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# BEST SIMULTANEOUS APPROXIMATION ON SMALL REGIONS BY RATIONAL FUNCTIONS

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Abstract. We study the behavior of best simultaneous $(l^q, L^p)$-approximation by rational functions on an interval, when the measure tends to zero. In addition, we consider the case of polynomial approximation on a finite union of intervals. We also get an interpolation result.

## 1. Introduction

Let $x_j \in \mathbb{R}, 1 \leq j \leq k, k \in \mathbb{N}$, and let $B_j$ be pairwise disjoint closed intervals centered at $x_j$ and radius $\beta > 0$. Let $n, m \in \mathbb{N} \cup \{0\}$ and we suppose that

$$n + m + 1 = kc + d, \quad c, d \in \mathbb{N} \cup \{0\}, \quad d < k.$$

We denote $\mathcal{C}^s(I)$, $s \in \mathbb{N} \cup \{0\}$, the space of real functions defined on $I := \cup_{j=1}^{k} B_j$, which are continuously differentiable up to order $s$ on $I$. For simplicity we write $\mathcal{C}(I)$ instead of $\mathcal{C}^0(I)$. We also denote $\mathrm{co}(I)$ the convex hull of $I$. Let $\Pi^n$ be the class of algebraic polynomials of degree at most $n$, and $\partial P$ the degree of $P \in \Pi^n$. We consider the set of rational functions

$$\mathcal{R}_m^n := \left\{ \frac{P}{Q} : P \in \Pi^n,\ Q \in \Pi^m,\ Q \neq 0 \right\}.$$

Clearly, we can assume $\frac{P}{Q} \in \mathcal{R}_m^n$ with $L^2$-norm of $Q$ equal to one on $I$. Recall that $\frac{P}{Q} \in \mathcal{R}_m^n$ is called *normal* if this expression is irreducible and either $\partial P = n$ or $\partial Q = m$, and the null function is called *normal* if $m = 0$ (see [10]).

If $h \in \mathcal{C}(I)$, we put

$$\|h\| := \left( \int_I |h(t)|^p \frac{dt}{|I|} \right)^{1/p}, \quad 1 \leq p < \infty,$$

where $|I|$ is the Lebesgue measure of $I$. If $p = \infty$, as it is usual, $\|\cdot\|$ will be the supreme norm. For each $0 < \epsilon \leq 1$, we also put $\|h\|_\epsilon = \|h^\epsilon\|$, where $h^\epsilon(t) = h(\epsilon(t - x_j) + x_j)$, $t \in B_j$.

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If $\chi_{B_j}$ is the characteristic function of the set $B_j$, we write $\|h\|_{B_j} = \|h\chi_{B_j}\|$. We denote $I_\epsilon = \cup_{j=1}^k [x_j - \epsilon\beta, x_j + \epsilon\beta]$.

Let $f_1, \ldots, f_l \in \mathcal{C}(I)$ and $1 \leq q < \infty$. The rational function $u_\epsilon \in \mathcal{R}_m^n$, $0 < \epsilon \leq 1$, is called a *best simultaneous* $(l^q, L^p)$*-approximation* ($(l^q, L^p)$*-b.s.a.*) *of* $f_1, \ldots, f_l$ *from* $\mathcal{R}_m^n$ *on* $I_\epsilon$ if

$$\left(\sum_{i=1}^{l} \|f_i - u_\epsilon\|_\epsilon^q\right)^{1/q} = \inf_{u\in\mathcal{R}_m^n} \left(\sum_{i=1}^{l} \|f_i - u\|_\epsilon^q\right)^{1/q}. \tag{1}$$

For $q = \infty$, we need to consider in (1) the supreme norm on $\mathbb{R}^l$.

If a net $\{u_\epsilon\}$ has a limit in $\mathcal{R}_m^n$ as $\epsilon \to 0$, it is called a *best simultaneous local* $(l^q, L^p)$*-approximation of* $f_1, \ldots, f_l$ *from* $\mathcal{R}_m^n$ *on* $\{x_1, \ldots, x_k\}$ ($(l^q, L^p)$*-b.s.l.a.*).

A pair $(P, Q) \in \Pi^n \times \Pi^m$ is a *Padé approximant pair of* $f$ *on* $\{x_1, \ldots, x_k\}$ if $Q \neq 0$ and

$$(Qf - P)(x) = o((x - x_j)^{c-1}), \quad \text{as } x \to x_j,\ 1 \leq j \leq k.$$

If $\left(f - \frac{P}{Q}\right)(x) = o((x - x_j)^{c-1})$, as $x \to x_j$, $1 \leq j \leq k$, then $\frac{P}{Q}$ is called a *Padé rational approximant of* $f$ *on* $\{x_1, \ldots, x_k\}$. This rational approximant may not exist. If $d = 0$ there is at most one, and we denote it by $\mathrm{Pa}(f)$ when it exists.

In [6] the author studied properties of interpolation of best rational approximation to a single function with respect to an integral norm, which includes the $L^p$-norm, $1 \leq p < \infty$. In [7] the authors proved that the best approximation to $l^{-1}\sum_{j=1}^l f_j$ from an arbitrary class of functions, $S$, is identical with the $(l^2, L^2)$-b.s.a. of $f_1, \ldots, f_l$ from $S$. However it is known that the $(l^q, L^p)$-b.s.a., in general, does not match with the best approximation to the mean of the functions $f_1, \ldots, f_l$ when $S = \Pi^n$ (see [8]). The $(l^\infty, L^p)$-b.s.l.a. from $\Pi^n$ was studied in [4] and [5]. In [2], the authors showed that the $(l^q, L^p)$-b.s.l.a. to two functions is the average of their Taylor polynomials.

In this paper, we prove an interpolation property of any $(l^q, L^p)$-b.s.a. to two functions from $\mathcal{R}_m^n$. As a consequence, we prove the existence and characterization of the $(l^q, L^p)$-b.s.l.a. when $q > 1$ and $k = 1$. Analogous results over $(l^q, L^p)$-b.s.l.a. were obtained, for $m = 0$, in several intervals. All our theorems generalize previous results for a single function.

## 2. Preliminary results

Henceforward we suppose that $1 < p < \infty$ and $1 \leq q < \infty$, except in Lemma 4.3 and Theorem 4.4 where we assume $q > 1$. First, we establish an existence theorem for the $(l^q, L^p)$- b.s.a.

**Theorem 2.1.** *Let* $f_1, f_2 \in \mathcal{C}(I)$ *and* $0 < \epsilon \leq 1$. *Then there exists a* $(l^q, L^p)$*- b.s.a. of* $f_1, f_2$ *from* $\mathcal{R}_m^n$ *on* $I_\epsilon$.

*Proof.* Let $\{v_r = \frac{P_r}{Q_r} \in \mathcal{R}_m^n : r \in \mathbb{N}\}$ be such that

$$\sum_{i=1}^{2} \|f_i - v_r\|_\epsilon^q \to \inf_{v \in \mathcal{R}_m^n} \sum_{i=1}^{2} \|f_i - v\|_\epsilon^q := b \quad \text{as } r \to \infty.$$

It is easy to see that $\{\|v_r\|_\epsilon : r \in \mathbb{N}\}$ is a bounded set. As the sequence $\{Q_r\}_{r\in\mathbb{N}}$ is uniformly bounded on compact sets, $\{\|P_r\|_\epsilon : r \in \mathbb{N}\}$ is a bounded set. Now, following the same patterns of the proof of existence for best rational approximation to a single function (see [11, Theorem 2.1]), we can find a subsequence $v_{r'}$ which converges to $v \in \mathcal{R}_m^n$ verifying $\sum_{i=1}^{2} \|f_i - v\|_\epsilon^q = b$, i.e., $v$ is a $(l^q, L^p)$- b.s.a. □

The following two lemmas can be proved analogously to [6, p. 88] and [1, p. 236], respectively.

**Lemma 2.2.** *Let $f_1, f_2 \in \mathcal{C}(I)$ and $0 < \epsilon \leq 1$. Suppose that $u_\epsilon = \frac{P_\epsilon}{Q_\epsilon} \in \mathcal{R}_m^n$ is a $(l^q, L^p)$-b.s.a. of $f_1, f_2$, from $\mathcal{R}_m^n$ on $I_\epsilon$, and $f_j \neq u_\epsilon$ on $I_\epsilon$, $1 \leq j \leq 2$. Then*

$$\sum_{j=1}^{2} \beta_j \left( \int_{I_\epsilon} |f_j - u_\epsilon|^{p-1} \operatorname{sgn}(f_j - u_\epsilon) \frac{P_\epsilon Q - P Q_\epsilon}{Q_\epsilon^2} \right) \geq 0, \quad \frac{P}{Q} \in \mathcal{R}_m^n, \tag{2}$$

*where $\beta_j = \beta_j(\epsilon) := \frac{q}{p} \|f_j - u_\epsilon\|_\epsilon^{p(\frac{q}{p}-1)}$.*

*Remark* 2.3. If $q \geq p$, the constraints $f_j \neq u_\epsilon$ on $I_\epsilon$, $1 \leq j \leq 2$, are not necessary. Moreover, if $q = p$ we observe that $\beta_j = 1$, $1 \leq j \leq 2$.

**Lemma 2.4.** *Let $\gamma \in \mathcal{C}(\mathrm{co}(I))$ be a strictly monotone function. If $f \in \mathcal{C}(I)$ and $\int_I f\gamma^n = 0$ for all $n \in \mathbb{N} \cup \{0\}$, then $f = 0$.*

**Lemma 2.5.** *Let $f_1, f_2 \in \mathcal{C}(I)$ and $0 < \epsilon \leq 1$. Suppose that $u_\epsilon = \frac{P_\epsilon}{Q_\epsilon} \in \mathcal{R}_m^n$ is a $(l^q, L^p)$-b.s.a. of $f_1, f_2$, from $\mathcal{R}_m^n$ on $I_\epsilon$ and $f_j \neq u_\epsilon$ on $I_\epsilon$, $1 \leq j \leq 2$. If $u_\epsilon$ is not normal then*

$$\sum_{j=1}^{2} \beta_j |f_j - u_\epsilon|^{p-1} \operatorname{sgn}(f_j - u_\epsilon) = 0 \quad \text{on } I_\epsilon,$$

*where $\beta_j$ was introduced in Lemma 2.2.*

*Proof.* Suppose that $u_\epsilon$ is not normal. Let $\mathcal{S} = \{S \in \Pi^1 : S(x) = x - a, a \in \mathbb{R} \setminus \mathrm{co}(I)\}$. For $\lambda \in \mathbb{R}$ and $S \in \mathcal{S}$, let $P = P_\epsilon S - \lambda$ and $Q = Q_\epsilon S$. Since $u_\epsilon = \frac{P_\epsilon S}{Q_\epsilon S}$ is a $(l^q, L^p)$-b.s.a., by Lemma 2.2,

$$\sum_{j=1}^{2} \beta_j \left( \int_{I_\epsilon} |f_j - u_\epsilon|^{p-1} \operatorname{sgn}(f_j - u_\epsilon) \frac{\lambda}{Q_\epsilon S} \right) \geq 0.$$

Since $\lambda$ is arbitrary, then $\sum\limits_{j=1}^{2} \beta_j \left( \int_{I_\epsilon} |f_j - u_\epsilon|^{p-1} \operatorname{sgn}(f_j - u_\epsilon) \frac{1}{Q_\epsilon S} \right) = 0$.

Let $h := \sum\limits_{j=1}^{2} \beta_j |f_j - u_\epsilon|^{p-1} \operatorname{sgn}(f_j - u_\epsilon) \frac{1}{Q_\epsilon} \in \mathcal{C}(I)$. Then

$$\int_{I_\epsilon} h \frac{1}{S} = 0, \quad S \in \mathcal{S}. \tag{3}$$

Let $\alpha < \min \operatorname{co}(I)$ and $\gamma(x) = \frac{a}{x-\alpha}$, $a > 0$. We choose $a$ sufficiently small such that $|\gamma(x)| < 1$, $x \in I$. For each $\lambda \in [-1, 0)$ let $S(x) = (x - \alpha) - \lambda a$. We observe that $\sum_{n=0}^{\infty} [\lambda\gamma(x)]^n$ uniformly converges to $\frac{1}{1-\lambda\gamma(x)}$ on $I$. Since

$$\begin{aligned}\int_{I_\epsilon} h(x)\frac{1}{S(x)}dx &= \int_{I_\epsilon} \frac{h(x)}{(x-\alpha)(1-\lambda\gamma(x))}dx \\ &= \sum_{n=0}^{\infty} \lambda^n \int_{I_\epsilon} \frac{h(x)}{x-\alpha}\gamma^n(x)dx,\end{aligned}$$

from (3) we conclude that $\int_{I_\epsilon} \frac{h(x)}{x-\alpha}\gamma^n(x)dx = 0$, $n \in \mathbb{N} \cup \{0\}$. As $h \in \mathcal{C}(I_\epsilon)$, using Lemma 2.4 for $I_\epsilon$ instead of $I$ we get the desired result. $\square$

The following result was proved in [6, Theorem 2] for a single function.

**Theorem 2.6.** *Let $0 < \epsilon \leq 1$ and $f_1, f_2 \in \mathcal{C}(I)$. Let $u_\epsilon \in \mathcal{R}_m^n$ be a non normal rational function. Then $u_\epsilon$ is a $(l^p, L^p)$-b.s.a. of $f_1$ and $f_2$ from $\mathcal{R}_m^n$ on $I_\epsilon$ if and only if $u_\epsilon = \frac{f_1+f_2}{2}$ on $I_\epsilon$.*

*Proof.* By Remark 2.3, $\beta_j = 1$, $j = 1, 2$. Lemma 2.5 implies

$$|f_1 - u_\epsilon|^{p-1} \operatorname{sgn}(f_1 - u_\epsilon) + |f_2 - u_\epsilon|^{p-1} \operatorname{sgn}(f_2 - u_\epsilon) = 0 \quad \text{on } I_\epsilon.$$

If $\operatorname{sgn}(f_1 - u_\epsilon)(x) = -\operatorname{sgn}(f_2 - u_\epsilon)(x)$, then $u_\epsilon(x) = \frac{(f_1+f_2)(x)}{2}$. Otherwise, $u_\epsilon(x) = f_1(x) = f_2(x) = \frac{(f_1+f_2)(x)}{2}$ on $I_\epsilon$. Reciprocally, suppose $u_\epsilon = \frac{f_1+f_2}{2}$ on $I_\epsilon$ and let $u \in \mathcal{R}_m^n$. Then

$$\begin{aligned}\|f_1 - u_\epsilon\|_\epsilon^p + \|f_2 - u_\epsilon\|_\epsilon^p &= 2\int_I \left|\frac{(f_1 - f_2)^\epsilon(x)}{2}\right|^p \frac{dx}{|I|} \\ &\leq 2\int_I \left(\left|\frac{(f_1 - u)^\epsilon(x)}{2}\right| + \left|\frac{(u - f_2)^\epsilon(x)}{2}\right|\right)^p \frac{dx}{|I|} \\ &\leq 2\left(\int_I \frac{|(f_1 - u)^\epsilon(x)|^p}{2}\frac{dx}{|I|} + \int_I \frac{|(f_2 - u)^\epsilon(x)|^p}{2}\frac{dx}{|I|}\right) \\ &= \|f_1 - u\|_\epsilon^p + \|f_2 - u\|_\epsilon^p.\end{aligned}$$

The proof is complete. $\square$

## 3. An interpolation property

Next, we introduce some notation to prove an interpolation result. Let $f_1, f_2 \in \mathcal{C}(I)$ and $0 < \epsilon \leq 1$. We write

$$y_i = y_i(\epsilon) := x_i + \epsilon\beta, \quad y^i = y^i(\epsilon) := x_{i+1} - \epsilon\beta, \quad 1 \leq i \leq k-1.$$

If $g \in \mathcal{C}(I_\epsilon)$, we denote

$$\mathcal{A}(g) = \{i : g(y_i(\epsilon))g(y^i(\epsilon)) < 0,\ 1 \leq i \leq k-1\}$$

and $k^\star(g)$ the cardinal of $\mathcal{A}(g)$. If $k = 1$, we put $k^\star(g) = 0$.

Let $\widetilde{f_1}, \widetilde{f_2} \in \mathcal{C}(\mathrm{co}(I))$ be extensions of $f_1$ and $f_2$, respectively. Now, we suppose that $\beta_j$, $1 \leq j \leq 2$, introduced in Lemma 2.2, is well defined. For a) $m = 0$ or b) $m \geq 1$, $k = 1$, the function

$$\widetilde{h_\epsilon} := \beta_1 |\widetilde{f_1} - u_\epsilon|^{p-1} \operatorname{sgn}(\widetilde{f_1} - u_\epsilon) + \beta_2 |\widetilde{f_2} - u_\epsilon|^{p-1} \operatorname{sgn}(\widetilde{f_2} - u_\epsilon) \tag{4}$$

is well defined on $\mathrm{co}(I_\epsilon)$. We write

$$\alpha_j(\epsilon) = (\beta_j)^{\frac{1}{p-1}} \left( \sum_{l=1}^{2} \beta_l^{\frac{1}{p-1}} \right)^{-1}. \tag{5}$$

Now, we establish the main result of this section.

**Theorem 3.1.** *Let $f_1, f_2 \in \mathcal{C}(I)$ and $0 < \epsilon \leq 1$. Suppose that $u_\epsilon \in \mathcal{R}^n_m$ is a $(l^q, L^p)$-b.s.a. of $f_1$ and $f_2$ from $\mathcal{R}^n_m$ on $I_\epsilon$. If $f_j \not\equiv u_\epsilon$ on $I_\epsilon$, $1 \leq j \leq 2$, and a) or b) holds, then $u_\epsilon$ interpolates to $\alpha_1(\epsilon)\widetilde{f_1} + \alpha_2(\epsilon)\widetilde{f_2}$, in at least $n + m + 1$ different points of $\mathrm{co}(I_\epsilon)$, where at least $n + m + 1 - k^\star(\widetilde{h_\epsilon})$ of them belong to $I_\epsilon$.*

*Proof.* Since $f_j \not\equiv u_\epsilon$ on $I_\epsilon$, $1 \leq j \leq 2$, the function $\widetilde{h_\epsilon}$ is defined. We consider two cases. First, suppose that $u_\epsilon$ is not normal, then Lemma 2.5 implies $\widetilde{h_\epsilon} = 0$ on $I_\epsilon$. Now, we assume that $u_\epsilon := \frac{P_\epsilon}{Q_\epsilon}$ is normal. It is well known that $P_\epsilon \Pi^m + Q_\epsilon \Pi^n = \Pi^{n+m}$ (see [1, p. 240]). Therefore by Lemma 2.2, we have

$$\int_{I_\epsilon} \frac{\widetilde{h_\epsilon}}{(Q_\epsilon)^2} v = 0, \quad v \in \Pi^{n+m}. \tag{6}$$

Suppose that $\widetilde{h_\epsilon}$ exactly changes of sign in $z_1, \ldots, z_s \in I_\epsilon$, with $s < n + m + 1 - k^\star(\widetilde{h_\epsilon})$. We can choose $r_1, \ldots, r_{k^\star(\widetilde{h_\epsilon})}$, with $r_i \in (y_i, y^i)$ such that $\widetilde{h_\epsilon}(r_i) = 0$, $i \in \mathcal{A}(\widetilde{h_\epsilon})$. Let $v := \eta \Pi_{i=1}^s (x - z_i) \Pi_{i \in \mathcal{A}(\widetilde{h_\epsilon})} (x - r_i)$, $\eta := \pm 1$ be such that $v$ satisfies $\widetilde{h_\epsilon} v \geq 0$ on $I_\epsilon$ and $\widetilde{h_\epsilon} v > 0$ on a positive measure subset of $I_\epsilon$. This contradicts (6), so $s \geq n+m+1-k^\star(\widetilde{h_\epsilon})$. In this way we have proved that $\widetilde{h_\epsilon}$ has at least $n+m+1$ different zeros in $\mathrm{co}(I_\epsilon)$, where at least $n + m + 1 - k^\star(\widetilde{h_\epsilon})$ of them belong to $I_\epsilon$.

Let $x \in \mathrm{co}(I_\epsilon)$ be such that $\widetilde{h_\epsilon}(x) = 0$, i.e.

$$\begin{aligned} 0 = {} & \beta_1 |(\widetilde{f_1} - u_\epsilon)(x)|^{p-1} \operatorname{sgn}((\widetilde{f_1} - u_\epsilon)(x)) \\ & + \beta_2 |(\widetilde{f_2} - u_\epsilon)(x)|^{p-1} \operatorname{sgn}((\widetilde{f_2} - u_\epsilon)(x)). \end{aligned}$$

Now, the proof follows analogously to the first part in the proof of Theorem 2.6. ☐

We denote $l_j(\epsilon)$, $1 \leq j \leq k$, the cardinal of the set of points of $B_j$, where $u_\epsilon$ interpolates to the function $\alpha_1(\epsilon)\widetilde{f_1} + \alpha_2(\epsilon)\widetilde{f_2}$, whenever $\alpha_j(\epsilon)$, $1 \leq j \leq 2$, are defined. The following corollary can be proved similarly to [5, Corollary 9].

**Corollary 3.2.** *Under the same hypotheses of Theorem 3.1, there exists $j$, $1 \leq j \leq k$, such that $l_j(\epsilon) \geq c$.*

## 4. Existence of $(l^q, L^p)$-b.s.l.a. from $\mathcal{R}_m^n$

First, in this section we obtain a general result about the asymptotic behavior of the error

$$\mathcal{E}_\epsilon := \|f_1 - u_\epsilon\|_\epsilon^q + \|f_2 - u_\epsilon\|_\epsilon^q.$$

**Theorem 4.1.** *Let $f_1, f_2 \in \mathcal{C}(I)$, $0 < \epsilon \leq 1$, $u_\epsilon \in \mathcal{R}_m^n$ a $(l^q, L^p)$-b.s.a. of $f_1$ and $f_2$ from $\mathcal{R}_m^n$ on $I_\epsilon$. If there exists a Padé rational approximant of $\frac{f_1+f_2}{2}$ on $\{x_1, \ldots, x_k\}$, then*

$$\mathcal{E}_\epsilon^{1/q} = 2^{\frac{1-q}{q}} \|f_1 - f_2\|_\epsilon + o(\epsilon^{c-1}), \text{ as } \epsilon \to 0.$$

*Proof.* Let $R$ be a Padé rational approximant of $\frac{f_1+f_2}{2}$ on $\{x_1, \ldots, x_k\}$. Consider the semi-norm on $\mathcal{C}(I) \times \mathcal{C}(I)$ defined by

$$\|(g_1, g_2)\|_\epsilon = (\|g_1\|_\epsilon^q + \|g_2\|_\epsilon^q)^{1/q}.$$

By the triangle inequality we have

$$\begin{aligned}
\|f_1 - u_\epsilon\|_\epsilon^q + \|f_2 - u_\epsilon\|_\epsilon^q &\leq \|(f_1 - R)\|_\epsilon^q + \|(f_2 - R)\|_\epsilon^q \\
&= \left\| \left( \frac{f_1 - f_2}{2}, \frac{f_2 - f_1}{2} \right) + \left( \frac{f_1 + f_2}{2} - R, \frac{f_1 + f_2}{2} - R \right) \right\|_\epsilon^q \\
&\leq \left( 2^{1/q} \left\| \frac{f_1 - f_2}{2} \right\|_\epsilon + 2^{1/q} \left\| \frac{f_1 + f_2}{2} - R \right\|_\epsilon \right)^q \\
&\leq 2 \left( \frac{\|f_1 - f_2\|_\epsilon}{2} + o(\epsilon^{c-1}) \right)^q \\
&= \frac{1}{2^{q-1}} \left( \|f_1 - f_2\|_\epsilon + o(\epsilon^{c-1}) \right)^q.
\end{aligned} \tag{7}$$

Since

$$(a+b)^q \leq 2^{q-1}(a^q + b^q), a, b \geq 0, \tag{8}$$

we get

$$\|f_1 - u_\epsilon\|_\epsilon^q + \|f_2 - u_\epsilon\|_\epsilon^q \geq \frac{1}{2^{q-1}} \|f_1 - f_2\|_\epsilon^q. \tag{9}$$

From (7) and (9) we obtain the theorem. □

*Remark* 4.2. If $m = 0$ and $f_1, f_2 \in \mathcal{C}^c(I)$, with an analogous proof we have

$$\mathcal{E}_\epsilon^{1/q} = 2^{\frac{1-q}{q}} \|f_1 - f_2\|_\epsilon + O(\epsilon^c), \text{ as } \epsilon \to 0.$$

For $c > 0$ and $h \in \mathcal{C}^{c-1}(I)$, we consider the set

$$\mathcal{H}(h) = \{P \in \Pi^n : P^{(i)}(x_j) = h^{(i)}(x_j),\ 0 \leq i \leq c-1,\ 1 \leq j \leq k\}.$$

We define

$$A_j = \{i : 0 \leq i \leq c-1,\ f_1^{(i)}(x_j) \neq f_2^{(i)}(x_j)\}, \quad 1 \leq j \leq k.$$

Let $m_j = \min A_j - 1$ if $A_j \neq \emptyset$, and $m_j = c - 1$ otherwise. Set

$$\overline{m} = \min\{m_j : 1 \leq j \leq k\}. \tag{10}$$

For $c = 0$, we put $\mathcal{H}(h) = \Pi^n$, and $\overline{m} = -1$. With these notations, we obtain the following lemma.

**Lemma 4.3.** *Let $q > 1$ and assume $c > 0$, $f_1, f_2 \in \mathcal{C}^{c-1}(I)$ and $-1 \leq \overline{m} \leq c-2$. Under the same hypotheses of Theorem 4.1, $f_j \neq u_\epsilon$ on $I_\epsilon$, $1 \leq j \leq 2$, for small $\epsilon$. Then $\alpha_1(\epsilon)$ and $\alpha_2(\epsilon)$ (see (5) above) are defined for small $\epsilon$ and $\lim\limits_{\epsilon\to 0} \alpha_1(\epsilon) = \lim\limits_{\epsilon\to 0} \alpha_2(\epsilon) = \frac{1}{2}$.*

*Proof.* For simplicity all subnets $\epsilon \to 0$ will be denoted in the same way. Let $g = \frac{1}{2}(f_1 + f_2)$, $H = \frac{1}{2}(H_1 + H_2)$ with $H_l \in \mathcal{H}(f_l)$, $l = 1, 2$, and $u_\epsilon = \frac{P_\epsilon}{Q_\epsilon}$. Then $(g - H)(x) = o((x - x_j)^{c-1})$, as $x \to x_j$, $1 \leq j \leq k$, and

$$(f_1 - u_\epsilon)(x) = \left(\frac{1}{2}(f_1 - f_2) + (g - H) + H - \frac{P_\epsilon}{Q_\epsilon}\right)(x).$$

Hence

$$\begin{aligned}\frac{Q_\epsilon(f_1 - u_\epsilon)}{\|Q_\epsilon\|_\epsilon \epsilon^{\overline{m}+1}}(x) = \frac{Q_\epsilon(x)}{\|Q_\epsilon\|_\epsilon}&\left(\frac{\frac{1}{2}(f_1 - f_2)(x) + (g - H(x)}{\epsilon^{\overline{m}+1}}\right)\\ &+ \frac{Q_\epsilon(x)H(x) - P_\epsilon(x)}{\|Q_\epsilon\|_\epsilon \epsilon^{\overline{m}+1}},\end{aligned} \tag{11}$$

for $x \in B_j$, $1 \leq j \leq k$. By Theorem 4.1 and the definition of $\overline{m}$ we obtain $\frac{\|f_1 - u_\epsilon\|_\epsilon}{\epsilon^{\overline{m}+1}} \leq \frac{\mathcal{E}_\epsilon^{1/q}}{\epsilon^{\overline{m}+1}} = O(1)$. Since $Q_\epsilon^\epsilon \in \Pi^m$ on each $B_j$, and $\|\cdot\|$ can be also considered as a norm in $(\Pi^m)^k$, the equivalence of norms in this space implies that there exists $K > 0$ such that $\|Q_\epsilon^\epsilon\|_\infty := \max\limits_{1\leq j\leq k} \max\limits_{B_j} |Q_\epsilon^\epsilon| \leq K\|Q_\epsilon^\epsilon\|$. As $\overline{m} \leq c-2$, by (11) we get

$$\begin{aligned}\left\|\frac{Q_\epsilon H - P_\epsilon}{\|Q_\epsilon\|_\epsilon}\right\|_\epsilon &\leq \frac{\|Q_\epsilon^\epsilon\|_\infty}{\|Q_\epsilon^\epsilon\|}\left(\|f_1 - u_\epsilon\|_\epsilon + \left\|\sum_{j=1}^k \left(\frac{1}{2}(f_1 - f_2) + (g - H)\right)\chi_{B_j}\right\|_\epsilon\right)\\ &\leq \frac{\|Q_\epsilon^\epsilon\|_\infty}{\|Q_\epsilon^\epsilon\|}\left(\mathcal{E}_\epsilon^{1/q} + \left\|\sum_{j=1}^k \left(\frac{1}{2}(f_1 - f_2) + (g - H)\right)\chi_{B_j}\right\|_\epsilon\right)\\ &= O(\epsilon^{\overline{m}+1}).\end{aligned} \tag{12}$$

From (12) we have a subnet such that $\frac{(Q_\epsilon H - P_\epsilon)^\epsilon}{\|Q_\epsilon\|_\epsilon \epsilon^{\overline{m}+1}} \to R$. Moreover, we can choose the subnet such that $\frac{Q_\epsilon^\epsilon}{\|Q_\epsilon\|_\epsilon} \to S$. Here, $R$ and $S$ are polynomials on each $B_j$. We denote

$$\lambda(x) = \sum_{j=1}^k \frac{(f_1 - f_2)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(x - x_j)^{\overline{m}+1}\chi_{B_j}(x) \quad \text{and} \quad T(x) = \frac{R(x)}{S(x)}.$$

As $-1 \leq \overline{m} \leq c-2$, $\lambda \neq 0$. Since $\frac{(g-H)^\epsilon}{\epsilon^{\overline{m}+1}} \to 0$, from (11) we obtain

$$\lim_{\epsilon\to 0} \frac{(f_1 - u_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}} = \frac{1}{2}\lambda + T \tag{13}$$

on $I$ except possibly by the zeros of $S$. Similarly, we have

$$\lim_{\epsilon\to 0}\frac{(f_2-u_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}=-\frac{1}{2}\lambda+T. \tag{14}$$

By Fatou's Lemma, (13) and (14), there exists a subnet such that

$$\left\|\frac{1}{2}\lambda+T\right\|\le\lim_{\epsilon\to 0}\frac{\|f_1-u_\epsilon\|_\epsilon}{\epsilon^{\overline{m}+1}}\quad\text{and}\quad\left\|\frac{1}{2}\lambda-T\right\|\le\lim_{\epsilon\to 0}\frac{\|f_2-u_\epsilon\|_\epsilon}{\epsilon^{\overline{m}+1}}.$$

Therefore, from (8) we have

$$\begin{aligned}\|\lambda\|^q&=\left\|\frac{1}{2}\lambda+T+\frac{1}{2}\lambda-T\right\|^q\le\left(\left\|\frac{1}{2}\lambda+T\right\|+\left\|\frac{1}{2}\lambda-T\right\|\right)^q\\&\le 2^{q-1}\left(\left\|\frac{1}{2}\lambda+T\right\|^q+\left\|\frac{1}{2}\lambda-T\right\|^q\right)\\&\le 2^{q-1}\lim_{\epsilon\to 0}\frac{\|f_1-u_\epsilon\|_\epsilon^q+\|f_2-u_\epsilon\|_\epsilon^q}{\epsilon^{(\overline{m}+1)q}}=\|\lambda\|^q,\end{aligned}$$

where the last equality holds by Theorem 4.1. So,

$$\left\|\frac{1}{2}\lambda+T+\frac{1}{2}\lambda-T\right\|=\left\|\frac{1}{2}\lambda+T\right\|+\left\|\frac{1}{2}\lambda-T\right\| \tag{15}$$

and

$$\frac{\|\frac{1}{2}\lambda+T\|^q+\|\frac{1}{2}\lambda-T\|^q}{2}=\left(\frac{\|\frac{1}{2}\lambda+T\|+\|\frac{1}{2}\lambda-T\|}{2}\right)^q. \tag{16}$$

As $\|\cdot\|$ is strictly convex, from (15) there exists $a\ge 0$ such that

$$\frac{1}{2}\lambda+T=a\left(\frac{1}{2}\lambda-T\right), \tag{17}$$

i.e., $T=\frac{(a-1)\lambda}{2(1+a)}$. Also, as $x^q$ is strictly convex, from (16) we get

$$\left\|\frac{1}{2}\lambda+T\right\|=\left\|\frac{1}{2}\lambda-T\right\|. \tag{18}$$

If $\frac{1}{2}\lambda-T=0$, then $\frac{1}{2}\lambda+T=0$ and $\|\lambda\|=0$, a contradiction. Therefore $\frac{1}{2}\lambda-T\ne 0$, so (17) and (18) imply $a=1$. Therefore $T=0$. Now, from (13) and (14), we have

$$\lim_{\epsilon\to 0}\frac{(f_1-u_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}=\frac{\lambda}{2}\quad\text{and}\quad\lim_{\epsilon\to 0}\frac{(f_2-u_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}=-\frac{\lambda}{2}$$

on $I$ except possibly by the zeros of $S$. Again, an application of Fatou's Lemma implies $\frac{\|\lambda\|}{2}\le\lim_{\epsilon\to 0}\frac{\|f_1-u_\epsilon\|_\epsilon}{\epsilon^{\overline{m}+1}}$ and $\frac{\|\lambda\|}{2}\le\lim_{\epsilon\to 0}\frac{\|f_2-u_\epsilon\|_\epsilon}{\epsilon^{\overline{m}+1}}$ for some subnet. Theorem 4.1 implies

$$\lim_{\epsilon\to 0}\left(\frac{\|f_1-u_\epsilon\|_\epsilon^q}{\epsilon^{(\overline{m}+1)q}}+\frac{\|f_2-u_\epsilon\|_\epsilon^q}{\epsilon^{(\overline{m}+1)q}}\right)=\frac{\|\lambda\|^q}{2^{q-1}}.$$

So,

$$\frac{\|\lambda\|}{2}=\lim_{\epsilon\to 0}\frac{\|f_1-u_\epsilon\|_\epsilon}{\epsilon^{\overline{m}+1}}=\lim_{\epsilon\to 0}\frac{\|f_2-u_\epsilon\|_\epsilon}{\epsilon^{\overline{m}+1}}. \tag{19}$$

Note that there exists $\epsilon_0>0$, such that for all $0<\epsilon\le\epsilon_0$, we have $\|f_j-u_\epsilon\|_\epsilon\ne 0$, $j=1,2$, because $\lambda\ne 0$. So, $f_j\ne u_\epsilon$ on $I_\epsilon$, $1\le j\le 2$, for

$0 < \epsilon \leq \epsilon_0$, and $\alpha_1(\epsilon)$ and $\alpha_2(\epsilon)$ are defined for $0 < \epsilon \leq \epsilon_0$. Finally, from (5) and (19) we conclude that $\lim_{\epsilon \to 0} \alpha_1(\epsilon) = \lim_{\epsilon \to 0} \alpha_2(\epsilon) = \frac{1}{2}$. □

Next, we prove the main result of this section, which extends [10, Theorem 1].

**Theorem 4.4.** *Let $q > 1$ and assume $k = 1$. Let $f_1, f_2 \in \mathcal{C}^{n+m}(I)$, $0 < \epsilon \leq 1$, and $u_\epsilon = \frac{P_\epsilon}{Q_\epsilon} \in \mathcal{R}_m^n$ a $(l^q, L^p)$-b.s.a. of $f_1$ and $f_2$ from $\mathcal{R}_m^n$ on $I_\epsilon$. Suppose that there exists $\operatorname{Pa}\left(\frac{f_1+f_2}{2}\right)$. Then there exists a subnet $\epsilon' \to 0$ such that $P_{\epsilon'} \to P_0$, $Q_{\epsilon'} \to Q_0$, and $(P_0, Q_0)$ is a Padé approximant pair of $\frac{f_1+f_2}{2}$ on $\{x_1\}$. In addition, if the Padé approximant pair is unique, then $u_\epsilon$ converges pointwise to $\frac{P_0}{Q_0}$ as $\epsilon \to 0$, in a neighborhood of $x_1$ except possibly at $x_1$. Moreover, if $\operatorname{Pa}\left(\frac{f_1+f_2}{2}\right)$ is normal then $u_\epsilon$ uniformly converges to $\operatorname{Pa}\left(\frac{f_1+f_2}{2}\right)$, in a neighborhood of $x_1$.*

*Proof.* Lemma 4.3 and Theorem 3.1 imply that for small $\epsilon$ there are $n + m + 1$ points in $\operatorname{co}(I_\epsilon) = I_\epsilon$, say $z_0(\epsilon), \ldots, z_{n+m}(\epsilon)$, such that

$$P_\epsilon(z_i(\epsilon)) = Q_\epsilon(z_i(\epsilon))(\alpha_1(\epsilon) f_1(z_i(\epsilon)) + \alpha_2(\epsilon) f_2(z_i(\epsilon))), \quad 0 \leq i \leq n + m.$$

Consider $g_\epsilon = \alpha_1(\epsilon) f_1 + \alpha_2(\epsilon) f_2$. By the uniqueness of the interpolation polynomial of degree at most $n + m$, we get

$$P_\epsilon = H_{\{z_0(\epsilon), \ldots, z_{n+m}(\epsilon)\}}(Q_\epsilon g_\epsilon),$$

where the right-hand side denotes the interpolation polynomial of $Q_\epsilon g_\epsilon$ of degree $n + m$ on $\{z_0(\epsilon), \ldots, z_{n+m}(\epsilon)\}$. For a subnet $\epsilon' \to 0$, we have

$$Q_{\epsilon'} \to Q_0 \quad \text{and} \quad P_{\epsilon'} \to T_{n+m,x_1}(Q_0 g) =: P_0,$$

where $g$ is the limit of $g_{\epsilon'}$, and $T_{n+m,x_1}(h)$ represents the Taylor polynomial of $h$ of degree $n + m$ at $x_1$. First, we assume that $-1 \leq \overline{m} \leq c - 2$. By Lemma 4.3, $g = \frac{f_1+f_2}{2}$ and

$$\left(Q_0 \frac{f_1 + f_2}{2} - P_0\right)^{(i)}(x_1) = 0, \quad 0 \leq i \leq n + m.$$

Now, we suppose that $\overline{m} = c - 1$. Theorem 4.1 implies $\left\| f_1 - \frac{P_\epsilon}{Q_\epsilon} \right\|_\epsilon = o(\epsilon^{n+m})$. As a consequence $\|Q_\epsilon f_1 - P_\epsilon\|_\epsilon = o(\epsilon^{n+m})$, so

$$\|Q_\epsilon T_{n+m,x_1}(f_1) - P_\epsilon\|_\epsilon = o(\epsilon^{n+m}).$$

By definition of $\overline{m}$ we can replace $f_1$ by $\frac{f_1+f_2}{2}$, and from a Pólya type inequality (see [3, Theorem 3]) we have

$$\left(Q_0 \frac{f_1 + f_2}{2} - P_0\right)^{(i)}(x_1) = 0, \quad 0 \leq i \leq n + m.$$

In any case, we conclude that $(P_0, Q_0)$ is a Padé approximant pair of $\frac{f_1+f_2}{2}$ on $\{x_1\}$. On the other hand, if $\operatorname{Pa}\left(\frac{f_1+f_2}{2}\right)$ is normal, then $(P_0, Q_0)$ is the unique Padé approximant pair of $\frac{f_1+f_2}{2}$ on $\{x_1\}$ and $Q_0(x_1) \neq 0$ (see [10, Lemma 3]). Therefore

$\mathrm{Pa}\left(\frac{f_1+f_2}{2}\right) = \frac{P_0}{Q_0}$ and $u_\epsilon$ uniformly converges to $\mathrm{Pa}\left(\frac{f_1+f_2}{2}\right)$ on a neighborhood of $x_1$. □

## 5. Existence of $(l^q, L^p)$-b.s.l.a. from $\Pi^n$

Next, we prove a result about uniform boundedness of a net of best simultaneous approximations from $\Pi^n$.

**Theorem 5.1.** *Let $f_1, f_2 \in \mathcal{C}^n(I)$, $0 < \epsilon \le 1$, and let $P_\epsilon \in \Pi^n$ be a $(l^q, L^p)$-b.s.a. to $f_1$ and $f_2$ from $\Pi^n$ on $I_\epsilon$. Then the net $\{P_\epsilon\}$ is uniformly bounded on compact sets as $\epsilon \to 0$.*

*Proof.* Without loss of generality we can assume that the extensions $\widetilde{f}_1, \widetilde{f}_2$ considered in page 61 belong to $\mathcal{C}^n(\mathrm{co}(I))$. By Theorem 3.1 there exists $z_0(\epsilon) < \cdots < z_n(\epsilon)$ in $\mathrm{co}(I)$ such that $P_\epsilon = H_{\{z_0(\epsilon),\ldots,z_n(\epsilon)\}}(\gamma_1(\epsilon)\widetilde{f}_1 + \gamma_2(\epsilon)\widetilde{f}_2)$, where as before $H_{\{z_0(\epsilon),\ldots,z_n(\epsilon)\}}(\gamma_1(\epsilon)\widetilde{f}_1+\gamma_2(\epsilon)\widetilde{f}_2)$ denotes the interpolation polynomial of $\gamma_1(\epsilon)\widetilde{f}_1+\gamma_2(\epsilon)\widetilde{f}_2$ of degree $n$ on $\{z_0(\epsilon),\ldots,z_n(\epsilon)\}$, $\gamma_1(\epsilon), \gamma_2(\epsilon) \ge 0$ and $\gamma_1(\epsilon)+\gamma_2(\epsilon) = 1$. Since the nets $\{(z_0(\epsilon),\ldots,z_n(\epsilon))\}$ and $\{(\gamma_1(\epsilon),\gamma_2(\epsilon))\}$ are bounded, we can find convergent subnets. Suppose that $\gamma_j(\epsilon') \to \gamma_j$, $j = 1, 2$, and $z_i(\epsilon') \to t_i$, $0 \le i \le n$, as $\epsilon' \to 0$. Clearly $t_0 \le \cdots \le t_n$. Using Newton's divided difference formula and the continuity of the divided differences we get $P_{\epsilon'} \to H_{\{t_0,\ldots,t_n\}}(\gamma_1\widetilde{f}_1 + \gamma_2\widetilde{f}_2)$, as $\epsilon' \to 0$. Therefore the net $\{P_\epsilon\}$ is uniformly bounded on compact sets as $\epsilon \to 0$. □

Now, we state results about the convergence of b.s.a. We consider a basis of $\Pi^n$, $\{u_{sv}\}_{\substack{1\le v\le k\\ 0\le s\le c-1}} \cup \{w_e\}_{1\le e\le d}$ which satisfies

$$u_{sv}^{(i)}(x_j) = \delta_{(i,j)(s,v)}, \quad w_e^{(i)}(x_j) = 0, \quad 0 \le i \le c-1, \quad 1 \le j \le k,$$

where $\delta$ is the Kronecker delta function.

In the next theorem we need to recall the number $\overline{m}$ which was defined in (10).

**Theorem 5.2.** *Assume $f_1, f_2 \in \mathcal{C}^c(I)$, $0 < \epsilon \le 1$. Let $P_\epsilon \in \Pi^n$ be a $(l^q, L^p)$-b.s.a. to $f_1$ and $f_2$ from $\Pi^n$ on $I_\epsilon$, and let $A$ be the cluster point set of the net $\{P_\epsilon\}$ as $\epsilon \to 0$. Then:*

a) *$A$ is contained in $\mathcal{M}(f_1, f_2)$, the set of solutions of the following minimization problem:*

$$\min_{P\in\Pi^n}\left(\sum_{l=1}^{2}\left(\sum_{j=1}^{k}\left|(f_l-P)^{(\overline{m}+1)}(x_j)\right|^p\right)^{q/p}\right) \tag{20}$$

*with the constraints $P^{(i)}(x_j) = \frac{(f_1+f_2)^{(i)}(x_j)}{2}$, $0 \le i \le \overline{m}$, $1 \le j \le k$.*

b) *If $f_1, f_2 \in \mathcal{C}^n(I)$, then $A \ne \emptyset$. In particular, if $\mathcal{M}(f_1, f_2)$ is unitary, there exists a unique $(l^q, L^p)$-b.s.l.a. of $f_1$ and $f_2$ from $\Pi^n$ on $\{x_1, \ldots, x_k\}$.*

*Proof.* a) Let $P_0 \in A$. By definition of $A$, there is a net $\epsilon \downarrow 0$ such that $P_\epsilon \to P_0$. We denote $U_\epsilon = \frac{H_1-P_\epsilon}{2}$ and $V_\epsilon = \frac{H_2-P_\epsilon}{2}$, where $H_l \in \mathcal{H}(f_l)$, $l = 1, 2$. Clearly,

$$\mathcal{E}_\epsilon \ge \left(\frac{\|f_1-P_\epsilon\|_\epsilon + \|f_2-P_\epsilon\|_\epsilon}{2}\right)^q.$$

Since $(H_l - f_l)(x) = O((x-x_j)^c)$, as $x \to x_j$, $l = 1,2$, $1 \le j \le k$, we obtain

$$\|U_\epsilon\|_\epsilon + \|V_\epsilon\|_\epsilon \le \mathcal{E}_\epsilon^{1/q} + O(\epsilon^c). \tag{21}$$

By Remark 4.2,

$$\frac{\mathcal{E}_\epsilon^{1/q}}{\epsilon^{\overline{m}+1}} = 2^{\frac{1-q}{q}} \left\| \frac{f_1 - f_2}{\epsilon^{\overline{m}+1}} \right\|_\epsilon + O(1). \tag{22}$$

Expanding $(f_1 - f_2)^\epsilon$ by its Taylor polynomial at $x_j$, $1 \le j \le k$, up to order $\overline{m}$, we have

$$\begin{aligned}\lim_{\epsilon \to 0} & \left\| \frac{f_1 - f_2}{\epsilon^{\overline{m}+1}} \right\|_\epsilon \\ &= \frac{1}{(\overline{m}+1)!} \left( \sum_{j=1}^k |(f_1 - f_2)^{(\overline{m}+1)}(x_j)|^p \|(t - x_j)^{\overline{m}+1}\|_{B_j}^p \right)^{1/p} =: L \end{aligned} \tag{23}$$

From (22) and (23) we obtain that $\frac{\mathcal{E}_\epsilon^{1/q}}{\epsilon^{\overline{m}+1}}$ is bounded as $\epsilon \to 0$. So, (21) implies that $\left\| \frac{U_\epsilon^\epsilon}{\epsilon^{\overline{m}+1}} \right\|_{B_j}$ and $\left\| \frac{V_\epsilon^\epsilon}{\epsilon^{\overline{m}+1}} \right\|_{B_j}$, $1 \le j \le k$, are bounded. Since $\frac{U_\epsilon^\epsilon}{\epsilon^{\overline{m}+1}}, \frac{V_\epsilon^\epsilon}{\epsilon^{\overline{m}+1}} \in \Pi^n$ on $B_j$, then $\frac{(U_\epsilon^\epsilon)^{(i)}(x_j)}{\epsilon^{\overline{m}+1}} = \frac{(f_1 - P_\epsilon)^{(i)}(x_j)}{2} \epsilon^{i - \overline{m} - 1}$ and $\frac{(V_\epsilon^\epsilon)^{(i)}(x_j)}{\epsilon^{\overline{m}+1}} = \frac{(f_2 - P_\epsilon)^{(i)}(x_j)}{2} \epsilon^{i - \overline{m} - 1}$ are bounded for all $0 \le i \le c-1$, $1 \le j \le k$. Therefore there exists $d_{ij}$ such that

$$\lim_{\epsilon \to 0} (f_l - P_\epsilon)^{(i)}(x_j) \epsilon^{i - \overline{m} - 1} = d_{ij}, \quad 0 \le i \le \overline{m},\ 1 \le j \le k,\ l = 1,2 \tag{24}$$

for some subnet, that we again denote by $\epsilon$. For $t \in B_j$ we have

$$\begin{aligned}\frac{(f_1 - P_\epsilon)^\epsilon(t)}{\epsilon^{\overline{m}+1}} &= \sum_{i=0}^{\overline{m}} \frac{(f_1 - P_\epsilon)^{(i)}(x_j)}{i!} \epsilon^{i - (\overline{m}+1)} (t - x_j)^i \\ &\quad + \frac{(f_1 - P_\epsilon)^{(\overline{m}+1)}(\epsilon(\xi_j(t) - x_j) + x_j)}{(\overline{m}+1)!} (t - x_j)^{\overline{m}+1},\end{aligned}$$

where $\xi_j(t)$ belongs to the segment with ends $t$ and $x_j$. From (24) we get

$$\lim_{\epsilon \to 0} \frac{(f_1 - P_\epsilon)^\epsilon(t)}{\epsilon^{\overline{m}+1}} = \sum_{i=0}^{\overline{m}} \frac{d_{ij}}{i!} (t - x_j)^i + \frac{(f_1 - P_0)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!} (t - x_j)^{\overline{m}+1},$$

uniformly on $B_j$. Therefore

$$\begin{aligned}\lim_{\epsilon \to 0} & \left\| \frac{(f_1 - P_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}} \right\|^p \\ &= \sum_{j=1}^k \left\| \sum_{i=0}^{\overline{m}} \frac{d_{ij}}{i!} (t - x_j)^i + \frac{(f_1 - P_0)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!} (t - x_j)^{\overline{m}+1} \right\|_{B_j}^p \qquad (25) \\ &\ge \sum_{j=1}^k \left| \frac{(f_1 - P_0)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!} \right|^p J_j^p,\end{aligned}$$

where $J_j = \inf_{Q\in\Pi^{\overline{m}}} \|(t-x_j)^{\overline{m}+1} - Q(t)\|_{B_j}$. Clearly (25) holds for $f_2$ instead of $f_1$. From (24) we can assume $P_0^{(i)}(x_j) = f_1^{(i)}(x_j)$ for $1 \le j \le k$, $0 \le i \le \overline{m}$, so we can write

$$P_0 = \sum_{v=1}^{k}\sum_{s=0}^{\overline{m}} f_1^{(s)}(x_v)u_{sv} + \sum_{e=1}^{d} \overline{b}_e w_e + \sum_{v=1}^{k}\sum_{s=\overline{m}+1}^{c-1} \overline{c}_{sv}u_{sv},$$

for some real numbers $\{\overline{b}_e\}_{1\le e\le d}$ and $\{\overline{c}_{sv}\}_{\substack{1\le v\le k\\ 0\le s\le c-1}}$. Given two sets of real numbers (independent of $\epsilon$), say $\{c_{sv}\}_{\substack{1\le v\le k\\ 0\le s\le c-1}}$ and $\{b_e\}_{1\le e\le d}$, consider the following net of polynomials in $\Pi^n$,

$$R_\epsilon = \sum_{v=1}^{k}\sum_{s=0}^{\overline{m}} (f_1^{(s)}(x_v) - c_{sv}\epsilon^{\overline{m}+1-s})u_{sv} + \sum_{e=1}^{d} b_e w_e + \sum_{v=1}^{k}\sum_{s=\overline{m}+1}^{c-1} c_{sv}u_{sv}.$$

We observe that $R_\epsilon^{(i)}(x_j) = f_1^{(i)}(x_j) - c_{ij}\epsilon^{\overline{m}+1-i}$, $1\le j\le k$, $0\le i\le \overline{m}$.
Let $h = \sum_{v=1}^{k}\sum_{s=0}^{\overline{m}} f_1^{(s)}(x_v)u_{sv} + \sum_{e=1}^{d} b_e w_e + \sum_{v=1}^{k}\sum_{s=\overline{m}+1}^{c-1} c_{sv}u_{sv}$. Expanding $(f_1 - R_\epsilon)^\epsilon$ by its Taylor polynomial at $x_j$ up to order $\overline{m}$, we obtain

$$\begin{aligned}
&\frac{(f_1-R_\epsilon)^\epsilon(t)}{\epsilon^{\overline{m}+1}}\\
&\quad= \sum_{i=0}^{\overline{m}} \frac{c_{ij}}{i!}(t-x_j)^i + \frac{(f_1-h)^{(\overline{m}+1)}(\epsilon(\xi_j(t)-x_j)+x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}\\
&\qquad+ \sum_{v=1}^{k}\sum_{s=0}^{\overline{m}} \frac{c_{sv}\epsilon^{\overline{m}+1-s}u_{sv}^{(\overline{m}+1)}(\epsilon(\xi_j(t)-x_j)+x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}, \quad t\in B_j,
\end{aligned}$$

where $\xi_j(t)$ belongs to the segment with ends $t$ and $x_j$. Since $\lim_{\epsilon\to 0} \frac{(f_1-R_\epsilon)^\epsilon(t)}{\epsilon^{\overline{m}+1}} = \sum_{i=0}^{\overline{m}} \frac{c_{ij}}{i!}(t-x_j)^i + \frac{(f_1-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}$, uniformly on $B_j$, we have

$$\lim_{\epsilon\to 0}\left\|\frac{(f_1-R_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}\right\|^p = \sum_{j=1}^{k}\left\|\sum_{i=0}^{\overline{m}} \frac{c_{ij}}{i!}(t-x_j)^i + \frac{(f_1-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}\right\|_{B_j}^p.$$

Let $c_{ij}$, $1\le j\le k$, $0\le i\le \overline{m}$, be such that $\sum_{i=0}^{\overline{m}} \frac{c_{ij}}{i!}(t-x_j)^i$ is the best approximation to $\frac{(f_1-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}(t-x_j)^{\overline{m}+1}$ with respect to $\|.\|_{B_j}$. Then

$$\lim_{\epsilon\to 0}\left\|\frac{(f_1-R_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}\right\|^p = \sum_{j=1}^{k}\left|\frac{(f_1-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p, \tag{26}$$

and similarly we get

$$\lim_{\epsilon\to 0}\left\|\frac{(f_2-R_\epsilon)^\epsilon}{\epsilon^{\overline{m}+1}}\right\|^p = \sum_{j=1}^{k}\left|\frac{(f_2-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p. \tag{27}$$

From (25)-(27) and the continuity of the function $|x|^{\frac{q}{p}} + |y|^{\frac{q}{p}}$, we have

$$\sum_{l=1}^{2}\left(\sum_{j=1}^{k}\left|\frac{(f_l-P_0)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p\right)^{q/p} \leq \liminf_{\epsilon\to 0}\frac{\mathcal{E}_\epsilon}{\epsilon^{(\overline{m}+1)q}} \leq \sum_{l=1}^{2}\left(\sum_{j=1}^{k}\left|\frac{(f_l-h)^{(\overline{m}+1)}(x_j)}{(\overline{m}+1)!}\right|^p J_j^p\right)^{q/p}, \quad (28)$$

for all $h = \sum_{v=1}^{k}\sum_{s=0}^{\overline{m}} f_1^{(s)}(x_v)u_{sv} + \sum_{e=1}^{d} b_e w_e + \sum_{v=1}^{k}\sum_{s=\overline{m}+1}^{c-1} c_{sv}u_{sv}$.

For all $1 \leq j \leq k$, $J_j = \inf_{Q\in\Pi^{\overline{m}}}\left(\int_{-\beta}^{\beta}|y^{\overline{m}+1}-Q(y)|^p\frac{dy}{|I|}\right)^{\frac{1}{p}} \neq 0$. Then $J_j$ does not depend on $j$. So, from (28) we obtain

$$\sum_{l=1}^{2}\left(\sum_{j=1}^{k}\left|(f_l-P_0)^{(\overline{m}+1)}(x_j)\right|^p\right)^{q/p} \leq \sum_{l=1}^{2}\left(\sum_{j=1}^{k}\left|(f_l-h)^{(\overline{m}+1)}(x_j)\right|^p\right)^{q/p}.$$

In addition, as $f_1^{(i)}(x_j) = f_2^{(i)}(x_j)$, $0 \leq i \leq \overline{m}$, $1 \leq j \leq k$, then $P_0^{(i)}(x_j) = \frac{(f_1+f_2)^{(i)}(x_j)}{2}$. The proof of a) is complete.

b) If $f_1, f_2 \in \mathcal{C}^n(I)$, by Theorem 5.1 the net $\{P_\epsilon\}$ is uniformly bounded on compact sets, then there exists $P_0 \in A$. From a), $P_0 \in \mathcal{M}(f_1, f_2)$. In particular, if $\mathcal{M}(f_1, f_2)$ is unitary, there exists a unique $(l^q, L^p)$-b.s.l.a. of $f_1$ and $f_2$ from $\Pi^n$ on $\{x_1, \ldots, x_k\}$. □

The following theorem gives sufficient conditions for $\mathcal{M}(f_1, f_2)$ to be a unitary set. Its proof is analogous to that of [5, Theorem 12].

**Theorem 5.3.** *Let $f_1, f_2 \in \mathcal{C}^c(I)$ and $q > 1$. If either a) $\overline{m} = c-2, d = 0$ or b) $\overline{m} = c-1$, then $\mathcal{M}(f_1, f_2)$ is a unitary set.*

The next theorem shows that there always exists a unique $(l^2, L^2)$-b.s.l.a. of $f_1$ and $f_2$ from $\Pi^n$ on $\{x_1, \ldots, x_k\}$.

**Theorem 5.4.** *Let $f_1, f_2 \in \mathcal{C}^c(I)$. Then there exists a unique $(l^2, L^2)$-b.s.l.a. of $f_1$ and $f_2$ from $\Pi^n$ on $\{x_1, \ldots, x_k\}$, and it is the best local approximation of $\frac{f_1+f_2}{2}$ from $\Pi^n$ on $\{x_1, \ldots, x_k\}$ with respect to the norm $L^2$.*

*Proof.* Let $0 < \epsilon \leq 1$ and let $\{P_\epsilon\}$ be a net of $(l^2, L^2)$-b.s.a. of $f_1$ and $f_2$ from $\Pi^n$ on $I_\epsilon$; then it is well known that $P_\epsilon$ is the best approximation to $\frac{f_1+f_2}{2}$ with respect to the norm $L^2$ (see [12, Theorem 3]). Hence, we deduce that $\{P_\epsilon\}$ converges to the best local approximation of $\frac{f_1+f_2}{2}$ (see [9, Theorem 4]). □

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