of a high-Q lossless cavity. Introducing the unitary time evolution operator

$$\mathcal{U}_I(t) = \exp(-i\mathcal{H}_I t), \tag{25}$$

along with the Hermitian operators $A(a, a^\dagger)=A^\dagger(a, a^\dagger)$, $A(a^\dagger, a)=A^\dagger(a^\dagger, a)$ (but $A(a, a^\dagger) \neq A(a^\dagger, a)$)

$$A(a, a^\dagger) = \sum_{i,j=1}^{2} G_i G_j a_i^2 a_j^{\dagger 2}, \qquad (26)$$

allows the even and odd powers of $\mathcal{H}_I$ to be written as

$$\mathcal{H}_I^{2l} = \sqrt{A(a, a^\dagger)}^{2l} |b\rangle\langle b| + \sqrt{A(a^\dagger, a)}^{2l} |a\rangle\langle a|, \qquad (27)$$

and

$$\mathcal{H}_I^{2l+1} = \frac{\sqrt{A(a, a^\dagger)}^{2l+1}}{\sqrt{A(a, a^\dagger)}} \sum_{j=1}^{2} G_j a_j^2 |b\rangle\langle a| + \frac{\sqrt{A(a^\dagger, a)}^{2l+1}}{\sqrt{A(a^\dagger, a)}} \sum_{j=1}^{2} G_j a_j^{\dagger 2} |a\rangle\langle b|, \qquad (28)$$

respectively, and where nilpotency and idempotency properties of the operators $\sigma_\pm$ and $\sigma_\pm\sigma_\mp$ (i.e., $\sigma_\pm^k = 0$, $k \geq 2$; $(\sigma_\pm\sigma_\mp)^k = \sigma_\pm\sigma_\mp$, $k \geq 1$) have been used. Thus, upon expanding Eq. (25) the unitary time evolution operator is found to be given by the 2×2 matrix

$$\mathcal{U}_I(t) = \cos(\sqrt{A(a, a^\dagger)}t)|b\rangle\langle b| + \cos(\sqrt{A(a^\dagger, a)}t)|a\rangle\langle a| - i\left[\frac{\sin(\sqrt{A(a, a^\dagger)}t)}{\sqrt{A(a, a^\dagger)}} \sum_{j=1}^{2} G_j a_j^2 |b\rangle\langle a| + \frac{\sin(\sqrt{A(a^\dagger, a)}t)}{\sqrt{A(a^\dagger, a)}} \sum_{j=1}^{2} G_j a_j^{\dagger 2} |a\rangle\langle b|\right]. \qquad (29)$$

The entangled interaction picture state vector at any time t emerges from the coherent state $|\psi(0)\rangle$ in Eq. (21) via the unitary time-evolution operator $\mathcal{U}_I(t)$, i.e.,

$$|\psi(t)\rangle = \sum_{M=-1/2}^{1/2} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} C_M(0) C_{m_1 m_2}(0) \mathcal{U}_I(t)|SM\rangle \otimes |m_1 m_2\rangle. \qquad (30)$$

Up to this point there are no approximations involved in the obtention of the probability amplitudes emerging from this state. In fact, the evolution operator in Eq. (29) is exact to all orders. However, to explicitly evaluate matrix elements of this operator, it is convenient to linearize the expansions of $A(a, a^\dagger)^l$ and $A(a^\dagger, a)^l$ implicit in the computation of the probability amplitudes $\langle a; n_1 n_2|\psi(t)\rangle$ and $\langle b; n_1 n_2|\psi(t)\rangle$, according to

$$A(a, a^\dagger)^l \to \sum_{i=1}^{2} [G_i^2 a_i^2 a_i^{\dagger 2}]^l + \text{highly nonlinear cross terms}. \qquad (31)$$

The higher-order cross terms in this equation involve powers of the bosonic operators a_j^p, $a_k^{\dagger q}$ which can be evaluated through

$$a_j^p |m_j\rangle = \sqrt{\frac{m_j!}{(m_j - p)!}} |m_j - p\rangle, \qquad (32)$$

and

$$a_k^{\dagger q} |m_k\rangle = \sqrt{\frac{(m_k + q)!}{m_k!}} |m_k + q\rangle, \qquad (33)$$

with $j, k = 1, 2$ and $p \leq m_j$. In resonant states, such as those considered in the present work, this is a reasonable approximation [9]. Here, therefore, we neglect these higher-order cross terms, which amounts to keeping in Eq. (26) only the diagonal terms, i.e., those terms with $i = j$. In this way, the double sum over m_1 and m_2 in Eq. (30) is restricted only to those terms involving the photon number operators $m_i = a_i^\dagger a_i$. Thus, invoking these approximations in Eq. (29) and projecting Eq. (30) successively over the ground $\langle a|$ and excited states $\langle b|$, the amplitudes are found to be given by

$$\langle a; n_1 n_2|\psi(t)\rangle = \cos\left(\frac{\theta}{2}\right) \sqrt{\rho_{n_1 n_1}^S(0)\rho_{n_2 n_2}^S(0)} \cos(\chi_n t) - i \sin\left(\frac{\theta}{2}\right) e^{-i\phi} \frac{\sin(\chi_n t)}{\chi_n} \sum_{j \neq k=1}^{2} G_j \sqrt{Q_{n_j} \rho_{n_{j-2} n_{j-2}}^S(0) \rho_{n_k n_k}^S(0)}, \qquad (34)$$

and

$$\langle b; n_1 n_2|\psi(t)\rangle = \sin\left(\frac{\theta}{2}\right) \cos\phi \cos(\chi_{n+2} t) \sqrt{\rho_{n_1 n_1}^S(0)\rho_{n_2 n_2}^S(0)} - i\left[\cos\left(\frac{\theta}{2}\right) \frac{\sin(\chi_{n+2} t)}{\chi_{n+2}} \sum_{j \neq k=1}^{2} G_j \sqrt{Q_{n_j+2} \rho_{n_j+2 n_j+2}^S(0) \rho_{n_k n_k}^S(0)} + \sin\left(\frac{\theta}{2}\right) \sin\phi \cos(\chi_{n+2} t) \sqrt{\rho_{n_1 n_1}^S(0)\rho_{n_2 n_2}^S(0)}\right], \qquad (35)$$

where Q_{n_j} and χ_n are given by

$$Q_{n_j} = n_j(n_j - 1), \qquad (36)$$

and

$$\chi_n = \sqrt{\sum_{j=1}^{2} G_j^2 Q_{n_j}}. \qquad (37)$$

Thus, the reduced density operator matrix elements in the two-dimensional atomic space are simply obtained as

$$\rho_{n_1 n_2}^{aa}(t) = |\langle a; n_1 n_2|\psi(t)\rangle|^2, \qquad (38)$$

$$\rho_{n_1 n_2}^{bb}(t) = |\langle b; n_1 n_2|\psi(t)\rangle|^2, \qquad (39)$$

$$\rho_{n_1 n_2}^{ab}(t) = \langle a; n_1 n_2|\psi(t)\rangle\langle\psi(t)|n_1 n_2; b\rangle, \qquad (40)$$

and the full density operator reads as

$$\rho(t) = \begin{pmatrix} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \rho_{n_1 n_2}^{aa}(t) & \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \rho_{n_1 n_2}^{ab}(t) \\ \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \rho_{n_1 n_2}^{ba}(t) & \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \rho_{n_1 n_2}^{bb}(t). \end{pmatrix}. \qquad (41)$$

Various nonclassical effects in the JCM can be generated by choosing different initial states of the spin system and when the cavity field is initially in a coherent state of photons, one finds that the level occupation probability of the system can display collapse and revivals of the Rabi oscillations in a field that is not in a pure number state. Thus, in the next section nonclassical effects such as the based Heisenberg uncertainty relation variance and Shannon information entropy squeezing factors are discussed through numerical simulations of the corresponding time evolution of the appropriate observables.

3. Results and discussion

The dynamics of the present model will be examined in the special case when both of the field modes are prepared in coherent

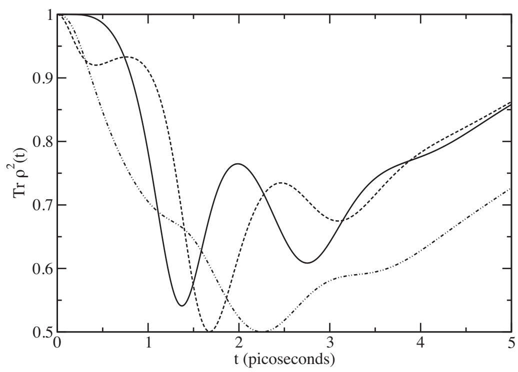

Fig. 1. Time evolution of purity measured by $Tr\rho^2(t)$. The initial coherent state of the field is given by a Poisson distribution. $\langle n_1\rangle = \langle n_2\rangle$=20. G_1 = 1.6 cm^{-1}; G_2 = 3.0 cm^{-1}. $\phi = \pi/2$. (—): $\theta = 0$; (- - -): $\theta = \pi$; (-.-.-): $\theta = \pi/2$.

states. Thus, in the initial state, i.e., at $t = 0$, the corresponding density matrix elements of the field are given by a Poisson distribution,

$$\rho^S_{n_i n_i}(0) = \frac{\langle n_i\rangle^{n_i} e^{-\langle n_i\rangle}}{n_i!}, \tag{42}$$

where $\langle n_i\rangle$ is the initial average number of photons in the *i*th mode. The quantum state *purity* may be used as a good tool to give information about the entanglement of the components of the system (spin plus field). In order to describe the most general spin state, i.e., pure states and statistical mixtures the density matrix formalism is used. Thus, the purity of the field can be determined from the quantity $Tr\,\rho(t)^2$. A necessary and sufficient condition for the ensemble to be described in terms of a pure state is that $Tr\,\rho(t)^2 = 1$, in this case clearly a state vector description of each individual system of the ensemble is possible. These pure qubit states lie on the periphery of the Bloch ball (*Poincaré–Bloch sphere*). For the case $Tr\,\rho(t)^2 < 1$, the field will be in a statistical mixture state. However, for a two-level spin system, a maximally mixed ensemble corresponds to $Tr\,\rho(t)^2 = 1/2$. This maximally mixed state lies at the center of the ball. Thus, it follows from Eq. (41) that

$$1/2 \leq Tr\,\rho^2(t) = 1 - 2Det\rho(t) \leq 1, \tag{43}$$

with

$$Det\rho(t) = [Tr^S \rho^{aa}_{n_1 n_2}(t)][Tr^S \rho^{bb}_{n_1 n_2}(t)] - |Tr^S \rho^{ab}_{n_1 n_2}(t)|^2, \tag{44}$$

and where the sums over the photon quantum numbers have been substituted by the corresponding partial traces. It is firstly observed that the initial state of the two-level atomic system can be characterized in terms of the rotation angle θ according to whether it is equal to 0 (excited state), π (ground state), or any other different value such that $0 < \theta < \pi$ (linear superposition of both states). It then follows from Eq. (13) that $C_{-1/2}(0) = \langle SM = -1/2|\theta, \phi\rangle = \cos(\theta/2)$ and $C_{1/2}(0) = \langle SM = 1/2|\theta, \phi\rangle = \sin(\theta/2)\, e^{-i\phi}$. Thus, the coherent atomic state $|\theta, \phi\rangle$ in Eq. (12), extracted from Eq. (21), can be expressed as

$$|\theta,\phi\rangle = \sum_{M=-1/2}^{1/2} C_M(0)|SM\rangle = \cos\left(\frac{\theta}{2}\right)\left|\frac{1}{2} - \frac{1}{2}\right\rangle + \sin\left(\frac{\theta}{2}\right) e^{-i\phi}\left|\frac{1}{2}\,\frac{1}{2}\right\rangle. \tag{45}$$

It is to be noted that the expression of $Tr\rho^2(t)$ in Eq. (41) is representation-independent since the trace is invariant under unitary transformations of the model. In Fig. 1, $Tr\,\rho^2(t)$ is plotted against t in the picosecond time scale, assuming that the field is prepared in a coherent state and the spin system is in the three possible different initial states, (i) ground state ($\theta = \pi$); (ii) excited state ($\theta = 0$); (iii) linear superposition of both states ($\theta = \pi/2$). This figure displays the time evolution of the purity for a bimodal cavity field interacting with an effective two-level spin system in resonant states. It is observed that the purity in general satisfies the inequality $1/2 \leq Tr\,\rho^2(t) \leq 1$. The system approaches the pure state showing weak entanglement over the whole time scale considered. In the meantime, the maximum value of the purity becomes nearly 0.99. This means that the interaction between the field and the spin system is almost disentangled (at those times where the maximum value of $Tr\,\rho^2(t) = 0.99$). It is observed that when the initial state is the excited state of the spin system ($\theta = 0$) the purity never reaches the maximally mixed ensemble. At times longer than 1 ps the behavior of the purity is strongly affected. In this case, the extrema (maxima and minima) of the purity function for $\theta = \pi$ and $\theta = \pi/2$ are less than those corresponding to the excited state $\theta = 0$. For these three different initial coherent states the field plus spin system become in a mixture state, but without reaching its maximal, and therefore leads to a large enough entanglement for times longer than 1 ps. Also, it can be observed that the maximum value of the purity function occurs at the onset of the interaction, and remains well above 0.9 for $t < 1$ ps when the spin system is in a definite initial state (ground or excited state). When the system is initially prepared either in the ground state ($\theta = \pi$) or in a linear superposition of both initial and excited states ($\theta = \pi/2$) the purity reaches the maximally mixed ensemble and never returns to a pure state, remaining well below of the level of 75% of purity in the latter case and above of 85% when the system is initially prepared in the ground state. These mixed qubit states lie in the interior of the Poincaré–Bloch sphere and are weighted convex combinations of pure states. It would be interesting to point out that although the maximum value of $Tr\,\rho^2(t)$ does not reach the value *one* (pure state showing disentanglement), it is greater for the ground or excited states than for the linear superposition of both states. However, in all the three cases the interaction between the spin system and the field remains maximally correlated or entangled and, as a result of this interaction, the whole system never returns to the pure state. Thus, it is concluded that the off-diagonal matrix elements of the density operator would strongly affect the interaction between the field mode and spin system. The pattern in Fig. 1 is reminiscent of damping processes which destroy coherence and reduce each of the three possible initial states to statistical mixtures. Since statistical mixtures have $Tr\,\rho(t)^2 < 1$, Fig. 1 shows that the present model does not preserve the modulus of the Bloch vector. This entanglement is a sort of accessible entanglement, i.e., the maximum value of the entanglement that could be extracted from the system and placed in quantum registers, from which it could be used to perform quantum information processing.

The Bloch states form minimum-uncertainty packets. The uncertainty relation can be defined in terms of the set of rotated operators

$$\begin{pmatrix} S_\chi \\ S_\xi \\ S_\zeta \end{pmatrix} = R_{\theta,\phi}\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} R^{-1}_{\theta,\phi}. \tag{46}$$

These three observables obey a commutation relation of the type $[A, B] = iC$ with $A = S_\chi$, $B = S_\xi$, $C = S_\zeta$, whence they have the uncertainty property $\Delta S_\chi \Delta S_\xi \geq 1/2|\langle S_\zeta\rangle|$ for any states. It is easy to show that the equality sign holds for the Bloch state $|\theta, \phi\rangle$, which is therefore a minimum-uncertainty state.

Fluctuations in the component S_α ($\alpha = \chi$, ξ or x, y) of the atomic dipole are said to be squeezed if S_α satisfies the condition

$$V(S_\alpha) = \Delta S_\alpha - \left(\frac{|\langle S_\zeta\rangle|}{2}\right)^{1/2} < 0 \tag{47}$$

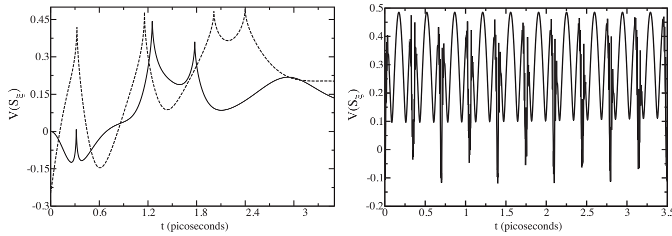

Fig. 2. Time evolution of the based Heisenberg uncertainty relation variance squeezing factor $V(S_\xi)$, with the initial coherent state of the field given by a Poisson distribution. (a) left: $\langle n_1\rangle=\langle n_2\rangle$=20; G_1 = 1.6 cm^{-1}; G_2 = 3.0 cm^{-1}; $\phi=\pi/2$; (—): $\theta=0$; (- - -): $\theta=\pi/3$. (b) right: $\langle n_1\rangle$=0; $\langle n_2\rangle$=20; G_1 = 160 cm^{-1}; G_2 = 300 cm^{-1}; $\phi=\pi/2$; $\theta=0$.

where $\Delta S_\alpha = [\langle S_\alpha^2\rangle - \langle S_\alpha\rangle^2]^{1/2}$ is the variance in the S_α direction. Spin squeezed states feature reduced noise in one of the spin directions but excess noise in another direction can be present either due to a non-Heisenberg-limited quantum state or due to an incoherent mixture of several quantum states. The former might limit precision in standard interferometry, but specific correlated quantum states enable even enhanced interferometric precision in a generalized interferometer [29]. The latter easily limits interferometric precision at a level above the standard quantum limit and experimentally requires a large effort to prevent decoherence due to technical noise from the environment or due to finite temperature effects in the system. Large noise-quantum or classical-even in a spin direction that is not directly measured has a degrading effect on interferometric precision in a standard interferometer which arises due to the curved surface of the Bloch sphere.

The time evolution of the variance squeezing factor $V(S_\xi)$ is shown in Fig. 2 for two different sets of parameters. It can be observed in Fig. 2(a) that $V(S_\xi)$ predicts a certain amount of squeezing in the variable S_ξ for transient times less than 0.8 ps when the initial state is prepared in a superposition of both ground and excited states. At longer times no variance squeezing has been observed at all. When the initial state is prepared in the excited state the period during which the spin squeezing is observed is appreciably longer as compared to that in which the initial state of the atomic system is prepared in a linear superposition of states. When one of the field modes is initially prepared in a vacuum state $\langle n_1\rangle$=0 and the coupling constants G_1 and G_2 are increased by a factor of one hundred, the variance squeezing takes place at rather regular intervals of time, as Fig. 2(b) shows. These intervals occurs in regions where the corresponding atomic inversion $W(t)$ [24]

$$W(t) = \langle S_z\rangle \cos\theta - \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} Re[e^{-i\phi}\rho_{n_1n_2}^{ab}(t)] \sin\theta, \tag{48}$$

displayed in Fig. 3, evolves to values very close to the revivals.

As an alternative to the Heisenberg uncertainty relation, several authors have studied quantum uncertainty by using quantum entropy theory, and obtained an entropic uncertainty relation for position and momentum which can overcome the limitations of the Heisenberg uncertainty relation [26–28]. Moreover, the entropic uncertainty relation can e generalized to the case of two Hermitian operators in a K-dimensional Hilbert space and an optimal entropic uncertainty relation for sets of $K+1$ complementary observables with nondegenerate eigenvalues in an even K-dimensional Hilbert space has been investigated [30,31]. By defining

$$\delta H(S_\alpha) \equiv \exp[H(S_\alpha)], \tag{49}$$

where $H(S_\alpha)$ is the Shannon information entropy, and fluctuations in the component S_α of the system are said to be "squeezed in

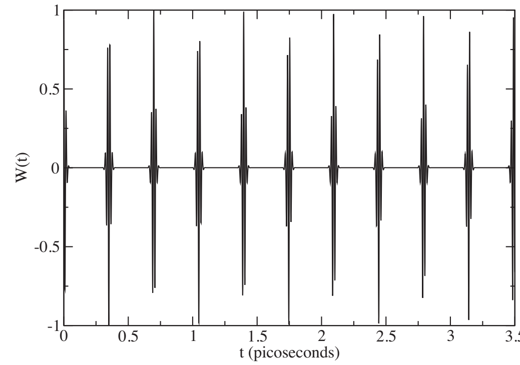

Fig. 3. Time evolution of the atomic population inversion $W(t)$, with the initial coherent state of the field given by a Poisson distribution and initial average photon numbers $\langle n_1\rangle$=0, $\langle n_2\rangle$=20. The atom is initially in the excited state $\theta=0$. G_1 = 160 cm^{-1}; G_2 = 300 cm^{-1}; $\phi=\pi/2$.

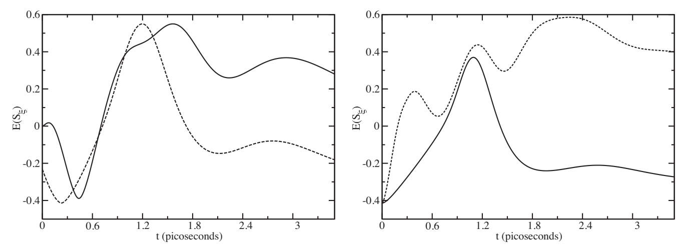

Fig. 4. Time evolution of the based Shannon information entropy squeezing factor $E(S_\xi)$, with the initial coherent state of the field given by a Poisson distribution. G_1 = 1.6 cm^{-1}; G_2 = 3.0 cm^{-1}; $\phi=\pi/2$. (a) left: $\langle n_1\rangle$=0; $\langle n_2\rangle$=20; (—): $\theta=0$; (- - -): $\theta=\pi/3$. (b) right: $\theta=\pi/2$; (—): $\langle n_1\rangle$=0; $\langle n_2\rangle$=20; (- - -): $\langle n_1\rangle=\langle n_2\rangle$=20.

entropy" if the information entropy $H(S_\alpha)$ of S_α satisfies the condition [26–28]

$$E(S_\alpha) = \delta H(S_\alpha) - \frac{2}{[\delta H(S_z)]^{1/2}} < 0, \tag{50}$$

The time evolution of the entropy squeezing factor $E(S_\xi)$ is displayed in Fig. 4. It is observed in Fig. 4(a) that for times less than 0.8 ps $E(S_\xi)$ predicts squeezing when the atom is initially either in the excited state ($\theta=0$) or in a linear superposition of states ($\theta=\pi/3$) with one of the field modes in the vacuum state. For times longer than 1.8 ps the latter initial state shows again a moderate amount of squeezing, but not such squeezing is observed when the system is prepared initially in the excited state. Fig. 4(b) shows a comparison of $E(S_\xi)$ when the initial state of the system is prepared in a linear superposition of states ($\theta=\pi/2$) and one of the field modes assuming two values, i.e., $\langle n_1\rangle$=0 (solid line) and $\langle n_1\rangle$=20 (dashed line). It is observed that when $\langle n_1\rangle$=0 (vacuum state) the squeezing is more pronounced and it reaches around 20% in the long time regime. No such behavior is observed when the initial state of the field is prepared in the excited state, far from the vacuum state. In fact, in this case the squeezing is only observed at transient times less than 0.4 ps. This different behavior may be due to the decoherence resulting from the energy exchange between the atom and the field modes in the long time regime during the interaction process.

4. Final remarks

A nonlinear generalized JCM involving a degenerate bimodal cavity and two photon transitions is presented. The validity of the RWA and a resonant interaction of the spin system with the electromagnetic field have been assumed. Initially the atomic system and the electromagnetic field are in a disentangled state, where the field modes are represented by Glauber coherent states via

Poisson distribution and the two-level system has been assumed to be prepared in three different Bloch coherent states (rotated states), according to the different values of the rotation angle θ. In all the three cases the interaction between the spin system and the field remains maximally entangled and, as a result of this interaction, the whole system never returns to the pure state. A detailed analysis of the purity function reveals that the present model does not preserve the modulus of the Bloch vector. In fact, analytical expression of the total density operator matrix elements at $t > 0$ shows this nonlinear model to be strongly entangled, where each of the chosen initial Bloch coherent states is reduced to statistical mixtures. These mixed qubit states lie in the interior of the Poincaré–Bloch sphere and are weighted convex combinations of pure states. The nonclassical Heisenberg uncertainty relation variance and Shannon information entropy squeezing factors have been discussed through numerical simulations of the corresponding time evolution of the appropriate observables. It was observed that the variance squeezing factor predicts a certain amount of squeezing in the variable S_ξ at rather small transient times when the initial state is prepared in a superposition of the ground and excited states. When the initial state of the atomic system is prepared in the excited state the period of the spin squeezing is appreciably longer as compared to that in which the initial state is prepared in a linear superposition of states. This effect is more pronounced when one of the field modes is initially prepared in the vacuum state and the coupling constants are increased by a factor of one hundred. The variance squeezing takes place in this case at rather regular intervals of time. Particularly interesting is the finding that these intervals occur in regions where the corresponding atomic inversion evolves to values very close to the revivals. On the other hand the time evolution of the Shannon information entropy squeezing factor $E(S_\xi)$ predicts squeezing when the atom is initially prepared either in the excited state or in a linear superposition of states with one of the field modes in the vacuum state. A comparison of $E(S_\xi)$ for different initial field states shows that when the initial state of the system is prepared in a linear superposition of states and one of the field modes is in the vacuum state the squeezing is more pronounced than that observed when the initial state of the field is prepared in excited states, far from the vacuum state. In this case the squeezing is only observed at very small transient times. This different behavior may be due to the decoherence resulting from the energy exchange between the atom and the field modes in the long time regime during the interaction process. Finally, it should be stressed that nonlinear Hamiltonians can provide some clues in the search for new squeezed atomic states which are based on phase-sensitive devices (e.g., degenerate-parametric amplifiers) and that can be used to construct interferometers of specific characteristics.

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