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Nonlinear Chebyshev approximation to set-valued functions

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ABSTRACT
In this paper, we give a characterization of best Chebyshev approximation to set-valued functions from a family of continuous functions with the weak betweeness property. As a consequence, we obtain a characterization of Kolmogorov type for best simultaneous approximation to an infinity set of functions. We introduce the concept of a set-sun and give a characterization of it. In addition, we prove a property of Amir–Ziegler type for a family of real functions and we get a characterization of best simultaneous approximation to two functions

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1. Introduction

Let X be a compact Hausdorff space, and let $C(X)$ be the space of continuous real functions defined on X, with the Chebyshev norm

$$\|h\| = \sup_{x \in X} |h(x)|, \quad h \in C(X).$$

Let $\mathcal{G} \subset C(X)$. We say that $\mathcal{G}$ has the *weak betweeness property* if for all $g, g_0 \in \mathcal{G}$ and any closed subset $D \subset X$ satisfying $\min_{x \in D} |g(x) - g_0(x)| > 0$, there exists $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{G}$ such that

(a) $\|g_n - g_0\| \to 0$, as $n \to \infty$;
(b) $(g(x) - g_n(x))(g_n(x) - g_0(x)) > 0$ for all $x \in D$ and $n \in \mathbb{N}$.

Families with the weak betweeness property (see [1]) are also referred to as having the *closed sign property* (see [2]) or *regular* (see [3]). The best known examples are linear families, convex families and admissible rational functions. Other families with this property include those satisfying: Haar condition, weak Haar condition,[4] betweeness property,[5] representation condition,[6] or those which are asymptotically convex,[7] Kolmogorov set of the second kind,[8] unisolvent [9,10] and sun.[3,11] The relationship between these properties and other examples can be found in [3].

We consider the Hausdorff space

$$\mathcal{H}(\mathbb{R}) := \{K \subset \mathbb{R} : K \neq \emptyset \text{ and } K \text{ is compact}\},$$

with the Hausdorff metric d_H (see [12]). Let $F : X \to \mathcal{H}(\mathbb{R})$ be a set valued function. For $g \in C(X)$, $x \in X$, and $y \in F(x)$ we write

$$\widehat{E}_F(g, x, y) = y - g(x), \quad E_F(g, x) = \sup_{y \in F(x)} \left|\widehat{E}_F(g, x, y)\right| \quad \text{and} \quad e_F(g) = \sup_{x \in X} E_F(g, x).$$

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We say that $g_0 \in \mathcal{G}$ is *a best approximation to F from $\mathcal{G}$* if

$$\inf_{g\in\mathcal{G}} e_F(g) = e_F(g_0). \tag{1}$$

If for all continuous set-valued function there exists a best approximation, we say that $\mathcal{G}$ is an *existence set*.

Observe that our definition extends the usual definition of a best Chebyshev approximation to a function $f \in C(X)$, when $F(x) = \{f(x)\}$, for all $x \in X$. More generally, if $\mathcal{D} \subset C(X)$ is a compact set, we can consider the function $F_D : X \to \mathcal{H}(\mathbb{R})$ defined by

$$F_D(x) = \{h(x) : h \in \mathcal{D}\}. \tag{2}$$

It is easy to see that

$$e_{F_D}(g) = \sup_{x\in X}\sup_{h\in\mathcal{D}} |h(x) - g(x)| = \sup_{h\in\mathcal{D}}\sup_{x\in X} |h(x) - g(x)| = \sup_{h\in\mathcal{D}} \|h - g\|.$$

So, in this case (1) means that g_0 is a best simultaneous Chebyshev approximation to $\mathcal{D}$ from $\mathcal{G}$.

Let $\mathcal{G} \subset C(X)$ be an existence set.

We say that $\mathcal{G}$ is *a set-sun* if for each continuous set-valued function $F : X \to \mathcal{H}(\mathbb{R})$, $g_0 \in \mathcal{G}$ is a best approximation to F from $\mathcal{G}$ implies that

$$e_F(g_0) \le e_F((1-\alpha)g_0 + \alpha g), \quad \text{for all} \quad g \in \mathcal{G},\ 0 < \alpha < 1. \tag{3}$$

Each best approximant with this property is said to be *a solar point of $\mathcal{G}$*.

Characterization of nonlinear best approximation has been studied extensively in the literature. In ([5], Theorem 1), Dunham proved a characterization of best approximation by families with the betweeness property to a function (see Theorem 2.1). A characterization of a best simultaneous approximation of Kolmogorov type when $\mathcal{G}$ has the weak betweeness property and the function F is as in (2) for a finite set, D, was established in ([1], Theorem 4.1).

The notion of suns has played important roles in nonlinear approximation theory. In ([13], Theorem 1), a characterization of a sun for simultaneous approximation to a numerable set of functions is given. Results about characterization of best simultaneous approximation to a bounded set from suns in different Banach spaces and their relationships with a Kolmogorov-type condition, were considered in [14,15]. Another results can be seen in [16].

The purpose of this paper was to show that the Dunham's method, given in ([5], Theorem 1), can be employed in more general cases, i.e. for approximation of continuous set-valued functions from families with the weak betweeness property. As a consequence we get a characterization of Kolmogorov type for approximating to continuous set-valued functions on any compact subset. Also, show a characterization of set-suns in $C(X)$. In particular, our results can be applied to simultaneous approximation to a set of continuous functions under certain conditions. In addition, we prove a property of Amir–Ziegler type for a suitable set of functions, possibly infinite and we get a characterization of best simultaneous approximation to two functions.

2. Characterization of best approximation

Let $g \in C(X)$ and $F : X \to \mathcal{H}(\mathbb{R})$ be a set valued function such that F is continuous. Then $Y_F := \{(x, y) : x \in X \text{ and } y \in F(x)\}$ is a compact subset of $X \times \mathbb{R}$. In fact, let $\{(x_\alpha, y_\alpha)\}_{\alpha\in D}$ be a net of Y_F. As X is a compact set, $X \times F(X)$ is a compact set. So, there are a subnet, which is denoted in the same way, and $(x, y) \in X \times F(X)$ such that (x_α, y_α) converges to (x, y). Since

$$\min_{z \in F(x)} |y - z| \leq d_H(F(x_\alpha), F(x)) + |y - y_\alpha|, \quad \alpha \in D,$$

$F(x)$ is a closed set, and $F(x_\alpha)$ and y_α converges to $F(x)$ and y, respectively, we have $y \in F(x)$. Therefore $(x, y) \in Y_F$ and so Y_F is a compact subset.

In addition, a straightforward computation shows that

$$E_F(g, x) - E_F(g, x') \leq d_H(F(x), F(x')) + |g(x) - g(x')|, \quad x, x' \in X.$$

Therefore, $E_F(g, \cdot)$ and $\widehat{E}_F(g, \cdot, \cdot)$ are continuous functions on X and Y_F, respectively. So, the sets

$$M_F(g) := \{x \in X : e_F(g) = E_F(g, x)\} \text{ and}$$
$$\widehat{M}_F(g) := \left\{(x, y) \in Y_F : y \in F(x) \text{ and } e_F(g) = \left|\widehat{E}_F(g, x, y)\right|\right\},$$

are non empty for all $g \in \mathcal{G}$. Moreover, $M_F(g)$ and $\widehat{M}_F(g)$ are compact subsets of X and Y_F, respectively.

We observe that if $p_1 : Y_F \to X$ is the canonical projection, then

$$p_1\left(\widehat{M}_F(g)\right) = M_F(g). \tag{4}$$

In ([5], Theorem 1), Dunham proved the following interesting result of characterization.

Theorem 2.1: *Let $\mathcal{G} \subset C(X)$ be a family with the betweeness property and let $f \in C(X)$. An element $g_0 \in \mathcal{G}$ is a best approximation to f from $\mathcal{G}$ if and only if there exists no element $g \in \mathcal{G}$ such that $E_f(g, x) < e_f(g_0)$ for all $x \in M_f(g_0)$.*

The next result extends Theorem 2.1 to best Chebyshev approximation of a set-valued function.

Theorem 2.2: *Let $\mathcal{G} \subset C(X)$ be a family with the weak betweeness property and let $F : X \to \mathcal{H}(\mathbb{R})$ be a continuous set-valued function. The following statements are equivalent:*

(a) *$g_0 \in \mathcal{G}$ is a best approximation to F from $\mathcal{G}$;*
(b) *there is no element $g \in \mathcal{G}$ such that $\left|\widehat{E}_F(g, x, y)\right| < e_F(g_0)$ for all $(x, y) \in \widehat{M}_F(g_0)$;*
(c) *there is no element $g \in \mathcal{G}$ such that $E_F(g, x) < e_F(g_0)$ for all $x \in M_F(g_0)$.*

Proof: *(a)* $\Rightarrow$ *(b)*. Suppose that there is $g \in \mathcal{G}$ with $\widehat{E}_F(g, x, y) < e_F(g_0)$ for all $(x, y) \in \widehat{M}_F(g_0)$. Since g, g_0, and F are continuous, there exists an open set $U_1 \subset Y_F$ such that

(i) $\widehat{M}_F(g_0) \subset U_1$;
(ii) $\left|\widehat{E}_F(g, x, y)\right| < e_F(g_0)$ for all $(x, y) \in U_1$;
(iii) $\left|g_0(x) - g(x)\right| \geq \alpha > 0$ for all $x \in p_1(U_1)$ and for some $\alpha > 0$.

In fact, if $r = \max_{(x,y) \in \widehat{M}_F(g_0)} \left|\widehat{E}_F(g, x, y)\right|$ and $r < s < s' < e_F(g_0)$, we consider the open set

$$U_1 = \left|\widehat{E}_F(g, ., .)\right|^{-1}((-\infty, s)) \cap \left|\widehat{E}_F(g_0, ., .)\right|^{-1}((s', +\infty)) \subset Y_F.$$

Then, clearly U_1 satisfies (i) and (ii). In addition, if $x \in p_1(U_1)$, there exists $y \in \mathbb{R}$ such that $(x, y) \in U_1$, therefore

$$\left|g_0(x) - g(x)\right| \geq \left|\widehat{E}_F(g_0, x, y)\right| - \left|\widehat{E}_F(g, x, y)\right| > s' - s =: \alpha > 0,$$

and so (iii) is true.

Now, $\widehat{M}_F(g_0)$ and $Y_F \setminus U_1$ are disjoint closed subsets of Y_F. Since Y_F is normal topological space, then there exists an open set $U_2 \subset Y_F$ such that $\widehat{M}_F(g_0) \subset U_2 \subset \overline{U_2} \subset U_1$, where $\overline{U_2}$ denotes the closure of U_2. As $p_1\left(\overline{U_2}\right)$ is a compact set, the continuity of g and g_0, and (iii) implies that $\min_{x\in p_1\left(\overline{U_2}\right)} \left|g_0(x) - g(x)\right| > 0$. Since $\mathcal{G}$ has the weak betweeness property there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{G}$ such that $\|g_n - g_0\| \to 0$ as $n \to \infty$, and

$$\min_{x\in p_1\left(\overline{U_2}\right)} (g(x) - g_n(x))(g_n(x) - g_0(x)) > 0 \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

We consider the compact set $W = Y_F \setminus U_2$. If $W = \emptyset$, then $Y_F = U_1$, and (ii) yields $\left|\widehat{E}_F(g, x, y)\right| < e(g_0)$ for all $x \in X$, $y \in F(x)$. Hence, g_0 is not a best approximation to F from $\mathcal{G}$. If $W \neq \emptyset$, let

$$\beta = e_F(g_0) - \max_{(x,y)\in W} \left|\widehat{E}_F(g_0, x, y)\right|.$$

Since W and $\widehat{M}_F(g_0)$ are disjoint sets, then $\beta > 0$. Let n_0 be such that $\|g_0 - g_{n_0}\| < \beta$. If $(x, y) \in W$, we obtain

$$\begin{aligned} \left|\widehat{E}_F(g_{n_0}, x, y)\right| &= |y - g_{n_0}(x)| \leq |g_0(x) - g_{n_0}(x)| + \left|\widehat{E}_F(g_0, x, y)\right| \\ &< \beta + e_F(g_0) - \beta = e_F(g_0). \end{aligned} \tag{6}$$

On the other hand, if $(x, y) \notin W$, then $(x, y) \in \overline{U_2}$. From (5) and (ii), we have

$$\begin{aligned} \left|\widehat{E}_F(g_{n_0}, x, y)\right| &= |y - g_{n_0}(x)| < \max\{|y - g_0(x)|, |y - g(x)|\} \\ &= \max\left\{\left|\widehat{E}_F(g_0, x, y)\right|, \left|\widehat{E}_F(g, x, y)\right|\right\} \leq e_F(g_0). \end{aligned} \tag{7}$$

From (6) and (7) we get $\left|\widehat{E}_F(g_{n_0}, x, y)\right| < e_F(g_0)$ for all $x \in X$, $y \in F(x)$. So, g_0 is not a best approximation to F from $\mathcal{G}$.
Finally, $(b) \Rightarrow (c)$ is obvious and $(c) \Rightarrow (a)$ immediately follows from definition of a best approximation of F. ☐

Remark 2.3: We note that for any equicontinuous family of functions, $\mathcal{D} \subset C(X)$, the function $F(x) = \{f(x) : f \in \mathcal{D}\}$ is continuous, so we can apply Theorem 2.2. The results established in Theorema 2.2 are unknown even in the case $\mathcal{D}$ finite.

Next, we obtain a characterization of Kolmogorov type for best Chebyshev approximation to a continuous set-valued function.

To prove the next theorem, we use the following property of real numbers.

$$|a - b| < |a - c| \quad \text{implies} \quad (a - c)(b - c) > 0. \tag{8}$$

Theorem 2.4: *Let $\mathcal{G} \subset C(X)$ be a family with the weak betweeness property and let $F : X \to \mathcal{H}(\mathbb{R})$ be a continuous set-valued function. Then $g_0 \in \mathcal{G}$ is a best approximation to F from $\mathcal{G}$ if and only if for all $g \in \mathcal{G}$,*

$$\min_{(x,y)\in \widehat{M}_F(g_0)} \widehat{E}_F(g_0, x, y)(g(x) - g_0(x)) \leq 0. \tag{9}$$

Proof: Let $g \in \mathcal{G}$ and suppose that

$$\widehat{E}_F(g_0, x, y)(g(x) - g_0(x)) > 0 \quad \text{for all } (x, y) \in \widehat{M}_F(g_0). \tag{10}$$

From (4) and (10), we have $\min_{x\in M_F(g_0)} |g(x) - g_0(x)| > 0$. Since $\mathcal{G}$ has the weak betweeness property, there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subset \mathcal{G}$ such that $\|g_n - g_0\| \to 0$ as $n \to \infty$, and

$$\min_{x\in M_F(g_0)} (g(x) - g_n(x))(g_n(x) - g_0(x)) > 0 \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

We choose $n_0 \in \mathbb{N}$ such that $\|g_0 - g_{n_0}\| < \frac{e_F(g_0)}{2}$.

We claim that

$$\left|\widehat{E}_F(g_{n_0}, x, y)\right| < e(g_0) \quad \text{for all } (x, y) \in \widehat{M}_F(g_0). \tag{12}$$

In fact, let $(x, y) \in \widehat{M}_F(g_0)$, by (4), $x \in M_F(g_0)$. If $g(x) > g_0(x)$, from (10) we have $g_0(x) < y$. Further, (11) implies that $g_0(x) < g_{n_0}(x) < g(x)$. For $y \geq g_{n_0}(x)$, we have $\left|\widehat{E}_F(g_{n_0}, x, y)\right| = |y - g_{n_0}(x)| < |y - g_0(x)| \leq e_F(g_0)$. Otherwise, the condition $\|g_0 - g_{n_0}\| < \frac{e_F(g_0)}{2}$ implies that $\left|\widehat{E}_F(g_{n_0}, x, y)\right| \leq |g_0(x) - g_{n_0}(x)| < e_F(g_0)$, so (12) holds. The same conclusion can be obtained for $g(x) < g_0(x)$. Now, (12) contradicts Theorem 2.2.
Reciprocally, we suppose that $g \in \mathcal{G}$ and (9) is true. Then there exists $(x, y) \in \widehat{M}_F(g_0)$ such that $(y - g_0(x))(g(x) - g_0(x)) \leq 0$. From (8) we get $e_F(g_0) = |y - g_0(x)| \leq |y - g(x)| \leq e_F(g)$. As g is arbitrary, g_0 is a best approximation to F from $\mathcal{G}$. □

The previous theorem extends Kolmogorov's characterization theorem of best Chebyshev approximation proved in ([5], p.153). In fact, it is sufficient to take $\mathcal{D}$ an unitary set.

3. Characterization of set-suns in $C(X)$

Let $\mathcal{G} \subset C(X)$ and let $F : X \to \mathcal{H}(\mathbb{R})$ be a set-valued function. We claim that

$$(1 - \alpha)e_F(h) + \alpha e_F(g) \leq e_F((1 - \alpha)h + \alpha g) \quad \text{for all} \quad h, g \in \mathcal{G}, \ \alpha \geq 1. \tag{13}$$

Indeed, for $x \in X$ and $y \in F(x)$ we have

$$\begin{aligned}(1 - \alpha)e_F(h) + \alpha|y - g(x)| &= -|1 - \alpha|e_F(h) + |\alpha||y - g(x)| \\ &\leq -|1 - \alpha||y - h(x)| + |\alpha||y - g(x)| \\ &\leq \left|y - \big((1 - \alpha)h(x) + \alpha g(x)\big)\right|.\end{aligned}$$

Therefore, $(1 - \alpha)e_F(h) + \alpha e_F(g) \leq e_F((1 - \alpha)h + \alpha g)$.

Hence, if $g_0 \in \mathcal{G}$ is a best approximation to F from $\mathcal{G}$, we get

$$e_F(g_0) \leq e_F((1 - \alpha)g_0 + \alpha g) \quad \text{for all} \quad g \in \mathcal{G}, \ \alpha \geq 1. \tag{14}$$

The definition of a set-sun is an extension of the notion given in ([17], p.31) to the case of approximation of a set-valued function on $C(X)$, as it is proved in the next lemma.

Lemma 3.1: *Let $\mathcal{G}$ be an existence set in $C(X)$ and $g_0 \in \mathcal{G}$. Then g_0 is a solar point of $\mathcal{G}$ if and only if for each continuous set-valued function $F : X \to \mathcal{H}(\mathbb{R})$, g_0 is a best approximation to F from $\mathcal{G}$ implies that g_0 is a best approximation to $F_\alpha := g_0 + \alpha(F - g_0)$ from $\mathcal{G}$ for all $\alpha > 0$.*

Proof: The proof follows immediately from (14) and the equality

$$\alpha\, e_{F_{\frac{1}{\alpha}}}(h) = e_F((1 - \alpha)g_0 + \alpha h) \quad \text{for all} \quad h \in C(X). \tag{15}$$

□

Definition 3.2: Let $\mathcal{G} \subset C(X)$ be an existence set, $F : X \to \mathcal{H}(\mathbb{R})$ a continuous set-valued function, and $g_0 \in \mathcal{G}$. We say that g_0 is a local best approximation to F from $\mathcal{G}$, if there exists $r > 0$ such that g_0 is a best approximation to F from $\mathcal{G} \cap B(g_0, r)$, where $B(g_0, r) = \{g \in C(X) : \|g - g_0\| < r\}$.

The following result generalizes ([17], Theorem 2.6).

Theorem 3.3: *Let $\mathcal{G} \subset C(X)$ be an existence set. The following statements are equivalent:*

(a) *$\mathcal{G}$ is a set-sun;*
(b) *Each local best approximation from $\mathcal{G}$ to any F is a best approximation to F from $\mathcal{G}$;*
(c) *$\mathcal{G}$ is a family with the weak betweeness property.*

Proof: $(a) \Rightarrow (b)$. Let $F : X \to \mathcal{H}(\mathbb{R})$ be a continuous set-valued function. Assume that g_0 is a local best approximation from $\mathcal{G}$ to F from a set-sun $\mathcal{G}$. Then there exists $r > 0$ such that g_0 is a local best approximation to F from $\mathcal{G} \cap B(g_0, r)$, i.e.

$$e_F(g_0) \leq e_F(g) \quad \text{for all} \quad g \in \mathcal{G}, \quad \text{such that} \quad \|g - g_0\| < r. \tag{16}$$

If $e_F(g_0) > 0$, let $0 < \alpha < \min\left\{\frac{r}{3e_F(g_0)}, 1\right\}$. For $g \in \mathcal{G}$, $x \in X$ and $y \in F(x)$, we observe

$$\begin{aligned} |g_0(x) - g(x)| - \frac{r}{3} &\leq |g_0(x) - g(x)| - \frac{r}{3e_F(g_0)}|y - g_0(x)| \leq |g_0(x) - g(x)| - \alpha|y - g_0(x)| \\ &\leq \left|g_0(x) + \alpha(y - g_0(x)) - g(x)\right| \\ &= \alpha\left|y - \left(\left(1 - \frac{1}{\alpha}\right)g_0(x) + \frac{1}{\alpha}g(x)\right)\right|, \end{aligned}$$

and consequently,

$$\|g_0 - g\| - \frac{r}{3} \leq \alpha e_F\left(\left(1 - \frac{1}{\alpha}\right)g_0 + \frac{1}{\alpha}g\right).$$

Therefore, for $\|g_0 - g\| \geq \frac{2r}{3}$ we have

$$e_F(g_0) \leq e_F\left(\left(1 - \frac{1}{\alpha}\right)g_0 + \frac{1}{\alpha}g\right). \tag{17}$$

On the other hand, for $\|g_0 - g\| < \frac{2r}{3}$, it follows from (13) and (16) that

$$\begin{aligned} e_F(g_0) &= \left(1 - \frac{1}{\alpha}\right)e_F(g_0) + \frac{1}{\alpha}e_F(g_0) \leq \left(1 - \frac{1}{\alpha}\right)e_F(g_0) + \frac{1}{\alpha}e_F(g) \\ &\leq e_F\left(\left(1 - \frac{1}{\alpha}\right)g_0 + \frac{1}{\alpha}g\right). \end{aligned} \tag{18}$$

According to (15), (17) and (18), we have

$$e_{F_\alpha}(g_0) \leq e_{F_\alpha}(g) \quad \text{for all} \quad g \in \mathcal{G}.$$

If $e_F(g_0) = 0$, the last inequality also holds. So, g_0 is a best approximation to F_α from $\mathcal{G}$. Since $\mathcal{G}$ is a set-sun, g_0 is a solar point of $\mathcal{G}$. Lemma 3.1 shows that g_0 is a best approximation to $\left(F_\alpha\right)_{\frac{1}{\alpha}}$ from $\mathcal{G}$. As $\left(F_\alpha\right)_{\frac{1}{\alpha}} = F$, the proof is complete.

$(b) \Rightarrow (c)$. The proof of this implication is the same as in ([17], Theorem 2.6).

$(c) \Rightarrow (a)$. Assume that $\mathcal{G}$ has the weak betweeness property. Let $G : X \to \mathcal{H}(\mathbb{R})$ be a continuous set valued function and let g_0 be a best approximation to G from $\mathcal{G}$. Suppose that there exist $g \in \mathcal{G}$ and $0 < \alpha < 1$ such that $e_G((1-\alpha)g_0 + \alpha g) < e_G(g_0)$. Set $F = g_0 + \frac{1}{\alpha}(G - g_0)$. Replacing in (15), F, $F_{\frac{1}{\alpha}}$ and h by G, F and g, respectively, we obtain

$$\alpha e_F(h) = e_G((1-\alpha)g_0 + \alpha h), \quad h \in C(X). \tag{19}$$

Hence, $e_F(g) < e_F(g_0)$ and so $\delta := \frac{1}{2}(e_F(g_0) - e_F(g)) > 0$. Put

$$A = \{(x,y) \in Y_F : |y - g_0(x)| \geq e_F(g_0) - \delta\}.$$

For $(x,y) \in A$, we have

$$|y - g_0(x)| \geq e_F(g_0) - \delta = \frac{e_F(g_0) + e_F(g)}{2} > e_F(g) \geq |y - g(x)|,$$

and from (8) we get

$$(y - g_0(x))(g(x) - g_0(x)) > 0. \tag{20}$$

Obviously, A is a compact set and therefore $p_1(A)$ is a compact set. In addition, by (20) it follows that $\min_{x \in p_1(A)} |g(x) - g_0(x)| > 0$. So, by the weak betweeness property there is a $v \in \mathcal{G}$ such that

$$\|v - g_0\| < \delta\,\alpha, \quad \text{and} \quad (g(x) - v(x))(v(x) - g_0(x)) > 0 \quad \text{for all} \quad x \in p_1(A). \tag{21}$$

Let $(x,y) \in A$. If $y - g_0(x) > 0$, from (20) we get $g(x) - g_0(x) > 0$ and by (21), $g_0(x) < v(x) < g(x)$. If $y - g_0(x) < 0$, by a similar argument we have $g(x) < v(x) < g_0(x)$. So, we deduce that

$$(y - g_0(x))(v(x) - g_0(x)) > 0 \quad \text{for all} \quad (x,y) \in A.$$

Therefore, for $(x,y) \in A$ we have $sgn((y - g_0(x)) = sgn(v(x) - g_0(x))$ and

$$\begin{aligned}\left|y - \left(\left(1 - \frac{1}{\alpha}\right)g_0(x) + \frac{1}{\alpha}v(x)\right)\right| &= \frac{1}{\alpha}\left|\alpha(y - g_0(x)) - (v(x) - g_0(x))\right| \\ &= \frac{1}{\alpha}\left|\alpha|y - g_0(x)| - |v(x) - g_0(x)|\right| \\ &< |y - g_0(x)| \leq e_F(g_0).\end{aligned}$$

On the other hand, for $(x,y) \in Y_F \setminus A$ it follows from the definition of A that

$$\begin{aligned}\left|y - \left(\left(1 - \frac{1}{\alpha}\right)g_0(x) + \frac{1}{\alpha}v(x)\right)\right| &= \frac{1}{\alpha}\left|\alpha(y - g_0(x)) - (v(x) - g_0(x))\right| \\ &\leq |y - g_0(x)| + \frac{1}{\alpha}\|v - g_0\| \\ &< \big(e_F(g_0) - \delta\big) + \delta = e_F(g_0).\end{aligned}$$

Consequently, $e_F\left(\left(1 - \frac{1}{\alpha}\right)g_0 + \frac{1}{\alpha}v\right) < e_F(g_0)$. Finally, from (19) we conclude that $e_G(v) < e_G(g_0)$, a contradiction. Therefore

$$e_G(g_0) \leq e_G((1-\alpha)g_0 + \alpha g), \quad \text{for all} \quad g \in \mathcal{G},\ 0 < \alpha < 1, \tag{22}$$

and so $\mathcal{G}$ is a set-sun. □

4. Best simultaneous Chebyshev approximation

A necessary condition for best simultaneous Chebyshev approximation is established by the next theorem.

Theorem 4.1: *Let $\mathcal{G} \subset C(X)$ be a family with the weak betweeness property and let $\mathcal{D} \subset C(X)$. Let $g_0 \in \mathcal{G}$ be a best simultaneous Chebyshev approximation to $\mathcal{D}$ from $\mathcal{G}$. If there exists $f \in \mathcal{D}$ satisfying*

(a) $\mathcal{D} \setminus \{f\}$ *is a compact set,*
(b) $\|h - g_0\| < \|f - g_0\|$ *for all* $h \in D \setminus \{f\}$,
(c) $F := F_D$ *is continuous,*

then g_0 is a best approximation to f from $\mathcal{G}$.

Proof: We claim that

$$M_F(g_0) = \{x \in X : |f(x) - g_0(x)| = \|f - g_0\|\}. \tag{23}$$

Indeed, if $x \in M_F(g_0)$, from (b) we get $\sup_{h \in \mathcal{D}} |h(x) - g_0(x)| = \|f - g_0\|$. By (*a*), $\mathcal{D}$ is a compact set, thus $|g(x) - g_0(x)| = \|f - g_0\|$ for some $g \in \mathcal{D}$. According to (b) we have $g = f$, and so $|f(x) - g_0(x)| = \|f - g_0\|$. On the other hand, if $|f(x) - g_0(x)| = \|f - g_0\|$ and $h \in D \setminus \{f\}$, then (b) implies that $|h(x) - g_0(x)| \le \|h - g_0\| < \|f - g_0\| = |f(x) - g_0(x)|$. Hence, $\sup_{h \in \mathcal{D}} |h(x) - g_0(x)| = \sup_{h \in \mathcal{D}} \|h - g_0\|$ and consequently, $x \in M_F(g_0)$.

Suppose that g_0 is not a best approximation to f from $\mathcal{G}$. As F is a continuous function, Theorem 2.2 implies that there is $g \in \mathcal{G}$ with

$$|f(x) - g(x)| < \|f - g_0\| \quad \text{for all} \quad x \in M_F(g_0). \tag{24}$$

If $x \in M_F(g_0)$ and $g_0(x) = g(x)$, from (23) we get $\|f - g_0\| = |f(x) - g_0(x)| = |f(x) - g(x)|$, which contradicts (24). Therefore for $x \in M_F(g_0)$ we have $g_0(x) \neq g(x)$.

Since $M_F(g_0)$ is a closed set and

$$\alpha = \min_{x \in M_F(g_0)} |g(x) - g_0(x)| > 0, \tag{25}$$

the weak betweeness property implies that there exists a sequence $\{g_n\}_{n \in \mathbb{N}} \subset \mathcal{G}$ such that $\|g_n - g_0\| \to 0$, as $n \to \infty$, and

$$(g_n(x) - g_0(x))(g(x) - g_n(x)) > 0 \quad \text{for all } x \in M_F(g_0), \;\; n \in \mathbb{N}.$$

As

$$\left| \sup_{h \in D \setminus \{f\}} \|h - g_n\| - \sup_{h \in D \setminus \{f\}} \|h - g_0\| \right| \le \|g_n - g_0\|,$$

from (*a*) and (*b*) we can choose $n_0 \in \mathbb{N}$ such that

$$\|g_{n_0} - g_0\| < \frac{\alpha}{2}, \quad \text{and} \quad \|h - g_{n_0}\| < \|f - g_0\| = e_F(g_0) \quad \text{for all} \quad h \in D \setminus \{f\}. \tag{26}$$

Let $x \in M_F(g_0)$. We claim that

$$g_0(x) < g_{n_0}(x) < \frac{g_0(x) + g(x)}{2} < f(x) \quad \text{if} \quad g_0(x) < g_{n_0}(x) < g(x).$$

In fact, from (25)–(26) we get $g_{n_0}(x) - g_0(x) < \frac{g(x) - g_0(x)}{2}$ and so $g_{n_0}(x) < \frac{g_0(x) + g(x)}{2}$. According to (24) we have $g_0(x) < f(x)$. If $g(x) < f(x)$, clearly $\frac{g_0(x) + g(x)}{2} < f(x)$. Otherwise, $g_0(x) < f(x) \le g(x)$, so (24) implies that $g(x) - f(x) < f(x) - g_0(x)$ and thus $\frac{g_0(x) + g(x)}{2} < f(x)$.

A similar argument shows that

$$f(x) < \frac{g_0(x) + g(x)}{2} < g_{n_0}(x) < g_0(x) \quad \text{if} \quad g(x) < g_{n_0}(x) < g_0(x).$$

Hence,

$$|f(x) - g_{n_0}(x)| < |f(x) - g_0(x)| = e_F(g_0) \quad \text{for all } x \in M_F(g_0). \tag{27}$$

Now, (26) and (27) imply that

$$E_F(g_{n_0}, x) < e_F(g_0) \quad \text{for all} \quad x \in M_F(g_0).$$

In consequence, according to Theorem 2.2 we have that $g_0 \in \mathcal{G}$ is not a best simultaneous approximation to $\mathcal{D}$ from $\mathcal{G}$, a contradiction. □

Remark 4.2: Let $\mathcal{D} \subset C(X)$ and $f \in \mathcal{D}$. The conditions (a) and (b) of Theorem 4.1 are equivalent to the conditions: $\inf_{h \in \mathcal{D}\setminus\{f\}} (\|f - g_0\| - \|h - g_0\|) > 0$ and D a compact set.

The following result is an immediate consequence of Theorem 4.1. It gives a necessary condition for best simultaneous Chebyshev approximation from a family with the weak betweeness property similar to those discovery by Amir and Ziegler for convex sets and two functions (see [18]). It is unknown for approximation from no convex sets and for a set of functions with cardinality greater than two.

Theorem 4.3: *Let $\mathcal{G} \subset C(X)$ be a family with the weak betweeness property and let $\mathcal{D} \subset C(X)$ be such that $\mathcal{D} \setminus \{h\}$ is a compact set for all $h \in \mathcal{D}$ and F_D is continuous. Let $g_0 \in \mathcal{G}$ be a best simultaneous Chebyshev approximation to $\mathcal{D}$ from $\mathcal{G}$. Then either*

(a) *there exist $f, h \in \mathcal{D}$ such that $\|f - g_0\| = \|h - g_0\|$, or*
(b) *there exists $f \in \mathcal{D}$ such that $\|h - g_0\| < \|f - g_0\|$ for all $h \in D \setminus \{f\}$ and g_0 is the best approximation to f from $\mathcal{G}$.*

Given $f, g \in C(X)$, we will denote by $\gamma^+(f, g)$ the one-sided Gateaux derivative of the norm at f in the direction g, i.e.

$$\gamma_+(f, g) = \max\{sgn(f(x))g(x) : x \in X \text{ and } |f(x)| = \|f\|\}. \tag{28}$$

The next result extends ([19], Theorem 5) for Chebyshev approximation and a class $\mathcal{G}$ more general than a closed subspace.

Theorem 4.4: *Let $\mathcal{G} \subset C(X)$ be a family with the weak betweeness property, and $f_1, f_2 \in C(X)$. Consider the following conditions:*

(a) $\gamma_+(f_1 - g_0, g_0 - g) \geq 0$ *or* $\gamma_+(f_2 - g_0, g_0 - g) \geq 0$, *for every* $g \in \mathcal{G}$;
(b) $\|f_1 - g_0\| = \|f_2 - g_0\|$;
(c) $\|f_2 - g_0\| < \|f_1 - g_0\|$ *and g_0 is the best approximation to f_1 from $\mathcal{G}$;*
(d) $\|f_1 - g_0\| < \|f_2 - g_0\|$ *and g_0 is the best approximation to f_2 from $\mathcal{G}$.*

Then $g_0 \in \mathcal{G}$ is a best simultaneous Chebyshev approximation to $\{f_1, f_2\}$ from $\mathcal{G}$ if and only if (a) and exactly one of (b), (c) or (d) hold.

Proof: Assume that $g_0 \in \mathcal{G}$ is a best simultaneous Chebyshev approximation to $\{f_1, f_2\}$ from $\mathcal{G}$ and let $M_j := \{(x, f_j(x)) : |f_j(x) - g_0(x)| = \|f_j - g_0\|\}$, $j = 1, 2$. It follows easily that

$$\widehat{M}_F(g_0) = \begin{cases} M_1 \cup M_2 & \text{if } \|f_1 - g_0\| = \|f_2 - g_0\| \\ M_1 & \text{if } \|f_2 - g_0\| < \|f_1 - g_0\| \\ M_2 & \text{if } \|f_1 - g_0\| < \|f_2 - g_0\| \end{cases}. \tag{29}$$

If $g \in \mathcal{G}$, by Theorem 2.4 we have

$$\max_{(x,y)\in\widehat{M}_F(g_0)} \widehat{E}_F(g_0,x,y)(g_0(x)-g(x)) \geq 0. \tag{30}$$

Since $\widehat{E}_F(g_0,x,y) = e_F(g_0)\mathrm{sgn}(y-g_0(x))$ for all $(x,y) \in \widehat{M}_F(g_0)$, from (28) and (29) we have $\max_{j=1,2}\{\gamma_+(f_j-g_0,g_0-g)\} \geq 0$. Therefore (a) holds.

On the other hand, according to Theorem 4.3, exactly one of (b), (c) or (d) hold.

Reciprocally, we assume that (a) and (b) (or (c)) holds and let $g \in \mathcal{G}$. If $\gamma_+(f_1-g_0,g_0-g) \geq 0$, from Proposition 1.4 in [20], we have $\|f_1-g\| - \|f_1-g_0\| \geq \gamma_+(f_1-g_0,g_0-g) \geq 0$. Hence,

$$\max\{\|f_1-g_0\|,\|f_2-g_0\|\} = \|f_1-g_0\| \leq \|f_1-g\| \leq \max\{\|f_1-g\|,\|f_2-g\|\}.$$

Otherwise, $\gamma_+(f_2-g_0,g_0-g) \geq 0$ and similarly we obtain $\|f_2-g_0\| \leq \|f_2-g\|$. Now, (b) or (c) implies

$$\max\{\|f_1-g_0\|,\|f_2-g_0\|\} = \|f_1-g_0\| \leq \max\{\|f_1-g\|,\|f_2-g\|\}.$$

As $g \in \mathcal{G}$ is arbitrary, $g_0 \in \mathcal{G}$ is a best simultaneous Chebyshev approximation to $\{f_1,f_2\}$ from $\mathcal{G}$. The same reasoning applies to the case (a) and (d). □

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