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# Symmetries and exact solutions of the BPS Skyrme model

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## Abstract

The BPS Skyrme model is a specific subclass of Skyrme-type field theories which possesses both a BPS bound and infinitely many soliton solutions (skyrmions) saturating that bound. A related property, the existence of a large group of symmetry transformations, allows for solutions of rather general shapes, which is in contrast to the situation for the original Skyrme model, where soliton solutions usually have some fixed shapes. We study here the classical symmetries of the BPS Skyrme model, applying them to construct soliton solutions with some prescribed shapes, which constitutes a further step to understand the similarities and differences with the original Skyrme model.

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## 1. Introduction

The Skyrme model [1] (SM), a nonlinear field theory for an  $SU(2)$ -valued field, is meant to be a low-energy effective theory, describing some interesting aspects of strong interaction physics. In this model, pions play the role of primary fields (excitations around the trivial vacuum), whereas nucleons and nuclei are, on the other hand, represented by topological solitons, collective excitations which are part of the nonperturbative spectrum of the theory.

The application of the SM to nuclear and hadronic physics has been quite successful at a qualitative level [2–5], but it encounters some problems once a more detailed, quantitative agreement is required. The main obstacle for this is the absence of (almost) BPS solutions in the original SM, as well as in its standard generalizations. Indeed, although there exists a BPS bound already in the original model, as proposed by Skyrme, nontrivial soliton solutions cannot saturate this bound. As a consequence, higher solitons, meant to describe larger nuclei, are strongly bound, in striking contrast to the weak binding energies of physical nuclei.

Some versions of the Skyrme model which do support BPS solutions have been proposed recently. Basically they imply the extension of the symmetries of the Skyrme-type theory to conformal transformations [6] or volume preserving diffeomorphisms [7]. It is the aim of this paper to further elaborate on one of them, namely the model of [7].

The SM may be generalized in a rather straightforward way, by simply adding some judiciously chosen extra terms to its defining Lagrangian [8–12]. Indeed, the addition of extra terms becomes a quite natural step when one recalls the fact that the SM is an effective theory, supplemented with the condiment of some simplicity and symmetry constraints. In fact, assuming, as one usually does, that we want to maintain the field content of the original model, as well as its Poincaré invariance and the standard Hamiltonian interpretation (Lagrangian quadratic in time derivatives), the number of possible terms is in fact quite restricted. One may then just have a potential term (no derivatives), a standard kinetic term (the nonlinear sigma model term) quadratic in first derivatives, the ‘Skyrme term’ originally introduced by Skyrme (quartic in derivatives) and, finally, a term which is the square of the baryon number current (topological current), which is sextic in derivatives.

As has been demonstrated in [7], there is a submodel, termed ‘BPS Skyrme model’ (BPSSM), defined by a Lagrangian consisting of just the potential and sextic terms, which satisfies some quite interesting properties. Indeed, it possesses a BPS bound, and infinitely many BPS solutions saturating this bound. Besides, it has been also shown in [7] that the static energy functional of the model is invariant under an infinite number of symmetry transformations, a fact that is obviously related to the properties enunciated in the previous sentence. Among the symmetry transformations, an interesting type are the volume preserving diffeomorphisms (VPDs), since they are precisely the symmetries of an ideal fluid.

We want to emphasize at this point that the BPSSM by itself should not be considered as a full low-energy effective field theory, first of all, because the absence of terms with lower derivatives (the sigma model and Skyrme terms) cannot be justified from this effective field theory point of view. Still, properties such as the BPS solutions or the VPD symmetries constitute appealing features for nuclear physics, so this submodel might be useful for the description of some properties of static nuclei (see, for example, [7, 13]). A further issue is that for certain initial data the BPSSM does not have a well-defined Cauchy problem. The inclusion of additional terms is, therefore, mandatory once dynamical, time-dependent problems are considered.

Because of the above, it would be important to relate the properties of solutions of the BPSSM to the corresponding solutions of more general Skyrme-type models. A stumbling block which immediately pops up when attempting this task is the different sizes of the respective spaces of solutions, which are in turn due to the different symmetry groups of the models. The solutions of the BPSSM may have almost any symmetry, due to the huge symmetry group of the field equations. In particular, there are spherically symmetric solutions (i.e. with spherically symmetric energy densities) for all the possible values of the baryon charge,  $Q_B$ . This is not the case, on the other hand, for the original SM and its non-BPS generalizations. Typically, the  $Q_B = 1$  skyrmion is spherically symmetric, the  $Q_B = 2$  skyrmion has cylindrical symmetry, while higher charge skyrmions have, at most, a set of discrete symmetries. Indeed, their energy densities are invariant under some discrete subgroup of the rotation group  $SO(3)$  (see, for example, [14–16]). A skyrmion of the BPSSM with the same set of discrete symmetries would, therefore, be a good starting point for a comparison with the original SM and for the understanding of effects induced by adding other extra terms to the Lagrangian. Because of this, it would be important to find a method for the systematic construction of solutions of the BPS Skyrme model with some prescribed symmetries.

It is the purpose of this paper to investigate the space of BPS solutions further, making explicit use of its symmetries as a tool to generate new solutions. To that end, we shall take the spherically symmetric ones as a starting point for the construction. Finally, we shall show that all local solutions may, in fact, be constructed in this way.

This paper is organized as follows. In section 2, we define the model and introduce our notation and conventions. Then, we construct the classical Hamiltonian in section 3. The BPS bound is considered in section 4. In section 5, we explore the issue of symmetries, within both the Lagrangian and Hamiltonian contexts. In section 6, we derive and discuss the main properties of the BPS solutions of the model. We also construct several explicit classes of solutions with some prescribed symmetries, including the important case of discrete symmetries. In section 7 we summarize our results and conclusions.

## 2. The model

The Lagrangian density  $\mathcal{L}$ , which has an  $SU(2)$  valued field  $U$  as dynamical variable, may be written as follows:

$$\mathcal{L} = \mathcal{L}_{06} = -\lambda^2 \pi^2 B_\mu B^\mu - \mu^2 \mathcal{V}(U, U^\dagger), \quad (1)$$

where  $\lambda$  is a positive constant,  $B^\mu$  denotes the topological current:

$$B^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{tr}(L_\nu L_\rho L_\sigma), \quad L_\mu \equiv U^\dagger \partial_\mu U, \quad (2)$$

and  $\mathcal{V}$  is a potential density. The current  $B^\mu$  is ‘topologically conserved’, namely it can be shown to be conserved, regardless of the equations of motion. The resulting conserved charge,  $Q_B$ , is therefore given by

$$\begin{aligned} Q_B &= \int d^3x B_0 = \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{tr}(L_i L_j L_k) \\ &= \frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} \text{tr}(U^\dagger \partial_i U U^\dagger \partial_j U U^\dagger \partial_k U), \end{aligned} \quad (3)$$

the degree of the map  $\mathbb{R}^3 \rightarrow S^3$ , an integer which is invariant under arbitrary globally well-defined coordinate transformations, as well as under global isospin rotations of  $U$ . It is, in fact, invariant under the much bigger group of target space transformations leaving invariant a certain target space volume form, see below.

To proceed to the classical equations of motion, it is convenient to introduce a specific parametrization for the three degrees of freedom of  $U$ .

Following [7], we use a real scalar field  $\xi$  plus a three-component unit vector  $\hat{\mathbf{n}}$ , so that

$$U(x) = e^{i\xi(x)\hat{\mathbf{n}}(x)\cdot\boldsymbol{\tau}}, \quad (4)$$

where  $\boldsymbol{\tau}$  are the three Pauli matrices. The real scalar  $\xi$  runs from 0 to  $\pi$ , while the two independent parameters defining  $\hat{\mathbf{n}}$  may be taken as the two components of a complex variable  $u$ , by means of a stereographic projection:

$$\hat{\mathbf{n}} = \frac{1}{1 + |u|^2} (u + \bar{u}, -i(u - \bar{u}), |u|^2 - 1). \quad (5)$$

In this way, one obtains for the Lagrangian density an expression in terms of  $\xi$ ,  $u$  and  $\bar{u}$ :

$$\mathcal{L} = \frac{\lambda^2 \sin^4 \xi}{(1 + |u|^2)^4} (\epsilon^{\mu\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma)^2 - \mu^2 \mathcal{V}(\xi), \quad (6)$$

where the lower indices in those variables denote partial derivatives with respect to the spatial coordinates, and we have assumed that the potential may only depend on  $U$  through  $\text{tr } U$ .

With the notation  $\mathcal{V}_\xi \equiv \partial_\xi \mathcal{V}$ , the Euler–Lagrange equations read

$$\begin{aligned} \frac{\lambda^2 \sin^2 \xi}{(1 + |u|^2)^4} \partial_\mu (\sin^2 \xi H^\mu) + \mu^2 \mathcal{V}_\xi &= 0 \\ \partial_\mu \left( \frac{K^\mu}{(1 + |u|^2)^2} \right) &= 0, \end{aligned} \quad (7)$$

where

$$H_\mu = \frac{\partial (\epsilon^{\alpha\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma)^2}{\partial \xi^\mu}, \quad K_\mu = \frac{\partial (\epsilon^{\alpha\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma)^2}{\partial \bar{u}^\mu}.$$

These objects satisfy, by construction, the relations

$$H_\mu u^\mu = H_\mu \bar{u}^\mu = 0, \quad K_\mu \xi^\mu = K_\mu u^\mu = 0, \quad H_\mu \xi^\mu = K_\mu \bar{u}^\mu = 2(\epsilon^{\alpha\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma)^2, \quad (8)$$

which are often useful.

### 3. Hamiltonian and static energy

In order to construct the Hamiltonian, we first introduce a more compact notation, in terms of three real fields  $\xi^{(a)}$ , with  $a = 1, 2, 3$ , such that  $u = \xi^{(1)} + i\xi^{(2)}$  and  $\xi^{(3)} \equiv \xi$ .

Then,  $\mathcal{L}$  may be written as follows ( $\xi_0^{(a)} \equiv \partial_0 \xi^{(a)}$ ):

$$\mathcal{L} = \frac{1}{2} \xi_0^{(a)} G_{(ab)} \xi_0^{(b)} - \frac{4\lambda^2 \sin^4 \xi^{(3)} (\epsilon^{ijk} \xi_i^{(1)} \xi_j^{(2)} \xi_k^{(3)})^2}{[1 + (\xi^{(1)})^2 + (\xi^{(2)})^2]^4} - \mu^2 \mathcal{V}(\xi), \quad (9)$$

where the kinetic term is determined by a metric  $G_{(ab)}$ , given by

$$G_{(ab)} = \frac{2\lambda^2 \sin^4 (\xi^{(3)})}{[1 + (\xi^{(1)})^2 + (\xi^{(2)})^2]^4} \mathcal{Q}_i^{(a)} \mathcal{Q}_i^{(b)}, \quad (10)$$

where

$$\mathcal{Q}_i^{(a)} = \epsilon_{ijk} \epsilon^{abc} \xi_j^{(b)} \xi_k^{(c)}. \quad (11)$$

In order to see whether the system defined by  $\mathcal{L}$  is regular or not, we note that  $\mathbb{Q} \equiv [\mathcal{Q}_i^{(a)}]$  the  $3 \times 3$  matrix defined by the nine elements  $\mathcal{Q}_i^{(a)}$  ( $i = 1, 2, 3; a = 1, 2, 3$ ) is proportional to the cofactor matrix of the matrix  $\mathbb{X} \equiv [\xi_i^{(a)}]$ :

$$\mathbb{Q} = 2 \operatorname{cof}(\mathbb{X}). \quad (12)$$

Thus, we see that the metric  $[G_{(ab)}]$  (hence, the Lagrangian system) is regular if and only if  $\det[\xi_i^{(a)}] \neq 0$ . In other words, the regularity of the system is equivalent to the non-vanishing of the Jacobian determinant:

$$\mathcal{J} \equiv \det[\mathbb{X}] = \det \left[ \frac{\partial \xi^{(a)}}{\partial x_i} \right] \neq 0, \quad (13)$$

for the mapping between the sphere (i.e. one-point compactified  $\mathbb{R}^3$ ) in coordinate space and the one in  $SU(2)$ .

Under the assumption that (13) holds true, the inverse of  $\mathbb{G} = [G_{(ab)}]$  may be found by elementary algebra. Indeed,

$$[\mathbb{G}^{-1}]^{(ab)} = \frac{[1 + (\xi^{(1)})^2 + (\xi^{(2)})^2]^4}{8\lambda^2 \mathcal{J}^2 \sin^4 (\xi^{(3)})} \xi_i^{(a)} \xi_i^{(b)}. \quad (14)$$

Thus, the Hamiltonian density in terms of the variables  $\xi^{(a)}$ , its spatial derivatives, and their canonical momenta  $\Pi^{(a)}$ , becomes

$$\mathcal{H} = \frac{[1 + (\xi^{(1)})^2 + (\xi^{(2)})^2]^4}{16\lambda^2 \mathcal{J}^2 \sin^4(\xi^{(3)})} \Pi^{(a)} \xi_i^{(a)} \xi_i^{(b)} \Pi^{(b)} + \frac{4\lambda^2 \sin^4 \xi^{(3)} (\epsilon^{ijk} \xi_i^{(1)} \xi_j^{(2)} \xi_k^{(3)})^2}{[1 + (\xi^{(1)})^2 + (\xi^{(2)})^2]^4} + \mu^2 \mathcal{V}(\xi), \quad (15)$$

which, for a Lagrangian like the one we are considering, coincides with the energy density of the system. In particular, for the static configuration case to be considered in the forthcoming sections, the total energy  $E$  is

$$E = \int d^3x \left\{ 4\lambda^2 \frac{\sin^4 \xi^{(3)} (\epsilon^{ijk} \xi_i^{(1)} \xi_j^{(2)} \xi_k^{(3)})^2}{[1 + (\xi^{(1)})^2 + (\xi^{(2)})^2]^4} + \mu^2 \mathcal{V}(\xi) \right\}. \quad (16)$$

We have shown that the regularity of the system depends on the field configurations considered. Specifically, the system is singular in regions where the fields take their vacuum values ( $\xi^{(a)} = \text{const}$  such that  $\mathcal{V}(\xi^{(3)}) = 0$ ). This already demonstrates that, while the system may provide a good approximation to the description of static properties of nucleons and nuclei via solitons (Skyrmions) and for the dynamics in regions with nonzero baryon charge density (where it is regular by construction), its fully consistent application to dynamical nuclear physics requires additional structures such as, e.g., quantum corrections, or the inclusion of further terms in the Lagrangian.

#### 4. BPS bound

The static energy functional in (16), or, in terms of the variables  $\xi$  and  $u$  introduced previously,

$$E = \int d^3x \left[ \frac{\lambda^2 \sin^4 \xi}{(1 + |u|^2)^4} (\epsilon^{mnl} i \xi_m u_n \bar{u}_l)^2 + \mu^2 \mathcal{V}(\xi) \right] \quad (17)$$

obeys a Bogomolny bound. Indeed,

$$\begin{aligned} E &= \int d^3x \left( \frac{\lambda \sin^2 \xi}{(1 + |u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l \pm \mu \sqrt{\mathcal{V}} \right)^2 \mp \int d^3x \frac{2\mu \lambda \sin^2 \xi \sqrt{\mathcal{V}}}{(1 + |u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l \\ &\geq \mp \int d^3x \frac{2\mu \lambda \sin^2 \xi \sqrt{\mathcal{V}}}{(1 + |u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l \\ &= \pm (2\lambda \mu \pi^2) \left[ \frac{-i}{\pi^2} \int d^3x \frac{\sin^2 \xi \sqrt{\mathcal{V}}}{(1 + |u|^2)^2} \epsilon^{mnl} \xi_m u_n \bar{u}_l \right] \equiv 2\lambda \mu \pi^2 \langle \sqrt{\mathcal{V}} \rangle |B|, \end{aligned} \quad (18)$$

where  $\langle \sqrt{\mathcal{V}} \rangle$  is the average value of  $\sqrt{\mathcal{V}}$  on the target space  $S^3$ . The corresponding Bogomolny (first-order) equation is

$$\frac{\lambda \sin^2 \xi}{(1 + |u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l = \mp \mu \sqrt{\mathcal{V}}. \quad (19)$$

The static second-order field equations may be derived from the squared Bogomolny equation by applying a gradient  $\partial_k$  and by projecting onto  $\epsilon_{ijk} \partial_j \xi^{(a)}$  where  $\xi^{(a)} \equiv (\xi, u, \bar{u})$ . We remark that a completely analogous BPS bound can be found for the BPS baby Skyrme model in one lower dimension [17–20].

Another interesting observation is that the BPS equation can be formulated in the language of a nonlinear generalization of the static (vacuum) Nambu–Poisson equation. Indeed, the left-hand side can be recast into the Nambu–Poisson three-bracket [21]

$$\{X^A, X^B, X^C\} = \epsilon^{mnl} \frac{\partial X^A}{\partial x^m} \frac{\partial X^B}{\partial x^n} \frac{\partial X^C}{\partial x^l}, \quad (20)$$

where the target space embedding coordinates  $X^A$ ,  $A = 1, 2, 3, 4$  form a 3-sphere  $S^3$  (i.e.  $(X^A)^2 = 1$ ) and are related to the previous coordinates such as  $X^a = n^a \sin \xi$ ,  $a = 1, 2, 3$  and  $X^4 = \cos \xi$ . Then, the generalized Nambu–Poisson dynamics is given by

$$\frac{dX^A}{dt} = \epsilon^{ABCD} \{X^B, X^C, X^D\} + X^A \sqrt{\mathcal{V}(X^4)}, \quad (21)$$

which differs from the standard case by the additional factor  $\sqrt{\mathcal{V}}$  in the last term [21]. Obviously, although the dynamics of the BPS Skyrme model is profoundly different, the BPS equation provides static solutions to this generalized Nambu–Poisson equation. Such solutions may be interpreted as vacuum configurations of the underlying hyper-membrane Lagrangian [22]. We remark that if one assumes from the outset that the target space variables  $X^A$  span a 3-sphere, as we do in this paper, then there is no dynamics in equation (21), i.e.  $\frac{dX^A}{dt} = 0$ , as follows from the fact that the rhs of (21) is proportional to  $X^A$  in this case. This just corresponds to the well-known result that the static vacuum equations for the hyper-membrane imply that the brane embedding coordinates  $X^A$  span a 3-sphere [22]. So, our model generalizes the static hyper-membrane action, with a correspondence between the BPS solitons and the vacuum membrane configurations, but with completely different dynamics.

It may be instructive to compare the BPS bound arising above with a  $(1+1)$ -dimensional analogue: the search for (non-trivial) static minimum energy configurations for the sine-Gordon model. Here, a real scalar field  $\varphi$  is in the presence of a potential density  $\mathcal{U}(\varphi)$  which allows for non-trivial topology. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 - \mathcal{U}(\varphi) \quad (22)$$

$$\mathcal{U}(\varphi) = \frac{m^4}{\lambda} \left[ 1 - \cos\left(\frac{\sqrt{\lambda}}{m} \varphi\right) \right]. \quad (23)$$

The static energy is then

$$E_\varphi = \int_{-\infty}^{+\infty} dx_1 \left[ \frac{1}{2} (\partial_1 \varphi)^2 + \mathcal{U}(\varphi) \right]. \quad (24)$$

The non-negative potential has non-trivial minima for

$$\varphi = \varphi_N = \frac{2\pi m}{\sqrt{\lambda}} N, \quad N \in \mathbb{Z}, \quad (25)$$

all of them having zero energy. Finite energy vacuum configurations must tend to one of the minima when  $x_1 \rightarrow \pm\infty$ .

The topologically conserved current is  $j^\mu = \frac{\sqrt{\lambda}}{2\pi m} \epsilon^{\mu\nu} \partial_\nu \varphi$  ( $\mu, \nu = 1, 2$ ), which obviously satisfies  $\partial \cdot j = 0$ . Its associated topological charge is quantized:

$$Q_\varphi = \frac{\sqrt{\lambda}}{2\pi m} \int_{-\infty}^{+\infty} dx_1 \partial_1 \varphi(x) = N; \quad (26)$$

it is a constant of motion, and it is akin to a winding number, if one interprets  $\varphi$  as an angular variable.

Note the striking similarity with the BPS Skyrme model, when one writes the energy as follows:

$$E_\varphi = \int_{-\infty}^{+\infty} dx_1 \left[ \frac{1}{2} \left( \frac{2\pi m}{\lambda} \right)^2 (j_0)^2 + \mathcal{U}(\varphi) \right]. \quad (27)$$

The static energy then also verifies a Bogomolny-like bound, since

$$E_\varphi = \int_{-\infty}^{+\infty} dx_1 \left[ \frac{1}{\sqrt{2}} \frac{d\varphi(x_1)}{dx_1} \pm \sqrt{\mathcal{U}(\varphi)} \right]^2 \mp \sqrt{2} \int_{-\infty}^{+\infty} dx_1 \frac{d\varphi(x_1)}{dx_1} \sqrt{\mathcal{U}(\varphi)}. \quad (28)$$

Thus,

$$\begin{aligned} E_\varphi &\geq \pm \sqrt{2} \int_{-\infty}^{+\infty} dx_1 \frac{d\varphi(x_1)}{dx_1} \sqrt{\mathcal{U}(\varphi)} \\ &= \pm \sqrt{2} \frac{2\pi m}{\sqrt{\lambda}} |\langle \mathcal{U} \rangle| |Q_\varphi| \\ &= 2\sqrt{2} \pi \frac{m^3}{\lambda} |Q_\varphi|, \end{aligned} \quad (29)$$

where

$$|\langle \mathcal{U} \rangle| = \frac{1}{\varphi_1 - \varphi_0} \int_{\varphi_0}^{\varphi_1} d\varphi \sqrt{\mathcal{U}(\varphi)}, \quad (30)$$

the average of  $\sqrt{\mathcal{U}(\varphi)}$  over the fundamental region.

Of course, the first-order equations that result from saturating the bound may be found by other methods; they lead to the well-known static solutions by a single quadrature. What we learn from the comparison with this model is that the particular form of the Lagrangian of the BPS Skyrme model involving the square of the topological current is what makes it produce quite powerful constraints on the solution. It is interesting to note that the kinetic term in this  $(1+1)$ -dimensional example allows for two different interpretations, either as a standard kinetic term or as the topological current squared, which is no longer true in higher dimensions. In other words, the simple sine-Gordon-type soliton model in  $1+1$  dimensions allows for two different generalizations to higher dimensions, generalizing either the standard kinetic term or the topological current, and the model studied in this paper just corresponds to the second case.

## 5. Symmetries

The Lagrangian certainly has the standard Poincaré symmetries. Besides, the sextic term is the square of the pull back of the target space volume form on  $S^3$ ,

$$dV = -i \frac{\sin^2 \xi}{(1 + |u|^2)^2} d\xi \, du \, d\bar{u}, \quad (31)$$

so this sextic term is invariant under target space diffeos which do not change this form (the volume preserving diffeos (VPDs) on  $S^3$ ). The potential only depends on  $\xi$ , so it is still invariant under those diffeomorphisms which do not change  $\xi$ , i.e. under the diffeos which obey

$$\xi \rightarrow \xi, \quad u \rightarrow \tilde{u}(u, \bar{u}, \xi), \quad (1 + |\tilde{u}|^2)^{-2} d\tilde{u} \, d\bar{\tilde{u}} = (1 + |u|^2)^{-2} d\xi \, du \, d\bar{u}.$$

The symmetries mentioned so far are symmetries of the action, i.e. Noether symmetries.

The static energy functional has some further symmetries. Indeed, it is invariant under volume preserving diffeos on the base space  $\mathbb{R}^3$ , as can be seen easily. The Bogomolny equation



has even more symmetries as we want to demonstrate now. For this purpose, we introduce the new target space coordinates

$$u = g e^{i\Phi} = \tan(\chi/2) e^{i\Phi}, \quad H(g) = \frac{1}{1+g^2},$$

(for later convenience we also introduced  $\chi$ , which together with  $\xi$  and  $\Phi$  provides the standard hyperspherical coordinates on the target  $S^3$ ), and

$$F(\xi) = \frac{\lambda}{\mu} \int d\xi \frac{\sin^2 \xi}{\sqrt{\mathcal{V}(\xi)}}, \quad (32)$$

and rewrite the Bogomolny equation as

$$\nabla F(\xi) \cdot \nabla H(g) \times \nabla \Phi = \pm 1 \quad (33)$$

or, in terms of differential forms

$$dF dH d\Phi = \pm dx^1 dx^2 dx^3 \quad (34)$$

from which it is obvious that the Bogomolny equation has as its symmetries all the VPDs both in base space and in a modified target space defined by the volume form  $dF dH d\Phi$ . The above equation implies, in fact, that all local VPDs on base space produce local solutions of the BPS equation. The problem is that, in general, a local solution cannot be extended to a global one, because of the different geometry and topology of the base space and the modified target space. The modified target space is defined by the volume form

$$dF dH d\Phi = -\frac{\lambda}{\mu} \frac{\sin^2 \xi}{\sqrt{\mathcal{V}(\xi)}} \sin \chi d\xi d\chi d\Phi \quad (35)$$

and differs from the volume form on  $S^3$  by the additional factor  $1/\sqrt{\mathcal{V}}$ . There does not exist a unique Riemannian metric giving rise to this volume form, but a natural choice which assumes that the  $S^2$  spanned by  $u$  (i.e.  $\chi$  and  $\Phi$ ) remains intact is

$$ds^2 = d\xi^2 + \frac{\sin^2 \xi}{\sqrt{\mathcal{V}(\xi)}} (d\chi^2 + \sin^2 \chi d\Phi^2). \quad (36)$$

For  $\mathcal{V} = 1$ , this is just the round metric on  $S^3$  in hyperspherical coordinates, but for nontrivial potentials the resulting target space manifold is different. Indeed, potentials which may support finite energy skyrmion solutions must have vacua  $\xi = \xi_0$  where  $\mathcal{V}(\xi_0) = 0$ , and the above metric is singular at the vacuum values  $\xi_0$ . These singularities may either be integrable (i.e. the function  $F$  defined in (32) is well defined and finite even at vacuum values  $\xi = \xi_0$ ), in which case the total volume of the modified target space is still finite. In the opposite case, the total volume is infinite. One further conclusion may be drawn immediately by integrating equation (34). If the total volume of the modified target space is finite, then any skyrmion solution of the BPS equation must have compact support (i.e. be a ‘compacton’). Further, its volume must be equal to  $|B|$  times the total volume of the modified target space, where  $B$  is the winding number. For equivalent results for the case of the BPS baby Skyrme model in one lower dimension, we refer to [20].

We remark that for  $\mathcal{V} = \sin^4 \xi$  the metric on the target space describes in fact a three-dimensional cylinder with a very simple skyrmion solution (see below).

## 6. Solutions

As already said, locally, any VPD on base space will provide a solution of the BPS equation, but this solution will, in general, not be extendible to a global, genuine one (i.e. a skyrmion),

because of the nontrivial topology one should have on the modified target space. A more promising strategy is the following: start from a simple known solution which may follow from a simple ansatz. Then, one may generate new solutions by composing the given solution with a VPD on base space  $\mathbb{R}^3$ . If the VPD is well defined on the whole of  $\mathbb{R}^3$ , then it will map genuine skyrmions into genuine skyrmions. In the case of compactons, we may even relax this condition, since it is then sufficient for the VPD on base space to be well defined in the region of the compacton.

To proceed, let us first find some simple solutions with the help of an ansatz in spherical polar coordinates

$$\xi = \xi(r), \quad \chi = \chi(\theta), \quad \Phi = n\varphi \quad (37)$$

which inserted into the BPS equation yields

$$-\frac{\lambda}{\mu} \frac{\sin^2 \xi}{\sqrt{\mathcal{V}(\xi)}} \sin \chi \, d\xi \, d\chi \, d\Phi = \mp r^2 \sin \theta \, dr \, d\theta \, d\varphi, \quad (38)$$

leading to  $\chi = \theta$  and

$$-\frac{n\lambda}{\mu} \frac{\sin^2 \xi}{\sqrt{\mathcal{V}}} d\xi = \mp r^2 dr$$

or, after the coordinate transformation

$$y = \frac{\mu}{3\sqrt{2\lambda n}} r^3 \quad (39)$$

to the autonomous ODE

$$\frac{\sin^2 \xi}{\sqrt{2\mathcal{V}(\xi)}} \xi_y = -1. \quad (40)$$

We have chosen the sign which leads to a negative  $\xi_y$ , which is compatible with the boundary conditions  $\xi(r=0) = \pi$ ,  $\xi(r=\infty) = 0$  for a potential which takes its vacuum at  $\xi_0 = 0$ .

Let us consider now the symmetries of these solutions. This issue depends on the criterion used to characterize that symmetry. Note that a given solution will not be invariant under any rotation, because it depends on the two angular coordinates  $\theta$  and  $\varphi$ . The energy density, on the other hand, depends only on the radial coordinate  $r$  and is, therefore, spherically symmetric. Note, however, that there exists another symmetry criterion, often used for solitons, whereby there is spherical symmetry when the effect of a base space rotation on a solution can be undone by a corresponding target space rotation. Under this criterion, only the solution with topological charge  $n = 1$  is spherically symmetric (i.e. all rotations can be undone). Solutions with a higher winding number  $n$  only have cylindrical symmetry, i.e. only a rotation about the  $z$  axis  $\varphi \rightarrow \varphi + \alpha$  can be undone by a target space rotation (a phase transformation  $u \rightarrow e^{-in\alpha} u$ ).

In any case, we shall call all solutions of the spherically symmetric ansatz ‘spherically symmetric solutions’ in what follows. We shall first review some general properties of these spherically symmetric solutions and, in a next step, construct solutions with lesser symmetries.

### 6.1. Solutions with spherical symmetry

Many qualitative aspects of solutions may be easily derived from the particular form of the potential, which should be contrasted with the typical situation in general Skyrme models, where similar results usually require a full three-dimensional numerical simulation.

First of all, depending on the form of the potential in the vicinity of the vacuum, one can distinguish three types of solitonic configurations: compactons (where the solution approaches its vacuum value at a strictly finite distance) and exponentially as well as power-like localized

solutions. Using the BPS equation and expanding the potential at a vacuum (e.g., at  $\xi = 0$ ),  $\mathcal{V} = \mathcal{V}_0 \xi^\alpha + \dots$ , one easily finds that for  $\alpha < 6$  one gets compactons. There is also one exponentially localized solution for  $\alpha = 6$ , while for  $\alpha > 6$  we find power-like localized solitons.

Another important feature of solutions reflects the number of vacua of the potential. It is easy to prove that for one-vacuum potentials the BPS solutions are of the nucleus type (no empty regions in the interior), while two-vacuum potentials lead to shell-like configurations.

Let us present some particular examples. For the most elaborated family of one vacuum potentials, the so-called old potentials

$$\mathcal{V}_{\text{old}} = \left( \text{Tr} \left( \frac{1-U}{2} \right) \right)^a \rightarrow \mathcal{V}(\xi) = (1 - \cos \xi)^a \quad (41)$$

(where  $a$  is a real positive parameter), we find (besides the previously known compacton) a solution with exponential tail ( $a = 3$ ) in implicit form

$$\cos \frac{\xi}{2} + \ln \tan \frac{\xi}{4} = -\frac{y}{2},$$

and power-like localized solutions. E.g., for  $a = 6$  we obtain

$$\xi = 2 \arccot \sqrt[3]{3\sqrt{2}y}.$$

A family of two-vacuum potentials is given by

$$\mathcal{V}_{\text{shell I}} = \left( \text{Tr} \left( \frac{1-U}{2} \right) \text{Tr} \left( \frac{1+U}{2} \right) \right)^a \rightarrow V(\xi) = (1 - \cos^2 \xi)^a, \quad (42)$$

which is the chiral counterpart of the so-called new baby potential. The vacua exactly coincide with the boundary values for the scalar field i.e.  $\xi = 0, \pi$ . From the BPS property of the solution, one can immediately see that the energy density should have a shell structure with two zeros: one at the center of the soliton, while the second (outer zero) can be located at a finite distance (compact shells) or approached asymptotically at infinity. Without losing generality (the potential is symmetric under the change of the vacua), we assume that  $\xi = 0$  is the outer vacuum. Of course, the inner vacuum can only be reached at a finite point as  $y \geq 0$ . This implies that only compact solitonic shells are acceptable. Specific examples of exact solutions are, for  $a = 1$

$$\xi = \begin{cases} \arccos(\sqrt{2}y - 1) & y \in [0, \sqrt{2}] \\ 0 & y \geq \sqrt{2}, \end{cases}$$

and for  $a = 2$

$$\xi = \begin{cases} \pi - \sqrt{2}y & y \in \left[0, \frac{\pi}{\sqrt{2}}\right] \\ 0 & y \geq \frac{\pi}{\sqrt{2}}. \end{cases}$$

The latter solution is, in fact, a solution for the case when the target space is a three-dimensional cylinder, as  $\sin^2 \xi / \sqrt{\mathcal{V}} = \text{const}$ .

In order to deal with non-compact shell skyrmions, we need to modify our potential in such a way that one vacuum (say, the inner vacuum at  $\xi = \pi$ ) is always approached in a compacton manner. A simple choice is

$$\mathcal{V}_{\text{shell II}} = \text{Tr} \left( \frac{1+U}{2} \right) \left( \text{Tr} \left( \frac{1-U}{2} \right) \right)^a \rightarrow V(\xi) = (1 + \cos \xi)(1 - \cos \xi)^a. \quad (43)$$

Again, we find compact shell skyrmions  $a < 3$

$$\xi = \begin{cases} \arccos \left[ 1 - \left( 2^{\frac{3-a}{2}} - \frac{3-a}{\sqrt{2}} y \right)^{\frac{2}{3-a}} \right] & y \leq \frac{\sqrt{2}}{3-a} \\ 0 & z \geq \frac{\sqrt{2}}{3-a} \end{cases}$$

an exponentially localized skyrmion for  $a = 3$

$$\xi = \xi = \arccos[1 - 2e^{-\sqrt{2}y}],$$

and shell skyrmions which extend to infinity but are localized in a power-like manner ( $a > 3$ )

$$\xi = \arccos \left[ 1 - \left( 2^{\frac{3-a}{2}} + \frac{a-3}{\sqrt{2}} y \right)^{\frac{2}{3-a}} \right].$$

## 6.2. Solutions with cylindrical symmetry

Now we assume that a spherically symmetric solution has been found, and we want to use symmetry transformations to map them to new solutions. In the first step, we construct solutions with cylindrical symmetry, using the ansatz (in cylindrical coordinates)

$$\xi = \xi(\rho, z), \quad g = g(\rho, z), \quad \Phi = n\varphi, \quad (44)$$

where  $\rho^2 = (x^1)^2 + (x^2)^2$ ,  $z = x^3$ . The Bogomolny equation for this ansatz may be written as

$$dF^{(n)} dH = \pm dq dp, \quad (45)$$

where  $F^{(n)} = nF$  and

$$q = \frac{\rho^2}{2}, \quad p = z$$

or like the Poisson bracket

$$\{F^{(n)}, H\} \equiv \frac{\partial F^{(n)}}{\partial q} \frac{\partial H}{\partial q} - \frac{\partial F^{(n)}}{\partial p} \frac{\partial H}{\partial p} = \pm 1. \quad (46)$$

Further, we know that it has the spherically symmetric solution

$$g = g_s = \tan(\theta/2) = \frac{\rho}{\sqrt{\rho^2 + z^2} + z} = \frac{\sqrt{2q}}{\sqrt{2q + p^2} + p} \equiv g_s(q, p) \quad (47)$$

and (depending on the potential)

$$\xi = \xi_s(r) = \xi_s(\sqrt{2q + p^2}) \equiv \xi_s(q, p). \quad (48)$$

As a consequence, a general solution with cylindrical symmetry may be written as

$$\xi(q, p) = \xi_s(Q(q, p), P(q, p)), \quad g(q, p) = g_s(Q(q, p), P(q, p)), \quad (49)$$

where  $(Q, P)$  are related to  $(q, p)$  via a canonical transformation, i.e.  $\{Q, P\} = 1$ .

The first class of examples is given by

$$Q = U(q), \quad P = \frac{p}{U'(q)},$$

where  $U'(q) \neq 0 \forall q$  must hold. Further, it should hold that  $\lim_{q \rightarrow 0} U(q)/q = \text{const}$  to have a well-behaved function near  $\rho = 0$ . Among these examples, the scale transformation  $Q = a^2 q, P = a^{-2} p$  can be found, which corresponds to the scale transformation  $x^1 \rightarrow ax^1$ ,  $x^2 \rightarrow ax^2$  and  $x^3 \rightarrow a^{-2}x^3$ . Another class of examples is

$$Q = \frac{q}{U'(p)}, \quad P = U(p).$$

### 6.3. Solutions with discrete symmetries

Here, we want to construct a class of base space VPDs which transform solutions with spherical or cylindrical symmetry into solutions which only preserve symmetries w.r.t. to some discrete rotations about the  $z$  axis. Concretely, we want to consider solutions which may be written as

$$\xi = \xi(\rho, z) = \xi_s(\tilde{\rho}, z), \quad g = g(\rho, z) = g_s(\tilde{\rho}, z), \quad \Phi = n\tilde{\varphi}, \quad (50)$$

where  $\xi_s, g_s, \Phi = n\varphi$  constitute a known solution with either spherical or cylindrical symmetry. That is to say, we consider base space VPDs which act nontrivially only on  $\rho$  and  $\varphi$ , where for simplicity we restrict ourselves to the following transformations:

$$\tilde{\rho} = \tilde{\rho}(\rho, \varphi), \quad \tilde{\varphi} = \tilde{\varphi}(\varphi). \quad (51)$$

Using  $q = \rho^2/2$  as before, and  $\tilde{q} = \tilde{q}(q, \varphi)$ , the condition for the transformation to be a VPD simplifies to

$$d\tilde{q}d\tilde{\varphi} = dqd\varphi. \quad (52)$$

A class of formal solutions is given by

$$\begin{aligned} \tilde{q} &= (f')^{-1}q \\ \tilde{\varphi} &= f(\varphi) \end{aligned} \quad (53)$$

in close analogy to the results of the last section. In order to define genuine diffeomorphisms, however, the transformations have to obey some further conditions. In particular, for the new coordinates  $\tilde{q}$  and  $\tilde{\varphi}$  to define polar coordinates on  $\mathbb{R}^2$  they must satisfy the boundary conditions

$$\begin{aligned} \tilde{q}(q=0, \varphi) &= 0, & \tilde{q}(q=\infty, \varphi) &= \infty, \\ \tilde{\varphi}(\varphi=0) &= 0, & \tilde{\varphi}(\varphi=2\pi) &= 2\pi. \end{aligned} \quad (54)$$

In addition, the vector field generating the flow induced by the coordinate transformation must be well defined (nonzero and nonsingular) on the whole of  $\mathbb{R}^2$ . A class of examples fulfilling all the required conditions is given by  $f = \varphi + (c/m) \sin m\varphi$ , i.e. by the class of transformations

$$\begin{aligned} \tilde{q} &= (1 + c \cos m\varphi)^{-1} q & m &\in \mathbb{N} \\ \tilde{\varphi} &= \varphi + \frac{c}{m} \sin m\varphi & c &\in \mathbb{R}, \quad |c| < 1. \end{aligned} \quad (55)$$

Clearly, if a solution  $\xi_s^{(a)}(\rho, z, \varphi)$  is invariant under rotations about the  $z$  axis (in the sense that its energy density is invariant under these rotations), then the new solution  $\xi_s^{(a)}(\tilde{\rho}, z, \tilde{\varphi})$  is invariant only under the discrete set of rotations  $\varphi \rightarrow \varphi + (2\pi/m)$ .

## 7. Summary

We explored in detail the symmetries of the static energy functional of the BPSSM, and of its related BPS equation. Then we applied these symmetries to the systematic construction of new solutions, starting from known ones. This is in the spirit of the dressing methods of classical integrability [23], which is an open problem for higher dimensional generalizations [24], an initial motivation of this work. Specifically, this allowed us to construct solutions with some prescribed symmetries, what is quite relevant to the physical problem one wants to consider. We gave concrete examples of solutions with cylindrical symmetry and with symmetries w.r.t. some discrete subgroup of the group  $SO(2)$  of rotations about the  $z$  axis. In this context, it would be interesting to construct solutions with the symmetries of platonic bodies or other discrete subgroups of the full rotation group  $SO(3)$  (crystallographic groups), because solitons

with these symmetries frequently show up as true minimizers of the energy in the original Skyrme model or some of its generalizations [14–16]. The corresponding volume-preserving diffeomorphisms producing solutions with these symmetries will be more complicated than the ones constructed in this paper, and it almost certainly will be more difficult to find them.

This issue is currently under investigation.

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